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Relative Finite Determinacy and Relative Stability of Function-Germs

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Introduction

By observing the behavior of some unstable differentiable functions in the usual sense (C^∞-Stability), like:

$$f : R \to R, \quad f(x) = x^3 \quad \text{or}$$
$$g : R \to R, \quad g(x) = 0$$

we can verify that:

If we compare f with a function $h : R \to R$, whose restriction to $R_- = \{x \in R \mid x \leq 0\}$ coincides with the restriction of f to R_-, it is possible to cojugate f

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and g through a diffeomorphism ψ, $\psi : R \to R$ which, restricted to R_-, coincides with the identity. The same doesn't happen with g, since it is always possible to obtain a function which vanishes, when it is restricted to R_- but, which behaves (even topologically) quite differently from g.

This suggest us to define the "*S-relative equivalence*", where S is a submanifold with non empty boundary, having the same dimension as the manifold and, consequently, to introduce the "*S-Stability*".

This is our aim in Chapter I, particulary in § 2, where we obtain a sufficient condition for S-Stability. In § 1, we make a similar study, although assuming that S is a proper submanifold without boundary, and requering that the compared functions have the same k-jet at a point of S, besides they coincide in S.

It is also interesting to notice that for a Morse function $f : R^2 \to R$, there exists a very strong relation between the position of the tangents at the curve $f(x,y) = 0$ with the y-axis, and the "*2-determinacy*" of f relative to diffeomorphisms of R^2 which let that axis invariant. Such observation, has inspired the study realized in Chapter II, where we define a more general relation than the equivalence relative to S (here we only assume that the diffeomorphism of conjugacy preserves the submanifold S), and consequently we define the finite determinacy relative to S ($2^{\underline{nd}}$ definition).

In § 1 of Chapter II, we state a necessary and sufficient condition for the finite determinacy relative to S ($2^{\underline{nd}}$ definition), where S is a submanifold without boundary. In § 2, we solve the problem which has inspired this estudy. In § 3, we search reduced forms for Morse functions $f : R^2 \to R$, $f(0) = 0$, through diffeomorphisms which let the fixed submanifold invariant.

I.1. Relative Finite Determinacy

Since this paper is concerned with a local study, we shall usually be considering germs and jets at 0 of mappings of Euclidean spaces, and suitable sets of R^n containing the origin. We shall be using the following notations:

Let S be a subset of R^n which contains the origin and $g : (R^n,0) \to (R^p,0)$ a differentiable germ.

1\. 1. $\varepsilon(g,S;n,p)$ denotes the set of germs at $0 \in R^n$ of smooth mappings from R^n to R^p, whose restriction to S coincide with the restriction of g to S.

If $p = 1$ or $g \equiv 0$, we shall omit it in this notation. Hence, taking $S = \{0\} \times R^{n-s}$, $s \geqslant 1$, we have:

$\varepsilon(S;n)$, the set of germs in $m(n)$ which vanish when they are restricted to S. It is a finitely generated

$\varepsilon(n)$-module.

1. 2. $\mathcal{R}_S(n) \equiv \mathcal{R}_S$ is the set of germs at 0 of local diffeomorphisms of R^n, whose restriction to S coincides with the identity. We also observe that $\mathcal{R}_S$ is a subgroup of $\mathcal{R}$, which acts on the right, in a natural way, on $\varepsilon(f,S;n)$, by:

If $\psi \in \mathcal{R}_S$, $g \in \varepsilon(f,S;n)$, then:

$g.\psi$ is the germ, at 0, of the composition

$g \circ \psi : R^n \to R$.

1. 3. definition: Suppose k is a non negative integer. Then $f \in m(n)$ is k determined relative to $\mathcal{R}_S$ (abbreviated: k det. rel. $\mathcal{R}_S$), or k-S determined, if for any $g \in \varepsilon(f,S;n)$, such that $j^k g(0) = j^k f(0)$, the $\mathcal{R}_S$ orbit of f contains g.

1.4. definition: $f \in m(n)$ is finitely determined relative to $\mathcal{R}_S$ (abbreviated: f. d. rel. $\mathcal{R}_S$) if f is k det. rel. $\mathcal{R}_S$ for some positive integer k.

By denoting:

$\langle df \rangle = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \rangle$, we can prove the:

"*Algebraic Formulation of the Finite Determinacy Relative to* $\mathcal{R}_S$", which is stated now.

1.5. theorem: Let $f \in m(n)$ and $S = \{0\} \times R^{n-s}$, $s \geq 1$. Then f is k determined relative to R_S if and only if there exists a positive integer k, such that:

$$m^k(n)\varepsilon(S;n) \subset \varepsilon(S;n)<df>.$$

We use the following proposition to prove the theorem:

1.6. proposition: Let $f \in m(n)$ and $S = \{0\} \times R^{n-s}$, $s \geq 1$. If for any $\omega \in \varepsilon(S;n)$ such that $j^{k-1}\omega(0) = 0$ there exists ξ, $\xi \in \varepsilon(S;n,n)$, satisfying:

$$\omega(x) = f'(x)(\xi(x)), \text{ i.e.,}$$

$$\omega(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x).\xi_i(x), \text{ where } \xi_i \in \varepsilon(S;n);$$

$i = 1, 2, \ldots, n$.

Then,

f is k det. rel. R_S.

