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HYPERBOLIC MOTIONS OF CONICS

MENDES, C.M. & RUAS, M.A.S.

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HYPERBOLIC MOTIONS OF CONICS

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The geometry associated with Einstein's principle of relativity is Lorentzian geometry. In the simplest case, the Lorentzian plane is $\mathbb{R}^2$ with the Lorentz bilinear form $<x,y> = x_1y_1 - x_2y_2$. Lorentzian plane geometry consists of the study of properties invariant by the action of the group, called hyperbolic motions, which preserve this bilinear form. In general, a Lorentz geometry is the geometry of an affine space resulting from the existence of a symmetric bilinear form with signature $(+,+,\dots,+,-)$. We will follow the standard custom of calling the bilinear form above an inner product even though it is possible for $<x,x> = 0$ but x not be zero.

The references dealing with elementary properties of this geometry are rare and brief. An overall view of Lorentz geometry is found in the book of Beem and Ehrlich [1]. In a recent article, Birman and Nomizu [2] develop Lorentzian trigonometry. A natural question would be to look at the action of the hyperbolic group on conics. A modern exposition of the Euclidean motions of conics, using matrix techniques, can be found in [4]. We shall follow the development of that paper.

By analogy with the Euclidean case, we present the hyperbolic group and analyse its action on a conic defined by $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$. We show that certain polynomials in the coefficients of the conic are invariant under the hyperbolic group and that these are all of the invariants. We then study the geometry of the space of orbits of this action and normal forms are found.

§1. A GROUP ACTION

1.1. *The Hyperbolic Rotations*

The Euclidean group is the group generated by rotations (special orthogonal matrices) and translations of $\mathbb{R}^2$. A special orthogonal matrix is one which takes the form

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

for some angle θ. To obtain the hyperbolic analogy we shall first discuss hyperbolic rotations which are obtained by replacing the trigonometric functions by the hyperbolic ones in the above formula.

A 2×2 matrix N is called *special hyperbolic* if there exists θ such that

$$N = N(\theta) = \begin{bmatrix} ch\theta & sh\theta \\ sh\theta & ch\theta \end{bmatrix}, \quad \theta \in \mathbb{R},$$

where we are writing ch and sh for hyperbolic functions cosh and sinh.

The set G of all such matrices, with the usual multiplication constitutes a group, called the *group of hyperbolic rotations*, or *proper Lorentz group* $SO^+(1,1)$.

Each $N \in G$ defines a linear transformation

$$N : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x' \\ y' \end{bmatrix} = N \begin{bmatrix} x \\ y \end{bmatrix}$$

wich preserves the Lorentz inner product, defined by $\langle\vec{v},\vec{w}\rangle = x_1x_2 - y_1y_2$, where $\vec{v} = (x_1,y_1)$, $\vec{w} = (x_2,y_2)$.

It follows easily that the eigenvectors of N are $e_1 = (1,-1)$ and $e_2 = (1,1)$ and that the corresponding eigenvalues $\lambda_1 = ch\theta - sh\theta$ and $\lambda_2 = ch\theta + sh\theta$ are positive real numbers.

The sets I, II, III, IV, V and VI in Figure I remain G invariant, that is: if $(x,y) \in A$, then $N(x,y) \in A$, $\forall\, N \in G$ (where A denotes anyone of these sets).

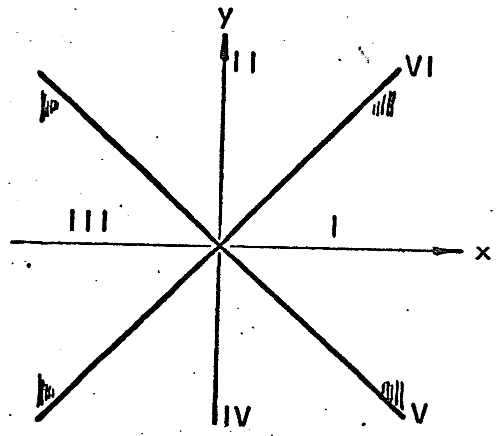


Figure 1

To see, for instance, that I is invariant, let $\vec{v} = (x,y)$ be any element in I. Then $\vec{v} = \alpha e_1 + \beta e_2$, where α and β are positive. Hence $N\vec{v} = N(\alpha e_1 + \beta e_2) = (\alpha\lambda_1)e_1 + (\beta\lambda_2)e_2$ with $\alpha\lambda_1$, $\beta\lambda_2$ positive and so $N\vec{v} \in I$.

Vectors (x,y) in I or III are called *spacelike*, in II or IV are *timelike*, and in V or VI, *null* [2].

Furthermore, the points of the invariant sets I, II, III, IV, slide along the hyperbolas $x^2 - y^2 =$ constant. Now, $\lambda_1\lambda_2 = 1$ for all θ. If $\theta > 0$, we have $\lambda_2 > 1$ and $\lambda_1 < 1$, and in this case, the plane shrinks λ_1 times to the straight line $x = -y$ and stretches orthogonal direction away from $x = y$ as shown in Figure 2. When $\theta < 0$, we just reverse the directions of stretching and compression, and the direction of motion of the points along the hyperbolas [3].

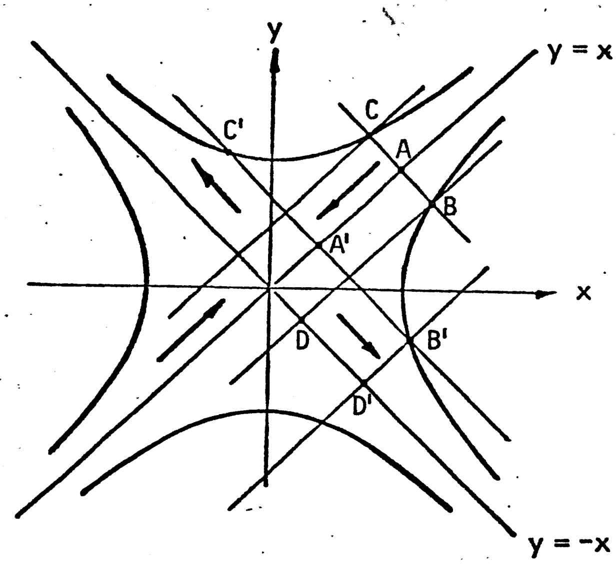


Figure 2

1.2. The Hyperbolic Group

In analogy with the Euclidean case (see Grosshans [4]), we introduce the group H consisting of all 3×3 real matrices of the form:

$$h(\theta) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{N(\theta)} & B \\ \hline 0 & 0 & 1 \end{array}\right] \qquad (1)$$

where $N(\theta)$ is a special hyperbolic matrix, and B is a 2×1 column matrix.

We can identify the plane $\mathbb{R}^2$ with the plane $z = 1$ in $\mathbb{R}^3$ that is,

$$p = \begin{bmatrix} x \\ y \end{bmatrix} \longleftrightarrow P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Each matrix $h \in H$ gives rise to a motion M_h, where

$M_h(p) = hP = Np + B$, that is a hyperbolic rotation N followed by a translation B. The 2-dimensional Lorentzian geometry is the study of properties invariant by these motions.

In this paper, we shall be interested in the action of M_h on the conics $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$, a, b, c do not simultaneously vanish.

1.3. *The Action of* H *on* V

For a moment, let G be any group and V be a set.

