I.C.M.S.C.

NOTAS DO INSTITUTO DE CIÊNCIAS MATEMÁTICAS DE SÃO CARLOS-USP

RELATIONSHIP BETWEEN LYAPUNOV FUNCTIONALS AND NONLINEAR PERTURBED NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Volume n.º 9

dezembro - 1980

UNIVERSIDADE DE SÃO PAULO
INSTITUTO DE CIÊNCIAS MATEMÁTICAS DE SÃO CARLOS
13560 - SÃO CARLOS - SÃO PAULO - BRASIL

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by

A. F. IZÉ

Universidade de São Paulo
Instituto de Ciências Matemáticas de São Carlos
Departamento de Matemática
13.560 - São Carlos - S.P.

and

A. A. FREIRIA and J. G. DOS REIS

Universidade de São Paulo
Faculdade de Medicina de Ribeirão Preto
Departamento de Matemática
14.100 - Ribeirão Preto - S.P.

This research was supported by CNPq, FAPESP and FINEP.

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1. Introduction.

In [3] Hale proved a comparison theorem that uses a combination of Lyapunov functionals and differential inequalities that proved to be very useful in the study of stability properties of perturbed retarded linear differential equations. Onuchic [5] used this theorem to prove an interesting theorem on uniform stability of perturbed delay equation. In [4] Izé and Vila proved some theorems on stability and asymptotic behavior of nonlinear perturbed neutral functional differential equations using an extension to neutral equations of Hale's theorem.

We prove here a more natural extension of Hale's theorem and apply it to extend Onuchic's theorem on uniform stability to a linear perturbed neutral functional differential equation.

2. Preliminary.

Suppose $r > 0$ is a given real number, $R = (-\infty,\infty)$, E^n is a real or complex n-dimensional linear vector space with norm $|\cdot|$, $C([-r,0],E^n)$ is the Banach space of continuous functions mapping the interval $[-r,0]$ into E^n with the topology of uniform convergence given by norm $||\phi|| = \sup_{-r\leq\theta\leq 0} |\phi(\theta)|$. If $\sigma \in R$, $A \geq 0$ and $x \in C = C([\sigma-r,\sigma+A],E^n)$ then for any $t \in [\sigma,\sigma+A]$ we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. If Ω is an open subset of $R \times C$ and $f,D : \Omega \to E^n$ are given functions we say that the relation

$$\frac{d}{dt}D(t,x_t) = f(t,x_t) \tag{1}$$

is a functional differential equation.

We say that (1) is a neutral functional differential equation if D is a linear, $D(\phi) = \phi(0)-g(t,\phi)$, where $g(t,\phi) = \int_{-r}^{0} d\mu(t,\theta)\phi(\theta)$, and μ is a $n \times n$ matrix function of bounded variation for $\theta \in [-r,0]$ continuous at zero, that is, there is a continuous scalar function $\ell(s)$ strictly increasing, $\ell(0) = 0$ such that

$$\left|\int_{-s}^{0} [d_\theta \mu(t,\theta)]\phi(\theta)\right| \leq \ell(s) \sup_{-s\leq\theta\leq 0} |\phi(\theta)|, \quad t \in R \tag{2}$$

If f takes bounded sets of $R \times C$ into bounded sets we have continuous dependence of solutions continuity and continuation of solutions to a maximal interval of existence of (1).

3. Main results.

Lemma 1. [2] Assume $L(t,\phi)$ satisfies a Lipschitz condition on ϕ, in a neighbourhood of the origin uniformly for $t \geq \sigma$ and if $x_t(\sigma,\phi)$ and $x_t(\sigma,\psi)$ are solutions of (1) then there exists constants $k_o > 0$ and $L_1 > 0$ such that

$$[x_t(\sigma,\phi)-x_t(\sigma,\psi)] \leq k_o e^{L_1(t,\sigma)} |\phi-\psi|, \quad t \geq \sigma. \tag{3}$$