In short, we want to prove that:

$$m^{k-1}(n)\varepsilon(S;n) \subset \varepsilon(S;n)<df> \rightarrow f \text{ is k det. rel.} R_S.$$

Proof:

Fix $t_o \in R$, arbitrarily and let $g \in \varepsilon(f,S;n)$, such that:

$$j^k g(0) = j^k f(0).$$

We denote by F, the germ at $(0,t_o)$, of the mapping:

$$F(x,t) = (1-t)f(x) + t.g(x), \text{ and}$$

$$F_t(x) = F(x,t).$$

Thus, we have to show that F_t is R_S equivalent to F_{t_o}, for any t sufficiently near t_o, in R.

Since $F_o = f$ and $F_1 = g$, by connection of $[0,1]$ we obtain the R_S equivalence between f and g.

Hence, we have to find a germ $H:(R^n \times R, 0 \times t_o) \to R^n$, satisfying the following conditions:

1. $H(x,t) = x$ for any t near t_o, in R and x near 0, in S.

2. $H_{t_o} = Id_{R^n}$

3. $F_t o H_t = F_{t_o}$ for any t near t_o in R, i.e,

$F(H(x,t),t) = F(x,t_o)$ for any t near t_o in R, and x near 0, in S.

Since the third condition is satisfied for $t = t_o$, it is enough to show that:

$$3'.\ \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(H(x,t),t) . \frac{\partial H_i}{\partial t}(x,t) +$$

$$+ \frac{\partial F}{\partial t}(H(x,t),t) = 0,$$

which is equivalent to condition 3, and which gives the independence of 3, relativly to the variable t.

To obtain such a germ H, it is sufficient to prove the existence of a germ ξ, $\xi : (R^n \times R, 0 \times t_o) \to R^n$, satisfying the following conditions:

I. $\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x,t).\xi_i(x,t) + \frac{\partial F}{\partial t}(x,t) = 0$ for any t near t_o and x near 0.

II. $\xi_i(x,t) = 0$ for any t near t_o, in R, and x near 0, in S.

This can be seen, by solving the differential equation:

$$\frac{\partial H}{\partial t}(x,t) = \xi(H(x,t),t),$$ with initial condition:

$$H_{t_o} = Id_{R^n}.$$

To find the germ ξ, we have to introduce some algebraic considerations:

Suppose:

$\varepsilon^*(n+1) = \{g : (R^n \times R, 0 \times t_o) \to R\}$ and

$$N = \{\omega \in \varepsilon^*(S \times R; n+1) \mid j^{k-1}\omega_t(0) = 0, t \text{ near } t_o\}.$$

In other words, N is the set of germs at $(0,t_o)$ of functions $\omega : R^n \times R \to R$ which vanish near $(0,t_o)$, in $S \times R$ and which also satisfy the condition:

$$j^{k-1}\omega_t(0) = 0 \quad \text{for any } t \text{ near } t_o, \text{ in } R.$$

Obviously, N is a finitely generated $\varepsilon^*(n+1)$-module.

Let $K = \{\frac{\partial F}{\partial x}(x,t)(\xi(x,t)) \mid \xi \in \varepsilon^*(S \times R;\ n+1,\ n)\}$.

K is also a $\varepsilon^*(n+1)$ module.

Denoting by $m^*(n+1)$ the maximal ideal of $\varepsilon^*(n+1)$ consisting of those germs which vanish in $(0,t_o)$, we can easily see that:

$$N \subset K + m^*(n+1)N.$$

By applying Nakayama's lemma, we have the inclusion:

$$N \subset K.$$

Since $\frac{\partial F}{\partial t} = g - f \in N$, it follows that:

$\frac{\partial F}{\partial t}(x,t) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x,t) . \xi_i(x,t)$, where ξ is the germ which satisfies I and II.

This concludes the proof of our proposition. #

Now, let $z \in J^q_o(n,1)$ and $f \in m(n)$ a representative of z. Let $J^q_o(f,S;n,1)$ denote the set of the q-jets at 0 of elements in $\varepsilon(f,S;n)$.

We denote by $\bar{\pi}_q : \varepsilon(f,S;n) \to J^q_o(f,S;n,1)$, the restriction of the canonical projection $\pi_q : \varepsilon(n,1) \to J^q_o(n,1)$.

If $R^q_S(n)$ is the set of the q-jets at the origin of elements in $R_S(n)$, then $R^q_S(n)$ is a subgroup of $R^q(n)$ which acts, at right, on $J^q_o(f,S;n,1)$, and this action is induced by the action of $R_S(n)$ on $\varepsilon(f,S;n)$.

Denoting by T_zA the tangent space to A, at z, we may prove the following lemma:

1.7 Lemma:

$$\bar{\pi}_q^{-1}(T_z z.R^q_S(n)) = \varepsilon(S;n)\langle df\rangle + m^q(n)\varepsilon(S;n).$$

The proof is very simple and follows a normal procedure, as we may see, for example, in |1| and |4|.

Let k and q be integers, $k \leq q$ and $f \in m(n)$ a representative of $z \in J^q_o(n,1)$.

