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B R A S I L

INTRODUCTION

Let R^n be a real or complex n-dimensional linear vector space with norm $|\cdot|$. For $r \geq 0$ let $C = C([-r,0], R^n)$ be the space of continuous function taking $[-r,0]$ into R^n with $||\phi||$ defined by $||\phi|| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$.

Suppose t_o is a real number and g, f are continuous functions taking $[t_o,\infty) \times C \to R^n$ and define the operators

$$D(\cdot): [t_o,\infty) \times C \to R^n$$

by

$$D(t)\phi = A(t)\ \phi(0) - g(t,\phi)$$

for $t \in [\tau,\infty)$, $\phi \in C$. A functional differential equation is a system of the form

$$\frac{d}{dt} D(t)x_t = f(t,x_t) \tag{1}$$

where $x_t \in C$ is defined by $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$.

For any $\phi \in C$, $\sigma \in [t_o,\infty)$ a function $x = x(\sigma,\phi)$ defined on $[\sigma - r, \sigma + A)$ is said to be a solution of (1) on $(\sigma, \sigma + A)$ with initial value ϕ at σ if x is continuous on $[\sigma - r, \sigma + A)$ and relation (1) is satisfied on $(\sigma, \sigma + A)$. We say that (1) is a neutral functional differential equation if det $A(t) \neq 0$, $t \in [t_o,\infty)$ and for every $\phi \in C$, $t \in [t_o,\infty)$ $g(t,\phi) = \int_{-r}^{0} d\mu(t,\theta)\ \phi(\theta)$ where μ is a $n \times n$ matrix of bounded variation for $\theta \in [-r,0]$ and there is a scalar function $\rho(s)$ continuous non decreasing for $s \in [0,r]$, $\rho(0) = 0$ such that

$$\left|\int_{-s}^{0} [d_\theta \mu(t,\theta)]\phi(0)\right| \leq \rho(s) \sup_{-s\leq\theta\leq 0} |\phi(\theta)| \qquad (2)$$

we assume that for each $(\sigma,\phi) \in (t_o,\infty) \times C$ (1) has a unique solution which depends continuously upon σ,ϕ and that every bounded solution of (1) is defined for $t \in [\sigma,\infty)$. We assume also that $f(t,0) = 0$.

Definition: We say that the solution $x = 0$ of (1) is uniformly stable if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $\sigma \in [t_o,\infty)$, $t_o > -\infty$, any solution $x(\sigma,\phi)$ of (1) with initial value ϕ at σ, $||\phi|| \leq \delta$ satisfies $||x_t(\sigma,\phi)|| \leq \varepsilon$ for $t \geq \sigma$. It is uniformly asymptotically stable if it is uniformly stable and for some fixed $\delta > 0$, for any $\eta > 0$ there exists a $T = T(\eta) > 0$ such that $||\phi|| \leq \delta$ implies $||x_t(\sigma,\phi)|| \leq \eta$ for $t \geq \sigma + T$.

1. *Lyapunov Functionals*

If $V: [t_o,\infty) \times C \to R$ is continuous we define

$$\dot{V}(t,\phi) = \overline{\lim_{h\to 0_+}} \frac{V[t+h, x_{t+h}(t,\phi)] - V(t,\phi)}{h}$$

Theorem 1: Let $f(t,0) = 0$, $f(t,\phi)$ locally Lipchitzian in ϕ, uniformly with respect to t, and for all ϕ in C, $|A(t)\phi(0)| \leq k(\phi)$ for $t \geq 0$. Assume that the null solution of (1) is uniform asymptotically stable and there exists a constant L such that

$$||x_t(t_o,\phi) - x_t(t_o,\Psi)|| \leq e^{L(t-t_o)}||\phi - \Psi|| \quad (3)$$

Then there exists positive definite functions $u(r)$, $b(r)$, $c(r)$ for $0 \leq r \leq r_1$, non decreasing, $u(0) = b(0) = c(0) = 0$ and a scalar functional $V(t,\phi)$ defined and continuous for $(t,\phi) \in [t_o,\infty) \times C$, $||\phi|| \leq r$, such that

a) $u(|D(t,\phi)|) \leq V(t,\phi)$

b) $c(||\phi||) \leq V(t,\phi) \leq b(||\phi||)$

c) $\dot{V}(t,\phi) \leq -\omega(||D(t,\phi)||)$

d) $|V(t,\phi) - V(t,\Psi)| \leq k||\phi - \Psi||$

for all $t \geq 0$, ϕ, Ψ in C, $||\phi||$, $||\Psi|| \leq r_1$.