Corollary 1. If the conditions of Lemma 1 are satisfied then there exists a scalar function $\alpha(t)$, with $\dot{\alpha}(t)$ continuous for $t \geq 0$ and a scalar function $k(t)$, continuous for $t \geq 0$ such that for every $t \geq \sigma \geq 0$ and $\phi \in C$ the solution $y_t(\sigma,\phi)$ of (1) satisfies the condition

$$|y_t(\sigma,\phi)| \le k(t)e^{-[\alpha(t)-\alpha(\sigma)]}|\phi| \tag{4}$$

Proof. Take in inequality (3) of Lemma 1

$$k(t) \equiv k, \quad e \quad \alpha(t) = -L_1 t.$$

Theorem 1. Assume that $L(t,\phi)$ satisfies a Lipschitz condition with respect to ϕ in a neighbourhood of the origin uniformly for $t \ge \sigma$, then there exists a Lyapunov functional $V(t,\phi)$ defined for $t \in [\sigma,\infty)$, $|\phi| \le \delta_o$, $\phi \in C$, satisfying the conditions:

1) $|\phi| \le V(t,\phi) \le k|\phi|$

2) $\dot{V}(t,\phi) \le -\dot{\alpha}(t)V(t,\phi)$

3) $|V(t,\phi) - V(t,\psi)| < k||\phi-\psi||$, $k > 0$, ϕ, $\psi \in C$, $||\phi||$, $||\psi|| \le \delta_o$.

Proof. Let $V(t,\phi)$ be defined by:

$$V(t,\phi) = \sup_{\tau\ge 0}||y_{t+\tau}(t,\phi)||e^{[\alpha(t+\tau)-\alpha(t)]}$$

Inequality (4) from Corollary 1 shows that (1) is satisfied.

$$\dot{V}(\sigma,\phi) = \overline{\lim_{r\to 0^+}}\,[\tfrac{1}{r}V(\sigma+r,y_{\sigma+r}(\sigma,\phi))-V(\sigma,y_\sigma(\sigma,\phi))] =$$

$$= \overline{\lim_{r\to 0^+}}\,\frac{1}{r}\{\sup_{\tau\ge 0}||y_{\sigma+r+\tau}(\sigma+r,y_{\sigma+r}(\sigma,\phi))||e^{[\alpha(\sigma+r+\tau)-\alpha(\sigma+r)]}$$

$$- \sup_{\tau\ge 0}||y_{\sigma+\tau}(\sigma,\phi)||e^{[\alpha(\sigma+\tau)-\alpha(\sigma)]}\} =$$

$$= \overline{\lim_{r\to 0^+}}\,\frac{1}{r}\{\sup_{\tau\ge r}||y_{\sigma+\tau}(\sigma+r,y_{\sigma+r}(\sigma,\phi))||e^{[\alpha(\sigma+\tau)-\alpha(\sigma+r)]}$$

$$- \sup_{\tau\ge 0}||y_{\sigma+\tau}(\sigma,\phi)||e^{[\alpha(\sigma+\tau)-\alpha(\sigma)]}\} \le$$

$$\le \overline{\lim_{r\to 0^+}}\,\frac{1}{r}\{\sup_{\tau\ge r}||y_{\sigma+\tau}(\sigma,\phi)||e^{\alpha(\sigma+\tau)-\alpha(\sigma+r)}$$

$$- \sup_{\tau\geq 0}||y_{\sigma+\tau}(\sigma,\phi)||e^{[\alpha(\sigma+\phi)-\alpha(\sigma)]}\} \leq$$

$$\leq \overline{\lim_{r\to 0^+}}\frac{1}{r}\{\sup_{\tau\geq r}||y_{\sigma+\tau}(\sigma,\phi)||e^{[\alpha(\sigma+\tau)-\alpha(\sigma+r)]}$$

$$- \sup_{\tau\geq 0}||y_{\sigma+\tau}(\sigma,\phi)||e^{[\alpha(\sigma+\tau)-\alpha(\sigma)]}\} \leq$$

$$\leq \overline{\lim_{r\to 0^+}}\frac{1}{r}\sup_{\tau\geq 0}\{||y_{\sigma+\tau}(\sigma,\phi)||e^{[\alpha(\sigma+\tau)-\alpha(\sigma)]}e^{[\alpha(\sigma)-\alpha(\sigma+r)]}-1\} =$$

$$= V(\sigma,\phi)\frac{d}{dr}\{e^{[\alpha(\sigma)-\alpha(\sigma+r)]}\}r = 0 = -V(\sigma,\phi)\dot{\alpha}(\sigma).$$

Then (2) is satisfied.