Denoting by:

$\bar{\pi}_{q,k} : J^q_o(f,S;n,1) \to J^k_o(f,S;n,1)$ the restriction of the projection $\pi_{q,k} : J^q(n,1) \to J^k(n,1)$, we say that:

z is k det. rel. $R_S^q(n)$, if for any $z' \in J_o^q(f,S;n,1)$ such that $\bar{\pi}_{q,k}(z') = \bar{\pi}_{q,k}(z)$, then z' belongs to the $R_S^q(n)$-orbit of z.

Thus, we can prove the following lemma:

1.8. lemma: Let $z \in J_o^q(n,1)$ be k determinated relative to $R_S^q(n)$, where k and q are integers such that $k \leqslant q$. Then:

$$m^k(n)\varepsilon(S;n) \subset \varepsilon(S;n)\langle df\rangle + m^q(n)\varepsilon(S;n).$$

Proof:

Let $A_S = \{z' \in J_o^q(f,S;n,1) \mid \bar{\pi}_{q,k}(z') = \bar{\pi}_{q,k}(z)\}$.

Thus A_S is an affin subspace of $J_o^q(n,1)$, and:

$$T_zA_S = \bar{\pi}_q(\varepsilon(S;n) \cap m^{k+1}(n)) = \bar{\pi}_q(\ker \bar{\pi}_k).$$

Since z is k det. rel. $R_S^q(n)$ it follows that:

$$A_S \subset z.R_S^q(n).$$

Consequently,

$$T_zA_S \subset T_z\ z.R_S^q\ .$$

This implies that:

$$m^k(n)\varepsilon(S;n) \subset \bar{\pi}_q^{-1}(T_z\ z.R_S^q(n)) = \varepsilon(S;n)\langle df\rangle +$$
$$+ m^q(n)\varepsilon(S;n),$$

which proves the lemma #.

Using these results, we may state that:

1.9. lemma: If $f \in m(n)$ is finitely determined relative to R_S, then there exists some positive integer k, such that:

$$m^k(n)\varepsilon(S;n) \subset \varepsilon(S;n)\langle df\rangle.$$

To prove it, we firstly observe that f, being f. d. rel. R_S, is k det. rel. R_S for some positive integer k. But this implies that $\bar{\pi}_{k+1}(f)$ is k-det. rel. R_S^{k+1}. From this, we have:

$$m^k(n)\varepsilon(S;n) \subset \varepsilon(S;n)\langle df\rangle + m^{k+1}(n)\varepsilon(S;n).$$

Using Nakayama's lemma, it follows that:

$$m^k(n)\varepsilon(S;n) \subset \varepsilon(S;n)\langle df\rangle.\ \#.$$

This finally concludes the proof of the "*Algebraic Formulation of the Finite Determinacy Relative to* R_S".

The following theorem show us that finite determinacy relative to R_S is only a new characterization of the usual (right) finite determinacy, when S is a proper submanifold without boundary.

1.10. theorem: Let $f \in m(n)$, $S = \{0\} \times R^{n-s}$, $s \geq 1$. Then f is (right) finitely determined if and only if it is finitely determined relative to R_S:

Proof:

Suppose f is (right) finitely determined. Then, for some positive integer k, we have:

$$m^{k+1}(n) \subset m(n)\langle df\rangle.$$

This immediately implies that:

$$m^{k+1}(n)\varepsilon(S;n) \subset \varepsilon(S;n)\langle df\rangle.$$

By theorem 1.5, f is f. d. rel. R_S.

Conversely, if f is f. d. rel. R_S, then for some positive integer k, we have:

$$m^{k}(n)\varepsilon(S;n) \subset \varepsilon(S;n)\langle df\rangle.$$

We want to show that f is (right) finitely determined.

As a matter of fact, we shall prove that f is, at least, $2k + 1$ determined. It is enough to prove that:

$$m^{2k}(n) \subset \langle df\rangle.$$

Since $2k - 1 \geq k$, it follows that:

$$m^{2k}(n) \cap \varepsilon(S;n) = m^{2k-1}(n)\varepsilon(S;n) \subset m^{k}(n)\varepsilon(S;n).$$

Hence,

$m^{2k}(n) \cap \varepsilon(S;n) \subset \varepsilon(S;n)\langle df\rangle$.

Therefore, the generators of $m^{2k}(n)$, which simultaneously belong to $\varepsilon(S;n)$ have to belong to $\langle df\rangle$.

Now, we are interested to analyse the generators of $m^{2k}(n)$ which do not belong to $\varepsilon(S;n)$.

For example, such a generator is:

x_α^{2k}, where $1 \leqslant \alpha \leqslant n$.

Obviously, $x_\beta . x_\alpha^k$ (with $1 \leqslant \alpha \leqslant n$ and $1 \leqslant \beta \leqslant s$), is an element of $m^k(n)\varepsilon(S;n)$.

