ON AMBIENTAL COBORDISM

Carlos Biasi

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INTRODUCTION

Let us consider M^m a closed submanifold of N^n. In this paper, we study the possibility of the existence of a submanifold $W^{m+1} \subset N^n$ such that $\partial W = N$. If $M = S^m$ and $N = S^{m+2}$, such a submanifold W is called a Seifert surface of knot S^m. In [5], Sato showed that every connected closed and oriented submanifold M^m of S^{m+2} is a boundary of an oriented surface of S^{m+2}.

In [4], Hirsch studies the following problem: "If a compact connected and oriented manifold M^m bounds, exists there an embedding from M^m into $\mathbb{R}^n$ which is a boundary on $\mathbb{R}^n$? ".

The answer is yes, if $n \geq 2m$.

The difference between the two problems is that in our case, the embedding from M into N is fixed.

Let us present a necessary condition to the existence of W, when M and N are oriented manifolds.

Let $\Omega_m(N)$ be the m-th oriented bordism group of N [2]. If $i : M \to N$ is the inclusion map, we can define element $[M,i]$ in $\Omega_m(N)$ and see that $[M,i] = 0$ if M bounds in N.

Generally, the converse is not true, but sometimes the vanishment of $[M,i]$ guarantees the existence of W, for example if the codimension $n-m$ is large.

We prove the following theorem.

THEOREM 5.2. *Let us suppose that* $M^m \subset N^n$, $n > m+1$, *is such that* $[M,i] = 0$ *in* $\Omega_m(N)$ *Then* M *bounds in* N *if one of the following conditions occurs:*

a) $n = m+2$

b) $m \leqq 3$

c) $m \leqq 4$ *and* $n \neq 7$.

In his Doctor thesis [1] the author proved that, when $n = 2m+1$, a submanifold $M \subset N$, where M and N are closed oriented manifolds, bounds in N if, and only if, $[M,i] = 0 \in \Omega_m(N)$.

§1. *A more general problem of ambiental bordism*

Let $G \subset O(n-m-1)$, $n > m+1$, be a closed transformation group and let $\gamma_G \to BG$ be the classifying fiber bundle of $(n-m-1)$ - vector bundles which have a G-structure.

Let us consider MG the Thom space of γ_G . We have:

$$\pi_i(MG) = \begin{cases} 0 , & i < n-m-1 \\ \mathbb{Z} , & i = n-m-1 \text{ and } G \subset SO(n-m-1) \\ \mathbb{Z}_2 , & i = n-m-1 \text{ and } G \not\subset SO(n-m-1) \end{cases}$$

Let us consider now N^n to be a closed connected manifold which we assume to be oriented if $G \subset SO(n-m-1)$

(If $G \not\subset SO(n-m-1)$ we drop the orientability hypothesis).

Let $M^m \subset N^n$ be a closed submanifold and let us suppose that the normal fiber bundle ν_M of M in N has a cross section s, nowhere zero, such that $\nu_M = \{s\} \oplus \xi$, where $\{s\}$ is a subbundle generated by s and ξ is a $(n-m-1)$ - vector bundle endowed with a G-structure.

We would say that a submanifold $W \subset N$ satisfies a condition ($\#$) if it has the properties:

" $\partial W = M$ and s is a cross-section inward to W, which is such that its normal fiber bundle ν_W has a G-structure which restricts to M, agreeing with the G-structure of ξ ".

(Observe that $\xi = \nu_W|M$).

§2. *Primary Obstruction to the existence of* W.

Let V be a closed tubular neighborhood of M in N, $A = \partial W$ and $X = N - \overset{\circ}{V}$. We can think s as an application $s : M \to A$. Then $s(M)$ is a submanifold of A, whose normal fiber bundle is isomorphic to ξ. By Thom's construction there exists an application $f : A \to MG$ such that, if ∞ is the point at infinity to MG, then f is differentiable on $A - f^{-1}(\infty)$, f is transversal to BG and $f^{-1}(BG) = s(M)$ [6].

We shall take $\pi_{n-m-1}(MG)$ as the cohomology coefficient group. Let $e \in H^{n-m-1}(MG)$ be the fundamental class of space MG. We know that $f^*(e) = \alpha$, where α is the dual class of $s_*(\mu_M)$ and μ_M is the fundamental class of M .

If $f : A \to MG$ extends to a map $\overline{f} : X \to M$, then we can suppose, up to homotopy, that $\overline{f}$ is differentiable in $X - \overline{f}^{-1}(\infty)$ and $\overline{f}$ transversal to BG ; then taking $W_1 = \overline{f}^{-1}(BG)$ we obtain a submanifold of X whose boundary is $s(M)$.

Let us observe that this submanifold can be extended to a submanifold W which satisfies condition (#) .