An *action* of G on V is given by a mapping

$$G \times V \rightarrow V \quad \text{denoted by} \quad (g,v) \rightarrow g.v$$

such that (i) $(hg).v = h.(g.v)$ and (ii) $1.v = v$ for all h, g in G and v in V. The *orbit* of an element v in V is

$$G.v = \{g.v \mid g \in G\}$$

In our case, V will be the vector space of 3×3 symmetric real matrices

$$Q = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}$$

and we shall be interested in the following action of H on V:

Given $h \in H$ and $Q \in V$, $h.Q = {}^t h^{-1} Q h^{-1}$. *(2)*

We can identify a matrix $Q \in V$ with the polynomial

$Q(p) = Q(x,y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f$. Note that ${}^tP\,Q\,P =$ $= Q(p)$.

Let $Q' = h.Q = {}^th^{-1}\,Q\,h^{-1}$ and let C and C' be the conics associated to Q and Q'; that is $C = \{p : Q(p) = 0\}$ and $C' =$ $= \{p : Q'(p) = 0\}$.

The equation

$$Q(p) = {}^tP\,Q\,P = {}^tP\,{}^thQ'hP = {}^t(hP)Q'(hP) = Q'(hP)$$

shows that p lies on C if and only if $hP = M_h(p)$ is on C'.

Hence, M_h sends C to C'. Thus, if Q' is in the orbit of Q under the action described above, there is a hyperbolic motion carrying the conic defined by Q to the conic defined by Q'. This means that the conics are equivalent under the action.

To study the orbit under H of an element Q in V, we shall denote a typical element in V by:

$$Q = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} = \left[\begin{array}{c|c} A(Q) & D(Q) \\ \hline {}^tD(Q) & f(Q) \end{array}\right] \qquad (3)$$

where $A(Q) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $D(Q) = \begin{bmatrix} d \\ e \end{bmatrix}$ and $f(Q) = f$.

To simplify notation, the matrix *(1)* will be denoted by h^{-1}. The corresponding matrix Q' in the orbit of Q, $Q' = {}^th^{-1}Qh^{-1}$ is obtained as follows:

$$A(Q') = N\,A\,(Q)\,N$$

$$D(Q') = N\,A\,(Q)\,B + N\,D(Q) \qquad (4)$$

$$f(Q') = {}^{t}B\,A\,(Q)\,B + 2\,{}^{t}D(Q)B + f(Q)$$

The following more explicit expression for $A(Q')$, will be useful later:

$$A(Q') = \begin{bmatrix} a\,ch^2\theta + b\,sh2\theta + c\,sh^2\theta & ((a+c)/2)\,sh2\theta + b\,ch2\theta \\ ((a+c)/2)\,sh2\theta + b\,ch2\theta & a\,sh^2\theta + b\,sh2\theta + c\,ch^2\theta \end{bmatrix} \qquad (5)$$

1.4. H-Invariant Polynomials

Let $R = \mathbb{R}[z_1,z_2,z_3,z_4,z_5,z_6]$ be the polynomial ring in six indeterminates over the reals. We can identify R with the ring of polynomial functions on V by defining $P(Q) = P(a,b,c,d,e,f)$, where $P = P(z_1,z_2,\ldots,z_6)$ is in R and Q is an element of V.

A polynomial P in R is called *invariant* with respect to H if $P(h.Q) = P(Q)$, $\forall\, h \in H$, $\forall\, Q \in V$. The set of such invariant polynomials, denoted by I, is an algebra over $\mathbb{R}$.

Now, we define three polynomials in R and show that they are H invariant:

Let $\delta(Q) = \det A(Q) = ac - b^2$, $\Delta(Q) = \det Q$ and $\tilde{\tau}(Q) = a - c$

<u>*Lemma*</u>: δ, Δ and $\tilde{\tau}$ are H-invariant.

Proof:

(i) $\delta(Q') = \det(A(Q')) = \det(N A(Q) N) = \det(A(Q)) = \delta(Q)$ since $\det N = 1$;

(ii) $\Delta(Q') = \det(Q') = \det({}^t h^{-1} Q h^{-1}) = \det Q$, since $\det h = 1$;

(iii) it follows from *(5)* that $\tilde{\tau}(Q') = a - c = \tilde{\tau}(Q)$.

Remark: Two non equivalent conics may have the same invariant polynomials δ, $\tilde{\tau}$ and Δ.

For instance, let $C_1 : x^2 + 2bxy = 1$ and $C_2 : x^2 - 2bxy = 1$, $b > 0$ as in Figure 3. It follows from the invariance of the sets I, II, III and IV in Figure 1 that a motion M_h can not send C_1 onto C_2

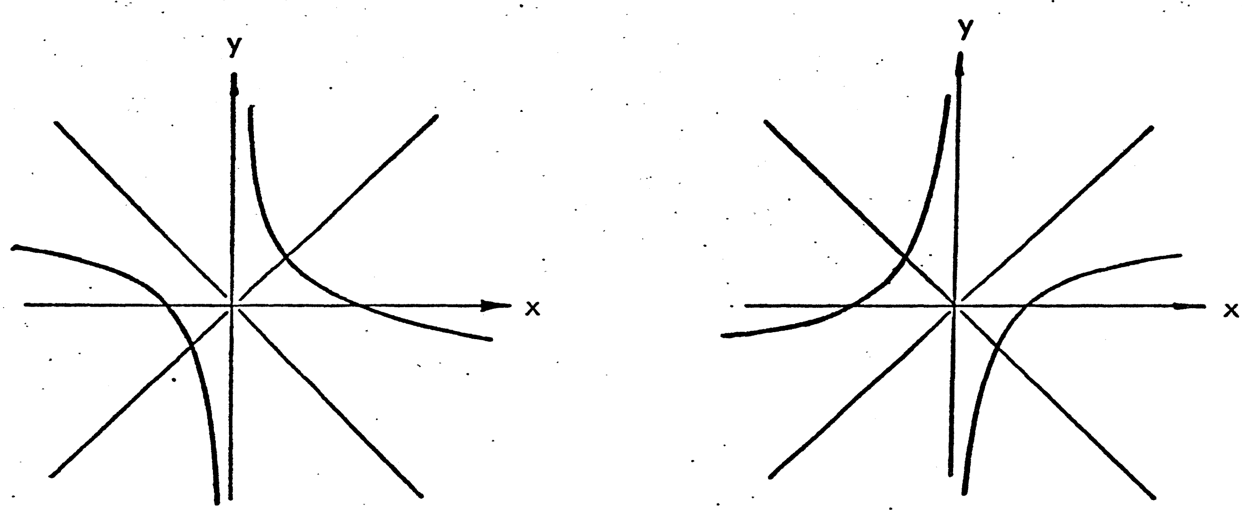


Figure 3

2. CANONICAL FORMS

2.1. *Normal Forms Under Rotations*

It is well known that under a Euclidean motion (rotations and translations) most conics (when $\delta \neq 0$) admit reduction to a sum of squares. This is not always the case for the action of the Hyperbolic group.

As before, let $Q = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}$ and $A(Q) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

The first step is to get rid of b, this is, we want to find conditions to reduce $A(Q)$ to a diagonal form. As we shall see the possibility of this reduction will depend upon the sign of

$$\Lambda(Q) = [\tau(Q)]^2 + 4\delta(Q) = (a-c)^2 + 4(ac-b^2) = (a+c)^2 - 4b^2$$

So Lemma follows.

<u>*Lemma:*</u> If $b \neq 0$, $\Lambda(Q) > 0$ if and only if $\left|\frac{a+c}{2b}\right| > 1$.

<u>*Proposition 1:*</u> If $\Lambda(Q) > 0$ then in the orbit of Q there exists a Q' which is of the following form

$$Q' = \left[\begin{array}{cc|c} a' & 0 & d' \\ 0 & c' & e' \\ \hline d' & e' & f' \end{array}\right]$$

Proof: We can assume $b \neq 0$.

