ON A FIXED POINT INDEX METHOD FOR THE ANALYSIS OF THE ASYMPTOTIC BEHAVIOR AND BOUNDARY VALUE PROBLEMS OF INFINITE DIMENSIONAL DYNAMICAL SYSTEMS AND PROCESSES

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1. Introduction.

Ważewski principle [23] plays an important role in the study of ordinary differential equations. Its applicability is largely due to the fact that in a finite dimensional euclidean space, the unity sphere is not a retract of the closed unit ball. Since this is no longer true in infinite-dimensional Banach spaces the direct extension of Ważewski's principle to processes or semidynamical systems on infinite dimensional Banach spaces has a very limited applicability.

Since in finite-dimensional spaces the fact that the unity sphere is not a retract of the closed unit ball is equivalent to the fact that every continuous mapping of the unity closed convex ball has a fixed point, the main idea of this work is to develop a method based on fixed point index properties instead of retraction properties.

Our fixed point formulation, Corollary 1, is essentially equivalent, in finite dimension, to Ważewski theorem. Although in infinite dimension, Ważewski theorem is no longer applicable, Theorem 1 and Corollary 1 are applicable and give deeper results since fixed point index methods have proved to be very useful in the solution of differential equations either in finite or infinite dimensional spaces. After that we go further and give a formulation of Theorem 1 using Leray-Schauder degree theory or the fixed point index theory for compact or condensing maps. These generalizations, theorems 2, 3, 4, 5 and 6 are stronger even in finite dimension than Ważewski theorem.

After Ważewski paper several papers arised applying Ważewski principle to the asymptotic behavior of ordinary differential equations, C. Olech [17], A. Pliss [20], Mikolajska [14], N. Onuchic [18], A. F. Izé [11] and others. Kaplan, Lasota and Yorke [12] applied Ważewski method to boundary value problem and C. Conley [3] also applied Ważewski method to a boundary value problem for a difusion equations in biology. Since our approach uses Ważewski basic ideas in connection with fixed point index theory it should give, even in finite dimensions, much better results and can be applied also to boundary problems in Hilbert spaces.

2. Dynamical systems and processes.

Let X be a topological space, $R_+ = [0,\infty)$, $A \subset X \times R^+$ a subset of $X \times R^+$ such that

$$\{0\} \times X \subset A$$

and let π be a mapping from A into X we put

$$I_x = \{t \in R^+ | (t,x) \in A\} \text{ , } w_x = \sup I_x$$

$w_x = \infty$ if $\sup I_x$ does not exist.

<u>Definition 1</u>: We say that (X,R^+,A,π) is a local semi-dynamical system if and only if

a) The map $x \to \omega_x$, $x \in X$, is lower semi-continuous in the sense that for every $x \in X$

i) If $\omega_x < \infty$, then for every $\eta > 0$ there exists a neighbourhood V of x such that

$$y \in V \Longrightarrow \omega_y > \omega_x - \eta$$

ii) If $\omega_x = \infty$ then for every $C \in R^+$ there exists a neighbourhood V of x such that

$$y \in V \Longrightarrow \omega_y > C$$

b) π is continuous

c) $\pi(x,0) = x$ for every $x \in X$

d) If $t \in I_x$ and $s \in I_{(x,t)}$ then $s + t \in I_x$

e) $\pi(\pi(t,x),s) = \pi(x,s+t)$ for every $t \in I_x$, $s \in I_{\pi(x,t)}$

Autonomous differential equations on Banach spaces, autonomous functional differential equations are examples of semi-dynamical systems. Dafermos [4] introduced a generalization of dynamical systems as to include also non autonomous differential equations in Banach spaces or nonautonomous functional differential equations.

<u>Definition 2</u>: [4]. Suppose X is a Banach space $R^+ = [0,\infty)$, $u: R \times X \times R^+ \to X$ is a given mapping and define $U(\sigma,t): X \to X$ for $\sigma \in R$, $t \in R^+$ by

$$U(\sigma,t)x = u(\sigma,x,t)$$

A process on X is a mapping $u:R \times X \times R^+ \to X$ satisfying the following property

i) u is continuous

ii) $U(\sigma,0) = I$ (identity)

iii) $U(\sigma+s,t)U(\sigma,s) = U(\sigma,s+t)$

A process is said to be an autonomous process or a semi dynamical system if $U(\sigma,t)$ is independent of σ, that is, $T(t) = U(0,t)$, $t \geq 0$. Then $T(t)x$ is continuous for $(t,x) \in R^+ \times X$.

Let $A \subset R \times X \times R^+$ and $u:A \to X$. We define

$$I_{(x,\sigma)} = \{t > \sigma | (\sigma,x,t) \in A\} \;,\; \omega_{(x,\sigma)} = \sup I_{(x,\sigma)}$$

$\omega_{(x,\sigma)} = \infty$ if $\sup I_{(x,\sigma)}$ does not exist.

Then if the map $(x,\sigma) \to \omega_{(x,\sigma)}$ is continuous in the sense of definition 1, u defines a local process. A local semi-dynamical system is an autonomous local process.

