INTEGRAL STABILITY FOR FUNCTIONAL DIFFERENTIAL EQUATIONS OF THE NEUTRAL TYPE

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by

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1. INTRODUCTION

In several problems of science it is important to study stability under some kind of disturbance, that is, if some system is stable, to know if it remains stable under the action of some disturbance. In some problems, for example, in control theory the function that represents the disturbance is not necessarily continuous and the smallness of the perturbation means that the integral of the function that represents the perturbation is small in some sence and we have the notion of integral stability. In [5] Yoshizawa studied stability of retarded equations under constantly acting disturbance. We study here integral stability by a different method using the technique of Lyapunov functionals and prove that for a general class of neutral functional differential equations uniform asymptotic stability implies integral asymptotic stability.

2. PRELIMINARY

Suppose $r > 0$ is a given real number, $R = (-\infty,\infty)$, E^n is a real or complex n-dimensional linear vector space with norm $|\cdot|$, $C = C([-r,0],E^n)$ is the Banach space of continuous functions mapping

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the interval $[-r,0]$ into E^n with the topology of uniform convergence given by the norm $||\phi|| = \sup_{-r\le\theta\le 0} |\phi(\theta)|$. If $\sigma \in R$, $A \ge 0$ and $x \in C([\sigma-r,\sigma+A],E^n)$ then for any $t \in [\sigma,\sigma+A]$ we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$. If Ω is an open subset of $R \times C$ and $f,D : \Omega \to E^n$ are given continuous function we say that the relation.

$$\frac{d}{dt}D(t,x_t) = f(t,x_t) \tag{1}$$

is a functional differential equation.

We say that (1) is a neutral functional differential equations if D is linear in ϕ, $D(t,\phi) = \phi(0) - g(t,\phi)$, where $g(t,\phi) = \int_{-r}^{0} d\mu(t,\theta)\phi(\theta)$, and μ is an $n \times n$ matrix function of bounded variation for $\theta \in [-r,0]$ and there is a continuous scalar function $\rho(s)$ strictly increasing, $\rho(0) = 0$ such that

$$\left|\int_{-s}^{0} [d_\theta\mu(t,\theta)]\phi(\theta)\right| \le \rho(s) \sup_{-s\le\theta\le 0} |\phi(\theta)|, \quad t \in R \tag{2}$$

If f takes bounded sets of $R \times C$ into bounded set we have continuous dependence of solution, continuity and continuation of solutions to a maximal interval of existence of (1), [2].

3. *INTEGRAL STABILITY*

We analyse in the following the relationship between uniform asymptotic stability, and integral stability.

Consider the system of functional differential equation of neutral type

$$\frac{d}{dt}D(t,x_t) = f(t,x_t), \quad x_\sigma = \phi \tag{3}$$

$$\frac{d}{dt}D(t,y_t) = f(t,y_t) + h(t,y_t), \; y_\sigma = \phi \qquad (4)$$

where $D : [\sigma,\infty) \times C \to R^n$ is a linear continuous operator, $f,h : \Omega \to R^n$, $\Omega \in R \times C$, are continuous and f takes bounded sets of $R \times C$ into bounded sets. We assume also that $f(t,\phi)$ satisfy a local Lipschitz condition with respect to ϕ uniformly for t in bounded sets of $[\sigma,\infty)$ and $f(t,0) = h(t,o) = 0$. Under the above conditions systems (3) and (4) have a unique solution $x_t(\sigma,\phi)$, $x_\sigma(\sigma,\phi) = \phi$ throught (σ,ϕ) and these solution can be continued to a maximal interval of existence as long as they exists and a bounded solution of (3) or (4) exists in $[\sigma,\infty)$.

In what follows we assume that D is a uniformly stable operator, that is, the solution $x = 0$ of the difference equation $\frac{d}{dt}D(t,x_t) = 0$, $x_\sigma = 0$ is uniform asymptotically stable. The following Lemma, [4], and Theorem 1 that is a Corollary of Theorem 1 of [3] is used to prove the following results of this paper on integral stability.

