SOME APPLICATIONS OF OBSTRUCTION THEORY

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INTRODUCTION

Two classic problems are studied in Topology: Extension of Functions and Homotopy.

In this paper these two problems are analysed using Obstruction Theory under a non conventional viewpoint and some generalizations of classical results are obtained.

Some results on the classification of homotopy classes of functions are obtained and as an application of this theory we prove a result of Adachi [1], about equivalence of vector bundles.

Consider the following problem: Given two embeddings $f,g : M^m \to N^n$ (M^m and N^n closed orientable C^∞ manifolds), $m \leq \frac{n-2}{2}$, when are f and g isotopic up to surgery on N ? Being more precise, knowing that the normal bundles ν_f and ν_g are equivalent as SO-bundles when are f and g isotopic up to surgery on N ? In [2] an afirmative answer was obtained when $M = S^m$ or N is a π-manifold.

Here we prove the following:

Theorem 3.6 :

If the closed orientable manifolds M^m and N^n, $m \leq \frac{n-2}{2}$, satisfies:

a) $H^{4k}(M,Z)$ are free groups, $k = 1,2,\ldots$

b) $H^{8k+1}(M,Z_2) = H^{8k+2}(M,Z_2) = 0$, $k = 1,2,\ldots$

then two embeddings f, g are isotopic up to surgery on N if and only if the normal bundles ν_f and ν_g are SO-equivalent.

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1. EXTENSION OF FUNCTIONS

Let (X,A) be a CW pair and Y a n-simple CW complex, $n \geq 1$. For an abelian group G, let $K = K(G,n)$ be an Eilenberg - MacLane space and $i_n \in H^n(K,G)$ a n-characteristic element.

Given $u \in H^n(Y,G)$ there exists a map $\psi : Y \to K$ such that $\psi^*(i_n) = u$. The induced homomorphism $\psi_* : \pi_n(Y) \to \pi_n(K)$ gives:

$$\psi_u^p : H^p(X,A,\pi_n(Y)) \longrightarrow H^p(X,A,\pi_n(K)), \text{ for all } p.$$

Let $f_A : A \to Y$ be a map, $X^{(p)}$ the p-skeleton of X and $X_p = X^{(p)} \cup A$.

Theorem 1.1 :

If ψ_u^{n+1} is a monomorphism and $f_n : X_n \to Y$ is an extension of f_A, then there exists $g : X_{n+1} \longrightarrow Y$ such that

$g|X_{n-1} = f|X_{n-1}$ if and only if $\delta f_A^*(u) = 0$, where δ is the coboundary operator.

Proof :

Let $\sigma(f) \in H^{n+1}(X,A,\pi_n(Y))$ be the obstruction to the extension of $f_n|X_{n-1}$ to X_{n+1} and let us observe that $\varphi_u^{n+1}(\sigma(f_n)) = \sigma(\varphi \circ f_n) = \sigma(\varphi \circ f_A)$ and that $\delta f_A^*(u) = \delta f_A^*(\varphi^*(i_n)) =$
$= -\sigma(\varphi \circ f_A)$.

Since φ_u^{n+1} is a monomorphism, then $\sigma(f_n) = 0$ if and only if $\delta f_A^*(u) = 0$.

□

Theorem 1.1 is a generalization of the known result: "If Y is $(n-1)$-connected ($\pi_1(Y)$ abelian if $n = 1$) then $f_A: A \to Y$ extends to X_{n+1} if and only if $\delta f_A^*(i_n) = 0$".

For each $n \geq 1$, let us consider abelian groups G_n and elements $u_n \in H^n(Y,G_n)$. We assume that Y is n-simple, for $n \leq \dim(X-A) - 1$ and let $J = \{n : H^{n+1}(X,A,\pi_n(Y)) \neq 0\}$.

Theorem 1.2 :

If $\varphi_{u_n}^{n+1}$ is a monomorphism for all n, then $f_A: A \to Y$ extends to X if and only if $\delta f_A^*(u_n) = 0$, for all $n \in J$.

The proof of this theorem follows by iterative applications of Theorem 1.1.

