UNIVERSIDADE DE SÃO PAULO
CAMPUS DE SÃO CARLOS
INSTITUTO DE CIÊNCIAS MATEMÁTICAS DE SÃO CARLOS

I.C.M.S.C.

On mixed type partial differential equations

RASSIAS, JOHN MICHAEL

Nº 4

Notas Didáticas do ICMSC - USP

ISSN 0103-2585

On mixed type partial differential equations

RASSIAS, JOHN MICHAEL

Nº 4

São Carlos (SP)
1989

- LECTURE NOTES -

ON MIXED TYPE PARTIAL DIFFERENTIAL EQUATIONS

JOHN MICHAEL RASSIAS

Professor of Mathematics,
The American College of Greece

SÃO CARLOS/SP./BRASIL

August 11, 1988 - September 9, 1988

CONTENTS

PREFACE

This series of lecture notes includes various parts of the theory of mixed type partial differential equations with boundary conditions such as: the classical dynamical equation of mixed type due to S.A. Chaplygin (1904), regularity of solutions in the sense of the first pioneer in the field F. G. Tricomi (1923) and in brief his fundamental idea leading to one-dimensional singular integral equations, the characteristic problem due to S. Gellerstedt (1935), the non-characteristic problem due to F. I. Frankl (1945), the mixed type equation due to A.V. Bitsadze and M.A. Lavrentjev (1950), the classical a,b,c, energy integral method for mixed type boundary value problems and quasi-regularity of solutions in the sense of M.H. Protter (1953), weak (or strong) solutions in the classical sense, well-posedness in the sense that there is at most one quasi-regular solution and a weak solution exists, a selection of new results, and open problems.

The present book is a revised and augmented version of a lecture course delivered by me at the I.C.M.S.C.-U.S.P., Brasil from August 11, 1988 to September 9, 1988.

Deep gratitude is due to all those who have contributed to this work and have generously helped me carry out this project. My very special thanks to Aldo Ventura for his great help and kindness while staying at the beautiful environment of the Instituto de Ciências Matemáticas de São Carlos - U.S.P., São Paulo, Brasil.

My special thanks to Antonio Fernandes Izé who along with Professor Ventura invited me here at I.C.M.S.C.-U.S.P. and to Luiz Antonio Favaro (Director of I.C.M.S.C.), to Hildebrando Munhoz Rodrigues (Head of the Mathematics Department of I.C.M.C.S.) and to the Foundations: Coordenação de Aperfeiçoamento do Pessoal de Ensino Superior (C.A.P.E.S.) and Financiadora de Estudos e Projetos (FIN.E.P.), and to the Instituto de Ciências Matemáticas de São Carlos (I.C.M.S.C.), Universidade de São Paulo/Brasil for their support.

Finally my appreciation to the typist, Luiz Carlos Franco, for his excellent job on my present work.

São Carlos, 1988

John Michael Rassias

A GAS DYNAMICAL EQUATION OF MIXED TYPE

See : S.A. Chaplygin (ON GAS JETS , Scientific Annals of the Imperial University of Moscow, Publication nº 21 , 1904 . Translation: Brown University, Providence, R.I., 1944) , M.H. Protter (J. Rat. Mech. Anal., 2 , 1953 , 721-732), and J.M. Rassias (MATHEMATICS & SPACE TECHNOLOGY, Athens, 1981).

Consider a two-dimensional adiabatic potential flow of a perfect gas. The stream function $\psi = \psi(x,y)$ satisfies the equation

$$L\psi = (\rho^2.a^2-\psi_y^2).\psi_{xx} + 2.\psi_x.\psi_y.\psi_{xy} + (\rho^2.a^2-\psi_x^2).\psi_{yy} = 0 , \quad (1)$$

where

a : = the local velocity of sound,

ρ : = the density of the gas.

Equation (1) is transformed to a linear equation of mixed type by applying the hodograph transformation:

$$u = \rho^{-1}.\psi_y , \quad v = -\rho^{-1}.\psi_x , \quad (2)$$

where

u,v : the rectangular velocity components as new independent variables.

The corresponding components in polar coordinates are:

$$r = (u^2+v^2)^{1/2} , \quad \theta = \tan^{-1}(v/u) . \quad (3)$$

To normalize r introduce

(4) $$t = r^2/r_o^2 \quad ,$$

which is dimensionless quantity, as new independent variable, where

r_o : the speed corresponding to zero density.

Therefore equation (1) becomes

(5) $$\frac{\partial}{\partial t} \{ \frac{2t}{(1-t)^{\beta}} . \psi_t \} + \frac{1-(2\beta+1).t}{2.(1-t)^{\beta+1}} . \psi_{\theta\theta} = 0 \quad ,$$

where

$\beta = c_v / (c_p - c_v)$,

c_p : the specific heat at constant pressure ,

c_v : the specific heat at constant volume .

Note

(6) $$r_o = k . \gamma . \rho_o^{-1} / (\gamma - 1)$$

where

$\gamma = c_p / c_v$,

ρ_o : the density of gas at zero speed ,

k : satisfies the weel-known relation (: = const.)

(7) $$p = k . \rho^{\gamma} \; .$$

Introduce new independent variables

(8) $$\xi = \theta \; , \quad \eta = - \int_{1/2\beta+1)}^{t} \frac{(1-u)^{\beta}}{2.u} . du \quad .$$

Then equation (5) takes the form

$$\frac{1-(2.\beta+1).t}{(1-t)^{2.\beta+1}} \cdot \psi_{\xi\xi} + \psi_{\eta\eta} = 0 \quad . \tag{9}$$

Set by $K = K(\eta)$ the coefficient of $\psi_{\xi\xi}$.

Note

β : is a positive constant (≈ 2.5 for air),

γ : is a positive constant (≈ 1.4 for air).

Besides

(10) $K(0) = 0$, because

$\eta = 0$ for $t = 1/(2.\beta+1)$.

This case corresponds to points where the velocity is equal to the local velocity of sound, and therefore (9) is parabolic. Moreover

(11) $K(\eta) > 0$, because

$\eta > 0$ for $t < 1/(2.\beta+1)$

corresponding to subsonic velocities, and (9) is elliptic. Finally

$K(\eta) < 0$, because

$\eta < 0$ for $t > 1/(2.\beta+1)$

corresponding to supersonic velocities, and (9) is hyperbolic. Therefore equation (9) is of mixed type.