From *(6)* we need θ such that

$$b' = \frac{a+c}{2} \cdot sh2\theta + bch2\theta = 0,$$

or equivalently

$$\text{cotgh}2\theta = \frac{a+c}{-2b}$$

This is possible if and only if $\left|\frac{a+c}{2b}\right| > 1$, or by the Lemma, $\Lambda(Q) > 0$.

Now, to obtain Q', we take

$$h^{-1} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{N(\theta)} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & 1 \end{array}\right], \quad \text{as in } (2).$$

The diagonal elements a' and c' are solutions of the system:

$$\begin{cases} a' - c' = \tilde{\tau}(Q) = a - c \\ a' \cdot c' = \delta(Q) = ac - b^2 \end{cases}$$

Now, from *(5)* we can deduce $(a')^2 - (c')^2 = \frac{(a-c)}{(a+c)} \Lambda\, ch2\theta$. Since $\Lambda > 0$, it follows that the sign of $(a')^2 - (c')^2$ is equal to sign of $a^2 - c^2$. These conditions determine a' and c' uniquely.

Proposition 2: If $\Lambda(Q) < 0$ $(b \neq 0)$, then Q is H-equivalent to:

$$Q' = \left[\begin{array}{cc|c} \tilde{\tau}(Q) & \pm\sqrt{-\delta(Q)} & d' \\ \pm\sqrt{-\delta(Q)} & 0 & e' \\ \hline d' & e' & f \end{array}\right]$$

(Where the sign of the square root is equal to the sign of b).

Proof: We want to make $c' = 0$.

It follows from *(5)* that we need θ such that

$$\begin{cases} a' = ach^2\theta + bsh2\theta + csh^2\theta \\ b' = \frac{a+c}{2} sh2\theta + bch2\theta \\ 0 = ash^2\theta + bsh2\theta + cch^2\theta \end{cases} \tag{6}$$

The invariance of $\tilde{\tau}$ and δ imply $a' = a - c = \tilde{\tau}(Q)$ and $b' = \pm\sqrt{-\delta(Q)}$. We shall be assuming that b' has the same sign of b, that is, $bb' > 0$.

Now, it is easy to see that the first and the last equation of *(6)* are linearly dependent. Hence, we obtain the equivalent system:

$$\begin{cases} b' = \frac{a+c}{2} sh2\theta + bch2\theta \\ 0 = ash^2\theta + bsh2\theta + cch^2\theta \end{cases} \quad \text{or:}$$

$$\begin{cases} 2b' = (a+c)sh2\theta + 2bch2\theta \\ a-c = 2bsh2\theta + (a+c)ch2\theta \end{cases} \tag{7}$$

For a moment, let us consider the system:

$$\begin{cases} 2b' = (a+c)y + 2bx \\ a-c = 2by + (a+c)x \end{cases} \tag{8}$$

Since the determinant $\Lambda(Q) = (a+c)^2 - 4b^2 < 0$, *(8)* has unique solutions $X = \dfrac{a^2 - c^2 - 4bb'}{\Lambda(Q)}$ and $Y = \dfrac{2b'(a+c) - 2b(a-c)}{\Lambda(Q)}$

It is not hard to show that $X^2 - Y^2 = 1$.

To finish the proof, we must show that $X \geq 0$.

In fact, since $\Lambda(Q) < 0$, it is only necessary to show that $(a^2 - c^2) - 4bb' < 0$.

Now, $\Lambda(Q) = [\tilde{\tau}(Q)]^2 + 4\delta(Q)$. *(9)*

From the invariance of $\delta(Q)$, we also have:

$$\Lambda(Q) = [\tilde{\tau}(Q)]^2 - 4(b')^2 \tag{10}$$

From *(9)* and *(10)*, it follows that the condition $\Lambda(Q) < 0$ is equivalent to each one of the inequalities:

(i) $|(a+c)| < 2|b|$

(ii) $|a-c| < 2|b'|$

By multiplying (i) and (ii) side by side, we obtain: $a^2 - c^2 \leq \leq |a^2 - c^2| < 4|b||b'|$ or, since $bb' > 0$, $a^2 - c^2 - 4bb' < 0$, as desired.

Proposition 3: If $\Lambda(Q) = 0$ then:

(i) If $a^2 - c^2 > 0$, Q is H-equivalent to

$$Q' = \left[\begin{array}{cc|c} \tilde{\tau}(Q) & \pm|\tilde{\tau}(Q)|/2 & d' \\ \pm|\tilde{\tau}(Q)|/2 & 0 & e' \\ \hline d' & e' & f \end{array}\right] \begin{array}{l} \text{plus sign if } b > 0 \\ \text{minus sign if } b < 0 \end{array}$$

(ii) If $a^2 - c^2 < 0$, Q is H-equivalent to

$$Q'' = \left[\begin{array}{cc|c} 0 & \pm|\tilde{\tau}(Q)|/2 & d' \\ \pm|\tilde{\tau}(Q)|/2 & -\tilde{\tau}(Q) & e' \\ \hline d' & e' & f \end{array}\right] \begin{array}{l} \text{plus sign if } b > 0 \\ \text{minus sign if } b < 0 \end{array}$$

(iii) If $a = c$, then $b = \pm a$ and $Q = \left[\begin{array}{cc|c} a & \pm a & d \\ \pm a & a & e \\ \hline d & e & f \end{array}\right]$

(iv) If $a = -c$, then $b = 0$ and $Q = \left[\begin{array}{cc|c} a & 0 & d \\ 0 & -a & e \\ \hline d & e & f \end{array}\right]$

Proof: First, let us suppose $a^2 - c^2 > 0$. In this case, we shall be assuming that b' has the same sign of b.

As in the proof of Proposition 2, we obtain the system:

$$\begin{cases} 2b' = (a + c)y + 2bx \\ a - c = 2by + (a + c)x \end{cases} \tag{11}$$

The determinant $\Lambda(Q) = (a + c)^2 - 4b^2$ is equal to zero. The system *(11)* admits a line of solutions, with slope $m = \pm 1$, which intercepts the x-axis in $x_o = \dfrac{a - c}{a + c} > 0$.

Thus, this line intercepts the hyperbola $x^2 - y^2 = 1, \quad x > 0$ and we can take θ such that $\begin{cases} x = ch2\theta \\ y = sh2\theta \end{cases}$

When $a^2 - c^2 < 0$, we shall make $a' = 0$. In this case, we proceed as before, just solving the system

$$\begin{cases} 2b' = (a + c)y + 2bx \\ c - a = 2by + (a + c)x \end{cases}$$

Of course, (iii) and (iv) follow directly from the hypothesis.

2.2. *Completing the Reduction*

Up to this point, we analysed the action of a hyperbolic rotation on the matrix Q. Now, we shall be using translations to bring Q to a normal form.

Case 1: If $\delta(Q) \neq 0$ and $\begin{cases} \text{(i) } \Lambda(Q) > 0 \text{ or} \\ \text{(ii) } \Lambda(Q) = 0, \text{ with } b = 0 \end{cases}$ then the orbit of Q contains the matrix:

$$\begin{bmatrix} a' & 0 & 0 \\ 0 & c' & 0 \\ 0 & 0 & \Delta(Q)/\delta(Q) \end{bmatrix} \tag{12}$$

In (ii), we also have $c' = -a'$.