If X is a bounded metric space we define the measure of non-compactness of A to be $\inf\{d > 0 | A$ can be cove - red by a finite number of sets of diameter less than or e- qual to $d\}$. If X is a Banach space and A a bounded subset of X, A inherites a metric from X and we can give the same definition of the measure of non compactness of A.

Let X_1 and X_2 be metric spaces and suppose $f:X_1 \to X_2$ is a continuous map. We say that f is a k-set-contraction if given any bounded set A in X_1, $f(A)$ is bounded and $\gamma_2(f(A)) \leq k\gamma_1(A)$. Of course, γ_i denotes the measure of non-compactness in X_i , $i = 1,2$. We assume that $0 \leq k < 1$. If f is a k-set contraction we define $\gamma(f) = \inf\{k \geq 0 | f$ is a k-set contraction$\}$. We say that $f:X \to X$ is a local strict set-contraction if for every $x \in X$ there is a neighbourhood $N(x)$ such that $f/N(x)$ is a k_x-set-contraction.

M. Furi and A. Vignoli [8] and B. N. Sadovskii gave a slight generalization of k-set-contraction. Given a continuous mapping $f:X_1 \to X_2$ we say that f is a condensing map if for every bounded set $A \subset X_1$ such that $\gamma_1(A) \neq 0$, $\gamma_2(f(A)) < \gamma_1(A)$. We say that f is a local condensing map if every $x \in X$ has a neighbourhood $N(x)$ such that $f/N(x)$ is a condensing map. If f is a k-set-contraction f is condensing but the converse is not true in general: see Nussbaum [16]. If f is linear the two concepts are equivalent.

There are several examples of processes described by functional differential equations and partial differential equations of the evolution type that are compact or α-set contractions.

Example 1 - Let $r > 0$, $C = C([-r,0],R^n)$ the space of continuous functions defined in $[-r,0]$. If $x \in C([\sigma-r,\sigma+A],R^n)$, $A > 0$, $\sigma \in R$ define $x_t(\theta) = x(t+\theta)$. Let $\Omega \subset R \times C$, Ω open, and let D , $f:\Omega \to R^n$ be a continuous functions, D is linear and $D(\phi) = \phi(0) - \int_{-r}^{0} d\mu(t,\theta)\phi(\theta)$ where μ is a matrix function of bounded variation for $\theta \in [-r,0]$. A functional differential equation of the neutral type is a relation of the form

$$\frac{d}{dt} D(t,x_t) = f(t,x_t) \quad , \quad x_\sigma = \phi \tag{1.1}$$

We say that D is an uniformly stable operator if there are constants $K > 1$, $\alpha > 0$ such that $|D(t,\phi)| \le Ke^{-\alpha(t-t_0)}$, $t \ge \sigma$. The solutions of this equations describes a process $U(\sigma,t)\psi = x_t(\sigma,\phi)$. If D is an uniformly stable operator and $t > r$, U is a weak α-set contraction, that is, for every bounded set $A \subset R \times C$ for which $U(A)$ is bounded $\gamma(U(A)) \le k\gamma(A)$. When $D(t,\phi) = \phi(0)$ equation (1.1) is the equation $\dot{x} = f(t,x_t)$ and if $t > r$, the process U is compact [9].

Another general form of a neutral equation for which there is a reasonable existence and continuation of solutions theory, [5], is the equation

$$\dot{x} = f(t,x_t,\dot{x}_t)$$

where $x_0(\theta) = \psi(\theta) \in C([-r,0],R^n)$, $\dot{x}_0(\theta) = \psi(\theta) \in L^p([-r,0],R^n)$ and f satisfies a uniform Lipchitz condition with respect to ψ in $L^p([-r,0],R^n)$, $1 \le p \le \infty$. The process described by these equations is also a k-set contraction.

The following example is given in [22].

Example 2 - Let X be a Banach space and $A:D(A) \to X$ be a closed, densely defined linear operator in X . A is called sectorial if there are constants $\phi,M,a,0 < \phi < \pi/2$, $M \ge 1$, $a \in R$ such that the sector $S_{\phi,a} = \{\lambda \in C | \lambda \ne a, \phi < \arg|\lambda - a| \le r\}$ is contained in $\rho(A)$ the resolvent set of A , and $||(\lambda-a)^{-1}|| \le M/(\lambda-a)$ for all $\lambda \in S_{\phi,a}$. If A is sectorial, then there is a $k \ge 0$ such that $\mathrm{Re}\,\sigma(A + kI) > 0$. Let $A_1 = A + KI$. For

$0 < \alpha < 1$ define

$$A_1^{-\alpha} = \frac{\sin\pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda - A_1)^{-1} d\lambda$$

$A_1^{-\alpha}$ is bounded and injective. Let X^α be the range of $A_1^{-\alpha}$, $X^0 = X$, $X' = D(A)$. Let $A_1^\alpha : X^\alpha \to X$ be the inverse of $A_1^{-\alpha}$, $A^0 = Id_X$, $A' = A$. X^α is dense in X. Define the norm $|| \; ||_\alpha$ on X^α by $||u||_\alpha = ||A_1^\alpha u||$ where $|| \; ||$ is the norm for X. X^α does not depend on the choice of k, and different choices of k yields equivalent norms on X^α. X^α is a Banach space under $||\cdot||_\alpha$.