Lemma 1: Assume that $f(t,\phi)$ satisfy a Lipschitz condition with respect to ϕ in a neighbourhood of the origin for $t \geq 0$ and $|f(t,\phi)| \leq M||\phi||$, $M \geq 0$.

If $x_t(\sigma,\phi)$ and $x_t(\sigma,\phi)$ are solutions of (1) then there exists constants $K_o > 0$, $L_o > 0$ such that

$$|x_t(\sigma,\phi) - x_t(\sigma,\psi)| \leq K_o e^{L_1(t-\sigma_o)} |\phi-\psi|, \quad t \geq \sigma.$$

Proof: Let $x(t)$ and $y(t)$ be solution of (1) with initial conditions ϕ and ψ respectively then

$$|x(t)-y(t)| \leq |(\phi(0)-\psi(0)) + g(\sigma,(\phi-\psi)) + g(t,x_t-y_t)$$

$$+ \int_\sigma^t [f(s,x_s)-f(s,y_s)]ds| .$$

Then there are constants L and $\alpha > 0$ such that $|x(t) - y(t)| \leq$ $\leq L||\phi-\psi||/2 + ||x_t-y_t||/2$, for $\sigma \leq t \leq \sigma+\alpha$. Since we can that $L \geq 1$ it is easy to see that $||x_t-y_t|| \leq L||\phi-\psi||/2 + ||x_t-y_t||/2$ and then $||x_t-y_t|| \leq L||\phi \; \psi||$. By icterating this inequality a number of times we have for $t \in [\sigma+(n-1)\alpha, \sigma+n\alpha]$ and $t \in [\alpha, \sigma+\tau]$.

$$||x_t(\sigma,\phi) - y_t(\sigma,\psi)|| \leq L^n||\phi \; \psi||.$$

If we choose K_o in such a way that $\log K_o \leq L_1\alpha$ and $L_1 \geq \frac{\log L}{\alpha}$, then

$$L^n \leq K_o e^{L_1(n-1)\alpha} \leq K_o e^{L_1(t-\sigma)}, \quad \sigma \leq t \leq \tau$$

and the Lemma is proved.

If $V : [\tau,\infty) \times C \to R$ is a continuous function we define the "derivative" $V(t,\phi)$ of V along the solution of (1) by

$$\dot V(t,\phi) = \dot V_{(1)}(t,\phi) = \overline{\lim_{h\to 0+}} \frac{1}{h}[V(t+h,x_{t+h}(t,\phi)-V(t,\phi))].$$

The following theorem is a Corollary of Theorem 1 of (3).

Theorem 1: If the zero solution of (1) is uniformly asymptotically stable then there are constants $\delta_o > 0$, $K = K(\delta_o) > 0$, $M > 0$ and continuous decreasing positive functions, $u(s)$, $c(s)$, $b(s)$, $w(s)$, $v(s)$, for $0 \leq s \leq \delta_o$, $u(0) = c(0) = b(0) = w(0) = v(0)$, such that:

i) $u(|D(t,\phi)|) \leq V(t,\phi)$

ii) $c(||\phi||) \leq V(t,\phi) \leq b(||\phi||)$

iii) $\dot V_{(4)}(t, \;) \leq \dot V_{(3)}(t, \;) + M|h(t,\phi)|$

iv) $\dot V(t,\phi) \leq -w(|D(t,\phi)|)$; $\dot V(t,\phi) \leq -v(||\phi||)$

v) $V(t,\phi) - V(t,\psi) \le K||\phi-\psi||$, for every $t \ge \tau$, ϕ, $\psi \in C$, $||\phi||$, $||\psi|| \le \delta_o$.