Proof:

(i) Since $\Lambda(Q) > 0$, the orbit of Q contains, from proposition 1, the matrix:

$$Q' = \begin{bmatrix} a' & 0 & d' \\ 0 & c' & e' \\ d' & e' & f \end{bmatrix} \qquad (13)$$

Since $a'.c' = \delta(Q') = \delta(Q) \neq 0$, there exist real numbers s and t such that:

$$a's + d' = 0 \quad \text{and} \quad c't + e' = 0$$

Next, we take h^{-1} in the form *(1)* with $N = I$ and ${}^tB = (s,t)$. Then, $Q'' = h.Q'$ is diagonal, with entries a', c' and f', say.

From $\Delta(Q) = \Delta(Q'') = a'c'f' = \delta(Q).f'$, it follows that Q'' is as we wanted.

(ii) When $b = 0$ and $\Lambda(Q) = 0$, it also follows that $c' = -a'$.

Case 2: If $\delta(Q) \neq 0$ and $\Lambda(Q) < 0$.

Then, the orbit of Q contains the matrix:

$$\begin{bmatrix} \mp\tau(Q) & \pm\sqrt{-\delta(Q)} & 0 \\ \pm\sqrt{-\delta(Q)} & 0 & 0 \\ 0 & 0 & -\Delta(Q)/\delta(Q) \end{bmatrix}$$

(sign $\sqrt{}$ equals to sign of b).

Proof: Let us suppose $b > 0$ (the case $b < 0$ is analogous).
The orbit of Q contains, from proposition 2, the matrix:

$$Q' = \begin{bmatrix} \tilde{\tau}(Q) & \sqrt{-\delta(Q)} & d' \\ \sqrt{-\delta(Q)} & 0 & e' \\ d' & e' & f \end{bmatrix}$$

Since $\delta(Q) \neq 0$, there exist real numbers s and t such that:

$$\tilde{\tau}(Q)s + \sqrt{-\delta(Q)}\, t + d' = 0 \quad \text{and} \quad \sqrt{-\delta(Q)}\, s + e' = 0$$

Now, we take h^{-1} in the form *(1)*, with $N = I$ and ${}^tB =$ $= (s,t)$. Then, $Q'' = h.Q'$ is as we wanted.

Case 3: If $\delta(Q) \neq 0$ and $\Lambda(Q) = 0$, then:

(i) If $a^2 - c^2 > 0$, the orbit of Q contains the matrix

$$\begin{bmatrix} \tilde{\tau}(Q) & \pm|\tilde{\tau}(Q)|/2 & 0 \\ \pm|\tilde{\tau}(Q)|/2 & 0 & 0 \\ 0 & 0 & -\Delta(Q)/\delta(Q) \end{bmatrix} \quad \begin{matrix} \text{plus sign if } b > 0 \\ \text{minus sign if } b < 0 \end{matrix}$$

(ii) If $a^2 - c^2 < 0$, the orbit of Q contains the matrix

$$\begin{bmatrix} 0 & \pm|\tilde{\tau}(Q)|/2 & 0 \\ \pm|\tilde{\tau}(Q)|/2 & -\tilde{\tau}(Q) & 0 \\ 0 & 0 & -\Delta(Q)/\delta(Q) \end{bmatrix} \quad \begin{matrix} \text{plus sign if } b > 0 \\ \text{minus sign if } b < 0 \end{matrix}$$

The proof is analogous to the proof of Case 2, and we shall omit it.

Case 4: If $\delta(Q) = 0$, $\tilde{\tau}(Q) \neq 0$ (hence $\Lambda(Q) > 0$) and $\Delta(Q) \neq 0$, then:

(i) If $\tilde{\tau}(Q).\Delta(Q) < 0$, the orbit of Q contains the matrix

$$\begin{bmatrix} \tilde{\tau}(Q) & 0 & 0 \\ 0 & 0 & e'' \\ 0 & e'' & 0 \end{bmatrix}, \text{ where } e'' = \pm\sqrt{\frac{-\Delta(Q)}{\tilde{\tau}(Q)}}$$

(ii) If $\tilde{\tau}(Q).\Delta(Q) > 0$, the orbit of Q contains the matrix

$$\left[\begin{array}{cc|c} 0 & 0 & d'' \\ 0 & -\tilde{\tau}(Q) & 0 \\ \hline d'' & 0 & 0 \end{array}\right], \text{ where } d'' = \pm\sqrt{\frac{\Delta(Q)}{\tilde{\tau}(Q)}}$$

Proof:

Case i: The hypothesis imply that $\Lambda(Q) > 0$.

Since $\delta(Q) = 0$, and $\tilde{\tau}(Q).\Delta(Q) < 0$, we may assume in *(13)* that:

$$Q' = \begin{bmatrix} \tilde{\tau}(Q) & 0 & d' \\ 0 & 0 & e' \\ d' & e' & f \end{bmatrix}$$

We note that $0 \neq \Delta(Q) = \Delta(Q') = -(e')^2.\tilde{\tau}(Q)$, so that $e' \neq 0$. Then, we can find s and t such that:

$$\tilde{\tau}(Q).s + d' = 0 \quad \text{and} \quad \tilde{\tau}(Q).s^2 + 2(d's + e't) + f = 0$$

To obtain the normal form, let $Q'' = h.Q'$, where h^{-1} has the form *(1)* with $N = I$ and ${}^{t}B = (s,t)$.

Case ii: Follows analogously.

Case 5: If $\delta(Q) = 0$, $\tilde{\tau}(Q) \neq 0$ (hence $\Lambda(Q) > 0$), and $\Delta(Q) = 0$ then, the orbit of Q contains either the matrix

$$\text{(i)} \quad \left[\begin{array}{cc|c} \tilde{\tau}(Q) & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & f' \end{array}\right] \quad \text{when } |c| < |b| \text{ or } (b = 0 \text{ and } a \neq 0),$$

$$\text{(ii)} \quad \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -\tilde{\tau}(Q) & 0 \\ \hline 0 & 0 & f' \end{array}\right] \quad \text{when } |c| > |b| \text{ or } (b = 0 \text{ and } c \neq 0).$$

In either case, $f' = f - \dfrac{d^2 - e^2}{\tilde{\tau}(Q)}$

Proof: From Proposition 1, we may assume $Q = \begin{bmatrix} a' & 0 & d' \\ 0 & c' & e' \\ d' & e' & f \end{bmatrix}$ where $a'.c' = 0$, since $\delta = 0$.

If $c' = 0$, $\Delta = 0$ implies $e' = 0$

Now, choosing $h^{-1} = \left[\begin{array}{cc|c} 1 & 0 & s \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$, where $\tilde{\tau}(Q).s + d' = 0$, we verify that:

$$Q' = h.Q = \begin{bmatrix} \tilde{\tau}(Q) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f' \end{bmatrix}$$

With the same procedure, when $a' = 0$, we obtain (ii).

Using the expression for $f' = f(Q')$ as in *(4)* and the invariance of $d^2 - e^2$ with respect to a hyperbolic rotation, we can find f'.

The decision about formulas (i) or (ii), can be made as follows:

Under the hypothesis, the conic C associated to Q is a pair of parallel lines or a line counted twice (or empty). An easy exercise shows that a normal vector to these lines is (b,c). Now, the invariance of the sets I to VI in Figure 1 gives the answer (Figure 4).