Proof: From Cruz and Hale [2], Lemma 7.1, there exists functions $\rho(u)$, $v(u)$ such that

i) $\rho(u)$ is continuous, monotone increasing for $u \geq 0$, $\rho(0) = 0$

ii) $v(u)$ continuous, monotone decreasing for $u \geq 0$, with $\lim_{u\to\infty} v(u) = 0$ such that if

$||\phi|| \leq r_1$, and $t \geq \sigma$

$$||x_t(\sigma,\phi)|| \leq \rho(||\phi||)\, v(t-\sigma),\ t \geq \sigma \qquad (4)$$

We can assume also that $v(u)$ has a continuous non negative derivative and $v(0) = 1$. From the properties of $v(u)$ there exists a function $q(u)$ defined, continuously differentiable and positive on $u \geq 0$, $q(0) = 0$, $q(u)$ strictly increasing $\lim_{u\to\infty} q(u) = \infty$ such that

$$v(u) = \exp[-\alpha(u)]$$

Suppose $q(u)$ is a bounded, continuously differentiable function on $[0,\infty)$ such that $q(0) = 0$, $q(u) > 0$, $q'(u) < \alpha'(u)$ for $u > 0$, q' monotone decreasing. Let $\beta > \ell(k) + k$ where $|A(t)\phi(0)| \leq k$ and define

$$V(t,\phi) = \sup_{u\geq 0} ||x_{t+u}(t,\phi)||e^{q(u)} \qquad (5)$$

Since D satisfies (2), $\frac{1}{\beta}||D(t,\phi)|| =$ $= u(||D(t,\phi)||) \leq V(t,\phi)$ and a) is satisfied. b), c) and d) follows step by step the argument given in [1] with the only change being that $V(t,\phi)$ is defined by (5).

Theorem 1 is an alternative for Theorem 7.1 of Cruz and Hale [2] which uses the hypotheses:

a') $||D(t,\phi)|| \leq V(t,\phi) \leq b(||\phi||)$

b') $\dot{V}(t,\phi) \leq -C(||D(t,\phi)||)$

c') $|V(t,\phi) - V(t,\Psi)| \leq K||\psi - \Psi||$

where $V(t,\phi) = \sup_{s\geq 0} ||D(t+s, x_{t+s}(t,\phi)||e^{as}$.

The principal advantage of this definition of $V(t,\phi)$ is condition c') which is very useful in the study of perturbed systems. By the other hand $||\phi|| \geq 0$ does not implies necessarily that $V(t,\phi) \geq a(||\phi||)$ for some positive increasing function a with $a(0) = 0$, what is quite often needed in the applications. With the definition of $V(t,\phi)$ given in Theorem 1 above this condition is satisfied. However, the Condition (3), that holds in general for linear systems, is not easy to verify for general non linear system and it is an open question to know if it is true for system (1) with f Lipchitzian.

2. *Stability of Perturbed Linear System*

Consider the system

$$\frac{d}{dt} D(t,y_t) = f(t,y_t) \tag{6}$$

where $D(t,\phi)$ and $f(t,\phi)$ are bounded linear continuous functionals and D is a uniformly stable operator. Consider now the perturbed system.

$$\frac{d}{dt} \bar{D}(t,x_t) = \bar{f}(t,x_t) \tag{7}$$

System (7) can be transformed into the system

$$\frac{d}{dt} D(t,x_t) = f(t,x_t) + h(t,x_t) \tag{8}$$

where only the f function is perturbed and

$$h(t,x_t) = \bar{f}(t,x_t) - f(t,x_t) - \frac{d}{dt}[\bar{D}(t,x_t) - D(t,x_t)]$$

To study the stability properties of system (8) we will need the following.