□

As an application of Theorem 1.2 let Y and J be as above and for each p, let $\chi_n^p : H^p(X,A,\pi_n(Y)) \to H^p(X,A,H_n(Y))$ be the coefficient change induced by the Hurewicz homomorphism.

Theorem 1.3 :

If $\mathcal{H}_{u_n}^{n+1}$ is a monomorphism and $H_{n-1}(A)$ is free for all $n \in J$ then $f_A : A \to Y$ extends to X if and only if for each $n \in J$, there exists a homomorphism $\psi : H_n(X) \to H_n(Y)$ such that $f_A^* = \psi \circ i_*$, where $i_* : H_n(A) \to H_n(X)$ is the induced homomorphism of the inclusion map $i : A \to X$.

Proof:

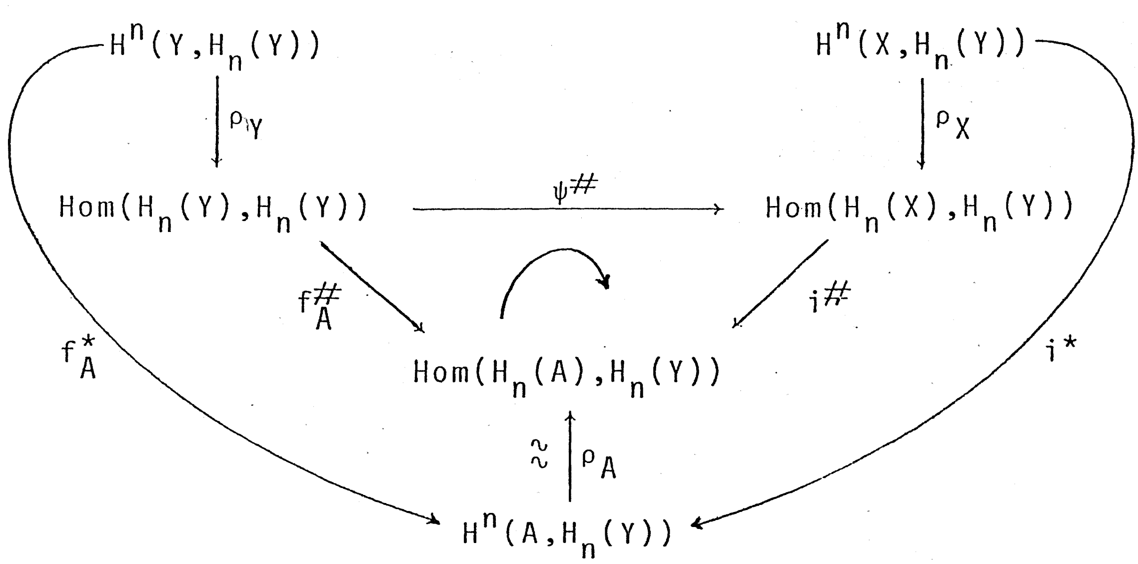
Let $u_n \in H^n(Y,H_n(Y))$ be an element satisfying $\rho_X(u_n)(x) = \langle u_n, x\rangle = x$ for all $x \in H_n(Y)$, where $\rho_Y : H^n(Y,H_n(Y)) \to \mathrm{Hom}(H_n(Y), H_n(Y))$ is the evaluation homomorphism and let $K = K(H_n(Y),n)$ be an Eilenberg Mac Lane space.

The map $\varphi : Y \to K$ such that $\varphi^*(i_n) = u_n$ induces the Hurewicz homomorphism $\mathcal{H}_Y : \pi_n(Y) \to \pi_n(K) = H_n(Y)$.

If $f_A : A \to Y$ extends to X, it follows from Theorem 1.2 the existence of $\alpha \in H^n(X,H_n(Y))$ satisfying $i^*(\alpha) = f_A^*(u_n)$ for all $n \in J$.

Then we define $\psi = \rho_X(\alpha)$.

Conversely let us consider the following commutative diagram:



and let $\alpha \in H^n(X,H_n(Y))$ be such that $\rho_X(\alpha) = \psi^{\#}\rho_Y(u_n)$. It follows then that $f_A^*(u_n) = i^*(\alpha)$ and so $\delta f_A^*(u_n) = 0$.