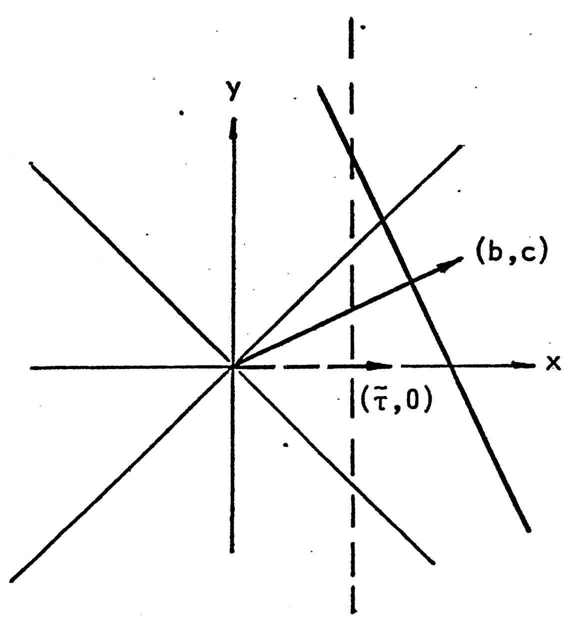


Figure 4

<u>*Case 6*</u>: If $\delta(Q) = \tilde{\tau}(Q) = 0$ (hence $\Lambda(Q) = 0$), then:

(i) If $b = a$, the orbit of Q contains either the matrix:

$$\left[\begin{array}{cc|c} a & a & d' \\ a & a & 0 \\ \hline d' & 0 & 0 \end{array}\right]$$ when $d \neq e$, or the matrix

$$\left[\begin{array}{cc|c} a & a & 0 \\ a & a & 0 \\ \hline 0 & 0 & f' \end{array}\right]$$ when $d = e$.

Furthermore, $d' = d - e$ and $f' = f - \frac{d^2}{a}$.

(ii) If $b = -a$, the orbit of Q contains either the matrix

$$\left[\begin{array}{cc|c} a & -a & d' \\ -a & a & 0 \\ \hline d' & 0 & 0 \end{array}\right]$$ when $d \neq -e$, or the matrix

$$\left[\begin{array}{cc|c} a & -a & 0 \\ -a & a & 0 \\ \hline 0 & 0 & f' \end{array}\right]$$ when $d = -e$.

Furthermore, $d' = d + e$ and $f' = f - \frac{d^2}{a}$.

The Proof is analogous to the previous cases, and we shall omit it.

Remark: The description of the canonical forms shows that the equation of a standard conic can be found in terms of the coefficients a, b, c, d, e and f, of $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$.

For instance, if $\delta(Q) \neq 0$ and $\Lambda(Q) < 0$ (with $b \neq 0$), the standard equation is:

$$\tilde{\tau}(Q)x^2 + 2\sqrt{-\delta(Q)}\,xy - \frac{\Delta(Q)}{\delta(Q)} = 0, \quad \text{if} \quad b > 0$$

or

$$\tilde{\tau}(Q)x^2 - 2\sqrt{-\delta(Q)}\,xy - \frac{\Delta(Q)}{\delta(Q)} = 0, \quad \text{if} \quad b < 0$$

3. *GEOMETRIC INTERPRETATION*

3.1. *The Sign of* Λ *and the Hyperbolas*

As we saw before, according to $\Lambda(Q) > 0$ or $\Lambda(Q) < 0$, a given hyperbola may or may not be reduced to one of the form $a'x^2 + c'y^2 + f' = 0$. Now, we want to find a geometric interpretation for this fact.

We recall here that the group of hyperbolic rotations leaves invariant the sets I to VI of Figure 1, in section 1.1. To simplify notation, we shall denote by $S = I \cup III$, the set of spacelike vectors and by $T = II \cup IV$ the set of timelike vectors.

Proposition 1: If $\delta(Q) < 0$ and $\Lambda(Q) > 0$, then both asymptotes of the hyperbola C associated to Q are timelike or both are spacelike. More precisely, if $a^2 - c^2 > 0$ they are both in T and if $a^2 - c^2 < 0$ they are in S.

Proof: Since S and T are invariant, we may analyse the position of the asymptotes of the reduced form.

First, let's assume $a^2 - c^2 > 0$. The standard equation of C after a hyperbolic rotation is $a'x^2 + c'y^2 = -\frac{\Delta(Q)}{\delta(Q)}$ where $(a')^2 - (c')^2 > 0$ (Case 1, §2.). Then, the asymptotes of this reduced conic are the lines L_1 and L_2, where:

$$L_1 : \sqrt{|a'|}\, x + \sqrt{|c'|}\, y = 0$$

$$L_2 : \sqrt{|a'|}\, x - \sqrt{|c'|}\, y = 0$$

L_1 and L_2 are symmetric with respect to the x-axis.

Since $a'^2 - c'^2 > 0$, $\frac{|a'|}{|c'|} > 1$ and it follows that L_1 lies in T (the same holding with L_2), (Figure 5).

The case $a^2 - c^2 < 0$ is analogous. In this case, L_1 and L_2 are in S (Figure 6).

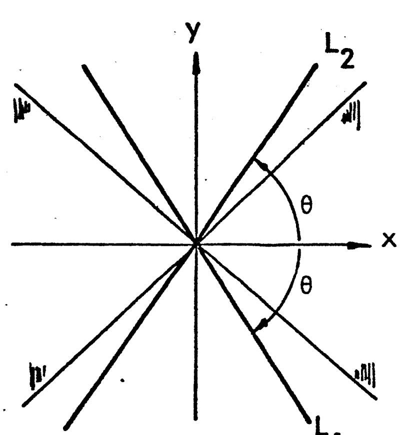


Figure 5

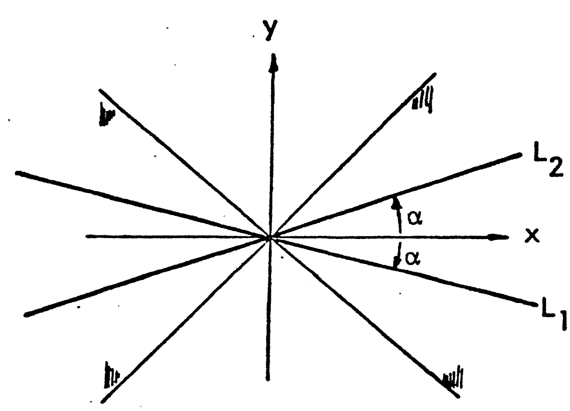


Figure 6

Proposition 2: If $\delta(Q) < 0$ and $\Lambda(Q) < 0$, the hyperbola C has one asymptote which is timelike and one spacelike.

Proof: By Case 2, §2., it follows that the standard equation of C is:

(i) $\bar{\tau}(Q)x^2 + 2\sqrt{-\delta(Q)}\,xy = \frac{\Delta(Q)}{\delta(Q)}$ or:

(ii) $\bar{\tau}(Q)x^2 - 2\sqrt{-\delta(Q)}\,xy = \frac{\Delta(Q)}{\delta(Q)}$

Let's assume that (i) holds. (Case (ii) is analogous). The asymptotes are the lines:

$$L_1 : x = 0 \quad \text{and} \quad L_2 : \bar{\tau}(Q)x + 2\sqrt{-\delta(Q)}\,y = 0$$

To finish the proof, we just observe that the slope of L_2 is in the interval $(-1,1)$.

Proposition 3: If $\delta(Q) < 0$ and $\Lambda(Q) = 0$, at least one of the lines $y = x$ or $y = -x$ is asymptote of C (may be both).

Proof:

(i) Let's assume $b = 0$. Then, from Case 1 of §2., the standard equation is:

$$a'x^2 - a'y^2 = -\frac{\Delta(Q)}{\delta(Q)}$$

which has $y = x$ and $y = -x$ as its asymptotes.

