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Summary

The basic idea of this work is to use Lyapunov functionals to show that for neutral functional differential equations, uniform asymptotic stability implies total stability.

1. Introduction.

In discussing the existence of almost periodic solutions of retarded functional differential equations connecting with boundedness, there are two ways. The one is to assume a separation condition for bounded solutions and other is to assume that an almost periodic system has a bounded solution with some kind of stability properties, uniform asymptotic stability, total stability and so on. In particular the existence of a bounded totally stable solution implies the existence of an almost periodic solution but even that the equation satisfies a local non uniform Lipschitz condition the existence of uniformly asymptotically stable solution does not imply the existence of an almost periodic solutions [4]. For neutral equations the above relationship between the existence of almost periodic solutions and some kind of stability is not well understood yet. It seems to be reasonable that if the operator D is stable in the sense defined in section 1, then the results obtained for retarded equations can be extended to neutral equations. We analyse in the following the relationship between uniform asymptotic stability, and total stability and show that if the equations satisfies a uniform Lipschitz condition and the D operator is stable then uniform asymptotic stability implies total stability.

2. Preliminary.

Suppose $r > 0$ is a given real number, $R = (-\infty,\infty)$, E^n is a real or complex n-dimensional linear vector space with norm $|\cdot|$, $C = C([-r,0],E^n)$ is the Banach space of continuous functions mapping the interval $[-r,0]$ into E^n with the topology of uniform convergence given by the norm $||\phi|| = \sup_{-r\le\theta\le 0} |\phi(\theta)|$. If $\sigma \in R$, $A \ge 0$ and $x \in C([\sigma-r,\sigma+A],E^n)$ then for any $t \in [\sigma,\sigma+A]$ we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$. If Ω is an open subset of $R \times C$ and $f,D : \Omega \to E^n$ are given continuous functions we say that the relation

$$\frac{d}{dt}D(t,x_t) = f(t,x_t) \tag{1}$$

is a functional differential equation

We say that (1) is a neutral functional differential equation if D is linear in ϕ, $D(t,\phi) = \phi(0) - g(t,\phi)$, where $g(t,\phi) = \int_{-r}^{0} d\mu(t,\theta)\phi(\theta)$, and μ is an $n \times n$ matrix function of bounded variation for $\theta \in [-r,0]$ and there is a continuous scalar function $\ell(s)$ strictly increasing, $\ell(0) = 0$ such that

$$\left|\int_{-s}^{0} [d_\theta\mu(t,\theta)]\phi(\theta)\right| \le \ell(s) \sup_{-s\le\theta\le 0} |\phi(\theta)|, \quad t \in R \tag{2}$$

If f takes bounded sets of $R \times C$ into bounded set we have continuous dependence of solutions, continuity and continuation of solutions to a maximal interval of existence of (1), [2].

3. Main results.

Consider the system of functional differential equations of neutral type

$$\frac{d}{dt}D(t,x_t) = f(t,x_t), \quad x_\sigma = \phi \tag{3}$$

and the perturbed system

$$\frac{d}{dt}D(t,y_t) = f(t,y_t) + h(t,y_t) \tag{4}$$

where $D : [\tau,\infty) \times C \to E^n (E^n = R^n$ or $C^n)$ is linear continuous and $f,h : [\tau,\infty) \times C \to E^n$ are continuous and take bounded sets of $R \times C$ into bounded sets. We assume also that f and h satisfy a Lipschitz condition with respect to ϕ in a neighborhood of the origin uniformly in t for t in bounded sets $f(t,0) = 0$, $h(t,0) = 0$. Under these conditions systems (3) and (4) have a unique solution $x_t(\sigma,\phi)$, $x_\sigma(\sigma,\phi) = \phi$ through (σ,ϕ).

Definition 1. The zero solution of (1) is uniformly stable if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that $||\phi|| < \delta$ implies that $||x_t(\sigma,\phi)|| \le \varepsilon$ for every $t \ge \sigma \ge 0$. The zero solution of (1) is uniformly asymptotically stable if it is uniformly stable and there is $\delta > 0$ and for any $\eta > 0$ there exists $T = T(\eta)$ such that $||\phi|| < \delta$ implies

$|x_t(\sigma,\phi)| \leq \eta$ for $t \geq \sigma + T(\eta)$.