□

2. *HOMOTOPY OF FUNCTIONS*

Let X and Y be two CW complexes, with Y n-simple for every integer $n \geq 1$ and let $\psi_u^p : H^p(X,\pi_n(Y)) \to H^p(X,\pi_n(K))$ be as defined before.

Theorem 2.1 :

Let $f,g : X^{(n+1)} \to Y$ be homotopic maps over the (n-1)-skeleton of X satisfying $f^*(u) = g^*(u)$. If ψ_u^n is a monomorphism, then f and g are homotopic maps on the n-skeleton of X .

Proof:

Let $H : X^{(n-1)} \times I \to Y$ be a homotopy between f_{n-1} and g_{n-1} and $d(f_n,H,g_n) \in H^n(X,\pi_n(Y))$, the obstruction to the extension of H to a homotopy between f_n and g_n .

Since $\psi_u^n(d(f_n,H,g_n)) = d(\psi \circ f_n, \psi \circ H, \psi \circ g_n)$ and $f^*(u) = g^*(u)$, then $(\psi \circ f)^*(i_n) = (\psi \circ g)^*(i_n)$. Hence, $d(f_n,H,g_n) = 0$ and so the result follows.

□

Theorem 2.2 :

Let $f : X^{(n+1)} \to Y$ be a map and $\alpha \in H^n(X^{(n+1)},G)$. There exists $g : X^{(n+1)} \to Y$ such that $g^*(u) = \alpha$ and $g|X^{(n-1)} = f|X^{(n-1)}$ if and only if $\alpha - f^*(u)$ belongs to the image of ψ_u^n .

Proof:

Let $c : X^{(n+1)} \to Y$ be a constant functions. If there exists $g : X^{(n+1)} \to Y$ satisfying the condition of Theorem 2.2, then $\psi_u^n(d(f_n,g_n)) = d(c,\psi \circ g_n) - d(c,\psi \circ f_n) = \alpha - f^*(u)$.

Conversely if $\alpha - f^*(u)$ belongs to the image of ψ_u^n, let $\beta \in H^n(X,\pi_n(Y))$ be such that $\psi_u^n(\beta) = \alpha - f^*(u)$.

It follows then the existence of $g : X^{(n+1)} \to Y$ such that $d(f_n,g_n) = \beta$ and $g^*(u) = \psi_u^n(d(f_n,g_n)) + d(c,\psi \circ f_n) = \alpha$.

□

Theorem 2.3 :

Let us consider $f : X^{(n)} \to Y$ and $\alpha \in H^n(X^{(n+1)},G)$. If ψ_u^n is an epimorphism and ψ_u^{n+1} is a monomorphism, then there exists $g : X^{(n+1)} \to Y$ such that $g|X^{(n-1)} = f|X^{(n-1)}$ and $g^*(u) = \alpha$.

Proof:

Let $o(f_n) \in H^{n+1}(X,\pi_n(Y))$ be the obstruction to the extension of f_{n-1} to $(n+1)$-skeleton. Since $\psi_u^{n+1}(o(f_n)) = o(\psi \circ f_n) = 0$ and ψ_u^{n+1} is a monomorphism we have that $o(f_n) = 0$. Then there exists an extension of f_{n-1} to $X^{(n+1)}$. Now applying Theorem 2.2 the proof is finished.

□

Proposition 2.4 :

Assuming that Y is n-simple for all $n \leq \dim X$; $H^i(X,\pi_i(Y)) = 0$, for $i \neq n$; $H^{i+1}(X,\pi_i(Y)) = 0$ for $i > n$ and ψ_u^n is a monomorphism, then the map $E : [X,Y] \to H^n(X,G)$ defined by $E([f]) = f^*(u)$ is injective and $\operatorname{im} E = \operatorname{im} \psi_n^u \approx H^n(X,\pi_n(Y))$.

