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A NEW PRIMAL-DUAL METHOD FOR LINEAR PROGRAMMING

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This paper presents a new primal-dual method for linear programming using a primal and a dual feasible solutions. The algorithm appears to be efficient when the feasible solutions are easily available.

Key words: Linear Programming, duality.

1. INTRODUCTION

In many classes of linear programming problems we have trivial primal and dual feasible solutions. Until the present moment the authors does not know any procedure that really explores these a priori furnished informations. The classic linear programming methods explore one or other feasible solution. The primal simplex method works with basic primal feasible solutions and it evolues considering only how violated is the complementary dual solutions and it does not consider any dual feasible solution that is available. The dual and classic primal-dual simplex methods proceed in the analogous way.

In this work we develop a method that explores primal and dual feasible solutions.

We also present an application of our method in the discrete linear L_1 approximation problem and we show numerical results for comparation with the primal simplex method.

2. BASIC RESULTS[5]

We consider the standard form of a linear programming problem:

$$(P) \qquad \begin{aligned} \min \quad & z(x) = cx \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

and it's dual problem:

$$(D) \qquad \begin{aligned} \max \quad & w(\lambda) = \lambda b \\ \text{subject to} \quad & \lambda A \leq c \end{aligned}$$

Without loss of generality we suppose that matrix A is full row rank.

Then we consider a partition in it's columns:

$A = (B, N)$ such that B is a non-singular matrix. We associate to the basis B the following two solutions:

primal basic: $\hat{x}_B = B^{-1}b$, $\hat{x}_N = 0$ (1)

complementary dual basic: $\hat{\lambda} = c_B . B^{-1}$ (2)

where $x^t = (x_B^t, x_N^t)$ and $c = (c_B, c_N)$

The basis B is said primal feasible if $\hat{x}_B \geq 0$, and dual feasible if $\hat{c}_N = c_N - \lambda N \geq 0$.

If B is a primal and dual feasible basis, then it is optimal, ie, the basic solutions (1) and (2) solve (P) and (D) respectively.

Consider B a primal feasible basis and $\bar{\lambda}$ any dual feasible solution (it doesn't need to be basic), and let $\bar{c} = c - \bar{\lambda}A$. The primal and dual objective values for $\hat{x}$ and $\bar{\lambda}$ solutions are related by:

$$z(\hat{x}) = c\hat{x} = (c - \bar{\lambda}A)\hat{x} + \bar{\lambda}b = \bar{c}\hat{x} + w(\bar{\lambda}) .$$

Since $\hat{x}_N = 0$, it follows that:

$$z(\hat{x}) = \bar{c}_B \hat{x}_B + w(\bar{\lambda}) .$$

Thus there is a slack between the objective values given by $\bar{c}_B \hat{x}_B \geq 0$.

If the slack vanishes, then the objective values become unique, and so $\hat{x}$ and $\bar{\lambda}$ are the optimal solutions for (P) and (D) respectively.

However, if the slack is positive, one of the solutions is not optimal. We describe in the following section a method whose iterations evolues in two steps, namely primal step and dual step. In the primal (dual) step, the primal (dual) feasible solution is perturbed and we prove that the slack between the objetive values decreases at each step.

3. A PRIMAL-DUAL METHOD

Let

i) B be a primal feasible basis,

ii) $\bar{\lambda}$ be a dual feasible solution.

If the slack $\bar{c}_B \hat{x}_B$ vanishes then $\hat{x}$ and $\bar{\lambda}$ will solve (P) and (D) respectively, otherwise we consider the following modifications:

DUAL STEP

We consider λ as a convex combination of $\bar{\lambda}$ and $\hat{\lambda}$:

$$\lambda = \bar{\lambda} + \varepsilon(\hat{\lambda}-\bar{\lambda}) \ , \quad 0 \leq \varepsilon \leq 1 \ .$$

The direction $d = \hat{\lambda}-\bar{\lambda}$ is a increase direction because

$$w(\lambda) = \lambda b = \bar{\lambda} b + \varepsilon(\hat{\lambda}-\lambda)b = w(\bar{\lambda}) + \varepsilon \bar{c}_B \hat{x}_B .$$

So, the dual objective value increases linearly with the rate $\bar{c}_B \hat{x}_B$, and the new slack between the objective values is:

$$(1-\varepsilon)\bar{c}_B \ \hat{x}_B \ .$$

Thus, we choose ε as great as possible so that λ does not violate the dual restritions.

$$\lambda A = \bar{\lambda} A + \varepsilon(\hat{\lambda}-\bar{\lambda})A \leq c \ .$$

Since $\lambda B \leq c_B$, $0 \leq \varepsilon \leq 1$, only the non-basic indexes associated to the columns of N can prevent the maximum value of ε to be one.

So,

$$\varepsilon \leq \frac{c_j - \bar{\lambda} N^j}{dN^j} = \frac{\bar{c}_j}{dN^j} \ , \quad \text{if} \quad dN^j > 0 \ ;$$

with $d = \hat{\lambda} - \bar{\lambda}$.