Proof: The derivative of V along the solutions of (4) is

$$\dot{V}_{(4)}(t,\phi) \le \dot{V}_{(3)}(t,\phi) + \frac{K}{1-\rho(s_o)}|h(t,\phi)|$$

$$= \dot{V}_{(4)}(t,\phi) + M|h(t,\phi)|, \quad t \ge \sigma, \quad |\phi| \le \delta_o$$

Since in the condition of atomicity (2), because D is linear, we can take $\rho(s)$ small enough in such a way that $0 < \rho(s_o) < 1$. $K > 0$ is the constant in condition (v) of Theorem 1. Condition (iv) follows easily since $|D(t,\phi)| \le K(||\phi||)$.

Definition I: The solution $x \equiv 0$ of (3) is integrally stable if given $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|\phi| < \delta, \int_\sigma^\infty \sup_{|\psi|\le\varepsilon} |h(t,\psi)|dt < \delta_2$$

implies $||y_t(\sigma,\phi)|| < \varepsilon$ for $t \ge \sigma$ where $y_t(\sigma,\phi)$ is a solution of (4).

Definition II: The solution $x = 0$ of (3) is asymptotically integrally stable if it is integrally stable and given $\varepsilon > 0$ there exists $\eta > 0$ and function $T = T(\eta,\varepsilon) > 0$ and $\gamma = \gamma(\eta,\varepsilon) > 0$ such that

$$|\phi| < \eta \quad \text{and} \quad \int_\sigma^\infty \sup_{|\psi|\le\varepsilon} |h(t,\psi)|dt < \gamma(\eta,\varepsilon)$$

$||y_t(\sigma,\phi)|| < \varepsilon$ for $t \ge \sigma + T(\eta,\varepsilon)$ where $y_t(\sigma,\phi)$ is a solution of (4).

Lemma 2: The solution $x \equiv 0$ of (3) is aymptotically integrally stable if only if it is integrally stable and there exists functions

$T(\eta,\varepsilon) > 0$, $\gamma(\eta,\varepsilon) > 0$, $0 < \eta < \eta_o$, $\varepsilon > 0$ such that for every continuous function $\alpha(t)$ with $\int_\sigma^\infty |\alpha(t)|dt < \gamma(\eta,\varepsilon)$, $||\phi|| < \eta$ and $t \geq \sigma + T(\eta,\varepsilon)$ the solution $y_t(\sigma,\phi)$ of system:

$$\frac{d}{dt}D(t,y_t) = f(t,y_t) + \alpha(t) \tag{5}$$

satisfies the inequality $||y_t(\sigma,\phi)|| < \varepsilon$.

Proof: If the zero solution of (3) is asymptotically integrally stable, the result follows immediately. Assume that the converse is not true, that is, there exists $\delta < \delta_o$ and $\varepsilon' > 0$ such that for any $T(\delta',\varepsilon') > 0$ and $\gamma(\delta',\varepsilon') > 0$ there exists ϕ with $||\phi|| < \delta'$ and $h(t,\psi)$ satisfyng

$$\int_\sigma^\infty \sup_{|\psi|\leq\varepsilon} |h(t,\psi)|dt < \gamma(\delta',\varepsilon')$$

and t_1, $t_1 > \sigma + T(\delta',\varepsilon')$ for which $||y_{t_1}(\sigma,\phi)|| \geq \varepsilon'$ where $y_t(\sigma,\phi)$ is a solution of (4).

If we take $h(t,y_t(\sigma,\phi)) = \alpha(t)$ fot $t \in [\sigma,t_1]$:

$$\int_\sigma^{t_1}|\alpha(t)|dt = \int_\sigma^{t_1}|h(t,y_t(\sigma\ \phi))|dt \leq \int_\sigma^{t_1} \sup_{|\psi|\leq\varepsilon} |h(t,\psi)|dt \leq$$

$$\leq \int_\sigma^\infty \sup_{|\psi|\leq\varepsilon} |h(t,\psi)|dt < \gamma(\delta',\varepsilon').$$

Let $z_t(\sigma,\psi)$ a solution of (5) with $\alpha(t)$ choosen as above. From the hypotheses, $\int_\sigma^\infty|\alpha(t)|dt < \gamma(\delta',\varepsilon')$ $||\phi|| < \delta'$ and $t_1 > \sigma + T(\delta',\varepsilon')$ imply that $|z_{t_1}(\sigma,\phi)| < \varepsilon'$. But for $t \in [\sigma,t_1]$, $y_t(\sigma,\phi) \equiv z_t(\sigma,\phi)$ and then $||y_{t_1}(\sigma,\phi)|| < \varepsilon'$ a contradiction.