Lemma 1: Let $x_t(t_o,\phi)$ be a solution of (8) and $y_t(t_o,\phi)$ a solution of (6), then there exists a constant L such that

$$||x_t(t_o,\phi) - y_t(t_o,\phi)|| \leq$$

$$\leq [e^{L(t-t_o)}-1] \; |h(t,x_t)|, \qquad t \geq t_o \tag{9}$$

It is obvious that (9) is satisfied for $t = t_o$. Suppose it is not true for every $t \geq t_o$. Let t_1 be the supremum of the t's for which inequality (9) is satisfied. From the continuity of both sides of (9) there exists a sequence $\{t_n\}$, $t_n \to 0$ for $n \to \infty$, t_n small enough such that

$$||x_{t_1+t_n}(t_o,\phi) - y_{t_1+t_n}(t_o,\phi)|| >$$

$$> [e^{L(t_1+t_n-t_o)}-1] \; |h(t,x_t)|$$

From Hale [3], p. 354, there is a h_o small enough such that if $0 \leq t_n \leq h_o$

$$||x_{t_1+t_n}(t_o,\phi) - y_{t_1+t_n}(t_o,\phi)|| \leq$$

$$\leq \frac{1}{1-\rho(h_o)} \int_{t_1}^{t_1+t_n} |h(s,x_s)|ds$$

Thus

$$\frac{1}{1-\rho(h_o)} \int_{t_1}^{t_1+t_n} |h(s,x_s)|\,ds >$$

$$> |(e^{L(t_1+t_n-t_o)}-1)\ h(t_1+t_n, x_{t_1+t_n})$$

Hence

$$\frac{t_n}{1-\rho(h_o)} \sup_{t_1 \leq s \leq t_1+t_n} |h(s,x_s)| \geq$$

$$\geq (e^{L(t_1+t_n-t_o)}-1)\ h(t_1+t_n, x_{t_1+t_n})$$

and

$$\overline{\lim_{t_n \to 0_+}} \frac{1}{1-\rho(h_o)} \sup_{t_1 \leq s \leq t_1+t_n} |h(s,x_s)| >$$

$$> \overline{\lim_{t_n \to 0_+}} \frac{(e^{L(t_1+t_n-t_o)}-1)}{t_n} (h(t_1+t_n, x_{t_1+t_n}))$$

Then

$$\frac{1}{1-\rho(h_o)} |h(t_1, x_{t_1})| \geq L\ e^{L(t_1-t_o)} |h(t_1, x_{t_1})|$$

and if we take $L > \dfrac{1}{1-\rho(h_o)}$ we have a contradiction.

Theorem 2: If the solution $y \equiv 0$ of (6) is uniformly asymptotically stable there exists a scalar functional $V(t,\phi)$ defined and continuous for $(t,\phi) \in [t_o,\infty) \times C$ such that

a) $|D(t,\phi)|^2 \leqslant V(t,\phi)$

b) $||\phi||^2 \leqslant V(t,\phi) \leqslant k_1||\phi||^2$

c) $|V(t,\phi_1) - V(t_1,\phi_2)| \leqslant k_2(||\phi_1+\phi_2||)||\phi_1-\phi_2||$

d) $\dot{V}(t,\phi) \leqslant - ||y_t(t_o,\phi)||^2$

Proof:

Let $\beta = \max\{(k+\rho(r))^2, 1\}$ and define

$$V(t,\phi) = \beta\Big[\int_0^\infty ||y_{t+s}(t,\phi)||^2 ds + \sup_{s\geqslant 0}||y_{t+s}(t,\phi)||^2\Big]$$

From the uniform asymptotic stability of the solution $y \equiv 0$, [3], there are constants α and k_o such that $||y_t(t_o,\phi)|| \leqslant k_o\, e^{-\alpha(t-t_o)}||\phi||$. It is obvious that $V(t,\phi) \geqslant ||\phi||^2$ and $V(t,\phi) \geqslant |D(t,\phi)|^2$, since

$|D(t,\phi)|^2 \leqslant k^2||\phi||^2 + \rho^2(r)||\phi||^2 \leqslant \beta||\phi||^2 \leqslant V(t,\phi)$.