Definition 2. We say that the operator D is uniformly stable if there exists constants $k \geq 0$, $M \geq 0$ such that the solution $x(\sigma,\phi,H)$ of $D(t,x_t) = D(\sigma,\phi) + H(t) - H(\sigma)$; $t \geq \sigma$, $x_\sigma = \phi$ satisfies $||x_t(\sigma,\phi,H)|| \leq k||\phi|| + \sup_{\sigma\leq\mu\leq t} |H(\mu)-H(\sigma)|$, $t \geq \sigma$. [1]

In what follows we assume that D is a uniformly stable operator. We will need the following Lemma:

Lemma. Assume that $f(t,\phi)$ satisfies a Lipschitz condition with respect to ϕ in a neighborhood of the origin uniformly for $t \geq \sigma$ and $f(t,0) = 0$. If $x_t(\sigma,\phi)$ and $y_t(\sigma,\psi)$ are solutions of (1) then there exists constants $k_o > 0$, $L_1 > 0$ such that

$$||x_t(\sigma,\phi)-y_t(\sigma,\psi)|| \leq k_o e^{L_1(t-\sigma)} ||\phi-\psi||, \quad t \geq \sigma.$$

Proof. Let $x(t)$ and $y(t)$ be solutions of (1) with initial conditions ϕ and ψ respectively then

$$|x(t)-y(t)| \leq |\phi(0)-\psi(0) + g(\sigma,(\phi-\psi)) + g(t,x_t-y_t) + \int_\sigma^t [f(s,x_s)-f(s,y_s)]ds|$$

Then there are constants L and $\alpha > 0$ such that $|x(t)-y(t)| \leq L||\phi-\psi||/2 + ||x_t-y_t||/2$, for $\sigma \leq t \leq \sigma+\alpha$. Since we can take $L \geq 1$ it is easy to see that $||x_t-y_t|| \leq L||\phi-\psi||/2 - ||x_t-y_t||/2$ and then $||x_t-y_t|| \leq L||\phi-\psi||$. By iterating this inequality a number of times we have for $t \in [\sigma+(n-1)\alpha, \sigma+n\alpha]$ and $t \in [\sigma, \sigma+\tau]$

$$||x_t(\sigma,\phi) - y_t(\sigma,\psi)|| \le L^n||\phi-\psi||$$

Let us show that there exist positive constants k_o and L_1 such that

$$L^n||\phi-\psi|| \le k_o e^{L_1(t-\sigma)}||\phi-\psi||, \quad \sigma \le t \le \tau$$

we have to show that there exists $k_o > 0$, $L_1 > 0$, independent of n, such that

$$k_o e^{L_1(n-1)\alpha} \ge L^n$$

or

$$\log k_o + L_1(n-1)\alpha \ge n\log L$$

we choose k_o in such a way that $\log k_o \le L_1\alpha$, then $L_1\alpha + L_1(n-1)\alpha \ge n\log L$ and $L_1 \ge \frac{\log L}{\alpha}$ satisfies the Lemma. Hence $||x_t-y_t|| \le k_o e^{L_1(t-\sigma)}||\phi-\psi||$, $\alpha \le t < \infty$.

If $V : [\tau,\infty) \times C \to R$ is a continuous function we define the *derivative* $V(t,\phi)$ of V along the solutions of (1) by

$$\dot{V}(t,\phi) = \dot{V}_{(1)}(t,\phi) = \overline{\lim_{h\to 0_+}} \frac{1}{h}[V(t+h, x_{t+h}(t,\phi) - V(t,\phi)].$$

The following theorem is a Corollary of Theorem 1 of [3].

<u>Theorem</u> 1. If the zero solution of (3) is uniformly asymptotically stable then there are constants $\delta_o > 0$, $k = k(\delta_o) > 0$, $M > 0$ and continuous nondecreasing, positive

definite functions $u(s)$, $c(s)$, $b(s)$, $w(s)$, $v(s)$ for $0 \le s \le \delta_o$, $u(0) = c(0) = b(0) = w(0) = v(0) = 0$ and a Lyapunov functional $V : [\sigma,\infty) \times C \to R^n$; such that

i) $u(|D(t,\phi)|) \le V(t,\phi)$

ii) $c(||\phi||) \le V(t,\phi) \le b(||\phi||)$

iii) $\dot{V}_{(4)}(t,\phi) \le \dot{V}_{(3)}(t,\phi) + M|h(t,\phi)|$

iv) $\dot{V}(t,\phi) \le -v(|D(t,\phi)|)$, $\dot{V}(t,\phi) \le -w(||\phi||)$

v) $|V(t,\phi) - V(t,\psi)| \le k||\phi-\psi||$, $t \ge \sigma$,

$\phi,\ \psi \in C[-r,0],E^n)$, $||\phi||,\ ||\psi|| \le \delta_o$

Proof. The existence of V satisfying conditions i), ii) iv) and v) is shown in [3, Theorem 1] by virtue of Lemma 1 and condition (2). The derivative of V along the solutions of (4) is