Corollary 1: If the solution $x \equiv 0$ of (3) is integrally stable and if for every δ, $0 < \delta < \delta_o$, $\varepsilon > 0$, there exists $T(\delta,\varepsilon) > 0$, $\gamma(\delta,\varepsilon) > 0$ such that for every continuous function $\alpha(t)$ on $[\sigma,t_1]$,

$t_1 > \sigma + T(\delta,\varepsilon)$ and such that $|\phi| < \delta$, $\int_\sigma^{t_1} |\alpha(t)|dt < \gamma(\delta,\varepsilon)$ implies $||y_t(\sigma,\phi)|| < \varepsilon$ on $[\sigma+T,t_1]$ where $y_t(\sigma,\phi)$ is a solution of (5), then the solution $x \equiv 0$ of (3) is asymptotically integrally stable.

Proof: Since the solution $x \equiv 0$ of (3) is integrally stable let $T(\sigma,\varepsilon)>0$, $\gamma(\delta,\varepsilon)>0$ and $\alpha(t)$ a continuous function satisfying $\int_\sigma^\infty |\alpha(t)|dt < \gamma(\delta,\varepsilon)$. We will show that if $||\phi|| < \delta$ the solution $y_t(\sigma,\phi)$ satisfies $||y_t(\sigma,\phi)|| < \varepsilon$ for $t \geq \sigma + T$.

Assume that this is not true, that is, there exists t_1, $t_1 \geq \sigma + T$, such that $||y_{t_1}(\sigma,\phi)|| \geq \varepsilon$.

Since $t_1 \geq \sigma + T$, from the hypotheses above we have

$$\int_\sigma^{t_1} |\alpha(t)|dt \leq \int_\sigma^\infty |\alpha(t)|dt < \gamma(\delta,\varepsilon)$$

and this implies that $||y_{t_1}(\sigma,\phi)|| < \varepsilon$ on $[\sigma+T,t_1]$ what is a contradiction.

Theorem 2: Let $V(t,\phi)$ a Lyapunov functional defined for $t \geq 0$, $||\phi|| \leq \delta_o$ satisfying:

1) $V(t,\phi) \geq a(||\phi||)$ where a is a continuous function, nonnegative, nondecreasing, $a(0) = 0$, $V(t,0) = 0$;

2) $|V(t,\phi) - V(t,\psi)| \leq M_1|\phi-\psi|$, $M_1 > 0$, $||\phi||$, $||\psi|| \leq \delta_o$;

3) $\overline{\lim_{h\to 0^+}} \frac{1}{h}[V(t+h,x_{t+h}(t,\phi)) - V(t,\phi)] \leq g(t)V(t,\phi)$;

where $\int_o^\infty g(t)dt < \infty$, $g(t) \geq 0$, $x_t(\sigma,\phi)$ is a solution of (3) then the solution $x \equiv 0$ of (3) is integrally stable.

Proof: Let $x_t(\sigma,\phi)$ and $y_t(\sigma,\phi)$ be solutions of (3) and (4), respectively. From Theorem 1, we have

$$\dot V_{(4)}(t,\phi) \le \dot V_{(3)}(t,\phi) + M|h(t,\phi)|$$

for $t \ge \sigma$, $||\phi|| \le \delta_o$, $M > 0$.

