I.C.M.S.C.

NOTAS DO INSTITUTO DE CIÊNCIAS MATEMÁTICAS DE SÃO CARLOS-USP

AN INFINITE DIMENSIONAL EXTENSION OF THEOREMS OF HARTMAN AND WINTNER ON MONOTONE POSITIVE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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1987

AN INFINITE DIMENSIONAL EXTENSION OF THEOREMS OF HARTMAN AND WINTNER ON MONOTONE POSITIVE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

A. F. Izé

Abstract. Consider the system of differential equations (1) $\dot{x}_i + \sum_{j=1}^{\infty} a_{ij}(t)x_j = 0$, $x_i(0) = x_i^o$, $i = 1,2 \ldots$, where $a_{ij}(t)$ are continuous fuctions of the real variable t for $0 \leq t < \infty$, $x^o = (x_1^o, x_2^o, \ldots) \in X$, sequence space with a Schauder basis. System (1) can be written in the form (2) $\dot{x} + A(t)x = 0$, $x(0) = x^o$ and we assume that for each $t \in [0,T)$, $-A(t)$ is the infinitesimal generator of a C^o-semigroup on the space X, the domain $D(A(t)) = D$ of $A(t)$ is independent of t, $0 \leq t < T$, is dense in X and we assume that the initial value problem (1) has a unique classical solution defined in $[0,\infty)$. It is proved that if the solution operator $K(t,0)x^o = x^o - \int_0^t A(s)ds$, is compact for $t > 0$, $\sum_{j=1}^{\infty} a_{ij}(t)x_j \geq 0$ for each i and for $x_j \geq 0$, $j = 1,2 \ldots$ then system (1) has at least on solution $x(t) = (x_1(t), x_2(t) \ldots)$, $x(t) \not\equiv 0$, such that $x_i(t) > 0$, $i = 1,2 \ldots$, $-\dot{x}_i \geq 0$, and $x_i(t)$ are consequently, monotone non-increasing for $t \geq 0$. Consider now the perturbed system (3) $\dot{x}_i + \sum_{j=1}^{\infty} a_{ij}(t)x_j = -f_i(t,x)$, $x_i(0) = 0$, $i = 1,2 \ldots$, $f_i(t,0) = 0$, that can be written in the form (4) $\dot{x} + A(t)x = -f(t,x)$ where $f : [0,\infty) \times U \to X$ is continuous, $U \subset X$, open, $f(t,0) = 0$ and we assume existence of a unique classical solution of (3) in $[0,\infty)$. Then if the solution operator $K(t,0)x^o = x^o - \int_0^t A(s)x(s)ds - \int_0^t f(s,x(s))ds$ is compact for $t > 0$, $\Sigma\, a_{ij}(t)x_j + f_i(t,x) \geq 0$ for each i and for $x_j \geq 0$, $j = 1,2 \ldots$ then system (3) has at least one solution $x(t) = (x_1(t), x_2(t), \ldots)$, $x(t) \not\equiv 0$ such that $x_i(t) \geq 0$, $-\dot{x}_i(t) \geq 0$, $i = 1,2 \ldots$, $t \geq 0$ and, consequently, $x_i(t)$ are monotone non-increasing for $t \geq 0$.

1 - *Introduction*

In [4] we presented a topological method that can be applied to study the asymptotic behavior of differential equations in Banach Space. The book of Krasnoselskii [5] is dedicated to the determination of positive solutions of operator equations. If X is a Banach Space with a cone C the linear operator A is called positive if it transforms the cone into itself. It follows from $x \geqq 0$ that $Ax \geqq 0$.

A closed convex subset D of a Banach Space is a wedge if $x \in D$, $t \geqq 0$, implies that $tx \in D$. A wedge is a cone C if $x \in C$, $x \neq 0$ implies that $-x \notin C$. If X is a Banach Space with a wedge the operator A is positive if it transforms the cone C into itself. It follows from $x \geqq 0$, that $Ax \geqq 0$. The operator A is negative if it transform the cone C into $-C$. It follows from $x \geqq 0$ that $-Ax \geqq 0$. Even when the operator $A: R^+ \times X \to X$ is negative, the differential equation

$$\dot{x} = A(t,x) \quad , \quad A(t,0) = 0$$

may have positive solutions. In this paper we apply the results proved in [3] to extend Theorems of Hartman and Wintner [2] and [3], to an infinite dimensional space. The proof shows the generality and simplicity of the method developed in [4] and also its technical difficulties and limitation.