$$\dot{V}_{(4)}(t,\phi) \le \dot{V}_{(3)}(t,\phi) + \frac{k}{1-\ell(s_o)}|h(t,\phi)| =$$

$$= \dot{V}_{(3)}(t,\phi) + M|h(t,\phi)|, \quad t \ge \sigma, \quad ||\phi|| \le \delta$$

where ℓ is the function defined in condition (2) with s_o small enough in such a way that $0 < \ell(s_o) < 1$, and $k > 0$ is the constant in condition v) what proves iii).

4. Total stability.

Definition 3. The solution $x \equiv 0$ of (3) is totally stable if for every $\varepsilon > 0$ there exists $\eta_1(\varepsilon) > 0$, $\eta_2(\varepsilon) > 0$ such that $||\phi|| < \eta_1$ and $|h(t,\phi)| < \eta_2$ implies that $||y_t(\sigma,\phi)|| < \varepsilon$, $t \geq \sigma$, where $y_t(\sigma,\phi)$ is a solution of (4).

<u>Theorem</u> 2. If the solution $x \equiv 0$ of (3) is uniformly asymptotically stable it is totally stable.

<u>Proof</u>. Assume that there is $t_1 > \sigma$ such that $||y_{t_1}(\sigma,\phi)|| \geq \varepsilon$. Choose $\eta_1 > 0$, $\ell > 0$ such that $b(\eta_1) \leq \ell < c(\frac{\varepsilon}{2})$ where b and c are the functions in ii) in Theorem 1, then

$$V(t_1,y_{t_1}(\sigma,\phi)) \geq c||y_{t_1}(\sigma,\phi)|| \geq c(\varepsilon) \geq c(\tfrac{\varepsilon}{2}) > \ell$$

$$V(\sigma,y_\sigma(\sigma,\phi)) \leq b(||\phi||) \leq b(\eta_1) \leq \ell$$

Since $V(t,\phi)$ is continuous there exists t_2, $\sigma < t_2 < t_1$ such that $V(t_2,y_{t_2}(\sigma,\phi)) = \ell$ and $V(t,y_t(\sigma,\phi)) > \ell$ for $t > t_2$. Hence $b(||y_{t_2}(\sigma,\phi)||) \geq V(t_2,y_{t_2}(\sigma,\phi)) \geq c(||y_{t_2}(\sigma,\phi)||)$.

If $\psi = y_{t_2}(\sigma,\phi)$ we have $b(||\psi||) \geq \ell \geq c(||\psi||)$ and $b(\eta_1) \leq \ell < c(\frac{\varepsilon}{2})$ what implies that $\eta_1 \leq ||\psi|| < \varepsilon$. From the inequalities iii) and iv) in Theorem 1 and choosing $\eta_2 < w(\eta_1)/M$ we have

$$\dot{V}_{(4)}(t_2,\psi) \leq -w(||\psi||) + M\eta_2 < 0 \qquad (5)$$

we have also that

$$\dot{V}_{(4)}(t_2,\psi) = \overline{\lim_{t\to 0^+}} \frac{1}{h}[V(t_2+h,y_{t_2+h}(t_2,\psi) - V(t_2,\psi)] \geq 0$$

Since $V(t_2,y_{t_2}(\sigma,\phi)) = V(t_2,\psi) = \ell$ and $V(t,y_t(\sigma,\phi)) > \ell$ for $t > t_2$ this is a contradiction, then $||y_t(\sigma,\phi)|| < \varepsilon$ for $t \geq \sigma$ and the proof is complete.

Example 1. If a $a > 0$ and $|c| < 1$, the solution $x = 0$ of the equation

$$\frac{d}{dt}[x(t) + cx(t-r)] = -ax(t) \tag{6}$$

is totally stable.

Proof. Since $|c| < 1$ it is well known [2] that the operator $D(\phi) = \phi(0) + c\phi(-r)$ is uniformly stable. It is proved in [1, Example 6.1] that the solution $x \equiv 0$ of (6) is uniformly asymptotically stable and from Theorem 2 it is totally stable.