Let $\sigma \geqslant 0$, then $||y_{t+s}(t,\phi)|| \leqslant K_o\, e^{\alpha\sigma}e^{-\alpha s}||\phi||$. Hence $V(t,\phi) = \beta\, k_o\, e^{2\alpha\sigma}||\phi||^2 \int_0^\infty e^{-2\alpha u}du + \beta K_o e^{2\alpha\sigma}||\phi||^2 =$
$= \beta(1 + \frac{1}{2\alpha})\, k_o\, e^{2\alpha\sigma}||\phi||^2 = k_1||\phi||^2$ and b) is proved.

$$\frac{1}{\beta}|V(t,\phi_1) - V(t,\phi_2)| \leqslant \Big|\int_0^\infty ||y_{t+s}(t,\phi_1)||^2 ds -$$
$$- \int_0^\infty ||y_{t+s}(t,\phi_2)||^2 ds\Big| + \Big|\sup_{s\geqslant 0}||y_{t+s}(t,\phi_1)||^2 -$$
$$- \sup_{s\geqslant 0}||y_{t+s}(t,\phi_2)||^2\Big| \leqslant \int_0^\infty |[||y_{t+s}(t,\phi_1)||^2 -$$
$$- ||y_{t+s}(t,\phi_2)||^2]ds + \sup|[||y_{t+s}(t,\phi_1)||^2 -$$

$$- \| y_{t+s}(t,\phi_2)\|^2]| \leq \int_0^\infty |(\|y_{t+s}(t,\phi_1)\| + \|y_{t+s}(t,\phi_2)\|) \cdot$$

$$\cdot (\|y_{t+s}(t,\phi_1) - \|y_{t+s}(t,\phi_2)\|) \, ds + \sup_{s\geq 0} |(\|y_{t+s}(t,\phi_1) +$$

$$+ \|y_{t+s}(t,\phi_2)\|)| \cdot \sup_{s\geq 0} |(\|y_{t+s}(t,\phi_1)\| - \|y_{t+s}(t,\phi_2)\|)| \leq$$

$$\leq \int_0^\infty k_o \, e^{\alpha\sigma} \, e^{-\alpha\sigma}(\|\phi_1\| + \|\phi_2\|) k_o \, e^{\alpha\sigma} e^{-\alpha s}(\|\phi_1 - \phi_2\|) ds +$$

$$+ k_o \, e^{\alpha\sigma}(\|\phi_1\| + \|\phi_2\|) \, k_o \, e^{\alpha\sigma} e^{-\alpha s}(\|\phi_1\| - \|\phi_2\|) \leq$$

$$\leq k_o^2 \, e^{2\alpha\sigma}(\|\phi_1\| + \|\phi_2\|)(\|\phi_1 - \phi_2\|)(1 - \frac{1}{2\alpha})$$

Hence

$$|V(t,\phi_1) - V(t,\phi_2)| \leq$$

$$\leq \beta \, k_o^2 \, e^{2\alpha\sigma}(\|\phi_1\| + \|\phi_2\|)(\|\phi_1\| - \|\phi_2\|)(1 - \frac{1}{2\alpha})$$

and c) is proved.

To prove d) since

$$V(t,y_t(t_o,\phi) = \int_0^\infty \|y_{t+s}(t,y_t(t_o,\phi))\|^2 ds +$$

$$+ \sup_{s\geq 0} \|y_{t+s}(t,y_t(t_o,\phi))\|^2 = \int_0^\infty \|y_{t+s}(t_o,\phi)\|^2 ds +$$

$$+ \sup_{s\geq 0} \|y_{t+s}(t_o,\phi)\| = \int_t^\infty \|y_s(t_o,\phi)\|^2 ds + \sup_{s\geq 0} \|y_{t+s}(t_o,\phi)\|^2$$

and $\|y_{t+s}(t_o,\phi)\|^2$ is a decreasing function it follows that

$$\dot{V}(t,y_t) \leq \overline{\lim_{h\to 0_+}} \, h^{-1}[\int_{t+h}^\infty \|y_s(t_o,\phi)\|^2 ds -$$

$$- \int_t^\infty \|y_s(t_o,\phi)\|^2 ds] = \frac{d}{dt} \int_t^\infty \|y_s(t_o,\phi)\|^2 ds = -\|y_t(t_o,\phi)\|^2$$

and the theorem is proved.

Theorem 3: Assume that

$$||D(t,\phi) - \bar{D}(t,\phi)|| + ||\bar{f}(t,\phi) - f(t,\phi)|| \leq \gamma||\phi|| \qquad (10)$$

where γ is choosen small enough in such a way that $\gamma\phi \in C_H$. Then if the solution $y \equiv 0$ of (6) is uniformly asymptotically stable, the solution $x \equiv 0$ of (7) is uniformly asymptotically stable.