REMARKS:

i. The velocity potential $\phi = \phi(x,y)$ and the stream function $\psi = \psi(x,y)$ satisfy the Cauchy-Riemann equations

(12) $$\phi_x = \rho^{-1} . \psi_y \quad , \quad \phi_y = - \rho^{-1} . \psi_x \quad .$$

ii. The discriminant of equation (1) is given by the formula

(13) $$\mathcal{D} = (\rho^2.a^2-\psi_y^2) . (\rho^2.a^2-\psi_x^2) - (\psi_x.\psi_y)^2 \quad .$$

Then

(14) $$\mathcal{D} = (\rho.a)^4 . (1-M^2) \quad ,$$

where

M : Mach number $:= r/a$,

$r = (\psi_x^2 + \psi_y^2)^{1/2} / \rho$.

iii. A flow is called subsonic, sonic, or supersonic at a point as the flow speed r is: $< a$, $=a$, or $> a$, respectively. These three cases correspond to: $\mathcal{D} > 0$, $= 0$, or < 0 .

iv. Transonic flows involve a transition from the subsonic to the supersonic region through the sonic. Therefore transonic flows are the most interesting.

v. Transition from subsonic to supersonic flow becomes possible: Two sections of cones or similarly shaped tubes with the same axis are placed opposite each other and connected, thus forming a de Laval nozzle with "entry section", "throat", and "exhaust section". Then

a subsonic expanding flow in the entry section, on passing via the throat, can change into a supersonic expanding flow in the exhaust section.

vi. Equation (1) is quasi-linear and is converted to the linear equation of mixed type (9). The corresponding equation to (1) is the quasi-linear equation

(15) $$L\phi = (a^2-\phi_x^2) \cdot \phi_{xx} - 2 \cdot \phi_x \cdot \phi_y \cdot \phi_{xy} + (a^2-\phi_y^2) \cdot \phi_{yy} = 0 \quad .$$

where $\phi = \phi(x,y)$ is the velocity potential.

vii. Equation (15) comes from the Euler continuity equation

(16) $$L\phi = (\rho \cdot \phi_x)_x + (\rho \cdot \phi_y)_y = 0 \quad .$$

THE TRICOMI PROBLEM

In 1923 F.G. Tricomi (Atti Accad. Naz. Lincei, 14, 1923, 133-247) initiated the work on boundary value problems for partial differential equations of mixed type and related equations of variable type.

The Tricomi Problem or Problem T : Consists in finding a function

$u = u(x,y)$ which satisfies equation

(*) $$y.u_{xx} + u_{yy} = 0 \qquad \text{(the Tricomi equation)}$$

in a mixed domain D which is simply connected and bounded by a Jordan (non-selfintersecting) "elliptic" arc g_1 (for $y > 0$) with endpoints $0 = (o,o)$ and $A = (1,0)$ and by the "real" characteristics

$$g_2 : \quad x + \frac{2}{3}.(-y)^{3/2} = 1 \quad ,$$

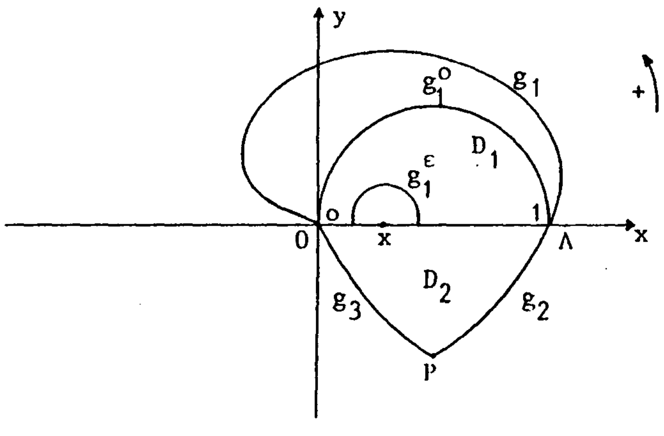
$$g_3 : \quad x - \frac{2}{3}.(-y)^{3/2} = 0$$

of (*) satisfying the characteristic equation

(17) $$y.(dy)^2 + (dx)^2 = 0$$

such that these characteristics meet at a point P (for $y < 0$), and assumes prescribed continuous boundary values

(**) $$\begin{cases} u = \phi(s) & \text{on} \quad g_1 \; , \\ u = \psi(x) & \text{on} \quad g_3 \; . \end{cases}$$



Denote

$D_1 = D \cap \{y > 0\}$: elliptic region , $D_2 = D \cap \{y < 0\}$: hyperbolic region and $OA := D \cap \{y = 0\}$: parabolic line of degeneracy . Consider the normal curve of Tricomi with equation

$$g_1^o : \quad (x - \tfrac{1}{2})^2 + \tfrac{4}{9} \cdot y^3 = \tfrac{1}{4}$$

or equivalently

$$g_1^o : \quad |z - \tfrac{1}{2}| = \tfrac{1}{2} \quad ,$$

where

$$z = x + i \cdot \tfrac{2}{3}\, y^{3/2} \quad , \quad i = \sqrt{-1} \quad ,$$

such that g_1 contains g_1^o in its interior.

REGULARITY OF SOLUTIONS

Definition 1 . A function $u = u(x,y)$ is a regular solution of Problem T in the sense of Tricomi if :

1) u is continuous in $\overline{D}$: closure of D ($:= D \cup \partial D$) ,

$$\partial D = g_1 \cup g_2 \cup g_3 \quad ,$$

2) u_x , u_y are continuous in $\overline{D}$
(except possibly points O , A , where they may have poles of order:

$< 2/3$, i.e. may go to infinity of order: $< 2/3$ as $x \to 0$ and $x \to 1$, i.e. $u_x = O(r^{-2/3+\varepsilon})$, $u_y = O(r^{-2/3+\varepsilon})$, where $r :=$ distance from O <u>or</u> A) ,

3) u_{xx}, u_{yy} are continuous in D (<u>except</u> possibly points on OA where they may <u>not</u> exist) ,

4) u satisfies equation (*) at all points of $D \setminus OA$ (i.e. D <u>except</u> OA) ,

5) u satisfies boundary conditions (**) .

<u>FUNDAMENTAL IDEA OF TRICOMI</u>

The fundamental idea of Tricomi of finding regular solutions for Problem T it was

FIRST: to solve <u>Problem N</u> (in D_1) : To find a regular solution of equation (*) satisfying the boundary conditions

$$(N) \qquad \begin{cases} u = \phi & \text{on } \quad g_1 \\ u_y = \nu & \text{on } \quad OA \end{cases}$$

where $\nu = \nu(x)$ is continuous for $x : 0 < x < 1$ and may go to infinity of order: $< 2/3$ as $x \to 0$ and $x \to 1$.

SECOND: to solve the Cauchy-Goursat Problem (in D_2) treating

$$\nu(x) = u_y(x,0)$$

as a known function,

i.e. First to solve the Cauchy Problem (in D_2) of equation (*) (in D_2) satisfying the boundary conditions

$$(C) \qquad \begin{cases} u = \tau & \text{on} \quad OA \\ u_y = \nu & \text{on} \quad OA \end{cases}$$

where $\tau = \tau(x)$ is continuous for $x : 0 < x < 1$ and may go to infinity of order: $< 2/3$ as $x \to 0$ and $x \to 1$, and

Second to take into account the boundary condition

$$u = \psi \quad \text{on} \quad g_3 \ ;$$

therefore we have the Goursat Problem (in D_2) of equation (*) (in D_2) satisfying the boundary conditions

$$(G) \qquad \begin{cases} u = \tau & \text{on} \quad OA \\ u = \psi & \text{on} \quad g_3 \end{cases}$$

Denote

$$\begin{aligned} \xi &= x - \frac{2}{3} \cdot (-y)^{3/2} \\ \eta &= x + \frac{2}{3} \cdot (-y)^{3/2} \end{aligned} \qquad \begin{pmatrix} \text{characteristic} \\ \text{coordinates} \end{pmatrix}$$

Then

$$g_2 : \eta = 1$$

$$g_3 : \xi = 0$$

Let

(18) $g_1 : x = x(s) , \quad y = y(s) ,$ (parametric equations of g_1)

where

s : arc length reckoned from A .