Suppose $0 \leq \alpha < 1$, V is open in X^α and $f: V \to X$ is a locally Lipchitz continuous mapping. Consider the equation

$$\frac{du}{dt} + Au = f(u) \qquad (1.2)$$

Let $u_0 \in V$. By a solution of (1.2) on $(0,A)$ through u_0 we mean a continuous mapping $u: [0,A) \to V$ such that $u(0) = u_0$, u is differentiable on $(0,A)$, $u(t) \in D(A)$ for $t \in (0,A)$, $t \to f(u(t))$ is locally Hölder continuous, $\int_0^a ||f(u(t))|| dt < \infty$ for some $a > 0$ and (1.2) holds for $t \in (0,A)$. In this definition "$t \to g(t)$" is locally Hölder-continous" means that for every t_0 there exists a neighbourhood W of t_0 and $L, \theta > 0$ such that $||g(t_1) - g(t_2)|| \leq L|t_1 - t_2|^\theta$ for $t_1, t_2 \in W$.

It follows from [7] that under the above assumptions, for every $u_0 \in V$ there exists a unique solution $u(u_0)$ of (1.2) through u_0, defined on a maximal interval $[0, W_{u_0})$. Defining $U(t)u_0 = u(u_0, t)$ for $t < W_{u_0}$ we obtain a local autonomous process or a local semidynamical system.

The most important example of a sectorial operator arises in the following way: Let Ω be an open, bounded set in R^n whose boundary is of class C^{2m}(m an integer). Let $X = L^2(\Omega)$, $D(A) = H^{2m}(\Omega) \cap H_0^m(\Omega)$.

$(Au)(x) = \sum_{|\alpha|\leq m} a_\alpha(x).D^\alpha u(x)$ where the $a_\alpha:\overline{\Omega} \to C$ are continuous mappings and $D^\alpha u$ is understood in the distributional sense. Suppose that A is uniformly strongly elliptic on Ω , i.e., there is a $C_0 > 0$ such that $(-1)^m \operatorname{Re}\{ \sum_{|\alpha|=m} a_\alpha(x).\xi^\alpha\} \geq C_0|\xi|^{2m}$ for all $\xi = (\xi_\alpha)_{|\alpha|\leq m}$ $\xi_\alpha \in R$, and all $x \in \Omega$, then equation (1.2) is called a semi linear parabolic P.D.E. Results in [10], imply that A is sectorial and $R(\lambda,A)$ is compact for every $\lambda \in \rho(A)$.

In the following we consider a process defined for all $t > \sigma$ but it becomes quite clear that the results are true for local processes or local semi dynamical systems.

<u>Definition 3</u>: Suppose u is a process on X . The trajectory $\tau^+(\sigma,x)$ through $(\sigma,x) \in R \times X$ is the set in $R \times X$ defined by

$$\tau^+(\sigma,x) = \{(\sigma+t,U(\sigma,t)x | t \in R^+\}$$

The orbit $\gamma^+(\sigma,x)$ through (σ,x) is the set in X defined by

$$\gamma^+(\sigma,x) = \{U(\sigma,t)x \text{ , } t \in R^+\}$$

<u>Definition 4</u>: If u is a process on X then an integral of the process on R is a continuous function $y:R \to X$ such that for any $\sigma \in R$,

$\tau^+(\sigma,y(\sigma)) = \{(\sigma+t,y(\sigma+t))|t \geq 0\}$. An integral y is an integral through $(\sigma,x) \in R \times X$ if $y(\sigma) = x$.

We assume in the following that the integral througn each $(\sigma,x) \in R \times X$ is unique.

We define

$$\tau^{-1}(x) = \{(\sigma,y) \in R \times X|\ \exists\ t > 0 \text{ such that } U(\sigma,t)y=x\}.$$

If $P_o = (\sigma,x) \in R \times X$ and $z \in \gamma^+(\sigma,x)$, we define

$$t_z = \inf\{t \geq 0|U(\sigma,t)x = z\}$$

$$Q_z = (\sigma+t_z,U(\sigma,t_z)x)$$

$$[P_o,Q_z] = \{(\sigma+t,U(\sigma,t)x|0 \leq t \leq t_z\}$$

$$[P_o,Q_z) = \{(\sigma+t,U(\sigma,t)x|0 \leq t < t_z\}$$

$$(P_o,Q_z] = \{(\sigma+t,U(\sigma,t)x|0 < t \leq t_z\}$$

$$(P_o,Q_z) = \{(\sigma+t,U(\sigma,t)x|0 < t < t_z\}$$

3. Main results

Let Ω be an open set of $R \times X$, ω an open set of Ω , $\omega \subset \Omega$, $\omega \neq \emptyset$ and $\partial\omega = \overline{\omega} \cap (\overline{\Omega-\omega})$ the boundary of ω with respect to Ω . We put:

$$S^o = \{P_o = (\sigma,x) \in \partial\omega|\ \exists\ t > 0 \text{ and } z \in \gamma^+(\sigma,x),$$

with $(P_o,Q_z) \neq \emptyset$ and $(P_o,Q_z) \cap \overline{\omega} = \emptyset\}$

$$S = \{Q \in \partial\omega|\ \exists\ P_o = (\sigma,x) \in \omega \text{ , with } Q \in \tau^+(\sigma,x)$$

and $[P_0, \Omega) \subset \omega\}$

$$S^* = S^0 \cap S .$$

The points of S are called egress points, the points of S^* are called strict egress points.