Then, an upper bound for ε is:

$$\hat{\varepsilon} = \frac{\bar{c}_k}{dN^k} = \min\{\frac{\bar{c}_j}{dN^j} \;/\; dN^j > 0\} \qquad (3)$$

We choose ε by:

$$\bar{\varepsilon} = \min\{\hat{\varepsilon}, 1\} \qquad (4)$$

From the considerations above and section 2 the next result follows.

Theorem 1 : If $\bar{\varepsilon} = 1$ given by (4), then $\lambda = c_B B^{-1}$ will be a dual feasible solution and this implies that basis B is optimal.

When $\bar{\varepsilon} = \hat{\varepsilon} < 1$, the slack doesn't vanish and so we can execute the primal step.

PRIMAL STEP

Let k be the index that causes the lower ratio in (3). We perturb the primal solution by the simplex strategy:

$$\begin{cases} x_k = \delta \geq 0 \\ x_j = 0 \quad \text{for non-basic } j \neq k. \end{cases}$$

To keep the primal feasibility we choose δ by:

$$\hat{\delta} = \frac{\hat{b}_k}{\hat{N}_{rk}} = \min_i\{\frac{\hat{b}_i}{\hat{N}_{ik}} \;/\; \hat{N}_{ik} > 0\}.$$

It remains to show that the primal step give us a primal solution with a lower slack between the objective values. This is next theorem.

Theorem 2 : The primal step furnishes a new primal feasible solution with a lower slack between the objective values.

Proof : By the choise of $\hat{\delta}$ the solution will be a primal feasible solution, and the new objective value will be:

$$c\hat{x} + \hat{\delta}(c_k - \hat{\lambda}N^k).$$

So, the slack between the objective values will be lower if $c_k - \hat{\lambda}N^k < 0$, but this is true because the way we have chosen $\hat{\varepsilon}$. It can be noted that:

$$0 = c_k - \lambda N^k = c_k - (\bar{\lambda} - \hat{\varepsilon}(\hat{\lambda} - \bar{\lambda}))N^k = c_k - \bar{\lambda}N^k - \hat{\varepsilon}(\hat{\lambda} - \bar{\lambda})N^k >$$

$$c_k - \bar{\lambda}N^k - 1(\hat{\lambda} - \bar{\lambda})N^k = c_k - \hat{\lambda}N^k.$$

and so the theorem follows.

4. GEOMETRIC INTERPRETATION OF THE METHOD

Consider a primal problem with only two constraints and so we can represent it's dual problem in R^2.

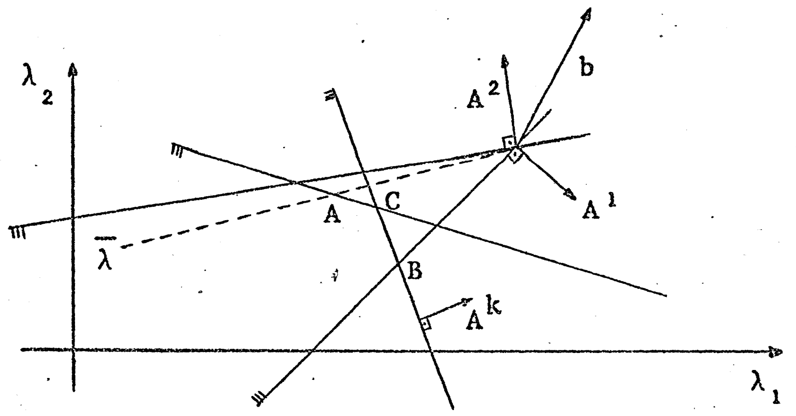

figure 1

The basis $B = (A^1, A^2)$ is primal feasible because b belongs to the convex cone generated by A^1 and A^2. The solution $\hat{\lambda}$ is determined by the intersection of the first and second restriction and so infeasible.

The dual step perturbs the dual feasible solution λ in the $\hat{\lambda}-\bar{\lambda}$ direction and it gives the solution pointed as A in the figure. The new primal basis (A^k, A^1) is choosen, in a way that keeps the primal feasibility. The dual complementary basic solution is represented by B in the figure. The next iteration will result the optimal solution C .

5. AN APPLICATION: DISCRETE L_1 LINEAR APPROXIMATION

The discrete linear L_1 approximation classic problem can be defined by:

$$P_{L_1} \begin{cases} \min \ \|r\| \\ \text{subject to} \ \ Ax + r = b \end{cases}$$

where $A \in R^{m \times n}$, $b \in R^m$, $x \in R^n$, $r \in R^m$ and $\|r\| = \sum_{i=1}^{m} |r_i|$.

We suppose that rank(A) = n.

The solution of this problem is also refered as a L_1 solution of the overdetermined linear equations: $Ax = b$.

Although P_{L_1} is a piecewise linear program, we can easily extend the linear program methods for it [4].The simplex method was utilized by Barrodale and Young [2] to solve P_{L_1}, and later it was best specialized by Barrodale and Roberts [1].