Assume the following conditions on g_1 :

i) The functions $x = x(s)$ and $y = y(s)$ have continuous derivatives $x'(s)$ and $y'(s)$ which do not vanish simultaneously on $[0,\ell]$, $\ell :=$ length of g_1 , and the second derivatives $x''(s)$ and $y''(s)$ satisfy a Hölder condition on $[0,\ell]$,

ii) In neighborhoods of the points O and A on g_1 ,

(19) $$\left|\frac{dx}{ds}\right| \leq C^2 . y^2(s) \quad ,$$

where

C : = constant .

REDUCTION OF CAUCHY PROBLEM (in characteristic coordinates) :

The Cauchy problem (*) and (C) is equivalent to the following problem in characteristic coordinates:

[*] $$Eu = u_{\xi\eta} + A(\xi,\eta) \cdot u_{\xi} + B(\xi,\eta) \cdot u_{\eta} = 0$$

(Euler-Darboux equation)

and

[C] $$\begin{cases} \lim\limits_{\eta-\xi\to o} u(\xi,\eta) = \tau(\xi) \\ \lim\limits_{\eta-\xi\to o} (\frac{3}{4})^{1/3} \cdot (\eta-\xi)^{1/3} \cdot (u_{\xi}-u_{\eta}) = \nu(\xi) \end{cases} ,$$

where

$$A = A(\xi,\eta) = \frac{1}{6} \cdot \frac{1}{\eta-\xi} ,$$

$$B = B(\xi,\eta) = -\frac{1}{6} \cdot \frac{1}{\eta-\xi} .$$

SOLUTION OF CAUCHY PROBLEM [*] and [C]

(20) $$u(\xi,\eta) = \int_{\xi}^{\eta} g(\xi,\eta;t) \cdot \tau(t) \cdot dt - \int_{\xi}^{\eta} h(\xi,\eta) \cdot \nu(t) \cdot dt$$

(K.I. Babenko: Doctoral Dissertation, Steklov Inst. Math. , Moscow , 1952) ,

where

$$h(\xi,\eta;t) = \gamma_2 \cdot (t-\xi)^{-1/6} \cdot (\eta-t)^{-1/6} ,$$

$$g(\xi,\eta;t) = \gamma_1 \cdot \frac{\Gamma(1/3)}{\Gamma^2(1/6)} \cdot \frac{(\eta-\xi)^{2/3}}{(t-\xi)^{5/6} \cdot (\eta-t)^{5/6}} ,$$

where

$$\gamma_1 = \frac{\Gamma(1/3)}{\Gamma^2(1/6)} \quad ,$$

$$\gamma_2 = \frac{1}{2} \cdot \left(\frac{4}{3}\right)^{1/3} \cdot \frac{\Gamma(2/3)}{\Gamma^2(5/6)} \quad .$$

IMPORTANT CONDITION:

Consider triangle OAP bounded by

i) the segment OP of the characteristic $g_3 : \xi = 0$,

ii) the segment PA of the characteristic $g_2 : \eta = 1$,

iii) an arc of the parabolic curve OA : $\eta = \xi$

i.e. $\Delta\, OAP := \{ \xi, \eta \in [0,1] \ , \ \eta > \xi \}$.

REDUCTION OF GOURSAT PROBLEM (in characteristic coordinates):

The Goursat problem (*) and (G) is equivalent to the following problem in characteristic coordinates: Equation (*) and

$$[G] \qquad \begin{cases} u\big|_{\eta = \xi} = \tau(\xi) \\ u\big|_{\xi = 0} = \psi(\eta) \ . \end{cases}$$

SOLUTION OF GOURSAT PROBLEM [*] and [G] (Special case: $\psi = 0$):

To solve the Goursat problem in this case we assume:

$u = u(\xi,\eta)$ to be a twice differentiable function of equation [*] in the triangle Δ OAP satisfying the boundary conditions [G].

Then

$$(21) \qquad u(\xi,\eta) = \frac{\Gamma(5/6)}{\Gamma(1/6)\,.\,\Gamma(2/3)}\,.\int_0^{\xi} \frac{(\eta-\xi)^{2/3}}{(\eta-t)^{5/6}\,.\,(\xi-t)^{5/6}}\,.\,\tau(t)\,.\,dt$$

satisfying equation [*] and boundary conditions [G] $(\psi = 0)$.

IN GENERAL (and combining Cauchy & Goursat Problem):

It is known G.H. Hardy and J.E. Littlewood (Math. Z. 27 (1927/28), 565-606) that:

If $f(x)$ satisfies a Hölder condition on $(0,1)$ with exponent α, it may be expressed as

$$(22) \qquad f(x) = f(0) + \int_0^x (x-t)^{\beta-1}\,.\,g(t)\,.\,dt \quad ,$$

where

$g = g(t)$ satisfies a Hölder condition with exponent $\alpha-\beta > 0$.

IMPORTANT CONDITIONS:

1) $\tau = \tau(t)$ satisfies a Hölder condition with exponent $\alpha_1 > \frac{5}{6}$ for $0 \leq t < 1$,

2) $\nu = \nu(t)$ satisfies a Hölder condition with exponent $\alpha_2 > \frac{1}{6}$ for $0 \leqq t < 1$.

Therefore

$\tau = \tau(t)$ and $\nu = \nu(t)$ may be expressed as

$$
(23)\qquad \begin{cases} \tau(t) = \tau(0) + \int_0^t (t-s)^{-1/6+\varepsilon} \cdot \phi(s) \cdot ds \\ \\ \nu(t) = \nu(0) + \int_0^t (t-s)^{-5/6+\varepsilon} \cdot \psi(s) \cdot ds \end{cases},
$$

where

$\varepsilon > 0$ is sufficiently small, and $\phi = \phi(s), \psi(s)$: are continuous functions for $0 \leqq s < 1$.