Given a point $P_0 = (\sigma, x) \in \omega$, if the trajectory $\tau^+(\sigma, x)$ of the process is contained in ω for every $t > 0$, we say that the trajectory is asymptotic with respect to ω; if the trajectory is not asymptotic with respect to ω then there is a $t > 0$ such that $(\sigma+t, U(\sigma,t)x) \in \partial\omega$. Taking

$$t_{P_0} = \{\min t > 0 \,|\, (\sigma+t, U(\sigma,t)x) \in \partial\omega\}$$

$$\Omega = (\sigma+t_{P_0}, U(\sigma, t_{P_0})x) = C(P_0)$$

we have

$$[P_0, \Omega) \subset \omega .$$

The point $C(P_0)$ is called the consequent of P_0.

Define G to be the set of all $P_0 = (\sigma, x) \in \omega$ such that there is $C(P_0)$ and $C(P_0) \in S^*$.

Consider the mapping

$$K: S^* \cup G \to S^*$$

$K(P_0) = C(P_0)$, if $P_0 \in \omega$ and $K(P_0) = P_0$, if $P_0 \in S^*$

The proof of the following is standard, see for example [23], [19].

<u>Lemma 1</u>: The mapping $K: S^* \cup G \to S^*$ is continuous.

To prove the following theorem, we will need to know the basic properties of the fixed point index theory as well as the extensions made by Nussbaum [16] for k-set contraction and condensing maps. We shall say that a topological space X is an absolute neighbourhood retract (ANR) if given any metric space M, a closed sub space $A \subset M$ and a continuous map $f: A \to X$ there exists an open neighbourhood U of A and a continuous map $F: U \to X$ such that $F(a) = f(a)$ for $a \in A$. X is called an absolute retract (AR) if F as above can be defined on all of M. A theorem of Dugundji [6] asserts that any convex subset of a locally convex topological space is an AR. Let $\mathcal{A}$ be the category of compact metric absolute neighbourhood retract (ANR_s). Let $A \in \mathcal{A}$, G be an open subset of A and $f: \overline{G} \to A$ be a continuous function which has no fixed points on ∂G. Then there is a unique integer valued function $i_A(f,G)$ which satisfies the following four properties: [1]

1 - If $f: \overline{G} \to A$ has no fixed point on ∂G and the fixed points of f lie in $G_1 \cup G_2$ where G_1 and G_2 are two disjoint open sets include in G, then $i_A(f,G) = i_A(f,G_1) + i_A(f,G_2)$. In particular if f has no fixed points in G, this is meant to say that $i_A(f,G) = 0$. (the additivity property)

2 - Let I denote the closed unit interval $[0,1]$. If $F: \overline{G} \times I \to A$ (A belongs to $\mathcal{A}$ of course) is a continuous map, and $F_t(x) = F(x,t)$ has no fixed points on

∂G for $0 \le t \le 1$ then $i_A(F_0,G) = i_A(F_1,G)$. (the homotopy property)

3 - If $\overline{G} = A$ then $i_A(f,G) = \Lambda(f)$, the Lefchetz number of f , equals $\Sigma(-1)^K$ trace (f_{*K}) , where $f_{*K}:H_K(A) \to H_K(A)$ is the vector space homomorphism of $H_K(A)$ to $H_K(A)$ and $H_K(A)$ is the Cech homology of A with rational coefficients. (the normalization property)

4 - Let A and B be two spaces which belongs to $\mathcal{A}$. Let $f:A \to B$ be a continuous map. Let V be an open subset of B and $g:\overline{V} \to A$ a continuous map. Assume fg has no fixed points on ∂V . Let $U = f^{-1}(V)$. Then gf has no fixed points on ∂U and $i_B(fg,V) = i_A(gf,U)$. (the commutativity property)

Let G be an open subset of a Banach space X and $g:\overline{G} \to X$ a continuous map such that $g(x) \neq x$ for $x \in \partial G$. Assume that g is compact, that is, $g(G)$ has compact closure. Leray and Schauder [1] defined a fixed point index for g and consequently a degree for $I - g$, I the identity function. We shall denote this degree by $\deg(I-g,G,0)$. It turns out that the Leray-Schauder degree satisfies all the four properties of the fixed point index listed above. So we can define the fixed point index by

$$i_X(g/X \cap G, X \cap G) = \deg(I-g,G,0)$$

where $X = \overline{co}\, g(G)$.