Example 2. The transmission line whithout loss with two differential elements in the terminals [5a], gives rise to a system of second order equations

$$\begin{aligned} C_1\frac{d}{dt}D(t,x_t) &= -\frac{1}{z}x(t) - \frac{q}{z}x(t-r) - g(D(t,x_t)) - i(t) \\ L_1\frac{d}{dt}i(t) &= -R_1 i(t) + D(t,x_t) \end{aligned} \tag{7}$$

where C_1, L_1, K_1, R_1, z are positive constants. If $|q| < 1$, $D(t,\phi) = \phi(0) - q\phi(-r)$ is uniformly stable and if there exists H such that $\inf\limits_{|x|>H} g(x)/x = M > -\frac{1}{z}\,\frac{1-|q|}{1+|q|}$ then the system above is totally stable.

Proof. We consider the operator

$$\overline{D}(\phi,j) = (\sqrt{C_1}(\phi(0) - q\phi(-r)), \sqrt{L_1}\, j) = (\sqrt{C_1}\, D\phi, \sqrt{L_1}\, j).$$

Define

$$V(\phi,j) = \frac{1}{2}[\overline{D}(\phi,j)]^2 + \beta\int_{-r}^{0} \phi^2(\theta)d\theta$$

where $\beta = |q|/z$, since $\frac{1}{2}|\overline{D}(\phi,j)|^2 = (C_1/2)(D\phi)^2 + (L_1/2)j^2$ we have

$$V(\phi,j) = (C_1/2)(D\phi)^2 + (L_1/2)j^2 + \beta\int_{-r}^{0}\phi^2(\theta)d\theta.$$

If $(x_t, i(t))$ is a solution of (7) then

$$\dot{V}(x_t,i(t)) \le F - R_1 i^2(t) \quad \text{where}$$

$$F = (Dx_t)(-1/z.x(t) - q/z.x(t-r))$$
$$+ \beta x^2(t) - \beta x^2(t-r) - (g(Dx_t)/Dx_t)(Dx_t)^2$$

choose γ such that $M > \gamma$, $\gamma > (-1/z)(1-|q|)/1+|q|$.

Adding and subtracting $\gamma(Dx_t)^2$ in $\dot{V}(x_t,i(t))$ we obtain

$$\dot{V}(x_t,i(t)) \le Q - (Dx_t)^2(g(Dx_t)/(Dx_t) - \gamma) - R_1 i^2(t)$$

$$Q = (Dx_t)(-1/z.x(t)-q/z.x(t-r))$$
$$+ \beta x^2(t) - \beta x^2(t-r) - \gamma(Dx_t)^2$$

Q is a negative definite quadratic form on $x(t)$ and $x(t-r)$ then

$$\dot{V}(x_t,i(t)) \le -(Dx_t)^2(g(Dx_t)/(Dx_t) - \gamma) - R_1 i^2(t)$$

For $|Dx_t| \ge H$ from the hypotheses on g we have

$$\dot{V}(x_t,i(t)) \le -(Dx_t)^2(M-\gamma) - R_1 i^2(t)$$
$$\le -\alpha(Dx_t)^2 - R_1 i^2(t)$$

where $\alpha = M - \gamma > 0$. Now we seek A such that

$$-\alpha(Dx_t)^2 - R_1 i^2(t) \le A|\overline{D}(x_t,i(t))|^2$$

for every Dx_t, $i(t)$. If $A = \alpha/2C_1 \leq R_1/L_1$ this condition is satisfied then $\dot{V}(x_t,i(t)) \leq -A|\overline{D}(x_t,i(t))|^2$ and the solution $x = 0$ of (3) is uniformly asymptotically stable and from Theorem 2 is totally stable.

REFERENCES

[1] - CRUZ, M.A. and HALE, J.K., Stability of functional differential equations of neutral type, J. Diff. Equations, Vol. 7, 2(1970), 334-355.

[2] - HALE, J.K., Functional Differential Equations, Springer-Verlag, N.Y. (1977), (Applied Mathematical Sciences 3).

[3] - IZÉ, A.F. and DOS REIS, J.G., Stability of perturbed neutral functional differential equations, J. of Nonlinear Analysis: Theory, Methods and Applications, Vol. 2, 5(1978), 563-571.

[4] - KATO, J. and SIBUYA, Y., Catastrophic deformation of a flow and non existence of almost periodic solution, J. Faculty of Sc. Univ. of Tokio, Ser. I A, 24(1977), 267-280.

[5] - LOPES, O.F., a) Stability and forced oscillations, J. of Mathematical Analysis and Applications, Vol. 55, 3(1976), September.

b) Periodic solutions of perturbed neutral differential equations, J. Diff. Equations, Vol. 15, 1(1974).

[6] - YOSHIZAWA, T., Stability theory by Lyapunov's. Second Method, The Mathematical Society of Japan, (1966).