Since these lines are themselves invariant, the result is also true for the original C;

(ii) Let $b \neq 0$. Then, $\Lambda(Q) = 0 \iff \bar{\tau}(Q) = \pm 2\sqrt{-\delta(Q)}$. From Case 3, §2., it follows that the standard equation is:

$$\tilde{\tau}(Q)x^2 \pm \tilde{\tau}(Q)xy = \frac{\Delta(Q)}{\delta(Q)} \quad \text{or} \tag{14}$$

$$\pm\tilde{\tau}(Q)xy - \tilde{\tau}(Q)y^2 = \frac{\Delta(Q)}{\delta(Q)} \tag{15}$$

If, for instance, *(14)* holds, then the asymptotes are

$$L_1 : x = 0 \quad \text{and} \quad L_2 : y = x \quad \text{or} \quad y = -x.$$

Of course, the asymptotes of the original C inherits the properties: L_1 lies in S and L_2 is $y = x$ or $y = -x$.

3.2. *The Geometry of Hyperbolic Rotations*

Two conics equivalent by a hyperbolic motion do not "look the same" in the Euclidean sense. We shall complete the study of §2., by giving geometric constructions relating a given conic with its standard form.

First, we note that a hyperbolic motion is given by a orthogonal matrix, hence preserves area of a plane figure.

Let us consider an ellipse C, with center at the origin. To obtain the standard ellipse C' equivalent to C, we shall determine its vertices by the following procedure: take the points of tangency between C and an invariant hyperbola $x^2 - y^2 = \text{const}$, and slide them along this hyperbola until they meet the coordinate axis. (Figure 7). Furthermore, C and C' have the same area.

A special case occours when the axes of the ellipse C coincide with the invariant lines $y = \pm x$. In this case, C' is always a circle (Figure 8).

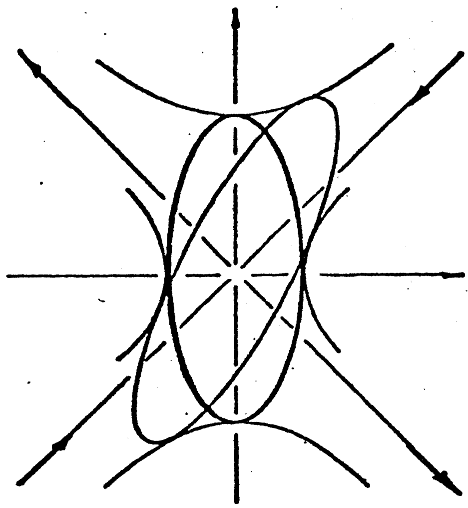


Figure 7

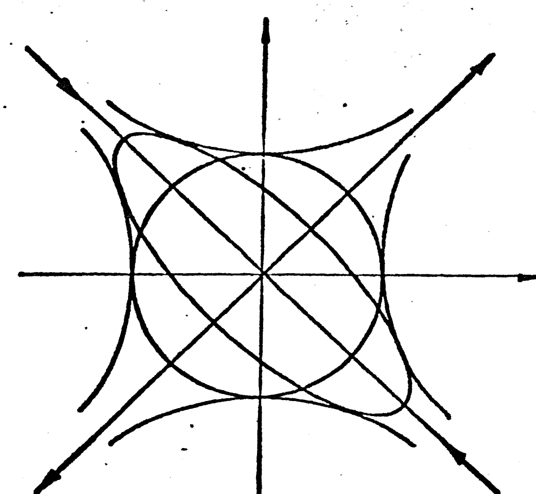


Figure 8

Now, let C be a hyperbola with center at the origin and such that both of its asymptotes are timelike or both are spacelike.

As above, we find a hyperbola $x^2 - y^2 = \text{const}$ tangent to C. Sliding the point P of tangency we determine the vertex P' of the standard hyperbola C'. The point Q, obtained by the intersection of $x^2 - y^2 = \text{const}$ with one of the asymptotes of C, will be sent to a point Q' in the corresponding asymptote of C'. This point Q' is uniquely determined by the condition that the triangles OPQ and OP'Q' have the same area. The other asymptote of C' is symmetric with respect to the x-axis (Figure 9).

When C has one asymptote which is timelike and one spacelike the standard hyperbola C' has the axis $x = 0$ as one of its asymptotes. In this case, sliding the point P given by the intersection of any hyperbola $x^2 - y^2 = k$ $(k < 0)$ with the asymptote of C, we find its image P' in the line $x = 0$.

Also, the intersection of C with the line $y = x$ determines a point Q which slides to Q' along this diagonal. As above, this point Q' is uniquely determined by an area argument.

We can find the other asymptote of C', by observing that the triangles OQR and OQ'R' have the same area, where R is the point in the intersection of any hyperbola $x^2 - y^2 = k$ $(k > 0)$ with the other asymptote of C (Figure 10).

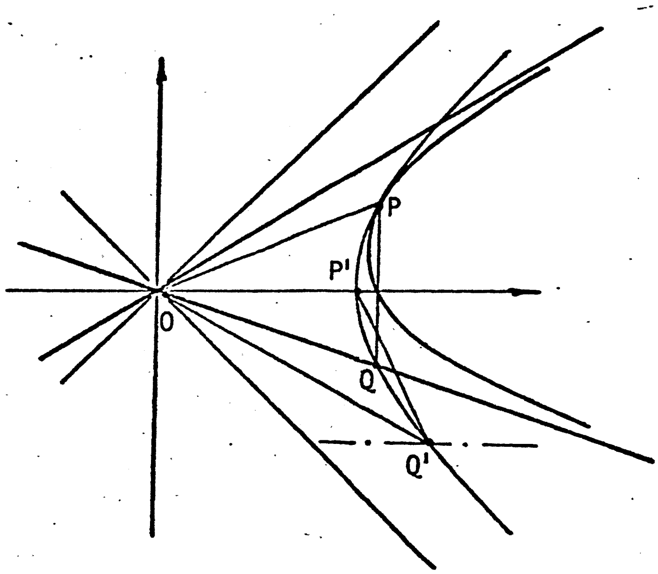


Figure 9

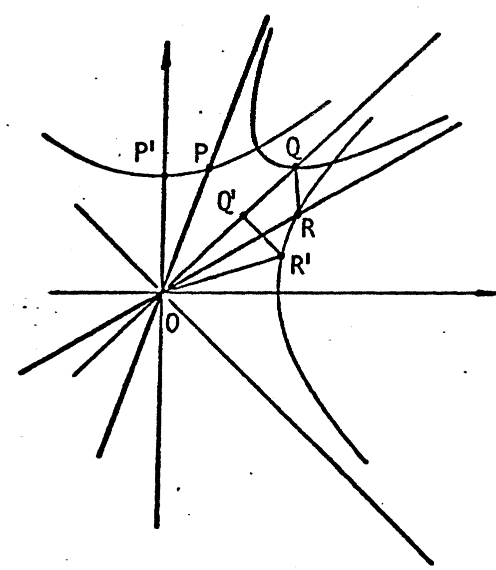


Figure 10

Finally, we note that a parabola C with vertex at the origin can be transformed by a hyperbolic rotation in a parabola C' with axis parallel to the coordinate axis, but with its vertex out of the origin. With similar argument, we can determine C'.

3.3. *Some Geometry of Orbits*

In this section we shall be studying the geometry of the equivalence classes, not on the whole space of conics, but rather in the subspace H^2 of all quadratic forms in two variables.

Now, a quadratic form can be written $ax^2 + 2bxy + cy^2$, so can be identified with the point (a,b,c) in $\mathbb{R}^3$.