Hence $\dot V_{(4)}(t,\phi) \le g(t)V(t,\phi) + M|h(t,\phi)|$. Let $y_t(\sigma,\phi)$ a solution of (4) and let us solve the inequality:

$$\dot V(t,y_t(\sigma,\phi)) - g(t)V(t,y_t(\sigma,\phi)) \le M|h(t,y_t(\sigma,\phi))|.$$

$$\cdot\ e^{-\int_\sigma^t g(s)ds} \cdot \dot V(t,y_t(\sigma,\phi)) + e^{-\int_\sigma^t g(s)ds}(-g(t)V(t,y_t(\sigma,\phi))) \le$$

$$\le Me^{-\int_\sigma^t g(s)ds}|h(t,y_t(\sigma,\phi))|.$$

The left side of the inequality above is less than or equal the derivative of

$$V(t,y_t(\sigma,\phi))e^{-\int_\sigma^t g(s)ds}$$

Then the solution of the differential inequality is

$$V(t,y_t(\sigma,\phi)) \le V(\sigma,\phi)e^{-\int_\sigma^t g(s)ds} + Me^{\int_\sigma^t g(s)ds}\int_\sigma^t e^{-\int_\sigma^t g(s)ds}|h(t,y_t(\sigma,\phi))|dt.$$

If $||\phi|| < \eta_1$, $\int_\sigma^\infty g(s)ds = k_1$, $\int_\sigma^t |h(s,y_s(\sigma,\phi))|ds < \eta_2$, and if we take $\eta = \max\{\eta_1,\eta_2\}$ and $\overline{M} = \max\{M_1,M\}$ we will have

$$a(||y_t(\sigma,\phi)||) \le V(t,y_t(\sigma,\phi)) \le M_1||\phi||e^{k_1} + Me^{k_1}\eta_2$$

$$a(||y_t(\sigma,\phi)||) \le V(t,y_t(\sigma,\phi)) \le M_1\eta_1 e^{k_1} + Me^{k_1}\eta_2$$

or

$$a(||y_t(\sigma,\phi)||) \le V(t,y_t(\sigma,\phi)) \le \overline{M}e^{k_1} + \overline{M}e^{k_1}\eta = 2\overline{M}e^{k_1}\eta.$$

Since $a(s)$ is a continuous function, positive, nondecreasing for $s > 0$, $a(0) = 0$, it follows that there exists a continuous positive monotonic function B such that

$$||y_t(\sigma,\phi)|| \le B(\eta), \quad \text{for} \quad t \ge \sigma.$$

Then the solution $x \equiv 0$ of (3) is integrally stable.

Theorem 3: Let $V(t,\phi)$ be a Lyapunov functional defined for $t \ge 0$, $||\phi|| \le \delta_o$, satisfying:

1) $V(t,\phi) \le a(||\phi||)$ where a is a continuous nonnegative, nondecreasing function, $a(0) = 0$, $V(t,0)$, $V(t,0) = 0$;

2) $|V(t,\phi) - V(t,\psi)| \le M_2||\phi-\psi||$, $M_2 > 0$, $||\phi||$, $||\psi|| \le \delta_o$;

3) $\overline{\lim_{h\to 0^+}} \frac{1}{h}[V(t+h,x_{t+h}(t,\phi)) - V(t,\phi)] \le -W(||\phi||)$,

where $W(s)$ is a continuous positive, nondecreasing, $W(0) = 0$ and $x_t(\sigma,\phi)$ is a solution of (3). Then the solution $x = 0$ of (3) is asymptotically integrally stable.