Let ϕ, $\psi \in C$,

$$|V(t,\phi)-V(t,\psi)| = \sup_{\tau\geq 0}||y_{t+\tau}(t,\phi)||e^{[\alpha(t+\tau)-\alpha(t)]}$$

$$- \sup_{\tau\geq 0}||y_{t+\tau}(t,\psi)||e^{[\alpha(t+\tau)-\alpha(t)]}| \leq$$

$$\leq \{\sup_{\tau\geq 0}||y_{t+\tau}(t,\phi)-y_{t+\tau}(t,\psi)||e^{[\alpha(t+\tau)-\alpha(t)]}\} \leq$$

$$\leq \sup_{\tau\geq 0}k||\phi-\psi||e^{-[\alpha(t-\tau)-\alpha(t)]}e^{[\alpha(t+\tau)-\alpha(t)]} = k||\phi-\psi||.$$

Then (3) is also satisfied.

Let $X(t,\phi)$ continuous for $t \geq 0$, $\phi \in C_H$, $0 \leq H \leq \infty$ locally Lipschitzian on ϕ.

Consider the perturbed system

$$\frac{d}{dt}D(t,x_t) = L(t,x_t) + X(t,x_t) \tag{5}$$

where X takes bounded sets of $R \times C$ into bounded sets.

<u>Theorem</u> 2. Let $V(t,\phi)$, defined as in Theorem 1, then

$$\dot{V}_{(5)}(t,\phi) \leq -\dot{\alpha}(t)V(t,\phi) + k(t)|X(t,\phi)|$$

for every $t \geq 0$, $\phi \in C$.

Proof. From Cruz and Hale [1] it there exists a functional $V(t,\phi)$ satisfying the condition of Theorem 1 if follows that

$$\dot{V}_{(5)}(t,\phi) \leq \dot{V}_{(1)}(t,\phi) + M|X(t,\phi)|$$

and then the inequality of Theorem 2 follows. We now can extend to neutral equations and for f nonlinear the comparison theorem proved by Hale in [3].

Theorem 3. Assume that there exists a scalar function $\omega(t,r)$, continuous for $t \geq 0$, $r \geq \sigma$, non decreasing in r such that

$$|X(t,\phi)| \leq \omega(t,|\phi|), \quad t \geq 0, \quad \phi \in C_H \tag{6}$$

Let $u(t,\sigma,\phi)$ the maximal solution of the scalar equation

$$\dot{u} = -\dot{\alpha}(t)u + k(t)\omega(t,u), \quad u(\sigma) = V(\sigma,\phi) \tag{7}$$

where $V(t,\phi)$ is the functional defined in Theorem 1 and $\phi \in C_H$. Let $u(t) = u(t,\sigma,\phi) < H$ for $\sigma \leq t < \infty$.

Then the solution $x_t(\sigma,\phi)$ of (5) is defined on $[\sigma,\infty)$ and

$$|X_t(\sigma,\phi)| \leq u(t,\sigma,\phi), \quad \sigma \leq t < \infty. \tag{8}$$

Proof. From (6) and Theorem 2 it follows that

$$\dot{V}_{(5)}(t,x_t(\sigma,\phi)) \leq -\dot{\alpha}(t)V(t,X_t(\sigma,\phi)) + k(t)|X(t,x_t(\sigma,\phi))| \leq$$

$$\leq -\dot{\alpha}(t)V(t,X_t(\sigma,\phi)) + k(t)\omega(t,||x_t(\sigma,\phi)||)$$

for $\sigma \leq t \leq t^+$, $t^+ \leq \infty$ where $[\sigma,t^+)$ is the maximal interval open to the right where $x_t(\sigma,\phi)$ is defined.