Substituting (23) into (20), changing the order of integration and using the integral representation of the hypergeometric function F we get

$$
\begin{aligned}
u(\xi,\eta) = & \int_0^\xi \phi_1(s) \cdot (\eta-s)^{-1/6+\varepsilon} \cdot F\left(\frac{1}{6}, \frac{1}{6}-\varepsilon, \frac{1}{3}; \frac{\eta-\xi}{\eta-s}\right) \cdot ds \\
& + \int_\xi^\eta \phi_2(s) \cdot (\eta-\xi)^{-1/6} \cdot (\eta-s)^{\varepsilon} \cdot F\left(\frac{1}{6}, \frac{5}{6}, 1+\varepsilon; \frac{\eta-s}{\eta-\xi}\right) \cdot ds \\
(24)\qquad & + \int_0^\xi \phi_3(s) \cdot (\eta-s)^{-1/6} \cdot (\xi-s)^{\varepsilon} \cdot F\left(\frac{1}{6}+\varepsilon, \frac{1}{6}, 1+\varepsilon; \frac{\xi-s}{\eta-s}\right) \cdot ds \\
& + \tau(0) - \left(\frac{3}{4}\right)^{2/3} \cdot (\eta-\xi)^{2/3} \cdot \nu(0) ,
\end{aligned}
$$

where

$$\phi_1(s) = \phi(s) - \sqrt{3}\,.\,\frac{\Gamma(\frac{2}{3})\,.\,\Gamma(\frac{1}{6}+\varepsilon)}{\Gamma(\frac{5}{6}+\varepsilon)}\,.\,\gamma_2\,.\,\psi(s) \quad ,$$

$$\phi_2(s) = \frac{\Gamma(\frac{1}{6})\,.\,\Gamma(\frac{5}{6}+\varepsilon)}{\Gamma(1+\varepsilon)}\,.\,\gamma_1\,.\,\phi(s) - \frac{\Gamma(\frac{5}{6})\,.\,\Gamma(\frac{1}{6}+\varepsilon)}{\Gamma(1+\varepsilon)}\,.\,\gamma_2\,.\,\psi(s) \quad ,$$

$$\phi_3(s) = \sqrt{3}\,.\,\frac{\Gamma(\frac{5}{6})\,.\,\Gamma(\frac{1}{6}+\varepsilon)}{\Gamma(1+\varepsilon)}\,.\,\gamma_2\,.\,\psi(s) \quad .$$

THE CAUCHY-GOURSAT PROBLEM

The Cauchy-Goursat Problem: consists in finding a solution of equation (*) in $\overline{D}_2$ ($= D_2 \cup \partial D_2$) and satisfying boundary conditions

$$(CG) \qquad \begin{cases} u|_{\xi=0} = \phi(\eta) \\ u_y|_{y=0} = \nu(x) \quad . \end{cases}$$

THE TRICOMI PROBLEM

The Tricomi Problem: consists in finding a solution of equation (*) in D assuming prescribed values on g_1 and on the characteristic g_3 :

$$(T) \qquad \begin{cases} u|_{g_1} = \phi(s) \\ u|_{\xi=0} = \psi(\eta) \quad , \end{cases}$$

where

$\phi(s)$: satisfies a Hölder consition with exponent α , and

$\psi(\eta)$: has a bounded first derivative satisfying a Hölder condi- tion with exponent β .

THE HYPERGEOMETRIC FUNCTION

The integral representation of the hypergeometric function F is

$$(25)\quad F(a,b,c;\mu) = \frac{\Gamma(c)}{\Gamma(a)\,.\,\Gamma(c-a)} \,.\, \int_0^1 t^{a-1}\,.\,(1-t)^{c-a-1}\,.\,(1-\mu.t)^{-b}\,.\,dt \quad ,$$

$$0 < Re(a) < Re(c) \; .$$

INTEGRALS OF FRACTIONAL ORDER

The expression

$$(26)\qquad f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \,.\, \int_0^x (x-t)^{\alpha-1}\,.\,f(t)\,.\,dt \;, \quad 0 \leqq x < 1 \;,$$

$$f = f(t) \; : \; \text{integrable on } (0,1) \;,$$

is called an integral of fractional order α (compare (26) with (22)).

We often use expressions of type (26) .

SOLUTION OF CAUCHY PROBLEM (*) AND (C)

(27)
$$u(x,y) = \gamma_1 . \int_0^1 \tau[x + \frac{2}{3} . (-y)^{3/2} . (2t-1)] . [t.(1-t)]^{-5/6} . dt$$
$$+ \frac{\Gamma(5/3)}{\Gamma^3(5/6)} . y . \int_0^1 \nu[x + \frac{2}{3}.(-y)^{3/2}.(2t-1)].[t.(1-t)]^{-1/6} . dt$$

Equating expression (27) on characteristic g_3 to function $\psi(x)$ (i.e. combining with Goursat Problem) and applying the well-known inversion formula

(28)
$$F(x) = \frac{\sin(\pi . \alpha)}{\pi} . \frac{d}{dx} \int_0^x \frac{\Phi(t)}{(x-t)^{1-\alpha}} . dt$$

to the Abel integral equation

(29)
$$\int_0^x \frac{F(t)}{(x-t)^{\alpha}} . dt = \Phi(x) \quad , \quad 0 < \alpha < 1$$

We get the First Fundamental Functional Relation for Problem T :

(30)
$$\tau(x) = \psi_1(x) + \gamma . \int_0^x \frac{\nu(t)}{(x-t)^{1/3}} . dt \quad ,$$

where

$$\psi_1(x) = \frac{1}{2 . \pi . \gamma_1} . x^{5/6} . \frac{d}{dx} \int_0^x \frac{\psi(t/2)}{t^{2/3} . (x-t)^{1/6}} . dt \quad ,$$

$$\gamma = \frac{3^{2/3} . \Gamma^3(1/3)}{4 . \pi^2} \quad .$$

SOLUTION OF PROBLEM N

By direct checking we see that

$$|\zeta - x|^{-1/3} : \quad \text{is a solution of} \quad (*) ,$$

where

$$\zeta = \xi + i \cdot \frac{2}{3}\,\eta^{3/2} \qquad (\zeta \neq x \;,\; \eta < 0) \;.$$

Denote

$$G = G(x\,;\,\xi\,,\,\eta) : \quad \underline{\text{the Green's function}}$$

i.e. the function of the form

$$\text{(31)} \qquad G(x\,;\,\xi\,,\,\eta) = |\zeta - x|^{-1/3} + G_o(x\,;\,\xi\,,\,\eta) \quad ,$$

where

$$G_o = G_o(x\,;\,\xi\,,\,\eta) : \quad \text{is a solution of} \quad (*)$$

regular with respect to ξ, and η in D_1, and satisfying the conditions

$$\text{(32)} \qquad \begin{cases} G_o = -\,|\zeta - x|^{-1/3} \;, & \zeta \in g_1 \\ \left.\dfrac{\partial G_o}{\partial \eta}\right|_{\eta = o} = 0 & \end{cases} \quad .$$

In the special case when g_1 coincides with the normal curve

$$g_1^o : \quad (\xi - \tfrac{1}{2})^2 + \frac{4}{9} \cdot \eta^3 = \frac{1}{4}$$

or equivalently

$$g_1^o : \quad \left|\zeta - \frac{1}{2}\right| = \frac{1}{2} \quad ,$$

the function

(33) $$G_o(x\,;\,\xi\,,\,\eta) = - \left|\zeta + x - 2\zeta\,.\,x\right|^{-1/3}$$

and therefore from (31) the function

(34) $$G(x\,;\,\xi\,,\,\eta) = \left|\zeta - x\right|^{-1/3} - \left|\zeta + x - 2\zeta\,.\,x\right|^{-1/3} \quad .$$