2 - *Preliminaries*

We begin by recalling a few definitions and results from [4].

Suppose X is a Banach Space, $R^+ = [0,\infty)$, $u: R \times X \times R^+ \to X$ is a given mapping and define $U(\sigma,t) : X \to X$ for $\sigma \in R$, $t \in R^+$

by $U(\sigma,t)x = u(\sigma,x,t)$. A process on X is a mapping $u : R \times X \times R^+ \to X$ satisfying the following properties:

i) u is continuous,

ii) $U(\sigma,0) = I$ (identity),

iii) $U(\sigma+s,t)U(\sigma,s) = U(\sigma,s+t)$.

A process is said to be an autonomous process or a semidynamical system if $U(\sigma,t)$ is independent of σ, that is, $T(t) = U(0,t)$, $t \geqq 0$. Then $T(t)x$ is continuous for $(t,x) \in R^+ \times X$.

Definition: Suppose u is a process on X. The trajectory $\tau^+(\sigma,x)$ through $(\sigma,x) \in R \times X$ is the set in $R \times X$ defined by

$$\tau^+(\sigma,x) = \{(\sigma+t,U(\sigma,t)x) \mid t \in R^+\}.$$

The orbit $\gamma^+(\sigma,x)$ through (σ,x) is the set in X defined by

$$\gamma^+(\sigma,x) = \{U(\sigma,t)x , \quad t \in R^+\}.$$

We assume in the following that the integral through each $(\sigma,x) \in R \times X$ is unique. We define $\tau^{-1}(x) = \{(\sigma,y) \in R \times X \mid \exists$ $t > 0$ such that $U(\sigma,t)y = x\}$. If $P_0 = (\sigma,x) \in R \times X$ and $z \in \gamma^+(\sigma,x)$ we define

$$t_z = \inf\{t \geqq 0 \mid U(\sigma,t)x = z\},$$

$$Q_z = (\sigma+t_z,U(\sigma,t_z)x),$$

$$[P_0,Q_z] = \{(\sigma+t,U(\sigma,t)x \mid 0 \leqq t \leqq t_z\}.$$

Let Ω be an open set of $R \times X$, ω an open set of Ω, $\omega \neq \emptyset$ and $\partial\omega = \overline{\omega} \cap (\overline{\Omega-\omega})$ the boundary of ω with respect to Ω. We put

$$S^0 = \{P_0 = (\sigma,x) \in \partial\omega \mid \exists\ t > 0 \text{ and } z \in \gamma^+(\sigma,x) \text{ with } (P_0,Q_z) \neq \emptyset \text{ and } (P_0,Q_z) \cap \overline{\omega} = \emptyset\} ,$$

$$S = \{Q \in \partial\omega \mid \exists\ P_0 = (\sigma,x) \in \omega \text{ with } Q \in \tau^+(\sigma,x) \text{ and } [P_0,Q) \subset \omega\} ,$$

$$S^* = S^0 \cap S .$$

The points of S are called egress points. The points of S^* are called strict egress points.

Given a point $P_0 = (\sigma,x) \in \omega$, if the trajectory $\tau^+(\sigma,x)$ of the process is contained in ω for every $t > 0$ we say that the trajectory is asymptotic with respect to ω . If the trajectory is not asymptotic with respect to ω then there is a $t > 0$ such that $(\sigma+t,U(\sigma,t)x) \in \partial\omega$. Taking:

$$t_{P_0} = \min\{t > 0 \mid (\sigma+t,U(\sigma,t)x) \in \partial\omega\}$$

$$Q = (\sigma+t_{P_0},U(\sigma,t_{P_0})x) = C(P_0)$$

we have

$$[P_0,Q] \subset \omega .$$

The point $C(P_0)$ is called the consequent of P_0 . Define G to be the set of all $P_0 = (\sigma,x) \in \omega$ such that there are $C(P_0)$ and $C(P_0) \in S^*$. G is called the left shadow of ω . Consider the mapping, the consequent operator:

$$K : S^* \cup G \to S^*$$

$K(P_0) = C(P_0)$ if $P_0 \in \omega$ and $K(P_0) = P_0$ if $P_0 \in S^*$.