Hence,

$$x_\beta . x_\alpha^k = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} . \xi_i ; \quad \xi_i \in \xi(S;n).$$

Thus,

$$\xi_i = \sum_{j=1}^{s} . \xi_i^j . x_j$$

In other way,

$$x_\beta . x_\alpha^k = x_\beta . \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} . \xi_i^\beta + \sum_{\substack{j=1 \\ j\neq\beta}}^{s} \Big(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} . \xi_i^j \Big) x_j .$$

Since:

$x_\beta . x_\alpha^k \in \langle x_\beta\rangle$ and

$$x_\beta . \sum_{i=1}^{n} . \frac{\partial f}{\partial x_i} . \xi_i^\beta \in \langle x_\beta\rangle ,$$

we immediately see that:

$$\sum_{j=1}^{s}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \xi_i^j\right) x_j \in \langle x_\beta \rangle .$$

In other words,

$$\sum_{\substack{j=1 \\ j \neq \beta}}^{s}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \xi_i^j\right) x_j = x_\beta \cdot \psi(x_1, \ldots, x_n).$$

Setting $x_1 = x_2 = \ldots = x_{\beta-1} = x_{\beta+1} = \ldots =$ $= x_s = 0$, it follows that:

$$0 = x_\beta \cdot \psi(0,0,\ldots,0,x_\beta,0,\ldots,x_s,x_{s+1},\ldots,x_n).$$

From this, we have:

$\psi(0,\ldots,0,x_\beta,0,\ldots,x_s,x_{s+1},\ldots,x_n) = 0$ for any $x_\beta \neq 0$.

By continuity, it implies that:

$$\psi(0,0,\ldots,0,x_s,x_{s+1},\ldots,x_n) = 0.$$

Equivalently,

$$\psi \in \varepsilon(S;n).$$

Then,

$$x_\beta \cdot x_\alpha^k = x_\beta \cdot \gamma_\beta + x_\beta \cdot \psi \quad \text{or}$$

$$x_\beta \cdot x_\alpha^{2k} = x_\beta \cdot x_\alpha^k \cdot \gamma_\beta + x_\beta \cdot \psi \cdot x_\alpha^k .$$

Since $\psi . x_\alpha^k \in m^k(n)\varepsilon(S;n)$, we easily prove that:

$$x_\alpha^{2k} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} (x_\alpha^k . \xi_i^\beta + \sum_{j=1}^{s} . \eta_i^j . x_j).$$

which means that:

$x_\alpha^{2k} \in \langle df \rangle$.

A generic element of $m^{2k}(n)$ which doesn't belong simultaneously to $\varepsilon(S;n)$ has the form:

$$g = x_{s+1}^{m_{s+1}} . x_{s+2}^{m_{s+2}} . \; \ldots \; . x_n^{m_n}, \quad \text{where}$$

$$m_{s+1} + m_{s+2} + \ldots + m_n = 2k.$$

Using a similar argument, we may prove that:

$g \in \langle df \rangle$.

This conclude s the proof of our proposition. #

I.2. Relative Stability

In this section, we assume that S is a submanifold with the same dimension as the manifold and having non empty boundary.

Since it will be an essentially local study, we may assume, without loss of generality that:

$S = R^n_-$, the half space whose elements have non positive first coordinate.

We will continue to work with germs in $m(n)$ and using the same notations as in the last section.

By defining the subgroup $R_S(n)$ and the R_S-equivalence, we may introduce the following:

1.11. definition: A germ $f \in m(n)$ is S-stable, if for any $g \in \varepsilon(f,S;n)$, the R_S orbit of f contains g.

We also observe that:

If S is a subset of R^n, containing the origin, it is possible to define the following set:

$C_S(R^n;0)$: the ring of germs in $\varepsilon(n)$ which are constant when they are restricted to S. It is a commutative ring with identity, 1. Moreover, $m_S(R^n;0)$, the set of germs in $C_S(R^n;0)$ whose restriction to S is constantly equal to zero, is the maximal ideal of $C_S(R^n;0)$ and has the property bellow:

$1 + \mu$ is invertible for any $\mu \in m_S(R^n;0)$.

We shall also use the following statement:

1.12.: Let $f \in m(n)$ and $S = R^n_-$. Assume either that $f \notin m_S(R^n;0)$. Then, unless suitable change of coordinates we claim that:

If $\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x).c_i = 0$ for any $x \in S$; c_i constant; $i = 1, 2, \ldots, n$.

Then

$c_i = 0$; $i = 1, 2, \ldots, n$.

Now we can prove the main result of this section:

1.13. proposition: Let $f \in m(n)$ and $S = R^n_-$. Suppose that for any $\omega \in \varepsilon(S;n)$, there exists $\xi \in \varepsilon(S;n,n)$ such that:

$$\omega(x) = f'(x)(\xi(x)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x).\xi_i(x),$$

$\xi_i \in \varepsilon(S;n)$; $i = 1, 2, \ldots, n$.

Then f is S-stable.

Proof:

It follows, by using the same ideas as in the proposition 1.6. In short, we have to integrate a suitable germ of vector field.

To obtain such a germ, we use some algebraic results, working with $C_{S\times R}(R^n \times R;\ 0 \times t_o)$-modules, like:

$$N = \{\omega : (R^n \times R,\ 0 \times t_o) \to R \mid \omega(x,t) =$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x).\xi_i(x,t) \mid \xi_i \in C_{S\times R}(R^n \times R;\ 0 \times t_o)\}.$$

$$K = \{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x,t).\xi_i(x,t) \,|\, \xi_i \in C_{S\times R}(R^n \times R;\ 0 \times t_o)\}.$$

Since both of them are $C_{S\times R}(R^n \times R;\ 0 \times t_o)$ modules and N is a finitely generated one, by using Nakayama's lemma, we prove that:

$N \subset K.$

On the other hand

$\frac{\partial F}{\partial t} = g - f \in N$ and $g|_S = f|_S$.