We say that a set X in $\mathbb{R}^3$ is invariant, when X is a union of G-orbits.

Figure 11 shows how the cone $\delta = ac - b^2 = 0$ and the pair of planes $\Lambda = (a - 2b + c).(a + 2b + c) = 0$ separate $\mathbb{R}^3$ in invariant sets.

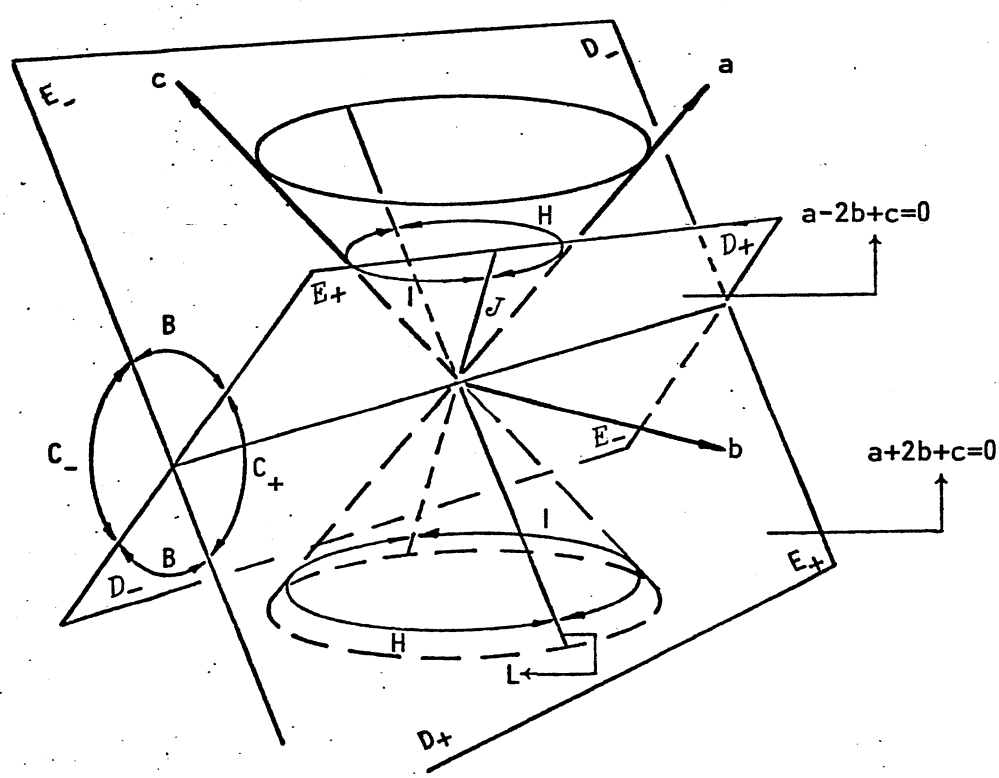


Figure 11

If $\delta > 0$ (hence $\Lambda > 0$), then (a,b,c) is in the interior of the cone $ac - b^2 = 0$, and all such points are of elliptic type (Case 1, §2.).

If $\delta < 0$, then (a,b,c) is in the exterior of the cone and all points are of hyperbolic type. More precisely, they fall in three cases:

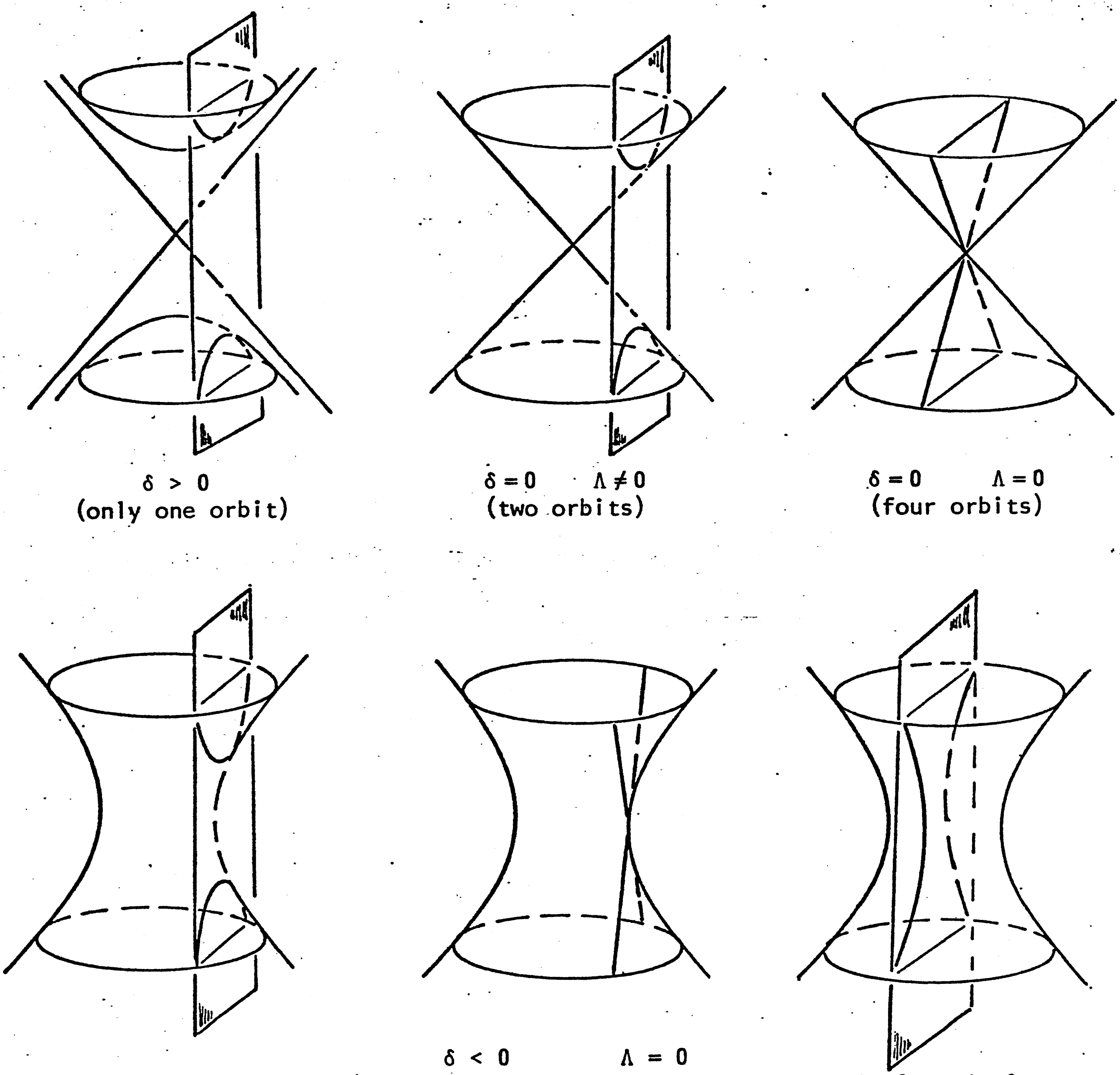
(i) $\Lambda > 0$. By Case 1, §2. it follows that these points correspond to hyperbolas that admit reduction to a sum of squares. They determine the invariant set B of Figure 11.

(ii) $\Lambda < 0$ corresponds to the hyperbolas that do not admit reduction to a sum of squares, as we saw in Case 2, §2. These are the sets C_+ and C_- in Figure 11.

(iii) $\Lambda = 0$. This is the limit case of the previous ones. From Case 3, §2., it follows that D_+, D_-, E_+ and E_- in Figure 11 are invariant sets.