Lemma: The consequent operator $K : S^* \cup G \to S^*$ is continuous. In [3] it is proved the following

Theorem 1 : Assume that there exist sets, $\omega \subset \Omega$, $S \subset \partial\omega$ and $Z \subset \omega \cup S$, $\omega \neq \emptyset$, $Z \neq \emptyset$ admissible satisfying the conditions:

i) $S = S^*$.

ii) Z is closed bounded convex and K is compact.

iii) $Z \cap S$ is a retract of S , that is , there exists a retraction $r : S \to Z \cap S$,

iv) There exists a continuous mapping $\Phi : Z \cap S \to Z \cap S$ such that $\Phi(P) \neq P$ for every $P \in Z \cap S$,

v) $\Phi . r . K$ is compact.

Then there exists at least one point $P = (\sigma,x) \in Z - S$ such that the trajectory $\tau^+(\sigma,x)$ through $P = (\sigma,x)$ is contained in $\omega - S$.

3 - *Main results consider the system*

$$\dot{x}_i + \sum_{j=1}^{\infty} a_{ij}(t)x_j = 0 \qquad i = 1,2,\ldots$$
$$x_i(0) = x_i^o \tag{1}$$

where $a_{ij}(t)$ are continuous functions of the real variable t for $0 \leq t < \infty$, $x_o = (x_1^o , x_2^o , x_3^o , \ldots) \in X$ sequence space with a schauder basis. Take, V.G., $X = c$ space of convergent sequences.

System (1) can be written in the form

$$\dot{x} + A(t)x = 0 , \quad x(0) = x^o . \tag{2}$$

We assume that for each $t \in [0,T]$, $-A(t)$ is the infini tesimal generator of a C^0 semigroup on the space X, the domain $D(A(t)) = 0$ is independent of t, is dense in X and that the initial value problem (1) has a unique classical solution defined in $[0,\infty)$.

Our purpose here is to apply the results of section (1) to prove existence of a positive solution of system (1).

Theorem 2 : Assume the hypotheses

i) The solution operator $K(t,0)x^0 = x^0 - \int_0^t A(s)x(s)ds$ is compact for $t > 0$.

ii) $\sum_{j=1}^{\infty} a_{ij}(t)x_j \geqq 0$ for every $i = 1,2,\ldots$, $x_j \geqq 0$; $j = 1,2,\ldots$.

Then system (1) has a monotone non-increasing solution $x(t) = (x_1(t), x_2(t)),\ldots)$, $x(t) \not\equiv 0$ such that $x_i(t) > 0$ and $-\dot{x}_i(t) \geqq 0$ for every $i = 1,2,\ldots$, $t \geqq 0$, and consequently $x_i(t)$ are monotone non-increasing for $t \geqq 0$.

Proof: Let us assume first that

$$\sum_{j=1}^{\infty} a_{ij}(t) > 0 , \qquad i = 1,2,\ldots . \tag{3}$$

Let for $\delta > 0$

$\omega = \{x = (x_1,x_2,\ldots) \in \ell^\infty \mid x_i > 0 , 0 < \sum_{i=1}^{\infty} x_i < \delta , i = 1,2,\ldots\}$

$Z = \{x = (x_1,x_2,\ldots) \in \ell^\infty \mid x_i \geqq 0 , \sum_{i=1}^{\infty} x_i = \delta \}$,

$S = \{x = (x_1,x_2,\ldots) \in \ell^\infty \mid x_i \geqq 0 , 0 < \sum_{i=1}^{\infty} x_i \leqq \delta , x_i = 0$ for at least one $i\}$.

The closure of ω, $\overline{\omega}$ is a solid truncated cone in the space ℓ^∞.