Proof:

Since $H^i(X,\pi_i(Y)) = 0$, $i < n$, any two maps $f,g : X \to Y$ are homotopic over the $(n-1)$-skeleton. If $f^*(u) = g^*(u)$ and ψ_u^n is a monomorphism it follows from theorem 2.1 that f and g are homotopic over the n-skeleton. Since $H^i(X,\pi_i(Y)) = 0$, $i > n$, then f and g are homotopic and so E is injective.

Given $f : X \to Y$ then f is homotopic to a constant map over the $(n-1)$-skeleton. It follows from Theorem 2.2 that $f^*(u)$ belongs to the image of ψ_u^n .

If $\alpha \in H^n(X,G) = H^n(X^{(n+1)},G)$ from Theorem 2.2 there exists $h : X^{(n+1)} \to Y$ such that $h^*(u) = \alpha$ and h is constant over the $(n-1)$-skeleton. Since $H^{i+1}(X,\pi_i(Y)) = 0$, $i > n$, there exists $f : X \to Y$ with $f^*(u) = \alpha$. Hence $\operatorname{im} E = \operatorname{im} \psi_u^n$.

□

For each $n \geq 1$, let G_n be an abelian group and u_n an element in $H^n(Y,G_n)$. We assume that Y is n - simple , $n \leq \dim X$ and we write $J = \{ n : H^n(X,\pi_n(Y)) \neq 0 \}$.

Theorem 2.5 :

If $\psi_{u_n}^n$ is a monomorphism for all $n \in J$, then the function $E : [X,Y] \to \prod_{n \in J} H^n(X,G_n)$ given by $E([f]) = \prod_{n \in J} f^*(u_n)$ is injective (Π denote the cartesian product) .

Proof:

Let p be the least positive integer such that $H^p(X,\pi_p(Y)) \neq 0$.

Since $H^i(X,\pi_i(Y)) = 0$, $i < p$, any two functions $f,g : X \to Y$ are homotopic over $X^{(p-1)}$.

Let us suppose that $f^*(u_p) = g^*(u_p)$. Since by hypothesis

φ_u^p is a monomorphism it follows from Theorem 2.1 that f and g are homotopic over the p-skeleton of X and so applying successively Theorem 2.1 the result follows.

□

Now we present an interesting consequence of Theorem 2.5 .

Let $\varkappa_Y : \pi_n(Y) \to H_n(Y)$ be the Hurewicz homomorphism and $\varkappa_n^p : H^p(X,\pi_n(Y)) \to H^p(X,H_n(Y))$ the change of coefficients.

Let $u_n \in H^n(Y,H_n(Y))$ be an element such that $\langle u_n, x\rangle = x$, for all $x \in H_n(Y)$.

Theorem 2.6 :

If $\varkappa_n^n$ is a monomorphism for all $n \in J$ then the function E is injective.

Proof:

Denoting $K = K(H_n(Y),G)$ and observing that the commutative diagram:

$$\begin{array}{ccc} \pi_n(Y) & \xrightarrow{\ \varkappa_Y\ } & H_n(Y) \\ \varphi_* \downarrow & \curvearrowright & \downarrow \varphi_* \\ \pi_n(K) & \xrightarrow{\ \varkappa_K\ } & H_n(K) \end{array}$$

induces for all n the commutative diagram:

$$\begin{array}{ccc} H^n(X,\pi_n(Y)) & \xrightarrow{\ \varkappa_n^n\ } & H^n(X,H_n(Y)) \\ \varphi_{u_n}^n \downarrow & \curvearrowright & \downarrow Id^n \\ H^n(X,\pi_n(K)) & \xrightarrow{\ \varkappa_n^n\ } & H^n(X,H_n(K)) \end{array}$$

Now applying Theorem 2.6 the result follows.

□

Corollary 2.7 :

Let us suppose that $\mathcal{H}_n$ is a monomorphism for all n and $H_{n-1}(X)$ a free group if $n \in J$. Then two maps $f,g : X \to Y$ are homotopic if and only if for each $n \in J$ the induced homomorphism $f_*, g_* : H_n(X) \to H_n(Y)$ are equal.