Proof: The solutions of (7) are the solutions of

$$\frac{d}{dt} D(t,x_t) = f(t,x_t) + h(t,x_t)$$

and from (10) $|h(t,\phi)| \leq \gamma||\phi||$.

Let $x_t(t_o,\phi)$ be a solution of (7) for $||\phi||$ small enough and $V(t,\phi)$ as in theorem 2, Define $V^*(t) = V(t,x_t(t_o,\phi))$. From a) and d) of Theorem 2 we have

$$\overline{\lim_{h\to 0_+}} \frac{V^*(t+h) - V^*(t)}{h} \leq$$

$$\overline{\lim_{h\to 0_+}} \frac{V(t+h,y_{t+h}(t,x_t(t_o,\phi))) - V(t,x_t(t_o,\phi))}{h} +$$

$$+ \overline{\lim_{h\to 0_+}} \frac{V(t+h,x_{t+h}(t,x_t(t_o,\phi))) - V(t+h,y_{t+h}(t,x_t(t_o,\phi)))}{h}$$

where $x_t(t_o,\phi)$ is a solution of (7), then.

$$\overline{\lim_{h\to 0_+}} \frac{V^*(t+h) - V(t)}{h} \leq -||y_t(t_o,\phi)||^2 +$$

$$+ \overline{\lim_{h\to 0_+}} \frac{(V(t+h,x_{t+h}(t,x_t(t_o,\phi)))) - V(t+h,y_{t+h}(t,x_t(t_o,\phi)))}{h}$$

From part c) of theorem 2 we have

$$V(t+h,x_{t+h}(t,x_t(t_o,\phi))) - V(t+h,y_{t+h}(t,x_t(t_o,\phi)))| \leqslant$$

$$\leqslant k(||x_{t+h}(t,x_t(t_o,\phi)) + y_{t+h}(t,x_t(t_o,\phi))||) \ .$$

$$. \ (||x_{t+h}(t,x_t(t_o,\phi) - y_{t+h}(t,x_t(t_o,\phi))||).$$

We know that

$$||y_{t+h}(t,x_t(t_o,\phi))|| \leqslant k_o||x_t(t_o,\phi)||.$$

From Lemma above and condition (10) we have

$$||x_{t+h}(t,x_t(t_o,\phi)) - y_{t+h}(t,x_t(t_o,\phi)) \leqslant$$

$$\leqslant (e^{Lh}-1)||g(t,x_t)|| \leqslant (e^{Lh}-1)\gamma||x_t(t_o,\phi)|| \quad \text{and then}$$

$$||x_{t+h}(t,x_t(t_o,\phi))|| \leqslant (e^{Lh}-1)||g(t,x_t)|| + k_o||x_t(t_o,\phi)||.$$

Hence

$$|V(t+h,x_{t+h}(t,x_t(t_o,\phi))) - V(t+h,x_{t+h}(t,x_t(t_o,\phi)))| \leqslant$$

$$\leqslant (K\gamma(e^{Lh}-1)||x_t(t_o,\phi) \quad +$$

$$+ 2k_o||x_t(t_o,\phi)||)(e^{Lh}-1)\gamma||x_t(t_o,\phi)) =$$

$$= ||x_t(t_o,\phi)||^2(K(e^{Lh}-1)\gamma + 2k_o)(e^{Lh}-1)\gamma =$$

$$= ||x_t(t_o,\phi)||^2(K(e^{Lh}-1)(e^{Lh}-1)\gamma^2 + 2k_o(e^{Lh}-1)\gamma).$$

Hence

$$\overline{\lim_{h\to 0_+}} \ h^{-1}|V(t+h,x_{t+h}(t,x_t(t_o,\phi))) - V(t+h,x_{t+h}(t,x_t(t_o,\phi))| \leqslant$$

$$\leqslant ||x_t(t_o,\phi)||^2 \overline{\lim_{h\to 0_+}} \ h^{-1}[k(e^{Lh}-1)(e^{Lh}-1)\gamma^2 + 2kk_o(e^{Lh}-1)\gamma] =$$

$$= 2kK_o \ L \ \gamma||x_t(t_o,\phi)||^2. \quad \text{Thus if we take} \quad \gamma < \frac{1}{2kk_oL}$$

$$\overline{\lim_{h\to 0_+}} \frac{V^*(t+h)-V^*(t)}{h} \leq -||x_t(t_o,\phi)||^2 + 2kk_oL\gamma||x_t(t_o,\phi)|| =$$

$$= ||x_t(t_o,\phi)||^2(2kk_oL\gamma-1) = -\gamma_1||x_t(t_o,\phi)||, \quad \gamma_1 > 0.$$

Then system (7) is uniformly asymptotically stable.