Hence,

$$0 = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x).\xi_i(x,t); \quad \xi_i \in C_{S\times R}(R^n \times R;\ 0 \times t_o).$$

But, $f \notin m_S(R^n;0)$, since it is impossible to satisfy this condition and the hipothesis of our proposition, simultaneously.

Using 1.12, we see that $\xi_i \in m_{S\times R}(R^n \times R;\ 0 \times t_o)$, $i = 1, 2, \ldots, n$.

Thus,

$$\frac{\partial F}{\partial t} = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x,t).\xi_i(x,t); \ \xi_i \in m_{S\times R}(R^n \times R;\ 0 \times t_o).$$

By the integration of germ ξ, we conclude the proof, in the usual way. #

Remark: If f satisfies the hipothesis of the theorem, it will be called "*S-infinitesimally stable*".

By using the last theorem, we can prove that:

1.14. theorem: Let $f \in m(n)$ be (right) finitely determined and $S = R^n_-$. Then f is S-stable.

Proof:

Let's show that f satisfies the hipothesis of proposition 1.13, (f is S infinitesimally stable).

Let $\omega \in \varepsilon(S;n)$.

Thus,

$\omega|_S = 0$, which implies that $\omega|_{\partial S} = 0$.

Hence,

$\omega(x) = x_1.\omega_1(x)$, where $\omega_1|_{S-\partial S} = 0 \rightarrow \omega_1|_{\partial S} = 0$, by continuity.

Since $\omega_1 \in \varepsilon(S;n)$, this reazoning may be repeated till we obtain:

$\omega = x_1^r.\omega_r$, and clearly $\omega_r \in \varepsilon(S;n)$.

But, (right) finite determinacy of f implies that:

$m^k(n)\varepsilon(\partial S;n) \subset \varepsilon(\partial S;n)\langle df\rangle$. (1.10 and 1.5).

Setting $r = k + 1$, we have:

$x_1^r = x_1^{k+1} \in m^k(n)\varepsilon(S;n)$.

Then,

$$x_1^r = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \xi_i, \quad \text{where} \quad \xi_i \in \varepsilon(\partial S;n).$$

Therefore,

$$\omega = x_1^r \cdot \omega_r = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \xi_i \cdot \omega_r = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \overline{\xi}_i, \quad \text{where}$$

$$\overline{\xi} = (\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\xi}_n) \in \varepsilon(S;n,n).$$

which proves the theorem. #

Remarks:

1. This theorem gives an answer to the problem which has suggested the main part of our study.

For, $f : R \to R$, $f(x) = x^3$ is (right) finitely determined, hence it is S-stable.

2. (Right) Finite Determinacy is a sufficient condition for S-stability, but it is not a necessary one as can be seen by the following examples:

a) The germ $f \in m(2)$, $f(x,y) = (x^2 + y^2)^2$ is not (right) finitely determined, but it is S-stable, where S is the half plane given by $x \leq 0$. To cheek this, it is enough to verify that f is S infinitesimally stable and this can be done by taking any $\omega = \omega(x,y) \in \varepsilon(S;2)$ and observing that there exist ξ_1, $\xi_2 \in \varepsilon(S;2)$ satisfying the following equation:

$$\frac{\omega(x,y)}{4(x^2+y^2)} = x\xi_1(x,y) + y\xi_2(x,y).$$

But this is a problem which has a well known solution.

b) The germ $g \in m(2)$, $g(x,y) = x^2$ is also not (right) finitely determined and in the analogous way, we can prove that is S-stable, where S is once more the same half plane.

It is very interesting to observe that the first germ, although being not (right) finitely determined, is determined by its infinite jet, while the second germ is neither (right) ifinitely determined and nor determined by its infinite jet.

3. We also observe that: If $f \in m(n)$ is S infinitesimally stable ($S = R^n_-$) and $g \in m(n)$ satisfies:

$$g|\mathcal{C}_S = f|\mathcal{C}_S,$$

then g will be also infinitesimally stable and so S-stable. The proof is very simple.

II.1. *Relative Finite Determinacy (2)*

In this section, we work again with germs in $m(n)$ and submanifolds without boundary, however using a more general equivalence relation as the one we have used before.

We denote by $R_S^*(n)$, the group of germs of diffeomorphisms which let that submanifold invariant. In other words as we mentioned before, we may assume, without loss of generality, that:

$$S = \{0\} \times R^{n-s}.$$

Set:

$$R_S^*(n) = \{h \in R(n) | h(S) \subset S\}.$$

If we also denote by S the germ of this subset, around the origin, it is not hard to see that:

$$R_S^*(n) = \{h \in R(n) | h(S) = S\}.$$

So, we may define:

2.1. definition: The germs f and g in $m(n)$ are R_S^*-equivalent (or equivalent relative to R_S^*) if there exists $h \in R_S^*$ such that:

$$g = f o h.$$

Furthermore, if k is an integer, we also define:

2.2. definition: $f \in m(n)$ is k determined relative to R_S^* if for any $g \in m(n)$ such that $j^k g(0) = j^k f(0)$, then f and g are equivalent relative to R_S^*.

The definition of the Finite Determinacy Relative to R_S^* follows the natural way.

Our next purpose is to search necessary and sufficient conditions for the finite determinacy relative to R_S^*.

First of all, we prove the following lemma.