If $\delta = 0$, the points lie on the cone. When $\Lambda > 0$, the subsets H and I of the cone are invariant, as we saw in Case 5, §2. When $\Lambda = 0$, the lines L and J, intersection of the cone with the planes $a + 2b + c = 0$ and $a - 2b + c = 0$ are also invariant sets (Case 6, §2.).

To complete our description, we observe that each orbit is contained in the curve intersection of the hyperboloid $ac - b^2 = \delta$ (or cone, when $\delta = 0$) with the plane $a - c = \tilde{\tau}$. Of course, this curve can be a hyperbola or a pair of concurrent lines. Their intersection with the invariant sets of Figure 11, determine the orbits. The several possibilities are shown in Figure 12.



$\delta > 0$ (only one orbit)

$\delta = 0 \quad \Lambda \neq 0$ (two orbits)

$\delta = 0 \quad \Lambda = 0$ (four orbits)

$\delta < 0 \quad \Lambda > 0$ (only one orbit)

$\delta < 0 \quad \Lambda = 0$ (four orbits lying respectively in D_+, D_-, E_+, E_-)

$\delta < 0 \quad \Lambda < 0$ (two orbits)

Figure 12

4. THE RING OF INVARIANTS

4.1. Proof of Theorem 1

The exposition of this paragraph is devoted to show that any polynomial in $R = \mathbb{R}[z_1,z_2,z_3,z_4,z_5,z_6]$ invariant under the action of the hyperbolic group H is a polynomial in $\bar{\tau}$, δ and Δ. In fact, we prove:

Theorem 1: If $\mathbb{R}[\bar{\tau},\delta,\Delta]$ denotes the sub algebra of R, generated by $\bar{\tau}$, δ and Δ, and I is the algebra of invariant polynomials, then $I = \mathbb{R}[\bar{\tau},\delta,\Delta]$. Moreover $\bar{\tau}$, δ and Δ are algebraically independent over $\mathbb{R}$.

To simplify notation, we shall denote a matrix

$$Q = \left[\begin{array}{cc|c} a & b & d \\ b & c & e \\ \hline d & e & f \end{array}\right] \quad \text{by} \quad (a,b,c,d,e,f).$$

We recall that R can be identified with the ring of polynomial functions over V, as we saw in (*1.4.*).

Now, we define the following subsets of V:

$$X_1 = \{(a,b,0,0,0,f);\ \ b \neq 0,\ \ a^2 - 4b^2 < 0\}$$

$$X_2 = \{(a,0,c,0,0,f);\ \ ac \neq 0,\ \ a \neq -c\}$$

$$U_1 = \{Q \in V : \delta(Q) < 0 \ \text{ and } \ \Lambda(Q) < 0\}$$

$$U_2 = \{Q \in V : \delta(Q) \neq 0 \ \text{ and } \ \Lambda(Q) > 0\}$$

Any element in U_1 is transformed by H in an element in X_1. Also, any element in U_2 is H-equivalent to an element in X_2.

We need the following lemma:

Lemma: Any invariant polynomial is even in the variable b.

Proof: First, we observe that if $(a,b,c,d,e,f) \in U_2$, then $(a,-b,c,d,e,f) \in U_2$.

Hence, given any invariant polynomial P, we define in U_2 the polynomial:

$$\tilde{P}(a,b,c,d,e,f) = P(a,b,c,d,e,f) - P(a,-b,c,d,e,f)$$

Now, we saw that any element (a,b,c,d,e,f) in U_2 is equivalent to an element $(a,0,c,0,0,f)$. The invariance of P implies $P(a,b,c,d,e,f) = P(a,0,c,0,0,f)$.

Hence, $\tilde{P} \equiv 0$ in U_2. Since U_2 is an open set in V, it follows that $\tilde{P} \equiv 0$ in V, that is, P is even in b.

We divide the proof of Theorem 1 in three main steps:

Step 1: Let $\mathcal{B}$ be the ring of the polynomial functions over X_1, and, as before, $\mathcal{I}$ be the set of invariant polynomials.

We define a mapping $H : \mathcal{I} \to \mathcal{B}$ by

$P \to H(P) = P|X_1$, restriction of P to X_1, that is $P(a,b,c,d,e,f) \longmapsto P(a,b,0,0,0,f)$.

It follows that H is a one-to-one homomorphism. In fact, from the invariance of R, $H(P) = P|X_1 = 0$ implies $P|U_1 = 0$. Since U_1 is open in V, the polynomial P must be identically zero in V.

We conclude this step by noting that:

$$H(\tilde{\tau}) = a, \quad H(\delta) = -b^2, \quad H(\Delta) = -b^2 f$$

Step 2: Let P be an invariant polynomial. There are unique polynomials $g_o(a,b),\dots,g_m(a,b)$, such that

$$H(P) = g_o(a,b)f^m + \dots + g_m(a,b)$$

From the previous Lemma, P is even in the variable b. Hence

$$H(P) = h_o(a,-b^2)f^m + \dots + h_m(a,-b^2)$$

Next, let us consider the invariant polynomials $\delta^m P$ and $h_o(\tilde{\tau},\delta)\Delta^m + h_1(\tilde{\tau},\delta)\delta\Delta^{m-1} + \dots + h_m(\tilde{\tau},\delta)\delta^m$. The image by H of each one of these polynomials is equal to

$$(-b^2)^m H(P)$$

As we saw in Step 1, H is 1 - 1, and we have the equality:

$$\delta^m P = h_o(\tilde{\tau},\delta)\Delta^m + h_1(\tilde{\tau},\delta)\delta\Delta^{m-1} + \dots + h_m(\tilde{\tau},\delta)\delta^m \tag{16}$$

Step 3: This part of the Proof follows as in [4]. We repeat it here for completeness.

We rewrite the right-hand side of *(16)* as:

$$\delta^m P = \delta^n(j_o(\tilde{\tau},\Delta)\delta^k + \dots + j_k(\tilde{\tau},\Delta)), \quad \text{where} \quad j_k(\tilde{\tau},\Delta) \neq 0$$

If $n \geq m$, it follows that P is a polynomial in $\tilde{\tau}$, δ and Δ. Now, assuming $n < m$, we obtain a contradiction. In fact,

$$\delta^{m-n}P = j_o(\tilde{\tau},\Delta)\delta^k + \ldots + j_k(\tilde{\tau},\Delta), \quad \text{where} \quad m-n > 0 \qquad (17)$$

Since $j_k(\tilde{\tau},\Delta) \neq 0$, there are real numbers α and β so that $j_k(\alpha,\beta) \neq 0$ and $\alpha.\beta < 0$. Now, we take $\gamma = \sqrt{-\frac{\beta}{\alpha}}$ and $Q = (\alpha,0,0,0,\gamma,0)$.

Evaluating both sides of *(17)* in Q, gives:

$$0 = j_k(\alpha,\beta) \neq 0$$

Finally, $\tilde{\tau}$, δ, Δ are algebraically independent since their images under H are.

4.2. *Final Remarks*

1. If Q_1 is not in the orbit of Q_2, we may ask if there is a polynomial $P \in I$ so that $P(Q_1) \neq P(Q_2)$. The remark at the end of section *1.4.* together with Theorem 1 imply that this is not always possible. For the action of the Euclidean Group on the nondegenerated conics, the answer is affirmative ([4]).

2. Another interesting plane geometry is the Galilean geometry, whose "motions" are the Galilean transformations of classical kinematics. A very complete account of this geometry can be found in [5]. (A brief reference for the orbits of the action of the Galileo Group on conics appears on page 300 of this book).

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