Isolating the point $(x,0)$ by the curve g_1^ε of the normal contour

$$g_1^\varepsilon : \quad \left|\zeta - x\right| = \varepsilon$$

from domain D_1 and applying Green's formula (to the remaining part of D_1) :

$$\int \eta\,.\,\left(\frac{\partial G}{\partial \xi} - G\,.\,\frac{\partial u}{\partial \xi} \right).\,d\eta - \left(u\,.\,\frac{\partial G}{\partial \eta} - G\,.\,\frac{\partial u}{\partial \eta} \right).\,d\xi = 0$$

we get (as $\varepsilon \to 0$) from (32) that

(35) $$\lim_{\varepsilon \to o} \left(\int_o^{x-\varepsilon} + \int_{x+\varepsilon}^{1} \right).\,G\,.\,\nu(\xi)\,.\,d\xi - \int_{g_1} u\,.\,\left(\eta\,.\,\frac{\partial G}{\partial \xi}\,.\,\frac{\partial \xi}{\partial N} + \frac{\partial G}{\partial \eta}\,.\,\frac{\partial \eta}{\partial N} \right).\,ds$$
$$- \lim_{\varepsilon \to o} \int_{g_1^\varepsilon} u\,.\,\left(\eta\,.\,\frac{\partial G}{\partial \xi}\,.\,\frac{\partial \xi}{\partial N} + \frac{\partial G}{\partial \eta}\,.\,\frac{\partial \eta}{\partial N} \right).\,ds = 0$$

where

N : is the inner normal of the boundary ∂D ,

$\frac{\partial G}{\partial N} := \eta\,.\,\frac{\partial G}{\partial \xi}\,.\,\frac{\partial \xi}{\partial N} + \frac{\partial G}{\partial \eta}\,.\,\frac{\partial \eta}{\partial N}$: is the co-normal derivative associated with Tricomi operator.

But

(36) $$\lim_{\varepsilon \to 0} \int_{g_1^\varepsilon} u \, . \frac{\partial G}{\partial N} . \, ds = [2^{-1/3} . 3^{-2/3} . \int_0^\pi (\sin \theta)^{1/3} . d\theta] . \tau(x) .$$

Therefore from condition (N) we have the Second Fundamental Functional Relation for Problem T :

(37) $$\tau(x) + \gamma . \int_0^1 G(x ; \xi , 0) . \nu(\xi) . d\xi = F^*(x) ,$$

where

$$\gamma = 2^{-1/3} . 3^{-2/3} . \int_0^\pi (\sin \theta)^{1/3} . d\theta = \frac{3^{2/3} . \Gamma^3(1/3)}{4 . \pi^2} ,$$

$$F^* = \gamma . \int_{g_1} \phi . \left(\frac{\partial G}{\partial \xi} . \frac{\partial \xi}{\partial N} + \frac{\partial G}{\partial \eta} . \frac{\partial \eta}{\partial N} \right) . ds .$$

SPECIAL CASE (when g_1 coincides with g_1^0) :

The expression (37) becomes simpler:

(38) $$\tau(x) + \gamma . \int_0^1 \left[\frac{1}{|t-x|^{1/3}} - \frac{1}{(t+x-2t.x)^{1/3}} \right] . \nu(t) . dt = F^*(x),$$

where

$F^* = F^*(x)$ is analytic function of x for $0 < x < 1$.

Eliminating $\tau = \tau(x)$ from functional relations (30) and (38) we get

(39) $$\int_0^x \frac{\nu(t)}{(x-t)^{1/3}} . dt + \int_0^1 \left[\frac{1}{|x-t|^{1/3}} - \frac{1}{(t+x-2t.x)^{1/3}} \right] . \nu(t) . dt = \phi_1^*(x) ,$$

where

$$\phi_1^* = \frac{1}{\gamma} \cdot [F^*(x) - \psi_1(x)] .$$

Applying the inversion formula (28) to the Abel integral equation (29) we write (39) , as follows:

(40)
$$\nu(x) + \frac{\sqrt{3}}{2.\pi} \cdot \frac{d}{dx} \int_0^1 \nu(t) . dt . [\int_0^x \frac{d\xi}{|\xi - t|^{1/3} . (x-\xi)^{2/3}} - \int_0^x \frac{d\xi}{(\xi + t - 2t.\xi)^{1/3} . (x-\xi)^{2/3}}] = \frac{3}{2} . F(x) ,$$

where

$$F(x) = \frac{1}{\pi . \sqrt{3}} \cdot \frac{d}{dx} \int_0^x \frac{\phi_1^*(t)}{(x-t)^{2/3}} . dt .$$

When x lies strictly inside the interval $(0,1)$ the second integral term on the left side of (40) can be written in the form

(41)
$$- \frac{\sqrt{3}}{2.\pi} \cdot \frac{d}{dx} \int_0^1 \nu(t) . dt . \int_0^x \frac{d\xi}{(\xi + t - 2t.\xi)^{1/3} . (x-\xi)^{2/3}}$$

$$= - \frac{3\sqrt{3}}{2.\pi} \cdot \frac{d}{dx} \{ \int_0^{1/2} \frac{\nu(t)}{(1-2.t)^{1/3}} . dt . \int_0^{[\frac{x.(1-2.t)}{t}]^{1/3}} \frac{dz}{1+z^3} + \int_{1/2}^1 \frac{\nu(t)}{(1-2t)^{1/3}} . dt . \int_0^{[\frac{x.(2t-1)}{t}]^{1/3}} \frac{dz_1}{1-z_1^3} \}$$

$$= - \frac{3}{2} \cdot \frac{1}{\pi.\sqrt{3}} \cdot \int_0^1 (\frac{t}{x})^{2/3} . \frac{\nu(t)}{t+x-2t.x} . dt ,$$

where

$$z_1^3 = \frac{x-\xi}{\xi - \frac{t}{2t-1}} \quad , \quad z_2^3 = \frac{x-\xi}{\frac{t}{2.t-1} - \xi} \quad .$$

Denote

$$I(x) = \int_o^1 \nu(t) \, . \, dt \, . \int_o^x \frac{d\xi}{|\xi-t|^{1/3} \, . \, (x-\xi)^{2/3}} \quad ,$$

$$I_\varepsilon(x) = \int_o^{x-\varepsilon} \nu(t) \, . \, dt \, . \left[\int_o^t \frac{d\xi}{(t-\xi)^{1/3} \, . \, (x-\xi)^{2/3}} + \int_t^x \frac{d\xi}{(\xi-t)^{1/3} . (x-\xi)^{2/3}} \right]$$

$$+ \int_{x+\varepsilon}^1 \nu(t) \, . \, dt \, . \int_o^x \frac{d\xi}{(t-\xi)^{1/3} \, . \, (x-\xi)^{2/3}} \quad .$$