Proof: From Theorem 2, the solution $x \equiv 0$ of (3) is integrally stable, then given $\varepsilon > 0$ we choose $\eta > 0$ in such a way that

$$||y_{t_2}(\sigma,\phi)|| < \eta, \qquad \int_{t_2}^{t_3} |h(t,y_t(\sigma,\phi)|dt < \eta$$

implies $||y_t(\sigma,\phi)|| < \varepsilon$ for t, $t_2 \le t \le t_3$. Let $\eta > 0$ such that

$B(\eta) = \varepsilon$, $\bar{\eta} = \min\{\delta,\eta\}$ $\ell = W(\bar{\eta})$, $T = T(\delta,\varepsilon) = \dfrac{M_2(\delta+\eta)}{\ell}$ and $\alpha(t) = h(t,y_t(\sigma,\phi))$ a continuous function on $[\sigma,t_1]$, $t_1 > \sigma + T$, $||\phi|| < \delta$ and $\int_\sigma^{t_1} |h(t,y_t(\sigma,\phi))|dt < \bar{\eta}$. If $||y_t(\sigma,\phi)|| \geq \bar{\eta}$ on $[\sigma,\sigma+T]$ we would have

$$\overline{\lim_{h\to 0^+}} \frac{1}{h}|V(t+h,y_{t+h}(t,y_t(\sigma,\phi))) - V(t,y_t(\sigma,\phi))|$$

$$\leq \overline{\lim_{h\to 0^+}} \frac{1}{h}|V(t+h,x_{t+h}(t,y_t(\sigma,\phi))) - V(t,y_t(\sigma,\phi))|$$

$$+ \overline{\lim_{h\to 0^+}} \frac{1}{h}|V(t+h,y_{t+h}(t,y_t(\sigma,\phi))) - V(t+h,x_{t+h}(t,y_t(\sigma,\phi)))|$$

$$\leq -W(||y_t(\sigma,\phi)||) + M_2|h(t,y_t(\sigma,\phi))|.$$

By integration of the inequality above on $[\sigma,\sigma+T]$ we have

$$V(\sigma+T,y_{\sigma+T}(\sigma,\phi)) \leq V(\sigma,\phi) + M_2\int_\sigma^{\sigma+T} |h(t,y_t(\sigma,\phi))|dt$$

$$- \int_\sigma^{\sigma+T} W(|y_t(\sigma,\phi)|)dt \leq M_2||\phi|| + M_2\bar{\eta} - \int_\sigma^{\sigma+T} W(\bar{\eta})dt \leq M_2(||\phi|| + \bar{\eta}) - T\ell \leq 0$$

what is a contradiction, then there exists $\bar{t}$, $\bar{t} \in [\sigma,\sigma+T]$ such that

$$||y_{\bar{t}}(\sigma,\phi)|| < \bar{\eta} \leq \eta.$$

Since

$$\int_t^{t_1} |h(t,y_t(\sigma,\phi))|dt \leq \int_t^{t_1} |h(t,y_t(\sigma,\phi))dt| < \bar{\eta} \leq \eta$$

from the choice of η it follows that $||y_t(\sigma,\phi)|| < \beta(\eta) = \varepsilon$ for t, $\bar{t} \leq t \leq t_1$, but this is true also for $t \in [\sigma+T,t_1]$. Then from

Corollary 1, the solution $x \equiv 0$ of (3) is asymptotically integrally stable.

Theorem 4: Assume that $f(t,\phi)$ satisfies a Lipschitz condition with respect to ϕ in a neighbourhood of the origin uniformly with respect to t in bounded sets, $f(t,0) = 0$, then if the solution $x = 0$ of (3) is uniform asymptotically stable it is asymptotically integrally stable.

Proof: The hypotheses of the Theorem implies the hypotheses of Theorem 1 and Corollary 1 and then of Theorem 3 and the solution $x \equiv 0$ of (3) is asymptotically integrally stable.