2.3. lemma: Let $f \in m(n)$ and k be a positive integer such that:

$$m^k(n) \subset m(n)\langle df \rangle.$$

Let $g \in m(n)$ such that $j^k f(0) = j^k g(0)$; let p be a non negative integer and fix $t_o \in R$, arbitrarily.

Then, for any $\omega : (R^n \times R, 0 \times t_o) \to (R,0)$, satisfying:

$j^{k+p-1}\omega_t(0) = 0$, for any t near t_o, there exists a germ $\xi : (R^n \times R, 0 \times t_o) \to (R^n,0)$, such that:

a) $j^p \xi_t(0) = 0$ for any t near t_o and

b) $\omega(x,t) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x,t).\xi_i(x,t)$, where

$\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and F denotes the germ, at $(0,t_o)$, of the mapping:

$$F(x,t) = (1-t)f(x) + t.g(x).$$

Proof:

It is well known for $p = 0$.

If $p > 0$, the proof follows from the previous case $(p = 0)$ using, in addition, that:

$$\omega(x,t) = \sum_{i=1}^{\ell} \lambda_i(x)\omega_i(x,t), \text{ where}$$

$$j^{p-1}\lambda_i(0) = 0 \text{ and}$$

$$j^{k-1}\omega_{i_t}(0) = 0 \text{ for any } t \text{ near } t_o;$$

$i = 1,2,\ldots,\ell$ #.

Now we prove the main theorem of this section.

2.4. theorem: Let $f \in m(n)$ and $S = \{0\} \times R^{n-s}$, satisfying: $m^k(n) \subset m(n)\langle df\rangle$ for some positive integer k and $m^{k_1}(n-s) \subset m(n-s)\langle df|_S\rangle$ for some positive integer k_1.

If q is a non negative integer such that:

$$j^q f(0) = 0,$$

(obviously this last condition is always verified at least for $q = 0$).

Setting:

$$d = 2k - q - 1 \text{ and}$$

$d_1 = k + \max\{k, k_1\} - q - 1.$

Then, for any $g \in m(n)$ such that:

$j^d f(0) = j^d g(0)$ and

$j^{d_1} f|_S(0) = j^{d_1} g|_S(0),$

there exists $h \in R(n)$ such that:

$g = f \circ h$, where $h(S) = S.$

Furthermore, h can be chosen in such way that its 1-jet at the origin is the same as the one of the identity.

(It follows, obviously, that f and g are R_S^* equivalent).

Proof:

We first observe that:

$d = k - 1 + k - q \rightarrow d \geqslant k \geqslant q$; furthermore, $d_1 \geqslant d$.

Fix $t_o \in R$ arbitrarily and denoting by F the germ, at $(0, t_o)$ of the mapping:

$F(x,t) = (1-t)f(x) + t.g(x),$

then, by the previous lemma, it follows that there exist germs:

$\xi : (R^n \times R, 0 \times t_o) \to R^n$ and

$\eta : (S \times R, 0 \times t_o) \to S,$ satisfying

$$(g-f)(x) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x,t).\xi_i(x,t) \quad \text{and}$$

$$(g-f)|_S(x) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x,t).\eta_i(x,t), \quad \text{where}$$

$j^{k-q}\xi_t(0) = 0$ and $j^{k-q}\eta_t(0) = 0$ for any t near t_o.

Extending η to $\bar{\eta} : (R^n \times R, 0 \times t_o) \to R^n$, trivially, we see that:

a) $$(g-f)(x) = \frac{\partial F}{\partial x}(x,t)(\bar{\eta}(x,t)) +$$

$$+ (g-f)(x) - \frac{\partial F}{\partial x}(x,t)(\bar{\eta}(x,t)).$$

Setting:

$$\omega(x,t) = (g-f)(x) - \frac{\partial F}{\partial x}(x,t)(\bar{\eta}(x,t)) =$$

$$= \frac{\partial F}{\partial x}(x,t)(\xi(x,t) - \bar{\eta}(x,t)).$$

we get:

b) $\omega(x,t) = 0$ for any $x \in S$ near 0, $t \in R$, near t_o.

c) $j^k\omega_t(0) = 0$, for any t near t_o.

It is enough to observe that:

$j^q F_t(0) = 0 \quad (t \text{ near } t_o).$

Thus,

$j^{q-1} \frac{\partial F}{\partial x}(0,t) = 0$ for any t near t_o.

Furthermore,

$j^{k-q}\xi_t(0) = j^{k-q}\eta_t(0)$, for any t near t_o.

Hence:

$\omega(x,t) = x_1 . \omega_1(x,t) + \ldots + x_s . \omega_s(x,t)$, where

$j^{k-1}\omega_{i_t}(0) = 0.$

By the lemma 2.3 $(p = 0)$, we have:

$\omega_i(x,t) = \frac{\partial F}{\partial x}(x,t)(\gamma_i(x,t))$, where

$\gamma_{i_t}(0) = 0$ for any t near t_o.

Therefore,

$$\omega(x,t) = \frac{\partial F}{\partial x}(x,t)\left(\sum_{i=1}^{s} . x_i . \gamma_i(x,t)\right) = \frac{\partial F}{\partial x}(x,t)(\gamma(x,t)).$$

Replacing in (a), we get:

$$(g-f)(x) = \frac{\partial F}{\partial x}(x,t)(\overline{\eta}(x,t) + \gamma(x,t)) = \frac{\partial F}{\partial x}(x,t)(\zeta(x,t)),$$

where:

$\zeta_t(x) = 0$ for any t near t_o, in R and x near 0, in S.