<u>Theorem 1</u>: Assume that there exists sets ω open, $S_1 \subset S \subset \partial\omega$ and $Z \subset \omega \cup S_1$, $Z \neq \emptyset$

satisfying the conditions:

i) $S = S^*$

ii) Z is a compact ANR

iii) There is a retraction $r: S_1 \to Z \cap S_1$

iv) There is a continuous map $\Phi: Z \cap S_1 \to Z \cap S_1$ such that $\Phi(P) \neq P$ for every $P \in Z \cap S_1$

v) $i_Z(\Phi.r.U, Z \cap \omega) \neq 0$

Then there exists at least one point $P_o = (\sigma, x) \in Z \cap \omega$ such that either $C(P_o) \in S - S_1$ or $C(P_o)$ does not exist, that is, $\tau^+(\sigma,x) \subset \omega$.

<u>Proof</u>: Assume that the theorem is not true. Then for every $P_o \in Z \cap \omega$, $C(P_o) \in S_1$ and then $Z \cap \omega \subset G$. Then $Z = (Z \cap S_1) \cup (Z \cap \omega) \subset S \cup G$. From (i) $S = S^*$ and from Lemma 1 the map K is continuous and the restriction of K to $Z \cup S_1$ is also continuous. From condition (iii) there is a retraction $r: S_1 \to Z \cap S_1$.

Then the map $R = r.K: Z \cup S_1 \to Z \cap S_1$ is continuous and takes P_o into $C(P_o) \in Z \cap S_1$.

From condition (iv) the map Φ takes $C(P_o)$ into $\Phi(C(P_o)) = C'(P_o) \neq C(P_o)$ and then the composite map $\Phi.r.K: Z \to Z$ is continuous and $\Phi.r.K(P) \neq P$ for every $P \in Z \cap \omega$. From property 2 of the fixed index $i_Z(\Phi.r.U, Z \cap \omega) = 0$ what is a contradiction with (v). Then there exists at least one point $P_o \in Z \cap \omega$ such that the trajectory through P_o is asymptotic with respect to ω , that is, $\tau^+(\sigma,x) \subset \omega$, or $C(P_o) \in S - S_1$.

<u>Remark</u>: When $S = S_1$ the only final conclusion is that the trajectory $\tau^+(\sigma,x)$ is asymptotic with respect to ω .

<u>Corollary 1</u>: Assume that there exists sets ω open $S \subset \partial\omega$ and $Z \subset \omega \cup S$, $Z \neq \emptyset$ satisfying the conditions

a) $S = S^*$

b) Z is compact and convex

c) $Z \cap S$ is a retract of S

d) There is a continuous map $\Phi : Z \cap S \to Z \cap S$ such that $\Phi(P) \neq P$ for every $P \in Z \cap S$

Then there exists at least one point $P_o \in Z \cap \omega$ such that the trajectory $\tau^+(\sigma,x)$ is contained in ω .

The proof follows easily since a compact convex set is an ANR and then (b) implies (ii). Since $\Phi.r.U : Z \to Z$ is continuous $\Phi.r.U$ has a fixed point in Z and then $i_Z(\Phi.r.U, Z \cap \omega) \neq 0$ what implies (v). In the applications, the following form of Theorem 1 is more useful since we can use the Ascoli-Arzela Theorem to prove the compactness of U .

<u>Theorem 2</u>: Assume that there exists sets, ω open $S_1 \subset S \subset \partial\omega$ and $Z \subset \omega \cup S_1$, $Z \neq \emptyset$, Z closed convex satisfying the conditions

i) $S = S^*$

ii) U is compact

iii) There is a retraction $r:S_1 \to Z \cap S_1$

iv) There is a continuous map $\Phi:Z \cap S_1 \to Z \cap S_1$ such that $\Phi(P) \neq P$ for every $P \in Z \cap S_1$

v) $i_A(\Phi.r.U, Z \cap \omega) \neq 0$, $A = \overline{Co}\ \Phi.r.U(Z)$

Then there exists at least one point $P_o = (\sigma,x) \in Z \cap \omega$ such that either $C(P_o) \in S - S_1$ or $C(P_o)$ does not exist, that is, $\tau^+(\sigma,x) \subset \omega$.

The proof follows as in Theorem 1, since Φ and r are continuous and then $\Phi.r.U$ is a compact map such that $\Phi.r.U(P) \neq P$ for every $P \in Z$ and then $i_A(\Phi.r.U, Z \cap \omega)=0$ what is a contradiction.

<u>Corollary 2</u>: Assume that there exists sets ω open $S \subset \partial\omega$ and $Z \subset \omega \cup S$, $Z \neq \emptyset$, such that

a) $S = S^*$

b) Z is closed bounded convex and U is compact

c) $Z \cap S$ is a retract of S

d) There is a continuous map $\Phi:Z \cap S \to Z \cap S$ such that $\Phi(P) \neq P$ for every $P \in Z \cap S$

Then there exists at least one point $P_o = (\sigma,x) \in Z \cap \omega$ such that $C(P_o)$ does not exist, that is, $\tau^+(\sigma,x) \subset \omega$.

Theorems 1 and 2 are general enough to cover most of the applications and if we restrict ourselves to finite dimension, Corollary 1 is essentially equivalent to Ważewski Theorem [23]. However we can give a more general formulation of Theorem 1 and 2. Actually Theorems 1 and 2 are Corollaries of Theorems 3 and 4 respectively although we prefered to prove them independently.