Since $\omega(t,r)$ is non decreasing in r, for fixed t it follows from (1) of Theorem 1 that

$$\omega(t,||x_t(\sigma,\phi)||) \le \omega(t,V(t,x_t(\sigma,\phi))$$

then

$$\dot{V}(t,x_t(\sigma,\phi)) \le -\dot{\alpha}(t)V(t,x_t(\sigma,\phi)) + k(t)\omega(t,x_t(\sigma,\phi)).$$

Since $u(t,\sigma,\phi)$ is the maximal solution of (7) it follows as a consequence of a theorem on differential inequalities that

$$V(t,x_t(\sigma,\phi)) \le u(t,\sigma,\phi), \quad \sigma \le t \le t^+.$$

Again from (1) of Theorem 1 it follows that

$$||x_t(\sigma,\phi)|| \le V(t,x_t(\sigma,\phi)) \le u(t,\sigma,\phi), \ \sigma \le t \le t^+$$

Since $u(t,\sigma,\phi) < H_L < H$ for $\sigma \le t < \infty$ we have $t^+ = \infty$ and then

$$||x_t(\sigma,\phi)|| \le u(t,\sigma,\phi) \quad \text{for} \quad \sigma \le t < \infty$$

and the proof is complete.

Remark 1. If $k(t)$ is constant, $\alpha(t) = -L_1 t$, then (7) has the form

$$\dot{u} = L_1 u + k\omega(t,u), \quad u(\sigma) = v(\sigma,\phi), \quad \phi \in C_H$$

what is a first order linear non homogeneous equation.

4. Uniform stability of perturbed linear systems.

Consider systems (1) and (5) with $L(t,\phi)$ and $D(t,\phi)$ linear in ϕ.

Theorem 4. The solution $y = 0$ of (1) is uniformly stable if only if there exists a constant $k \ge 0$ such that

$$||y_t(\sigma,\phi)|| \le k||\phi|| \quad \text{for every} \quad t \ge \sigma \ge 0, \quad \phi \in C.$$

Proof. Since D and L are linear in ϕ, if $y_t(\sigma,\phi)$ is a solution $cy_t(\sigma,\phi)$ is also a solution of (1) for every $c \ge 0$.

Assume that the solution $y = 0$ of (1) is uniformly stable then there exists $\delta > 0$ such that $|y_t(\sigma,\phi)| < 1$ for every $t \ge \sigma \ge 0$ and $|\psi| < \delta$. Hence for every $\phi \in C$.

$$|y_t(\sigma,\delta\frac{\phi}{||\phi||})| \le 1 \quad \text{or} \quad ||y_t(\sigma,\phi)|| \le k||\phi|| \quad \text{with}$$

$$k = \frac{1}{\delta}, \quad \text{for} \quad t \ge \sigma \ge 0.$$

The converse is imediate.

Theorem 5. Assume that the following conditions are satisfyed:

i) The solution $y \equiv 0$ of (1) is uniformly stable.

ii) For every $H > 0$ there exists a scalar continuous function:

$$h_H(t), \quad 0 \le t < \infty \quad \text{with} \int^{\infty} h_H(t)dt < \infty \quad \text{such that}$$

$$|X(t,\psi)| \le h_H \quad \text{for every} \quad t \ge 0 \quad \text{and} \quad |\psi| \le H.$$

Let $\phi \in C$, $\sigma \ge 0$ and $0 \le H < \infty$ given in such a way that

$$[k|\phi| + \int_\sigma^\infty h_H(s)ds] < H \quad \text{where } k \text{ is given by Theorem 2.}$$

Then the solution $x_t(\sigma,\phi)$ of (5) satisfies

$$|x_t(\sigma,\phi)| \le k\,|\phi| + \int_\sigma^t h_H(s)ds, \quad t \ge \sigma.$$

Proof. Take in Theorem 3 $\omega(t,r) = h_H(t)$, $k(t) = k$ and $\alpha(t) = 0$. Since the maximal solution of the scalar equation

$$\dot{u} = kh_H(t), \quad u(\sigma) = \sup_{s\ge 0}|y_{\sigma+s}(\sigma,\phi)| \le k|\phi|$$

is $u(t) = u(\sigma) + \int_\sigma^t kh_H(s)ds \le k[|\phi| + \int_\sigma^t h_H(s)ds] < H$ for $\sigma \le t < \infty$ it follows from Theorem 3 that

$$|x_t(\sigma,\phi)| \le u(t) \le k[|\phi| + \int_\sigma^t h_H(s)ds]$$

for $\sigma \le t < \infty$ and the proof is complete.

Theorem 6. Assume that condition i) and ii) of Theorem 4 are satisfied then every solution of (5) bounded on (σ,∞) is uniformly stable.