We conclude then that there exists W , satisfying (#), if and only if f extends to X .

The class $\delta f^*(e)$, where $\delta : H^{n-m-1}(A) \to H^{n-m}(X,A)$ is the obstruction to the extension of f to the $(n-m)$-skeleton of X .

Considering the commutative diagram:

$$\begin{array}{ccc} H^{n-m-1}(A) & \xrightarrow{\ \delta\ } & H^{n-m}(X,A) \\ \big\downarrow D & & \big\downarrow D \\ H_m(A) & \xrightarrow{\ s_*\ } & H_m(X) \approx H_m(N-M) \end{array}$$

We conclude that the primary obstruction to the extension of f , up to duality, is the element $s_*(\mu_M) \in H_m(N-M)$, taking s as an application from M into $N-M$.

Hence, we have:

PROPOSITION 2.1. *f extends to the $(n-m)$-skeleton of X if, and only if, $s_*(\mu_M) = 0$ in $H_m(N-M)$.*

□

Assuming that $s_*(\mu_M) = 0$, let us consider two cases:

1. $G = O(n-m-1)$.

Here, f extends up to the $(n-m+1)$-skeleton of X , because $\pi_{n-m}(MG) = 0$ and, if $n-m = 2$, then f extends ($MO(1)$ is a $K(\mathbb{Z}_2,1)$ space) .

2. $G = SO(n-m-1)$.

Since $\pi_{n-m+i}(MG) = 0$, $i = 0,1,2$, f extends up to the $(n-m+3)$ - skeleton of X . Hence, if $\dim M \leq 3$, f extends.

On the other hand, if $n-m = 2$ or 3 then MG is a $K(\mathbb{Z},1)$ or $K(\mathbb{Z},2)$, respectively. In any case , f extends globally.

§3. *The oriented ambiental bordism*

From now on, all manifolds and submanifolds will be considered to be oriented.

THEOREM 3.1. *Let us suppose that:*

a) $H_j(X) = 0$, $0 < j < m-3$

b) *The cannonical homomorphism* $\pi_{n-1}(MSO(n-m-1)) \xrightarrow{\psi} \Omega_m$ *is injective.*

There exists W *satisfying* ($\#$) *if , and only if ,* $s_*(\mu_M) = 0 \in H_m(X)$ *and* M *is a boundary.*

Proof. Let us use the notation $\pi_i = \pi_i(MSO(n-m-1))$. If $s_*(\mu_M) = 0$, then f extends to the $(n-m)$-skeleton of X .

From hypothesis *a)* and the Lefschetz duality, we conclude that $H^j(X,A,\pi_{j-1}) = 0$, $n-m < j < n$.

Let D be an open disk of $X-A$. Since X is orientable, $H^j(X-D,A,\pi_{j-1}) \cong H^j(X,A,\pi_{j-1}) = 0$, $n-m < j < n$. Hence, there exists an extension $\overline{f} : X-D \to Y$ of $f : A \to Y$, where $Y = MSO(n-m-1)$.

Let us consider $S = \partial D$ and $h = \overline{f} \mid \partial D : S \to Y$. We can suppose that h is transversal to $BSO(n-m-1)$ and let $M^m = h^{-1}(BSO(n-m-1))$.

Considering $\overline{W} = \overline{f}^{-1}(BSO(n-m-1))$ we have a bordism between M_1 and $s(M)$. Since $s(M)$ is a boundary, M_1 also is.

We have also that $\psi([h]) = [M_1] = 0$ and since ψ is a monomorphism, h is homotopic to a constant map and so h extends over D .

The converse is straighforward.

□

§4. *On existence of normal vector fields homologous to zero in* $N-M$.

In the next paragraph, we are showing that in certain situations it is possible to obtain a cross-section $s : M \to S(\nu_M)$ such that $s_*(\mu_M) = 0 \in H_m(N-M)$, where $S(\nu_M) \to M$ is the sphere normal bundle of M in N .

PROPOSITION 4.1. *The Euler class of the normal bundle of* M^m *in* N^n *is zero if and only if* $i_*(\mu_M) \subset im\ j_*$, *where* μ_M *is the fundamental class of* M *and* $i : M \to N$,

$j : N - M \to N$ are inclusion maps.

Proof. Let us consider $e \in H^{n-m}(M,\mathbb{Z})$ the Euler class of normal bundle ν_M and let $D_A : H^{n-m}(M;\mathbb{Z}) \to H_m(N,N-M;\mathbb{Z})$ be the Alexander duality. We have that $D_A(e) = \alpha_*(\mu_M)$ where α_* is induced by the inclusion map $\alpha : M \to (N,N-M)$.

Using the exact sequence of pair $(N,N-M)$ it follows that $\alpha_*(\mu_M) = 0$ if, and only if, $i_*(\mu_M) \subset im\, j_*$.