Then

(42) $$I(x) = \lim_{\varepsilon \to o} I_\varepsilon(x) \quad .$$

Using the substitution

(43) $$\xi = t + (x-t) \, . \, z \quad ,$$

we get

(44) $$\int_t^x \frac{d\xi}{(\xi-t)^{1/3} \, . \, (x-\xi)^{2/3}} = \int_o^1 z^{-1/3} \, . \, (1-z)^{-2/3} \, . \, dz$$

$$= \frac{2\pi}{\sqrt{3}} \quad , \quad x > t \quad .$$

Using the substitution

$$z^3 = \frac{x-\xi}{t-\xi}$$

we get

$$\int_0^t \frac{d\xi}{(t-\xi)^{1/3} . (x-\xi)^{2/3}} = -3 . \int_{(\frac{x}{t})^{1/3}}^{\infty} \frac{dz}{1-z^3} \quad , \quad x > t \quad ,$$

or

(45)

$$\int_0^t \frac{d\xi}{(t-\xi)^{1/3} . (x-\xi)^{2/3}} = -\ln(\zeta-1) + \ln\sqrt{1+\zeta+\zeta^2}$$
$$+ \sqrt{3} . \tan^{-1}(\frac{2.\zeta+1}{\sqrt{3}}) - \frac{3\pi}{2\sqrt{3}} \quad , \quad x > t$$

where

$$\zeta = (\frac{x}{t})^{1/3} \quad .$$

Besides

(46)

$$\int_0^x \frac{d\xi}{(t-\xi)^{1/3} . (x-\xi)^{2/3}} = -3 . \int_\zeta^0 \frac{dz}{1-z^3} = -\ln(1-\zeta) + \ln\sqrt{1+\zeta+\zeta^2}$$
$$+ \sqrt{3} . \tan^{-1}(\frac{2.\zeta+1}{\sqrt{3}}) - \frac{\pi}{\sqrt{3}} \quad , \quad x < t \quad .$$

Then substituting (44) - (46) into the integral expression of $I_\varepsilon(x)$ above we get

(47)

$$I_\varepsilon(x) = \frac{\pi}{2\sqrt{3}} . [\int_0^{x-\varepsilon} \nu(t) . dt - \int_{x+\varepsilon}^1 \nu(t) . dt]$$
$$+ (\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^1) . [-\ln|1-\zeta| + \ln\sqrt{1+\zeta+\zeta^2} + \sqrt{3}.\tan^{-1}(\frac{2.\zeta+1}{\sqrt{3}})].\nu(t). dt$$

Differentiating $I_\varepsilon(x)$ of (47) with respect to x we get

$$I'_\varepsilon(x) = \frac{\pi}{2\sqrt{3}}[\,\nu(x-\varepsilon) + \nu(x+\varepsilon)\,] + \left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^{1} \right) . \left(\frac{t}{x}\right)^{2/3} . \frac{\nu(t)}{t-x} . dt$$

(48)
$$+ \nu(x-\varepsilon) . \left[-\ell n(\zeta-1) + \ell n\sqrt{1+\zeta+\zeta^2} + \sqrt{3} . \tan^{-1}\left(\frac{2.\zeta+1}{\sqrt{3}}\right) \right] \Bigg|_{\zeta=\left(\frac{x}{x-\varepsilon}\right)^{1/3}}$$

$$+ \nu(x+\varepsilon) . \left[\ell n(1-\zeta) - \ell n\sqrt{1+\zeta+\zeta^2} - \sqrt{3} . \tan^{-1}\left(\frac{2.\zeta+1}{\sqrt{3}}\right) \right] \Bigg|_{\zeta=\left(\frac{x}{x+\varepsilon}\right)^{1/3}}$$

Letting $\varepsilon \to 0$ we get from (42) and (48) that

(49)
$$I'(x) = \lim_{\varepsilon \to 0} I'_\varepsilon(x) = \frac{\pi}{\sqrt{3}} . \nu(x) + \int_0^1 \left(\frac{t}{x}\right)^{2/3} . \frac{\nu(t)}{t-x} . dt$$

Therefore we find

(50)
$$\frac{\sqrt{3}}{2\pi} . \frac{d}{dx} \int_0^1 \nu(t) . dt . \int_0^x \frac{d\xi}{|\xi-t|^{1/3} . (x-\xi)^{2/3}} = \frac{1}{2} . \nu(x) + \frac{3}{2} . \frac{1}{\pi . \sqrt{3}} . \int_0^1 \left(\frac{t}{x}\right)^{2/3} . \frac{\nu(t)}{t-x} . dt \;.$$

From (40) , (41) and (50) we get

(51)
$$\nu(x) + \frac{1}{\pi . \sqrt{3}} . \int_0^1 \left(\frac{1}{t-x} - \frac{1}{t+x-2t.x} \right) . \nu(t) . dt = F(x)$$

which is <u>a one-dimensional singular integral equation with respect to ν for Problem T</u> . This equation (51) is equivalent to Problem T in the case when g_1 <u>coincides</u> with the normal curve g_1^o (the latter curve gives a simple Green's function) .

GENERAL CASE (When g_1 does not coincide with g_1^o) :

Eliminating the function $\tau = \tau(x)$ from expressions (30) and (37) as above we get

the general one-dimensional singular integral equation with respect to ν for Problem T

which is:

(52)
$$\nu(x) + \frac{1}{\pi . \sqrt{3}} . \int_o^1 \left(\frac{t}{x}\right)^{2/3} . \left(\frac{1}{t-x} - \frac{1}{t+x-2t.x}\right) . \nu(t) . dt + \int_o^1 K(x,t) . \nu(t) . dt = F(x) \quad ,$$

where

$K = K(x,t)$: is expressed through $G_o(x ; t , 0)$.

TRICOMI'S THEOREM :

If $F(x)$ satisfies a Hölder condition for $0 < x < 1$ and $F(x) \in L^p(0,1)$, $p > 1$, then the solution of the equation (51) in the class $\nu(x)$ such that $x^{2/3} . \nu(x) \in L^p(0,1)$ is given by

(53)
$$\nu(x) = \frac{3}{4} . \left\{ F(x) - \frac{1}{\pi . \sqrt{3}} . \int_o^1 \left[\frac{t(1-t)}{x.(1-x)}\right]^{1/3} . \left(\frac{1}{t-x} - \frac{1}{t+x-2t.x}\right) . F(t) . dt \right\} + A . x^{-4/3} . (1-x)^{-1/3} \quad ,$$

where A is an arbitrary constant.

REMARKS:

1) Since $\nu = \nu(x)$ may have a pole of order less than $\frac{2}{3}$ as $x \to 0$ we get $\Lambda = 0$ in (53).

2) With the help of the above Tricomi's theorem the singular equation (52) is regularized. Therefore we have Regularization of the Singular Equation (52).

LEMMA 1.

If $\phi = \phi(s)$ satisfies a Hölder condition with exponent α, and

$$(54) \qquad \begin{cases} \phi(s) = \phi(\ell) + \phi_1(s) \\ \phi(s) = \phi(0) + \phi_2(s) \end{cases},$$

where ℓ : length of g_1

$$(55) \qquad \begin{cases} |\phi_1(s)| < C.(\ell - s)^{4/3} \\ |\phi_2(s)| < C'.s \end{cases}$$

and $\psi'(\eta)$ satisfies a Hölder condition with exponent β,

then

F(x) satisfies a Hölder condition with exponent $\frac{1}{6} + \varepsilon$, $\varepsilon > 0$, for $0 \leq x < 1$.