The P_{L_1} dual problem is given by:

$$D_{L_1} \qquad \begin{cases} \max \quad \lambda b \\ \text{s.t.} \ \lambda A = 0 \\ -1 \leq \lambda_i \leq 1 \qquad i=1,\ldots,m \end{cases}$$

The D_{L_1} problem is a tipic linear program problem with lower and upper bounds variables, and so it can be solved directly applicating linear program methods. The direct application of the dual-simplex method furnished the Barrodale-Roberts' algorithm.

Now, basic solutions can be defined considering partitions in the A-rows (rather then A-columns):

$$A = \binom{B}{N} , \quad \text{such that} \quad B^{-1} \quad \text{exists}$$

with $B \in R^{n \times n}$ and $N \in R^{(m-n) \times n}$.

Then, we defined:

- primal basic solution

$$\hat{x} = B^{-1} b_B ,$$

$$\hat{r}_B = 0 , \quad \hat{r}_N = b_N - N\hat{x}$$

-complementary dual basic solution

$$\bar{\lambda}_B = -\bar{\lambda}_N \, NB^{-1} ,$$

$$\bar{\lambda}_N = \text{sgn}(b_N - NB^{-1} b_B)^T = \text{sgn}(\hat{r}_N)^T .$$

with $b^T = (b_B^T , b_N^T)$, $r^T = (r_B^T , r_N^T)$, $\lambda = (\lambda_B , \lambda_N)$.

Note 1 : For all basic partition, the primal basic solution is ever feasible. Then, if a complementary dual basic solution is dual feasible, ie, $-1 \leq \bar{\lambda}_i \leq 1$ for the basic indexes; the basis B will be optimal.

Note 2 : The trivial solution, $\bar{\lambda} = 0$, is a dual feasible one.

Therefore, we can apply the primal-dual method, developed in section 3, to the discrete linear L_1 approximation problem, considering any basic partition (in way of obtainning a primal feasible solution) and the dual feasible dual solution $\overline{\lambda} = 0$.

Some alterations could be hold. In the dual step, only the non-basic variables bounds could limit the ε-increasing. In the primal step we make a piecewise linear search.

Now, we compare the primal simplex method with our primal-dual one (both were specialized to the P_{L_1}). In the primal-simplex method we choose the variable to enter into the basis using the most negative relative cost criterion.

The table below shows the average time and the average number of iterations of 5 examples randonly generated. (We used the FORTRAN RAN function of the DEC-System. They were runned on PDP-11/45).

Table 1

Means of CPU time/iter. number for 5 examples.

		n = 2	n = 5	n = 10
m = 100	Primal	3.69/6.	13.97/20.	38.71/45.
	P-dual	3.13/5.	11.05/15.	34.75/37.
m = 200	Primal	21.20/8.6	54.97/26.2	121.34/49.4
	P-dual	16.11/7.6	51.54/23.6	113.18/43.2
m = 400	Primal	68.28/9.2	218.36/31.2	501.87/64.6
	P-dual	38.10/5.8	167.87/24.2	382.28/48.6

6. CONCLUSIONS AND FINAL REMARKS

In section 3 we described a new linear programming method that utilizes as much a primal feasible basic solution as a dual feasible one.

The numerical results of application in the discrete linear

L_1 approximation problem motivate us to believe that it is convinient to explore the primal and dual feasible solutions simultaneously, if they are available.

In large port problems, the slack between the objectives can be utilized as a stop criterion, since it furnished an improvement upper bound. Then a solution can be satisfatory if:

$$\frac{\bar{c}_B \ \hat{x}_B}{c_B \ \hat{x}_B} < \text{previous tolerance.}$$

The finite convergence of the method is obtained in the degeneracy absence, since the slack will decrease at each iteration, and so there is no chance to the cycle phenomenon . Since exists a finite basis number, the method will stop in a finite iterations number. In the degenerate case, a perturbation technique can be used to hold the finite convergence.

Many other linear programming problem classes offer a primal and a dual feasible solutions beyond the discrete linear L_1 approximation problem showed in section 5. For example, the classical maximal flows in network. A primal feasible solution can be given by null flow and a feasible dual solution can be obtained by any cut [5]. Then a new method for the maximal flow can be developed, updating our method to this problem.

Another problem for which it is possible to apply our method can be stated as:

$$\begin{aligned} &\max \ \ cx \\ &a \leq Ax \leq b \\ &0 \leq x \leq h \end{aligned}$$

where $a \leq 0 \leq b$, $A \in R^{m \times n}$. This problem occur in the determination of the load supplying capability of a transmission-generation system[3]. Here, the basic solutions can be obtained forming the basis B as m linearly independent rows from A and the identity matrix [6]. Two trivially basic solutions are given by $x=0$ and $x=h$. The first is a primal feasible solution and the other is a dual feasible solution.

It can be noted that both the feasible solutions to above examples are basic. In this cases, another algorithm could be developed to handle the basis associated with dual solution. And so, the primal solutions could move through the interior of the poliedral.

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