□

COROLLARY 4.2. *If $M^m \subset N^n$ is homologous to zero, $n-m = 2$ or $n \geq 2m$, then M has a normal vector field that is nowhere zero.*

□

Let $E \xrightarrow{\pi} B$ be a fiber bundle whose fibration is oriented, where the base space B is a CW complex and the fiber F is $(n-1)$-connected.

Let us suppose that there exists a cross-section $s : B^{n+1} \to E$ and let $i_n \in H^n(F,\pi_n(F))$ be a n-characteristic element.

Since $E \xrightarrow{\pi} B$ is oriented and $i^* : H^n(E,\pi_n(F)) \to H^n(F,\pi_n(F))$ is onto, there exists θ_s such that $i^*(\theta_s) = i_n$ and $s^*(\theta_s) = 0$.

Let us consider now a differentiable $SO(n+1)$ - fiber bundle with fiber S^n and base M^m (oriented manifold).

If $s : M \to E$ is a cross-section, then θ_s is the Poincaré dual to $\bar{s}_*(\mu_M)$, where $\bar{s} = -s$ is the opposite cross-section of s.

Having fixed a cross-section $s_0 : M \to E$, the following diagrams are commutative:

$$
\begin{array}{ccccc}
[M,E] & \xrightarrow{\quad\Psi\quad} & H^n(E) & & \\
\downarrow \xi & \nearrow & & & \\
H^n(M) & \xrightarrow{\quad\pi^*\quad} & H^n(E) & & \\
\downarrow D & & \downarrow D & & \\
H_{m-n}(M) & \xrightarrow{\quad\Delta\quad} & H_m(E) & \xrightarrow{\quad\pi_*\quad} & H_m(M)
\end{array}
$$

where $[M,E]$ is the set made up of cross-sections homotopy classes; $\xi([s]) = \bar{s}^*(\theta_{\bar{s}_0})$; $\Psi([s]) = \theta_{\bar{s}_0} - \theta_{\bar{s}}$; D is the Poincarē duality and the last line is a stretch of a Gysin generalized sequence.

We define $\psi : [M,E] \to H_m(E)$ by $\psi([s]) = s_{0_*}(\mu_M) - s_*(\mu_M)$ and let us observe that $\psi = D \circ \Psi$.

If $m \leq n+1$ or $n = 1$, then the application ξ is onto and so the image of ψ is the kernel of π_* .

This fact will be applied in the proof of proposition 4.3 below, where the fiber bundle to be considered is $S(\nu_M) \to M$.

PROPOSITION 4.3. *Let* $M^m \subset N^n$, $n = m+2$ *or* $n \geq 2m$, *be an oriented submanifold homologous to zero in an oriented manifold* N . *Then there exists a cross-section* $r : M \to S(\nu_M)$ *such that its image is homologous to zero in* $H_m(N-M)$.

Proof. Let $s_0 : M \to S(\nu_M)$ be a cross-section that is nowhere zero (Cor. 4.2) and let us consider the com mutative diagrams:

$$\begin{array}{ccccc} & \overset{s_{0*}}{\nearrow} & H_m(S(\nu_M)) & \xrightarrow{\pi_*} & H_m(M) \\ H_m(M) & & \downarrow \ell_* & & \downarrow i_* \\ & \underset{s_*}{\searrow} & H_m(N-M) & \xrightarrow{j_*} & H_m(N) \end{array}$$

where $s_* = \ell_* \, s_{0*}$ and ℓ_* is induced by the inclusion $S(\nu_M) \to (N-M)$.

We have that $j_* \, s_*(\mu_M) = i_* \, \pi_* \, s_{0*}(\mu_M) = 0$ which implies that $s_*(\mu_M)$ belongs to the kernel of j_* which is the image of $\partial : H_{m+1}(N, N-M) \to H_m(N-M)$.

Let us consider the following commutative diagram:

$$\begin{array}{ccc} H_{m+1}(D(\nu_M), S(\nu_M)) & \xrightarrow{\partial} & H_m(S(\nu_M)) \\ \downarrow exc & & \downarrow j_* \\ H_{m+1}(N, N-M) & \xrightarrow{\partial} & H_m(N-M) \end{array}$$

It follows then that there exists an element $\mu \in H_m(S(\nu_M))$ such that $\mu \in ker\, \pi_*$ and $j_*(\mu) = s_*(\mu_M)$.

Since $im\, \psi = ker\, \pi_*$, there exists a cross - section $r : M \to S(\nu_M)$ such that $\psi([r]) = \mu$.

But $\psi([r]) = s_{0*}(\mu_M) - r_*(\mu_M)$ and so $j_* \, r_*(\mu_M) = 0$ in $H_m(N-M)$. Hence, the image of $r : M \to S(\nu_M)$ is homologous to zero in $N - M$.