PROOF OF TRICOMI'S THEOREM

Set

(56) $$\begin{cases} \phi(x) = x^{2/3} \,.\, \nu(x) \quad , \\ f(x) = x^{2/3} \,.\, F(x) \quad . \end{cases}$$

Then singular integral equation (51) is reduced to

(57) $$\phi(x) + \frac{1}{\pi.\sqrt{3}} \,.\, \int_0^1 \left(\frac{1}{t-x} - \frac{1}{x+t-2.t.x}\right) . \phi(t) \,.\, dt = f(x) \quad .$$

Let z be an arbitrary point of the complex plane.

Following Carleman (Ark. Mat. Astr. Fys. 16, Nº 26, 1922), we set

(58) $$\Phi(z) = \frac{1}{2\pi.i} \,.\, \int_0^1 \left(\frac{1}{t-z} - \frac{1}{t+z-2t.z}\right) . \phi(t) \,.\, dt \quad .$$

It is clear that

$\Phi = \Phi(z)$: is holomorphic in both upper and lower half-planes, and for $0 < x < 1$:

(59) $$\begin{cases} \Phi^+(x) - \Phi^-(x) = \phi(x) \\ \Phi^+(x) + \Phi^-(x) = \frac{1}{\pi.i} \,.\, \int_0^1 \left(\frac{1}{t-x} - \frac{}{x+t-2t.x}\right) . \phi(t) \,.\, dt \end{cases}$$

Therefore (57) is reduced via (59) to the equation

(60) $$\left(1 + \frac{i}{\sqrt{3}}\right) . \Phi^+(x) - \left(1 - \frac{i}{\sqrt{3}}\right) . \Phi^-(x) = f(x) \quad , \quad 0 < x < 1 \quad .$$

Then from (58) we get

(61) $$\Phi\left(\frac{z}{2z-1}\right) = (2.z-1)\,.\,\Phi(z)\ .$$

<u>Note</u>: that <u>the Mobius transformation</u>:

$$\zeta = \frac{z}{2.z-1}$$

maps the upper half-plane onto the lower, and conversely ; the interval (0,1) is mapped onto the two rays $(0,\infty)$ and $(-\infty, 1)$.

Replacing x by $\frac{x}{2x-1}$ in (60) and using (61), we get

(63) $$\left(1+\frac{i}{\sqrt{3}}\right).\,\Phi^-(x) - \left(1-\frac{i}{\sqrt{3}}\right).\,\Phi^+(x) = \frac{1}{2x-1}\,.\,f\left(\frac{x}{2.x-1}\right)\ ,$$

$$-\infty < x < 0\ ,\quad 1 < x < \infty\ .$$

Denote

$$G(x) = \begin{cases} \dfrac{1-i\sqrt{3}}{2}\ ; & 0 < x < 1 \\[2ex] \dfrac{1+i\sqrt{3}}{2}\ , & -\infty < x < 0\ ,\ 1 < x < \infty\ , \end{cases}$$

$$H(x) = \begin{cases} \dfrac{\sqrt{3}}{\sqrt{3}+1}\,.\,f(x)\ , & 0 < x < 1 \\[2ex] -\dfrac{\sqrt{3}}{\sqrt{3}-i}\,.\,\dfrac{1}{2.x-1}\,.\,f\left(\dfrac{x}{2.x-1}\right)\ , & -\infty < x < 0\ ,\ 1 < x < \infty\ . \end{cases}$$

Then combining (60) and (63) we get the equation

(64) $$\Phi^+(x) - G(x)\,.\,\Phi^-(x) = H(x)\ ,\ -\infty < x < \infty\ ,\quad x \neq 0\ ,\quad x \neq 1\ .$$

Therefore the solution of the singular integral equation (57) is reduced to the following

PROBLEM OF COMPLEX ANALYSIS:

Find a function $\Phi = \Phi(z)$: holomorphic in both upper and lower half-planes and satisfying the boundary condition (64) .

SOLUTION:

The solution of this problem can be obtained in explicit form.

In fact, the function

$$X(z) = \exp \left[-\frac{1}{6} \cdot \int_0^1 \left(\frac{1}{t-z} - \frac{2.z-1}{t+z-2t.z}\right) . dt \right] \tag{65}$$

is a particular solution of the corresponding homogeneous problem:

$$X^+(x) = G(x) . X^-(x) \tag{66}$$

and satisfies the condition

$$X\left(\frac{z}{2.z-1}\right) = X(z) . \tag{67}$$

From (65) we have

(68) $$\begin{cases} X^+(x) = e^{-i.\pi/6} . \left(\frac{x}{1-x}\right)^{1/3} , & 0 < x < 1 , \\ X^-(x) = e^{i.\pi/6} . \left(\frac{x}{1-x}\right)^{1/3} , & 0 < x < 1 , \end{cases}$$

and, by (67) and (68) ,

(69) $$\begin{cases} X^+(x) = e^{i.\pi/6} . \left(\frac{x}{1-x}\right)^{1/3} , & -\infty < x < 0 , \quad 1 < x < \infty \\ X^-(x) = e^{-i.\pi/6} . \left(\frac{x}{1-x}\right)^{1/3} , & -\infty < x < 0 , \quad 1 < x < \infty . \end{cases}$$

Replacing $G(x)$ by $X^+(x) / X^-(x)$ in (64) and dividing by $X^+(x)$, we get :

(70) $$\frac{\Phi^+(x)}{X^+(x)} - \frac{\Phi^-(x)}{X^-(x)} = \frac{H(x)}{X^+(x)} .$$

One particular solution of this equation is

(71) $$\Phi(z) = \frac{X(z)}{2\pi.i} . \int_{-\infty}^{\infty} \frac{H(t)}{X^+(t)} . \frac{dt}{t-z}$$

To find the general solution, we consider the homogeneous equation

(72) $$\frac{\Phi^+(x)}{X^+(x)} - \frac{\Phi^-(x)}{X^-(x)} = 0 .$$

Note: This equation (72) shows that the function

(73) $$\Psi(z) = \frac{\Phi(z)}{X(z)}$$

is holomorphic in the whole plane, with the possible exception of $z = 0$ and $z = 1$, which can only be poles. Besides

$\Psi(z)$ vanishes at infinity.

Assuming that $\Phi(z)$ may have poles of order less than one as $z \to 0$ or $z \to 1$, we see that

$$\Psi(z) = a \cdot \frac{1}{z} ,$$

where $a :=$ arbitrary constant.

Thus the general solution of equation (64) is:

$$\Phi(z) = \frac{X(z)}{2\pi . i} \cdot \int_{-\infty}^{\infty} \frac{H(t)}{X^{+}(t)} \cdot \frac{dt}{t-z} + a \cdot \frac{X(z)}{z}$$

(74)

$$= \frac{X(z)}{2\pi . i} \cdot [\int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{\infty}] .$$

In the integrals

$$\int_{-\infty}^{0} \quad \text{and} \quad \int_{1}^{\infty}$$

above we replace t by $\frac{t}{2.t-1}$.