Theorem 3: Assume that there exists sets ω open in Ω,

$$S_1 \subset S \subset \partial\omega \quad \text{and} \quad Z \subset \omega \cup S_1 \ , \ Z \neq \emptyset$$

satisfying the conditions

i) $S = S^*$

ii) $Z \cup S_1$ is a compact ANR

iii) There exists a continuous map $\Phi : S_1 \to S_1$ such that $\Phi(P) \neq P$ for every $P \in S_1$

iv) $i_{Z \cup S_1}(\Phi U, Z \cap \omega) \neq 0$

Then there exists at least one point $P_o = (\sigma, x) \in Z \cap \omega$ such that either $C(P_o) \in S - S_1$ or $C(P_o)$ does not exist, that is, the trajectory $\tau^+(\sigma,x)$ is asymptotic with respect to ω .

The proof follows as in Theorem 1. If the theorem is not true $C(P_o) \in S_1$ for every $P_o \in Z \cap \omega$. Then, $Z = (Z \cap S_1) \cup (Z \cap \omega) \subset S \cup G$. From Lemma 1 the map K is continuous and the restriction of K to $Z \cup S_1$ is also continuous. The map $\Phi.K : Z \cup S_1 \to Z \cup S_1$ is continuous and $\Phi.K(P) \neq P$ for every $P \in Z \cup S_1$. Then $i_{Z \cup S_1}(\Phi.U, Z \cap \omega) = 0$ a contradiction and the theorem is proved.

Theorem 4: Assume that there exists sets ω open in Ω,

$$S_1 \subset S \subset \partial\omega \quad \text{and} \quad Z \subset \omega \cup S_1 \ , \ Z \neq \emptyset \ , \ Z \cup S_1$$

closed convex satisfying the conditions:

i) $S = S^*$

ii) There exists a continuous map $\Phi : S_1 \to S_1$ such that $\Phi(P) \neq P$ for every $P \in S_1$

iii) U is compact

iv) $i_A(\Phi U, Z \cap \omega) \neq 0$, $A = \overline{Co}(\Phi . U(Z \cup S_1))$

Then there exists at least one point $P_o = (\sigma, x) \in Z \cap \omega$ such that either, $C(P_o) \in S - S_1$ or $C(P_o)$ does not exist, that is, the trajectory $\tau^+(\sigma, P_o)$ is asymptotic with respect to ω .

Proof: Assume that the theorem is not true. Then

$C(P_o) \in S_1$ for every $P_o \in Z \cap \omega$ and then $Z \cap \omega \subset G$. Then $Z = (Z \cap S_1) \cup (Z \cap \omega) \subset S \cup G$. From Lemma 1 the map K is continuous and the restriction of K to $Z \cup S_1$ is also continuous. Since U is compact the map K that takes P_o into $C(P_o)$ is compact. The transformation ΦU is also compact and $\Phi U(P) \neq P$ for every $P \in Z \cup S_1$ . . Hence $i(\Phi U, Z \cap \omega) = 0$, what is a contradiction. Then there exists at least one point $P_o \in Z \cap \omega$ such that either $C(P_o) \in S - S_1$ or the trajectory through P_o is asymptotic with respect to ω .

For delay differential equations and some integral equations the operator U is compact. However most process described by neutral functional differential equations and evolution equations of the evolution type as in examples 1 of 2 of section 2, the process is not compact but is an α-set contraction or a condensing maps. We extend in the following theorems 1 and 2 for α-set contractions or condensing maps. Following Nussbaum [16] we will give an outline of the theory of fixed index for α-set-contractions and condensing maps.

Let X be a closed subset of a Banach space B . We shall say that $X \in F$ if we can write $X = \cup_{i=1}^{n} C_i$, where C_i are closed convex sets in B . The metric on X will always be that which it inherites from B .

Let $G \subset B$ and $g:G \to B$ a continuous map. Assume that the set $S = \{x \in G | g(x) = x\}$ is compact. Let us write $K_1(g,G) = \overline{Co}\, f(G)$, $K_n(g,G) = \overline{Co}\, (G \cap K_{n-1}(g,G))$ and $K_\infty(g,G) = \cap_{n\geq 1} K_n(g,G)$ where $\overline{Co}$ denotes convex closure. It is easy to see that $f:G \cap K_\infty(g,G) \to K_\infty(g,G)$ and $K_\infty(g,G)$ is closed and convex. If G is bounded and $g:G \to X$ is a k-set contraction, $k < 1$, Kuratowski's results [13] also implies that $K_\infty(f,G)$ is compact. Finally, assume that g is a local strict set contraction. By this we mean that for every point $x \in G$ there is a neighbourhood $N(x)$ and a real number $0 \leq k_x < 1$ such that $f/N(x)$ is k_x-set--contraction. Using these assumptions we can find a bounded open neighbourhood G_1 of S such that $g:G_1 \to X$ is a k-set contraction, $k < 1$. Let us write $K_\infty^* = K_\infty(f,G_1) \cap X$, K_∞^* is a compact metric ANR, $G_1 \cap K_\infty^*$ is an open subset of K_∞^* and $g:G_1 \cap K_\infty^* \to K_\infty^*$ is a continuous function satisfying the necessary condition, so $i_{K_\infty^*}(f,G_1 \cap K_\infty^*)$ is defined, R. Nussbaum [16]. We define $i_X(g,G) = i_{K_\infty^*}(g,G_1 \cap K_\infty^*)$. All the usual index property carry through to this setting.