It is also easily verified that:

$$j^1\zeta_t(0) = 0.$$

Now, we are able to apply the usual procedure of integrating the germ ζ.

Since $\zeta_t(x) \in S$ for any x near 0, in S and any t near t_o, we can also observe that the solution of the differential equation:

$$\frac{\partial H}{\partial t}(x,t) = \zeta(H(x,t), t),$$ with initial condition:

$$\zeta_{t_o} = Id_{R^n},$$ must satisfy:

$H(x,t) \in S$, for any x near 0, in S and t near t_o, in R, which follows by the unicity of the solution.

Analogous arguments which have been used, show us how to obtain a germ h of local diffeomorphism of R^n, whose 1-jet at the origin is equal the one of the identity and, $h(S) = S$ # .

Remark:

The statement of the last theorem is as better as bigger is the value of q. For example, take the case $q = k - 1$, which is verified for Morse functions.

2.5. corollary: On the hipothesis of the last theorem, if S has codimension 1 ($S = R^{n-1}$), then the germ h satisfies:

$h(\overline{S}) = \overline{S}$, where $\overline{S} = R^n_-$ and $S = \partial\overline{S}$. It follows immediately, since h is a diffeomorphism satisfying $h(S) = S$ and having its 1-jet equal to the 1-jet of identity.

2.6. theorem: Let $f \in m(n)$ and $S=\{0\}\times R^{n-s}$. Then f and $f|_S$ are (right) finitely determined if, and only if f is finitely determined relative to R^*_S.

Proof:

If f and $f|_S$ are (right) f. d., there exist k and k_1, non negative integers, such that:

$$m^k(n) \subset m(n)\langle df\rangle \quad \text{and}$$

$$m^{k_1}(n-s) \subset m(n-s)\langle df|_S\rangle.$$

Since $j^q f(0) = 0$ for some $q \geq 0$, by theorem 2.4 and using $d = 2\max\{k,k_1\} - q - 1$, we prove the first part.

If f is f. d. rel. R^*_S, thus f is d det. rel. R^*_S for some non negative integer d, which implies the d-determinacy of f.

It is not hard to prove that $f|_S$ is also d-determined. #

2.7. corollary: Let $f \in m(n)$ and $S=\{0\}\times R^{n-s}$. If f and $f|_S$ are Morse functions, then f is 2-det. rel. R^*_S.

2.8. corollary: Let $f \in m^2(n)$ and $S=\{0\}\times R^{n-s}$. Then f and $f|_S$ are Morse functions if and only if f is f. det. rel. R^*_S.

2.9. corollary: If $f \in m(n)$ is a Morse germ and its Hessian $Hf(0)$ is positive defined (negative defined), then f is 2-det. rel. R^*_S for any vector subspace S of R^n.

We can also show that:

2.10. theorem: Let $f \in m(n)$ be k determined. Then, the set of one-dimensional vector subspaces of R^n for which the restriction of f is k determined, is an open and dense set of $P^{n-1}(R)$.

To prove this, we use the following statement:

If $A : R^n \times R^n \times \ldots \times R^n \to R$ is a s-linear form and Q is the s-adic form associated, supposed non identically null, then the set $N_Q = \{v \in R^n | Q(v) \neq 0\}$ is an open and dense in R^n.

Now, if f is k determined, it follows that $f^{(s)}(0) \neq 0$ for some $s \leqslant k$. It is enough to take $Q^s_o(v) = f^{(s)}(0)\ (v, \ldots, v)$ and the proof will be conclude.

II.2. Applications

Here, we always consider a germ $f \in m(2)$, which is assumed to be finitely determined and is our aim to study the relative positions of the tangents at the origin to the curve $f(x,y) = 0$ and the y-axis.

We claim that, this is very related with the 2-determinacy of f, relative to R_S^*, where S is the straight given by $x = 0$.

We first observe that the curve $f(x,y) = 0$ is equivalent to an algebraic curve $P(x,y) = 0$, through a diffeomorphism of R^2, which preserves its tangents at the origin.

Of course, we shall also use the same notation, S, to design the germ at the origin, of the set S.

Then, we can prove the following theorem:

2.11. theorem: Let $f \in m(n)$ be a Morse germ. We claim that the tangents at the origin, to the branches of the curve $f(x,y) = 0$ are transverse to the y-axis if and only if, for any $g \in m(2)$ such that $j^2f(0) = j^2g(0)$, there exists $h \in R(2)$, satisfying:

$h(S) = S$ and

$g = f \circ h$.

(In other words, if and only if f is 2-det. rel. $R_S^*(2)$).

Proof:

It is enough to see that 0 is a double point of $f(x,y) = 0$ (therefore 0 is a double point of the algebraic curve $P_2(x,y) = 0$).

Since the transversality of the tangents to the y-axis implies $f_{yy}(0) \neq 0$, thus $\frac{\partial^2 f|_S}{\partial y^2}(0) \neq 0$ hence $f|_S$ is 2-det. rel. $R(1)$, it is enough to apply corollary 2.8.