Example: Consider a transmission line whitout loss with two differential elements in the terminal [6a]. The equation and boundary conditions that describes the system gives rise to a second order system of neutral equations

$$C_1\frac{d}{dt}D(x_t) = -\frac{1}{z}x(t) - \frac{q}{z}x(t-r) - g(D(t,x_t)) - i(t)$$
$$L_1\frac{d}{dt}i(t) = -R_1 i(t) + D(t,x_t) \tag{7}$$

$$\tilde{D}(\phi,j) = (\sqrt{C_1}(\phi(0)) - q(\phi(-r)), \sqrt{L_1}\, j) = (\sqrt{C_1}\, D(\phi), \sqrt{L_1}\, j).$$

Let $V(\phi,j) = \frac{1}{2}[\tilde{D}(\phi,j)^2] + \beta\int_{-r}^{0} \phi^2(\theta)d\theta$ where $\beta = \frac{|q|}{z}$,

It can be proved that $\dot{V}(x_t,i(t)) \le 0$ and then the hypotheses of Theorem 7.1 of [2] and Theorem 4 are satisfied and the solution $x \equiv 0$ of (7) is asymptotically uniformly stable and then asymptotically integrally stable.

Since $\frac{1}{2}[\tilde{D}(\phi,j)]^2 = \frac{C_1}{2}(D\phi)^2 + \frac{L_1}{2}j^2$, we have

$$V(\phi,j) = \frac{C_1}{2}(D\phi)^2 + \frac{L_1}{2}j^2 + \beta\int_{-r}^{o}\phi^2(\theta)d\theta.$$

If $(x_t,i(t))$ is a solution of (7) then

$$\dot{V}(x_t,i(t)) = \frac{C_1}{2}(Dx_t)^2 + \frac{L_1}{2}i^2 + \int_{-r}^{o}x(t+\theta)d\theta$$

then

$$\dot{V}(x_t,i(t)) = D(x_t)(-\frac{1}{z}x(t) - \frac{q}{z}x(t-r) - g(Dx_t) - i(t))$$
$$+ i(t)(-R_1 i(t) + Dx_t) + \beta x^2(t) - \beta x^2(t-r)$$
$$= (Dx_t)(-\frac{1}{z}x(t) - \frac{q}{z}x(t-r) + \beta x^2(t) - \beta x^2(t-r))$$
$$- D(x_t)^2\frac{g(Dx_t)}{Dx_t} - R_1 i^2(t)$$

then

$$\dot{V}(x_t,i(t)) = F - R_1 i^2(t) \tag{8}$$

where

$$F = (Dx_t)(-\frac{1}{z}x(t) - \frac{q}{z}x(t-r)) + \beta x^2(t) - \beta x^2(t-r) + \frac{g(Dx_t)}{Dx_t}(Dx_t)^2.$$

Assume that there exists H such that g satisfies the following condition:

$$\operatorname*{Inf}_{|x|\geq H}\frac{g(x)}{x} = M > -\frac{1}{z}\cdot\frac{1-|q|}{1+|q|}$$

We choose γ in such a way that $M > \gamma$ and

$$-\frac{1}{z}\cdot\frac{1-|q|}{1+|q|} < \gamma < M$$

adding and subtracting $(Dx_t)^2$ in (8) we have

$$\dot{V}(x_t,i(t)) \leq (Dx_t)(-\frac{1}{z}x(t) - \frac{q}{z}x(t-r)) + \beta x^2(t)$$
$$- \beta x^2(t-r) - \gamma(Dx_t)^2 - (Dx_t)^2(\frac{g(Dx_t)}{Dx_t} - \gamma) - R_1 i^2(t)$$

or

$$\dot{V}(x_t,i(t)) \leq Q - (Dx_t)^2(\frac{g(Dx_t)}{Dx_t} - \gamma) - R_1 i^2(t)$$

where

$$Q = (Dx_t)(-\frac{1}{z}x(t) - \frac{q}{z}x(t-r)) + \beta x^2(t) - \beta x^2(t-r) - \gamma(Dx_t)^2$$

Let us show that Q is a negative quadratic form on $x(t)$ and $x(t-r)$. Since $Dx_t = x(t) - qx(t-r)$ we have

$$Q = (x(t) - qx(t-r))(-\frac{1}{z}x(t) - \frac{q}{z}x(t-r) + \frac{|q|}{z}x^2(t)$$

$$- \frac{|q|}{z}x^2(t-r) - \gamma[x(t) - qx(t-r))^2]$$

$$= \frac{1}{z}[(-\gamma z - 1 + |q|)x^2(t) + 2\gamma zqx(t)x(t-r)+(-\gamma q - \gamma zq^2 + q^2)x^2(t-r)].$$