Remark :

If Y is (n-1)-connected ($\pi_1(Y)$ abelian) and $u \in H^n(Y,G)$ is the element corresponding to the inverse of Hurewicz homomorphism $\mathcal{H}_Y : \pi_n(Y) \to H_n(Y)$ then $\psi_* : \pi_n(Y) \to \pi_n(K)$ is the identity map, where $\psi : Y \to K$ is such that $\psi^*(i_n) = u$ and $G = \pi_n(Y)$.

It follows that ψ_u^p is an isomorphism for all X and p . This is true if u is any n-characteristic element.

3. *APPLICATIONS*

Given an O-stable vector bundle $\xi \to B$ where O is the stable orthogonal group let $w_i(\xi) \in H^i(B,Z_2)$ and $p_k(\xi) \in H^{4k}(B,Z)$ be respectively the Stiefel - Whitney and Pontrjagin classes of ξ .

Let $\gamma \to BO$ be the O-universal bundle and let $w_i = w_i(\gamma)$ and $p_k = p_k(\gamma)$ be respectively the universal Stiefel - Whitney and Pontrjagin classes.

Let us denote by $\pi_n = \pi_n(BO)$; $K_n = K(\pi_n, n)$ and $i_n \in H^n(K_n, \pi_n)$ the element corresponding to the inverse of Hurewicz homomorphism.

Let us consider also the maps $\psi_n : BO \to K_n$, $n = 1,2,\ldots$ satisfying

$$\psi_n^*(i_n) = \begin{cases} w_n , & n = 1,2 \\ P_k & n = 4k , \quad k \geq 1 \end{cases}$$

and $\psi_n^* : \pi_n(BO) \to \pi_n(K_n)$ the homomorphism induced by ψ_n that induces the change of coefficients.

$$\psi^n : H^n(X,\pi_n(BO) \longrightarrow H^n(X,\pi_n) \quad \text{for any } X .$$

LEMMA 3.1 :

Let X be a space satisfying:

a) If $x \in H^{4k}(X,Z)$, $k = 1,2,\ldots$, $x \neq 0$, then $(2k-1)!\, a_k x \neq 0$, where

$$a_k = \begin{cases} 1 , & k \text{ even} \\ 2 , & k \text{ odd} \end{cases}$$

b) $H^{8j+1}(X,Z_2) = H^{8j+2}(X,Z_2) = 0$, $j = 1,2,\ldots$

Under these conditions the changes of coefficients ψ^n are monomorphisms for $n = 1,2,4,8,\ldots 4k$

Proof:

For $n = 1,2,\ldots$ we have $\pi_n(BO) = Z_2$ and the change of coefficients are isomorphisms.

For $n = 4k$, $k = 1,2,\ldots$ the homomorphism $\psi : \pi_n(BO) \to Z$ is given by $\psi(\alpha) = (2k-1)!\, a_k$, where α is the generator of $\pi_n(BO)$. [1] .

Considering the long exact sequence:

$$\longrightarrow H^{n-1}(X,H) \xrightarrow{\beta} H^n(X,Z) \xrightarrow{\psi^n} H^n(X,Z) \longrightarrow H^n(X,H) \longrightarrow$$

where $H = Z/_{\mathrm{im}\,\psi}$ and β is the Bockstein operator, we see that there is no nonzero elements of $H^n(M,Z)$ belonging to the image of β ; because if there is such an element we would have $(2k-1)!\, a_k x = 0$.

Then ψ^n , $n = 4k$, $k = 1,2,\ldots$ are monomorphisms.

□

Theorem 3.2 (Adachi) :

Let X be a CW complex satisfying conditions a and b of Lemma 3.1 and let ξ_1 and ξ_2 be two 0 - stable vector bundles over X . Then ξ_1 and ξ_2 are equivalent if and only if $w_1(\xi_1) = w_1(\xi_2)$; $w_2(\xi_1) = w_2(\xi_2)$ and $p_k(\xi_1) = p_k(\xi_2)$, $k = 1,2,\ldots$

Proof:

Let $f_1, f_2 : X \to BO$ be the classifying maps for ξ_1 and ξ_2 .