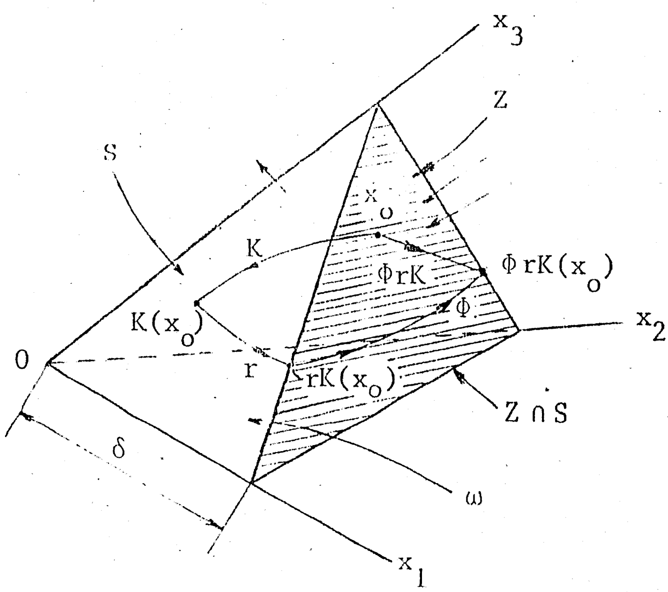

Fig. 1

In dimension 3 the sets ω , S and Z are shown in Fig.1 . ω is the interior of the trihedron, S is the sur face of the trihedron without the origin 0 and Z is the section of the trihedron.

From hypothesis (3) $\dot{x}_i < 0$, then the derivatives along the solutions of (1) on the faces of the infinitehedron are negative and the points of S are strict egress points. At the origin $\dot{x}_i = 0$ for every i , then the origin is not an egress point. Hence the boundary of Z , $\partial Z = \{x = (x_1, x_2, \ldots) \mid \sum_{i=1} x_i = \delta$, $\prod_{i=1}^{\infty} x_i = 0\}$ is a retract of S , that is, there exists a re traction $r : S \to Z \cap S = \partial Z$. The continuous function $\Phi : Z \cap S \to Z \cap S$ defined by $\phi(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ satisfies $\Phi(x) \neq x$ for every $x \in Z \cap S$. Assume that for any $x^o \in Z \cap \omega$, the solution through x^o does not remain in ω for $t > 0$, then $Z \cap \omega \subset G$.

The consequent operator $K : Z \to S$ defined by

$$K(t,0)(x^o)i = x_i^o - \int_0^{\overline{t}} \sum_{j=1}^{\infty} a_{ij}(s)x_j(s)ds \ , \quad i = 1,2,... \qquad (4)$$

is defined in $Z \cap \omega$. Since the points of S are strict egress, K is defined in $Z \cap S$ then in $\overline{Z}$, that is, $\overline{Z} \subset G \cup S$. From Lemma 1, K is continuous and from hypothesis i) , K is compact. Since Φ and r are continuous functions, $\Phi.r.K$ is compact. From Theorem 1 , there exists at least a $\overline{x} \in Z \cap \omega$ such that the solution of (1) $x(t) = (x_1(t),x_2(t)...)$ through $\overline{x}$ stay in ω for every $t \geq 0$ and since $\dot{x}_i < 0$, $i = 1,2,...$, each $x_i(t)$ decreases monotonically to zero as $t \to \infty$.

If

$$\sum_{j=1} a_{ij}(t) \geq 0 \ .$$

Consider the system

$$\begin{aligned} &\dot{y}_i + \sum_{j=1} (a_{ij}(t) + \varepsilon_{ij})y_i = 0 \\ &y_i(0) = y_i^o \ , \quad i = 1,2,... \end{aligned} \qquad (5)$$

$\varepsilon_{ij} > 0$ arbitrarily small. From the proof above (6) has a positive solution through some $\overline{y}^o$.

From the proof above there exists a positive solution of (5) through $y^o = (y_1^o,y_2^o,...)$. Now the proof is complete by choosing a subsequence $y_{o,k_j}(t)$, $y_{o,k_j}(0) = y_i^o$, which converges uniformily on every interval $0 \leq t \leq T < \infty$, as $\varepsilon_{ij} \to 0$ to the solution $\overline{y}(t)$, $\overline{y}(0) = y_i^o$, $i = 1,2...$ and from the assumed uniqueness of solution of (1) $\dot{x}(t) = \overline{y}(t)$. Furthermore since $\dot{y}_i \leq \dot{x}_i$ and $y_i \leq x_i$, $i = 1,2,...$, the solution $x(t)$ of (1) through y^o is positive.

Remark 1 : In finite dimension $X = R^n$ the operator K is compact and Theorem 2 is true without hypothesis (i) . This is a Theorem of Hartman and Wintner [2] .