□

§5. *A Theorem on ambiental bordism*

Let us consider $\Omega_j(N)$ to be the j-th oriented bordism group of N .

If $H_j(N) = 0$, $0 < j < m-3$, it is possible to show that, using spectral bordism sequence ([2]), the application $\Omega_m(N) \to H_m(N) \oplus \Omega_m$, which associates to each pair $[M,f]$ the element $\mu([M,f]) + [M]$, is an isomorphism, where μ is the cannonical homomorphism.

In the proof of Theorem 5.2, we are going to use the following lemma, which has been proved in [1] (the proof, if $q > m$, is due to Thom [6]) .

LEMMA 5.1. *The homomorphism* $\psi : \Omega_{q+m}(MSO(q)) \to \Omega_m$, $q \geq m$ *is an isomorphism.*

□

THEOREM 5.2. *Let us suppose* $M^m \subset N^n$, $n > m+1$ *is such that* $[M,i] = 0$ *in* $\Omega_m(N)$. *Then* M *bounds in* N *if one of the following conditions occurs:*

a) $n = m+2$

b) $m \leq 3$

c) $m \leq 4$ *and* $n \neq 7$.

Proof. Any one of the conditions *a)* , *b)* and *c)* , based on previous results, imply that normal bundle ν_M has a cross-section nowhere zero so that, if we consider s as an application from M into $N-M$, then $s_*(\mu_M) = 0 \in H_m(N-M)$.

If *a)* or *b)* occurs, the theorem results from case 2 , already discussed in paragraph 2 .

If $m = 4$ and $n \geq 8$, we apply Theorem 3.1.

Remark 1. If $n = m+2$ or $m \leq 3$, then $[M,i] = 0 \in \Omega_m(N)$ if, and only if, M is homologous to zero in N.

Remark 2. When $m = 4$ and $n = 7$, although we have proved that $[M,i] = 0$ implies the existence of a normal section nowhere zero (Theo 5.3) we cannot achieve to prove that there exists a normal vector field homologous to zero in $N - M$, which in this case should be sufficient to prove Theorem 5.2.

THEOREM 5.3. *Let us suppose* $M^4 \subset N^7$. *If* $[M,i] = 0$ *in* $\Omega_4(N)$ *then* ν_M *has a cross-section which is nowhere zero.*

Proof. There exists $W \subset N \times I$ such that $\partial W = M \times 0 \subset N \times I$ [1].

Let ν_W and ν_M be the normal fiber bundles of W in $N \times I$ and of M in N, respectively. We can also suppose that $\nu_W | M \times 0 = \nu_M$.

Let us consider $\overline{W} \subset N \times \mathbb{R}$ to be the double of W and let $i : \overline{W} \to N \times \mathbb{R}$ and $j : N \times \mathbb{R} \to \overline{W} \to N \times \mathbb{R}$ be inclusion maps.

Since $i_*(\mu_{\overline{W}}) \subset im\, j_*$, then $\overline{W}$ has a normal vector field which is nowhere zero in $N \times \mathbb{R}$ up to the 3-skeleton of $\overline{W}$.

Hence, there exists a dimensional - two - oriented fiber bundle ξ over M such that $\nu_M|M^{(3)} = \xi \oplus \varepsilon^1$.

Let us consider e to be the Euler class of ξ in $H^2(M^{(3)})$ and let $\bar{e} \in H^2(M)$ such that $i^*(\bar{e}) = e$, where $i : M^{(3)} \to M$ is the inclusion map.

Let $\bar{\xi}$ be a dimensional two fiber bundle over M such that its Euler class is $\bar{e}$. Let us observe that $\bar{\xi} \mid M^{(3)} = \xi$.

Let $f,g : M \to BSO(3)$ be classifying maps of $\bar{\xi} \oplus \varepsilon^1$ and ν_M , respectively.

Since the Euler classes of $\bar{\xi} \oplus \varepsilon^1$ and of ν_M are equal, then their second Stiefel-Whitney classes are equal.

Let us consider $\tilde{p}_1$ to be the Pontryagin class of classifying fiber bundle $\tilde{\gamma} \longrightarrow BSO(3)$ and let $\tilde{e}$ be the Euler class of $\tilde{\gamma}$. Since $f^*(\tilde{e}) = g^*(\tilde{e})$ and $\tilde{e} \cup \tilde{e} = \tilde{p}_1$, we have $f^*(\tilde{p}_1) = g^*(\tilde{p}_1)$. Hence, the fiber bundles $\xi \oplus \varepsilon^1$ and ν_M are equivalent [3] .

□

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ICMSC - USP
Departamento de Matemātica
Rua Dr. Carlos Botelho, 1465
Caixa Postal nº 668
13.560 - São Carlos - SP. - Brasil