Observing that $f_1^*(w_i) = w_i(\xi_1) = w_i(\xi_2) = f_2(w_i)$, $i = 1,2,$

$$f_1^*(p_k) = p_k(\xi_1) = p_k(\xi_2) = f_2^*(p_k) \quad , \quad k = 1,2,\ldots$$

and since $H^n(X,\pi_n) = 0$, $n \neq 1,2,4,\ldots$ and ψ^n is a monomorphism for $n = 1,2,4,\ldots$, then it follows from Theorem 2.5 that f_1 is homotopic to f_2 .

□

Let $f : S^p \to N^n$ be an embedding with trivial normal bundle. Consider N' the manifold obtained by surgery on N along the embedding f . We say that on N' was made a surgery of type $(p+1\ , n-p)$.

It follows from a result of [3] the:

Proposition 3.3 :

If N is a compact orientable manifold then it is possible obtain N' by surgeries of type $(p+1, n-p)$, where $p \leq [\frac{n-2}{2}]$ such that the map $\psi_p : \pi_p(N') \rightarrow \pi_p(BSO(n))$, induced by the classifying map for the tangent bundle of N is injective for $p \leq \frac{n-2}{2}$.

Definition 3.4 :

Given two embeddings $f, g : M^m \rightarrow N^n$ we say that they are isotopic up to surgery on N if it is possible to make a finite number of surgeries on N out of the images of f and g obtainyng a new manifold N' and maps $f', g' : M \rightarrow N'$ such that f' and g' are isotopic.

Let Y and Z be topological spaces $\psi : Y \rightarrow Z$ a map.

The following commutative diagram:

$$\begin{array}{ccc} \pi_n(Y) & \xrightarrow{\psi_*} & \pi_n(Z) \\ \downarrow{\scriptstyle \varkappa_Y} & \curvearrowright & \downarrow{\scriptstyle \varkappa_Z} \\ H_n(Y) & \xrightarrow{\psi_*} & H_n(Z) \end{array}$$

induces for all topological space X, the following commutative diagram:

$$\begin{array}{ccc} H^p(X, \pi_n(Y)) & \xrightarrow{\psi_n^p} & H^p(X, \pi_n(Z)) \\ \downarrow{\scriptstyle \varkappa_Y^p} & \curvearrowright & \downarrow{\scriptstyle \varkappa_Z^p} \\ H^p(X, H_n(Y)) & \xrightarrow{\psi_n^p} & H^p(X, H_n(Z)) \end{array}$$

Let us assume that X and Y are CW complexes and that Y is a n-simple, $n \leq \dim X$.

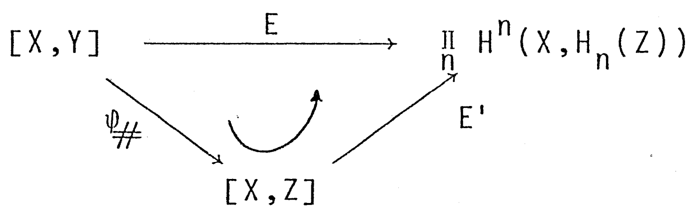
Theorem 3.5 :

If $\varkappa_Z^n \circ \varphi_n^n : H^n(X,\pi_n(Y)) \to H^n(X,H_n(Z))$ are monomorphisms for all n, then φ induces an injective map $\varphi_\# : [X,Y] \to [X,Z]$ defined by $\varphi_\#([f]) = [\varphi \circ f]$.

Proof:

Let us consider $K = K(H_n(Z),n)$, $u_Z^n \in H^n(Z,H_n(Z))$ for all n and $E' : [X,Z] \to \prod_n H^n(X,H_n(Z))$ defined by $E'([\varphi \circ f]) = \prod_n (\varphi \circ f)^*(u_Z^n)$.

Let us define $E : [X,Y] \to \prod_n H^n(X,H_n(Z))$ by $E([f]) = \prod_n f^*(u_Y^n)$ where u_Y^n equals $\varphi^*(u_Z^n)$. So we have the commutative diagram:



and since $\varkappa_Z^n \circ \varphi_n^n$ are monomorphisms for all n , it follows from Theorem 2.5 that E is injective and so, $\varphi_\#$ is injective.