Conversely, if f is 2-det. rel. $R_S^*(2)$, then, by theorem 2.6, $f|_S$ is 2-det. rel. $R(1)$; thus $f_{yy}(0) \neq 0$, which implies the transversality of the tangents with the y-axis. #

2.12. corollary: If $f \in m(n)$ is k det. rel. $R(2)$ and 0 is a multiple point of order k of $f(x,y) = 0$, then:

If the tangents to $f(x,y) = 0$, at the origin, are all transverse to the y-axis, there exists an integer d such that for any $g \in m(2)$ with $j^d f(0) = j^d g(0)$, then:

$$g = f \circ h, \text{ where } h \in R(2) \text{ and } h(S) = S.$$

(In other words, f is d-determined relative to R_S^*).

It is very easy to prove this; it is enough to use theorem 2.6.

To prove the next theorem, we use the contact order of manifold germs.

2.13. theorem: Let $f \in m(2)$ be (right) finitely determined. Then, the curve $f(x,y) = 0$ has contacts of finite order with the y-axis, at the origin, if and only if, f is finitely determined relative to $R_S^*(2)$.

Proof:

Suppose $f(x,y) = 0$ has only contacts of finite order with that axis, which are less, or equal to some integer s. Then, there exists a positive integer p, such that:

$$\frac{\partial^p f|_S}{\partial y^p}(0) \neq 0.$$

If there is not such an integer, then:

$$j^\infty f|_S(0) = 0 \qquad (f|_S \in m^\infty(1)).$$

Set $g(x,y) = f(x,y) - f(0,y)$.

It is obvious that $j^\infty g(0) = j^\infty f(0)$.

By the finite determinacy of f, it is possible to find an integer q, such that $s < q$ and:

$$j^q \varphi(0) = j^q Id.$$

From this, $g(x,y) = 0$ would have contacts of finite order with the y-axis. This would be a contradiction, since $g(x,0) = 0$, thus $g(x,y)$ must contain the germ S and so g would have contacts of infinite order with that axis.

Conversely, if f is finitely determined relative to $R_S^*(2)$, there exists an integer d, such that: For any g, $g \in m(2)$ such that $j^d g(0) = j^d f(0)$, there exists $h \in R_S^*(2)$, satisfying: $g = f \circ h$.

It follows that $f|_S$ is finitely determined relative to $R(1)$, thus:

$$(1) \quad \frac{\partial^p f|_S}{\partial y^p}(0) \neq 0 \quad \text{for some positive integer } p.$$

Moreover, by the finite determinacy relative to $R_S^*(2)$, it is possible to find an integer d, bigger enough, such that the curve $f(x,y) = 0$ has contacts of the same order as the curve $P_d(x,y) = 0$, at the origin, with the y-axis.

But $P_d(x,y) = 0$ doesn't contain S, thus it doesn't factor x.

However, if one of the branches of $f(x,y) = 0$ would have contact of infinite order with the y-axis, then $P_d(x,y) = 0$, (as algebraic curve) would have to contain the germ S. # .

Remark:

By corollary 2.5, we see that the results which we have obtained in this section (about the germs of diffeomorphisms which let S invariant) may be extended to the case when S is the half plane $x \leq 0$, whose boundary is the straight $x = 0$, the submanifold which we have always considered here.

II.3. *Searching Normal Forms*

By "*Admissible Change of Coordinates*", we mean any germ $h \in R_S^*(n)$, where S is the following set:

$$S = \{(x_1, x_2, \ldots, x_n) \mid x_1 = 0\}.$$

In this section, we assume that f is a Morse germ in $m(2)$, and S denotes the set of points in R^2 whose first coordinates vanish. S may also denote the germ of that set, around the origin.

We know that f can be written in the form:

$$\text{(a)} \quad f(x,y) = a(x,y)x^2 + 2b(x,y)xy + c(x,y)y^2.$$

The germ h has the form:

$$\begin{cases} x = \theta_1(X,Y) \\ y = \theta_2(X,Y) \end{cases}$$

with

$$\theta_1(0,Y) = 0 \quad \text{and} \quad \theta_2(0,0) = 0.$$

In other words,

$$\theta_1(X,Y) = X(m + \rho(X,Y)), \quad \rho \in m(2)$$

$$\theta_2(X,Y) = p.X + q.Y + \sigma(X,Y), \quad \sigma \in m(2)$$

and $m.q \neq 0$.

Observing (a), we see that:

If $c(x,y) = 0$ in a neighbourhood of the origin, then, by suitable change of coordinates (admissible ones) we get the new form:

$$f(X,Y) = X.Y.$$

Otherwise, if $c(0,0) \neq 0$ (therefore $c(x,y) \neq 0$ in a neighbourhood of $(0,0)$), then, by using admissible change of coordinates, we have:

$$f(X,Y) = X^2 \pm Y^2.$$

Finally, if $c(0,0) = 0$, but c doesn't vanish on a neighbourhood of $(0,0)$, then:

$f(X,Y) = Y(X + Y^r)$, where $r - 1$ is the order of contact of curve branch $f(X,Y) = 0$, which is tangent to the y-axis, with this axis.

Remark:

1. Here, we may do the same observation that we have done in the last section. In other words, the change of coordinates can be assumed letting the half plane $x_1 \leq 0$ invariant.

2. For Morse functions, in higher dimensions, is not hard to have an analogous statement to the first and the second. Although, the analogous to the third statement is not so simple and the solution has to envolve more complex results or different technics.

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