Another simple proof that implies in particular that the solutions of (7) are exponentially stable is the following. From conditions b) and d) of Theorem 2,

$$\overline{\lim} \frac{V^*(t+h) - V^*(t)}{h} \leq -\gamma_1 V^*(t), \quad \gamma_1 = \frac{1}{K_1}$$

$$\frac{1}{V^*(t)} \overline{\lim} \frac{V^*(t+h) - V^*(t)}{h} \leq -\gamma_1$$

or

$$\overline{\lim_{h\to 0_+}} \frac{\log V^*(t+h) - \log V^*(t)}{h} \leq -\gamma_1$$

$$\log V^*(t) - \log V^*(t_o) \leq -\gamma_1(t-t_o)$$

or

$$||x_t(t_o,\phi)||^2 \leq V^*(t) \leq V^*(t_o)Ke^{-\gamma_1(t-t_o)} \leq Ke^{-\gamma_1(t-t_o)}||\phi||^2$$

and the theorem is proved.

Theorem 4: Assume that

$$|\overline{f}(t,\phi)-f(t,\phi)| + |\overline{D}(t,\phi)-D(t,\phi)| \leq g(t)||\phi||, \quad \int g(t)dt < \infty \quad (11)$$

Then if the solution $y \equiv 0$ of (6) is uniform asymptotically stable the solution $x = 0$ of (7) is uniform asymptotically stable.

Proof: As in Theorem 4 we have from inequality (9)

$$||x_{t+h}(t,x_t(t_o,\phi)) - y_{t+h}(t,x_t(t_o,\phi))|| \leq$$

$$\leq (e^{Lh}-1)||h(t,x_t)|| \leq (e^{Lh}-1)g(t)||x_t(t_o,\phi)||$$

and

$||y_{t+h}(t,x_t(t_o,\phi)|| \leq k_o||x_t(t_o,\phi)||$ and therefore

$$||x_{t+h}(t,x_t(t_o,\phi))|| \leq |(e^{Lh}-1)g(t)||x_t(t_o,\phi)|| +$$

$$+ k_o||x_t(t_o,\phi)||$$

Thus we have

$$|V(t+h,x_{t+h}(t,x_t(t_o,\phi)) - V(t+h,y_{t+h}(t,x_t(t_o,\phi))| \leq$$

$$\leq (k(e^{Lh}-1)g(t)||x_t(t_o,\phi)|| +$$

$$+ 2k_o||x_t(t_o,\phi)||)(e^{Lh}-1)g(t)||x_t(t_o,\phi)|| =$$

$$= ||x_t(t_o,\phi)||^2(K(e^{Lh}-1)(e^{Lh}-1)g^2(t) + 2K_oK(e^{Lh} - 1)g(t))$$

Hence

$$\overline{\lim_{h\to 0_+}}\ h^{-1}[|V(t+h,x_{t+h}(t,x_t(t_o,\phi) - V(t+h,y_{t+h}(t,x_t(t_o,\phi)|] \leq$$

$$\leq k_1\ g(t)\ |\ ||x_t(t_o,\phi)||^2.$$

Therefore

$$\overline{\lim_{h\to 0_+}}\ \frac{V^*(t+h) - V^*(t)}{h} \leq - ||x_t(t_o,\phi)||^2+||x_t(t_o,\phi)||^2k_1g(t) =$$

$$= (k_1g(t) - 1)||x_t(t_o,\phi)||^2 \leq (k_1g(t) - 1)\ V^*(t).$$

Thus we have

$$||x_t(t_o,\phi)|| \leq V^*(t) \leq$$

$$\leq V^*(t_o) \exp[k_1 \int_{t_o}^{\infty} g(s)ds] \exp(-(t-t_o)) \leq$$

$$\leq K \exp - (t-t_o)||\phi||.$$

what implies that the solution $x \equiv 0$ of (7) is exponentially stable and therefore uniform asymptotically stable.

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