□

Let now $\varphi_\# : [M,N] \to [M,BSO(n)]$ be the map defined by $\varphi_\#([f]) = [\varphi \circ f]$, where $\varphi : N \to BSO(n)$ is the classifying map for the tangent bundle of N .

Theorem 3.6 :

If M^m and N^n , $m \leq \frac{n-2}{2}$, are two closed orientable manifolds such that

a) $H^{4k}(M,Z)$ are free groups for $k = 1,2,\ldots$

b) $H^{8k+1}(M,Z_2) = H^{8k+2}(M,Z_2) = 0$, $k = 1,2,\ldots$

Then two embeddings $f,g : M^m \to N^n$ are isotopic up to surgery on N if and only if the normal bundles ν_f and ν_g are equivalent as orientable vector bundles.

Proof:

Let N' be the manifold obtained by surgery on N' (of type $(p+1, n-p)$, where $p \leq [\frac{n-2}{2}]$) out of the images of f and g such that the induced of classifying map of N', $\psi_i : \pi_i(N') \to \pi_i(BSO(n))$ is injective for $i \leq \frac{n-2}{2}$ (Prop 3.3).

We shall now that the change of coefficients:

$$\mathcal{X}^i \circ \psi^i : H^i(M,\pi_i(N')) \to H^i(M,H_i(BSO(n)))$$

is a monomorphism for $i \leq \frac{n-2}{2}$.

If $i = 8k+1$ or $8k+2$, $k = 1,2,\ldots$, $i \leq \frac{n-2}{2}$ then $\pi_i(BSO(n)) = Z_2$ and since that ψ_i is injective it follows that $\pi_i(N')$ is the trivial group or Z_2 .

If $\pi_i(N') = Z_2$ we have that $H^{8k+1}(M,Z_2) = H^{8k+2}(M,Z_2) = 0$, $k = 1,2,\ldots$. Then $\mathcal{X}^i \circ \psi^i$ are monomorphisms.

For $n \geq 6$, $\mathcal{X}_2$ and ψ_2 are isomorphisms because $\pi_2(BSO(n) = Z_2$ and $\pi_1(BSO(n)) = 0$.

Let us consider $h : S^{4k} \to BSO(n)$, a generator of $\pi_{4k}(BSO(n))$, and p_k a Pontrjagin class of $h^*(\gamma_n)$.

We have that $h^*(p_k) = (-1)^{k+1}(2k-1)!\, a_k . s$, where s is a generator of $H_{4k}(S^{4k})$.[1] .

Then $h_*(u_s) \neq 0$, where $u_s \in H_{4k}(S^{4k})$ is the fundamental class of S^{4k} . So, if β is a non zero element of

$\pi_{4k}(BSO(n))$ then $\mathcal{H}(\beta)$ is a multiple of $h_*(\mu_s)$. It follows then that the Hurewicz homomorphisms $\mathcal{H}_{4k}: \pi_{4k}(BSO(n)) \to H_{4k}(BSO(n))$ are monomorphisms $k = 1,2,\ldots$.

Now, if $i = 4k$, $i \leq \frac{n-2}{2}$, we have $\pi_i(BSO(n)) = Z$, $H_{4k-1}(M)$ is without torsion and ψ_{4k} and $\mathcal{H}_{4k}$ are monomorphisms. So we conclude that $\mathcal{H}^{4k} \circ \psi^{4k}$ are monomorphisms, $4k \leq \frac{n-2}{2}$.

In the other cases $\pi_i(BSO(n))$ are trivial.

From Theorem 3.5 we conclude that $\psi: N' \to BSO(n)$ induces the injective map $\psi_{\#}: [M,N'] \to [M,BSO(n)]$ and since $\psi \circ f: M \to \to BSO(n)$ classifies $f^*(\tau N)$ then f' is homotopic to g' if and only if the normal bundles ν_f and ν_g are equivalent as orientable vector bundles.

□

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