Since

$$\gamma z > -\frac{1-|q|}{1+|q|} > -1 + |q| \qquad \text{we have} \qquad -\gamma z - 1 + |q| < 0$$

and then the coefficient of $x^2(t)$ is negative. Furthermore

$$N = (\gamma zq)^2 - (-\gamma z - 1 + |q|)(-|q| - \gamma zq^2 + q^2)$$

$$N = |q|(1 - |q|)^2(-z\frac{1+|q|}{1-|q|} - 1)$$

Since $-\gamma z\frac{1+|q|}{1-|q|} < 1$, $N < 0$ and $Q < 0$ then

$$\dot{V}(x_t,i(t)) \le -(Dx_t)^2(\frac{g(Dx_t)}{Dx_t} - \gamma) - R_1 i^2(t).$$

Now for $|Dx_t| \ge H$ from the hypotheses on g we have

$$\dot{V}(x_t,i(t)) \le -(Dx_t)^2(M-\gamma) - R_1 i^2(t) \le -\alpha(Dx_t)^2 - R_1 i^2(t)$$

where $\alpha = M - \gamma > 0$.

Let us try to get a constant A for which

$$-\alpha(Dx_t)^2 - R_1 i^2(t) \le A|\overline{D}(x_t,i(t))|^2$$

for every $Dx_t, i(t)$, that is

$$-\alpha(Dx_t)^2 - R_1 i^2(t) \le -A[C_1(Dx_t)^2 + L_1 i^2(t)]$$

$$= -AC_1(Dx_t)^2 - AL_1 i^2(t), \quad \text{then}$$

$$(AC_1 - \alpha)(Dx_t)^2 + (AL_1 - R_1)i^2(t) \leq 0$$

for every $Dx_t, i(t)$.

We should have $AL_1 - R_1 \leq 0$, $0 < A \leq \frac{R_1}{L_1}$, $AC_1 - \alpha \leq 0$, $AC_1 \leq \alpha$. If we take $AC_1 = \frac{\alpha}{2}$, $A = \frac{\alpha}{2C_1} \leq \frac{R_1}{L_1}$ and then

$$\dot{V}(x_t, i(t)) \leq -A|\overline{D}(x_t, i(t))|^2.$$

Hence the hypotheses of Theorem 7.1 of [2] and Theorem 4, are satisfied and the solution $x = 0$ of (7) is asymptotically uniformly stable and, asymptotically integrally stable.

REFERENCES

[1] - J.K. Hale, Forward and backward continuation for neutral functional differential equations, J. Differential Equations, 9(1971), 168-181.

[2] - J.K. Hale, Functional Differential Equations, Springer-Verlag, N.Y., Applied Mathematical Sciences, 3(1977).

[3] - A.F. Izé and J.G. dos Reis, Stability of perturbed neutral functional differential equations, J. of Nonlinear Analysis: Theory, Methods and Applications, Vol. 2, 5(1978), 563-571.

[4] - A.F. Izé and A.A. Freiria, Total stability for neutral functional differential equations, Proceedings of The American Mathematical Society.

[5] - J. Kato and T. Yoshizawa, Ordinary Differential Equations and Volterra Equations, Dynamical System an International Symposium, edited by Cesari, Hale, La Salle, Academic Press, 1976.

[6] - O.F. Lopes, a) Stability and forced oscillations, Journal of Mathematical Analysis and Applications, Vol. 55, 3(1976).
b) Periodic solutions of perturbed neutral differential equations, J. Differential Equations, Vol. 15, 1(1974).

[7] - A.M. Zverkin, Dependence of the stability of the solutions of differential equations with a delay on the choice of the initial instant, Vestinik Moscov. Uni. Ser. I Mat. Mech., 5(1969), 15-20.

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