Let G be a bounded open subset of a Banach space B , $g:\overline{G} \to B$, I the identidy on X and $g:\overline{G} \to B$ a k-set--contraction $k < 1$. Assume that $g(x) \neq x$ on ∂G , $A = K_\infty(g,G)$ is compact convex so we can define the Leray--Schauder degree for g as

$$\deg(I-g,G,0) = i_A(g,G \cap A)$$

A similar definition of index can be given for condensing maps. Suppose $X \in \mathcal{F}$, G is an open subset of X and $g:G \to X$ is a continuous map. We shall say that g is an admissible map if only if

(1) $S = \{x \in G | g(x) = x\}$ is closed and bounded

(2) There exists a bounded open neighbourhood U of S with $\overline{U} \subset G$ and a locally finite

covering $\{C_j | j \in J\}$ of X by closed convex sets $G \subset X$ such that a) $g/\overline{U}$ is condensing, b) $I-g/\overline{U}$ is a closed map, c) $g(\overline{U}) \cap C_j$ is empty except for finitely many $j \in J$. If S is empty, U may be empty. If g, U and $\{C_j | j \in J\}$ are as above we shall say that $\langle g,U,\{C_j | j \in J\}\rangle$ is an admissible triple.

Now let g be an admissible map and let $\langle g,U,\{C_j | j \in J\}\rangle$ be an admissible triple. Since $(I-g)(x) \neq 0$ for $x \in \partial U$ and since $(I-g)/\overline{U}$ is a closed map, $\inf\{||(I-g(x))|| \;\big|\; x \in \partial U\} = \delta > 0$. If $f:\overline{U} \to X$ is a continuous map we shall say that f is an admissible approximation with respect to $\langle g,U,\{C_j, j \in J\}\rangle$ if:

1) f is a k-set-contraction, $k < 1$

2) $||f(x) - g(x)|| < \delta$ for $x \in \overline{U}$,
$\delta = \inf\{||I-g|| \;\big|\; x \in \partial U\}$

3) For all $j \in J$ and $x \in \overline{U}$, if $g(x) \in C_j$ then $f(x) \in C_j$

Let now G be an open subset of a space $X \in \mathcal{F}$ and let $g:G \to X$ be a continuous function which is admissible. Let $\langle g,U,\{C_j | j \in J\}\rangle$ be an admissible triple and let f be an admissible approximation with respect to this triple. We define $i_X(g,G) = i_X(f,U)$. In [16] is proved that this definition is well defined.

Let X be a closed convex subset of a Banach space B, G is an open subset of X and $f:\overline{G} \to X$ is a continuous condensing map such that $g(x) \neq x$ for $x \in \partial G$. Then the fixed point index can be described in terms of the Leray--Schauder degree. First it is not hard to show that there exists $\delta > 0$ such that $||x - f(x)|| \geq \delta$ for $x \in \partial G$. Select any fixed $x_o \in X$ and define $g_t(x) = tg(x) + (1-t)x_o$ for $0 < t < 1$ and take t so close to 1 that $||g(x) - g_t(x)|| < \delta$ for $x \in \partial G$. Define $K_1 = \overline{Co}\, f_t(G)$, $K_n = \overline{Co}\, f_t(G \cap K_{n-1})$ for $n > 1$ and $K_\infty = \cap_{n\geq 1} K_n$. One can prove that K is compact (possibly empty) and convex and that $g_t(G \cap K_\infty) \subset K$. If K_∞ is empty define $i_X(g,G) = 0$. If K_∞ is not empty let K be any compact convex set such that $K \supset K_\infty$ and $g_t(G \cap K) \subset K$. K_∞ is itself such a set so the collection of such K is non empty. Let ρ be any retraction of B onto K . (A result of Dugundji [6] guarantees the existence of such a retraction), and let H be any bounded open neighbourhood of the (compact) fixed point set of g_t in $\overline{G}$ such that $\overline{H} \subset \rho^{-1}(G \cap K)$. Then one can prove that $i_X(g,G) = \deg(I-f_t \cdot \rho, H, 0)$. In particular the integer on the right hand side is independent of the particular K chosen, the retraction ρ, H, t and x_o . We say that a set A is admissible if $A \subset \mathcal{F}$. For example A is closed convex.

Theorem 5: Assume that there exists sets ω open in Ω, $S \subset \partial\omega$, $S_1 \subset S$ and $Z \subset \omega \cup S_1$, $Z \neq \emptyset$, $Z \cup S_1$ closed convex, satisfying the conditions:

i) $S = S^*$

ii) There exists a continuous map $\Phi: S_1 \to S_1$ such that $\Phi(P) \neq P$ for every $P \in S_1$

iii) U is condensing

iv) $i_{Z \cup S_1}(\Phi U, Z \cap \omega) \neq 0$

Then either $P_o \in S - S_1$ or the trajectory $\tau^+(\sigma,x)$ through (σ,x) is contained in ω. The proof follows as in Theorem 2.