Proof. Given $\phi \in C$ let $H > k|\phi|$. Then if we choose $\tau = \tau(\phi)$ in such a way that $k|\phi| + \int_\sigma^\infty h_H(s)ds < H$ it follows from Theorem 4 that

$$|x_t(\sigma,\phi)| \le [k|\phi| + \int_\sigma^\infty h_H(s)ds] < H \quad \text{for} \quad t \ge \sigma \ge \tau, \text{ and then}$$

$x_t(\sigma,\phi)$ is bounded on (σ,∞).

Changing variables

$$z = x - x_t(\sigma,\phi) \quad \text{in} \quad \frac{d}{dt}D(t,x_t) = L(t,x_t) + X(t,x_t) \quad \text{we have}$$

$$\frac{d}{dt}D(t,z_t) = \frac{d}{dt}D(t,x_t-x_t(\sigma,\phi)) = \frac{d}{dt}D(t,x_t) - \frac{d}{dt}D(t,x_t(\sigma,\phi)) =$$

$$= L(t,x_t) + X(t,x_t) - L(t,x_t(\sigma,\phi)) - X(t,x_t(\sigma,\phi)).$$

Then

$$\frac{d}{dt}D(t,z_t) = L(t,x_t-x_t(\sigma,\phi)) + X(t,x_t) - X(t,x_t(\sigma,\phi)) =$$

$$= L(t,z_t) + X(t,z_t+x_t(\sigma,\phi)) - X(t,x_t(\sigma,\phi)) =$$

$$= L(t,z_t) + Y(t,z_t)$$

where $Y(t,\psi) = X(t,\psi+x_t(\sigma,\phi)) - X(t,x_t(\sigma,\phi))$ satisfy hypothesis ii) of Theorem 4 and $Y(t,0) = 0$

This shows that there is no loss of generality to assume $X(t,0) = 0$ and it is enough to prove the uniform stability of the solution $x = 0$ of (5).

Given $\varepsilon > 0$ there exists $T = T(\varepsilon)$ and $\tau = \tau(\varepsilon)$ such that

$$k[|\phi| + \int_{t_1}^{\infty} h_\varepsilon(s)ds] < \varepsilon \quad \text{for every } t_1 \geq T \text{ and } |\phi| < \tau.$$

From Theorem 5 the solution $x_t(t_1,\phi)$ of (5) where $t_1 \geq T(\phi)$ and $|\phi| < t$, satisfies the condition

$$|x_t(t_1,\phi)| \leq k[|\phi| + \int_{t_1}^{t} h_\varepsilon(s)ds] < \varepsilon, \quad t \geq t_1 \geq T(\varepsilon).$$

Then it follows that given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|\phi| < \delta$, $T(\varepsilon) \geq t \geq t_1 \geq 0$ implies $|x_t(t_1,\phi)| < \varepsilon$ and then the solution $x \equiv 0$ of (5) is uniformly stable.

<u>Corollary</u> 2. Assume that the solution $y = 0$ of (1) is uniformly stable and $|X(t,\phi)| \leq h(t)|\phi|$, $t \geq 0$, $\phi \in C$, with $\int_0^{\infty} h(t)ds < \infty$, then every solution of (5) is bounded on $[\sigma,\infty)$ and uniformly stable.

<u>Proof</u>. Since X takes bounded sets of $R \times C$ into bounded sets the solutions of (5) can be continued to the boundary of Ω, then it is enough to prove that there is a $T \geq 0$ such that for every $\sigma \geq T$ and $||\phi|| < H$, $x_t(\sigma,\phi)$ is bounded on $[\sigma,\infty)$.

Take $T > 0$ such that $k\int_T^{\infty} h(t)dt < 1$.

Let $\sigma \geq T$ and $\phi \in C$.

We can choose H big enough in such a way that $k[|\phi| + \int^{\infty} Hh(t)dt] < H$.

If we take in Theorem 4 $h_H(t) = Hh(t)$ it follows that the solution $x(t,\sigma,\phi)$ of (5) satisfies $|x_t(\sigma,\phi)| < H$.

Since every solution of (5) is bounded on $[\sigma,\infty)$ and from the above theorem it is uniformly stable.

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