Remark 2 : Let $A = A(t)$ be a constant $n \times n$ matrix . Theorem 1 can be considered as a generalization of the algebraic theorem of Perron which states that a non-negative (constant) matrix A possesses at least one non-negative eigenvalue λ corresponding to which there is a non-negative eigenvector c (If $A > 0$ then $\lambda > 0$ and $c > 0$). (see Remark 2 in [2]) .

Consider now the nonlinear system

$$\begin{aligned} \dot{x}_i + \sum a_{ij}(t)x_j &= -f_i(t,x) \\ x_i(0) &= x_i^0 , \quad i = 1,2 \ldots \end{aligned} \tag{6}$$

System (6) can be written in the form

$$\begin{aligned} \dot{x} + A(t)x &= -f(t,x) \\ x(0) &= x^0 \end{aligned} \tag{7}$$

We assume that for each $t \in [0,T]$, $-A(t)$ is the infinitesimal generator of a C^0-semigroup on the space X , the domain $D(A(t) = D$ is independent of t , is dense in X , $f_i[0,\infty) \times U \to X$ is continuous $U \subset X$, open $f_i(t,0) = 0$ and we assume existence of a unique classical solution of (3) in $[0,\infty)$.

Theorem 3 : Assume the hypotheses

i) The solution operator

$$K(t,0)x^0 = x^0 - \int_0^t A(s)x(s) - \int_0^t f(t,x)$$

is compact for $t > 0$.

ii) $\sum_{j=1}^{\infty} a_{ij}(t)x_j + f_i(t,x) \geq 0$ for every $i = 1,2\ldots,$ $x_j \geq 0$, $j = 1,2\ldots$.

Then system (6) has a monotone non-increasing solution $x(t) = (x_1(t), x_2(t),\ldots)$ such that $x_i(t) \geq 0$ for every $i = 1,2\ldots,$ $t \geq 0$, and consequently $x_i(t)$ are monotone non-increasing.

The proof follows as in Theorem 2, assuming that $\sum_{j=1}^{\infty} a_{ij}(t)x_i + f_i(t,x) > 0$ and the conclusion is that there exists a positive solution of (6) through some point $x_o = (x_1^o, x_2^o,\ldots) \in \omega - S$. Consider then the system

$$\dot{y}_i + \sum (a_{ij}(t) + \varepsilon_{ij})x_j + f_i(t,x) = 0$$

$$y_i(0) = y_i^o$$

and a subsequence $y_{o,k_j}(t)$, $y_{o,k_j}(0) = y_i^o$ which converges, as $\varepsilon_{ij} \to 0$, uniformly on every interval $0 \leq t \leq T < \infty$. For system (6) the solution $x(t)$ can become zero after a finite time. Theorem 3 generalizes result of Hartman and Wintner [3].

Example: Let $\{a_i\} \in c$ space of convergent sequences with norm $\|a\| = \sup_i |a_i|$. Assume that $\lim_{i\to\infty} a_i = a_\infty \neq 0$ and define $\alpha^i = (0,0,\ldots,a_i,0,\ldots)$. Define $T(t)\alpha^i = \{e^{\lambda_i} \alpha^i\}$, $-\infty < \mathrm{Re}\,\lambda_i \leq \omega < \infty$. $T(t)$ is a strongly continuous semigroup with infinitesimal generator A given by $A\alpha^i = \{\lambda_i \alpha^i\}$. $T(t)$ is compact if and only if $\lim \mathrm{Re}\,\lambda_i = -\infty$. Consider the system

(8) $\dot{x}_i = -\lambda_i x_i - \sum_{j=1}^{\infty} g_{ij}(t)x_j$, $x_i(t_o) = x_i^o$ with $\lambda_i > 0$, $\sum_{i,j} \|g_{ij}\| < \infty$. This system can be written in the form

$\dot{x} = Ax + G(t)x$, $x(t_o) = x^o$. Since A is compact and G(t) is bounded, the consequent operator $K(t,t_o)x^o = x^o + \int_{t_o}^{t} (A+G(s))ds$ is compact. From Theorem 2 there exist at least one monotone solution $x(t) = (x_1(t), x_2(t), \ldots)$, of (8) such that $\lim_{t\to\infty} x(t) = 0$, $x(t) \geq 0$ and $-\dot{x}(t) \geq 0$.

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