Theorem 6: Assume that there exists sets ω open in Ω, $S_1 \subset S \subset \partial\omega$ and $Z \subset \omega \cup S_1$, $Z \neq \emptyset$, Z closed convex satisfying the conditions:

i) $S = S^*$

ii) $Z \cap S_1$ is a retract of S_1, that is, there exists a retraction $r: S_1 \to Z \cap S_1$

iii) There exists a continuous map $\Phi: Z \cap S_1 \to Z \cap S_1$ such that $\Phi(P) \neq P$ for every $P \in Z \cap S_1$

iv) $\Phi.r.U$ is condensing

v) $i_Z(\Phi r U, Z \cap \omega) \neq 0$

Then there exists at least one point $P \in Z \cap \omega$ such thateither $C(P_o) \in S - S_1$ or the trajectory $\tau^+(\sigma,x)$ through (σ,x) is contained in ω. The proof follows as in Theorem 5.

<u>Corollary 3</u>: Assume that there exists ω open, $S \subset \partial\omega$ and $Z \subset \omega \cup S$, $Z \neq \emptyset$, such that

a) $S = S^*$

b) Z is closed convex bounded and U is condensing

c) $Z \cap S$ is a retract of S

d) There is a continuous map $\Phi: Z \cap S \to Z \cap S$ such that $\Phi(P) \neq P$ for every $P \in Z \cap S$

Then there exists at least one point $P_0 = (t,x) \in Z \cap \omega$ such that $C(P_0)$ does not exist, that is, $\tau^+(\sigma,x) \subset \omega$.

In the applications of the theorems above we must give criteria to verify the condition $i_X(f,G) \neq 0$. In most applications G is a closed convex subset of a Banach space X and the index can be described in terms of the Leray--Schauder degree. If D is a closed convex subset of a locally convex topological vector space X we say that D is a wedge if $x \in D$ implies $tx \in D$ for $t \geq 0$. We call D a cone (with vertex at 0) if D is a wedge and $x \in D$, $x \neq 0$ implies that $-x \notin D$. The following result is given by Nussbaum [15].

Assume that D is a wedge in a Banach space X, r and R are unequal positive numbers,
$G_1 = \{x \in D \mid ||x|| < \rho_1 = \max(r,R)\}$ and $f:\overline{G}_1 \to D$ is a condensing map. Assume that there exists $h \neq 0$ such that $x - f(x) \neq th$ for all $x \in D$ with $||x|| = R$ and all $t \geq 0$ and suppose that $x - tf(x) \neq 0$ for $x \in D$, $||x|| = r$ and $0 \leq t \leq 1$.

Then a) If $r < R$, one has $i_D(f,U) = -1$ and b) If $r < R$, $i_D(f,U) = +1$.

A more general formulation is given in [16] for local condensing maps and some other condition are also given to verify $i_D(f,U) \neq 0$.

Some other criteria are known. For example, suppose that $h: R^2 \to R^2$ is of class C^1, $h(0) = 0$, and $x = 0$ is an isolated zero of h. Then $i(h,0) = 1 + \frac{1}{2}(E - H)$ where E and H are integers associated with the flow $\dot{x} = h(x)$, that is, E is equal to the number of elliptic regions and H is equal to the number of hyperbolic regions. This formula for the index is due to Poincaré.

Several authors applied Ważewski method to delay differential equations and partial differential equations [18], [19], [21], considering the solutions of these equations as elements of a finite dimensional euclidean space and defining properly the concepts of egress and ingress points. The reason because they considered the solutions of these equations as elements of an euclidean space is that they used Ważewski retract theorem in its original form, that is not applicable, in a natural way, in infinite dimension. Rybakowski [22] following an idea already used by Razumikhin [9] to extend Lyapunov stability theorems to delay equations, considered the space of continuous functions $\phi \in C = C([-r,0],R^n)$ such that $||\phi(\theta)|| \leq |\phi(0)|$ and considered Ważewski theorem, in its original form, for what is called a system of curves in R^n. One problem that arises in the application of Rybakowski approach is that we have to prove that the "delay" of the curve stay inside ω, that is, if $P:\overline{Z} \cap (Z \cup S) \to C$ is such that $P = (t,x)' \in \overline{Z} \cap (Z \cup S)$ then $(t,p(P)) \in \Omega$ and $(t+\theta,p(P)(\theta)) \in \omega$ for $\theta \in [-r,0)$.

If we try, for example, to extend the results obtained by Izê [11] to delay equations, the set ω_p will be defined by:

$$\omega_p = \{(t,x) \in R^+ \times R^n \,\big|\, |x_i|^2 - |x_p|^2 \psi^2(t) < 0 ,$$

$$i = 1,2,\ldots,p-1 , \ |x_j|^2 - |x_p|^2 \Psi^2(t) < 0 ,$$

$$j = p+1,\ldots,q , \ t > \sigma\}$$

where ψ and Ψ are continuous functions to be determined. If $n = 2$, for example, a section of ω_p for $t = \tau$ is a conic region bounded by two straight lines passing through the origin. If (t,x) is inside the region and close to the origin $(t+\theta,p(P)(\theta))$ may not stay inside ω_p since ω_p varies with t.

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