Then from (67) we get:

(75) $$\Phi(z) = \frac{X(z)}{2\pi . i} \cdot \int_{0}^{1} \frac{\sqrt{3}}{\sqrt{3}+i} \cdot \frac{f(t)}{X^{+}(t)} \cdot \left(\frac{1}{t-z} - \frac{1}{t+z-2z.t}\right) . dt + a \cdot \frac{X(z)}{z}$$

Define

(76) $$\phi(x) = \Phi^{+}(x) - \Phi^{-}(x) \;, \quad 0 < x < 1 \;.$$

Then from (75) - (76) we get

$$\phi(x) = \frac{3}{4} . \left[f(x) - \frac{1}{\pi . \sqrt{3}} . \left(\frac{x}{1-x}\right)^{1/3} . \int_0^1 \left(\frac{1-t}{t}\right)^{1/3} . \right.$$

(77) $$\left. . \left(\frac{1}{t-x} - \frac{1}{t+x-2t.x}\right) . f(t) . dt \right]$$

$$+ A . x^{-2/3} . (1-x)^{-1/3} \;,$$

where

A : = arbitrary constant.

Returning to the previous functions we get (53) ; thus completing the proof of Tricomi's theorem.

<u>REMARKS</u>:

I. The integral

$$\int_0^1 \left(\frac{t}{x}\right)^{2/3} . \left(\frac{1}{t-x} - \frac{1}{t+x-2t.x}\right) . \nu(t) . dt$$

is <u>one-dimensional singular integral</u>

i.e. it is the principal value of a divergent integral.

Singular integral equations were studied for the <u>first</u> time by T. Carleman (1922) (Ark. Mat. Astr. Fys. 16 , Nº 26 , 1922) .

The kernel of the integral equation (51) contains the addend $:= \frac{1}{t+x-2t.x}$ which is neither Cauchy kernel nor Summable. Therefore equation (51) does not belong to the class of Carleman's equations.

To solve (51) F.G. Tricomi thought as follows:

Denote the kernel

(78) $$L(t,x) \; := \; \left(\frac{t}{x}\right)^{2/3} . \left(\frac{1}{t-x} - \frac{1}{t+x-2t.x}\right)$$

and kernels $L_{n+1}(t,x)$:

(79) $$\begin{cases} L_1(t,x) = L(t,x) \\ L_{n+1}(t,x) = \frac{1}{n} . \sum_{i=o}^{n-1} L_{i+1}(t,w) . L_{n-i}(w,x) . dw \, , \quad n \geq 1 \, . \end{cases}$$

If $L(t,x)$ were an ordinary Fredholm kernel, then all the addenda in (79) would be equal and we would have the ordinary iterated kernels.

But these addenda are different in Tricomi's case because:

the ordinary formula of integration order
fails for a repeated singular integral.

Tricomi has shown that

(80) $$L_{n+1}(t,x) = \frac{1}{n!} . L(t,x) . \left[\, 2.\ell n \frac{(1-t).x}{t.(1-t)} \,\right]^n \, , \quad n \geq 1$$

which is valid for the kernel L of (78).

II. The two solutions (one of Problem N in D_1 and another of the Cauchy-Goursat Problem in D_2) obtained are then matched, together with their first derivatives on OA .

III. Let the function A(t) : be continuous on [a,b] and have a continuous first derivative on [a,b] , which may have poles of order: < 1 at the endpoints of [a,b] , and let c : be an interior point of [a,b] .

Then

(81)
$$\int_a^b \frac{A(t)}{t-c} \cdot dt = A(b) \cdot \ell n(b-c) - A(a) \cdot \ell n(c-a) - \int_a^b A'(t) \cdot \ell n|t-c| \cdot dt$$

Under the same consitions, we have

$$\int_a^b \frac{A(t)}{t-c} \cdot dt = \lim_{\varepsilon \to o} \left[- \int_a^c \frac{A(t)}{(c-t)^{1-\varepsilon}} \cdot dt + \int_c^b \frac{A(t)}{(t-c)^{1-\varepsilon}} \cdot dt \right]$$

by using (81) and recalling that

(83)
$$\lim_{\varepsilon \to o} \frac{x^{\varepsilon} - 1}{\varepsilon} = \ell n\, x$$

LEMMA 2.

If

$$T_1(\alpha\ ,\ \beta\ ;\ x)\ =\ \int_o^\infty \frac{t^{\alpha-1}}{[x+(1-x).t]^\beta\ .\ (1-t)}\ .\ dt\ ,$$

$$T_2(\alpha\ ,\ \beta\ ;\ x)\ =\ \int_o^\infty \frac{t^{\alpha-1}}{[x+(1-x).t]^\beta\ .\ (1+t)}\ .\ dt\ ,$$

where

$\alpha\ ,\ \beta$: constants : $0 < \alpha < \beta+1\ ,\quad \alpha > \beta\ ,$

then

$$(84)\qquad \begin{cases} T_1(\alpha\ ,\ \beta\ ;\ x)\ =\ \pi.\cot(\alpha-\beta)\pi\ +\ x^{\alpha-\beta}\ .\ \dfrac{\Gamma(\alpha)\ .\ \Gamma(\beta-\alpha)}{\Gamma(\beta)}\ .\ F_1\ , \\[2ex] T_2(\alpha\ ,\ \beta\ ;\ x)\ =\ (1-x)\ .\ \dfrac{-\beta.\Gamma(\alpha).\Gamma(\beta-\alpha+1)}{\Gamma(1+\beta)}\ .\ F_2\ , \end{cases}$$

where

$F_1\ ,\ F_2$: hypergeometric functions:

$$F_1\ =\ F(\alpha\ ,\ \alpha-\beta\ ,\ \alpha-\beta+1\ ;\ x)\ ,$$

$$F_2\ =\ F(1-\alpha+\beta\ ,\ \beta\ ,\ 1+\beta\ ;\ \frac{1-2.x}{1-x})\ .$$

LEMMA 3.

If

$$J(x)\ =\ \int_o^1 [\frac{t.(1-t)}{x.(1-x)}]^{1/3}\ .\ (\frac{1}{t-x}\ -\ \frac{1}{x+t-2t.x})\ .\ dt$$

then

(85)
$$J(x) = -\frac{\pi}{\sqrt{3}} + \frac{2}{3} \cdot \frac{\Gamma(1/3) \cdot \Gamma(-2/3)}{\Gamma(2/3)} \cdot F(\frac{7}{3}, \frac{2}{3}, \frac{5}{3}; x)$$
$$+ \frac{2}{5} \cdot \frac{\Gamma^2(1/3)}{\Gamma(2/3)} \cdot F(\frac{1}{3}, \frac{5}{3}, \frac{8}{3}; \frac{1-2x}{1-x}) \cdot (1-x)^{-5/3} \quad .$$

PROOF OF LEMMA 3 :

Replacing t by the new variable

$$\xi = \frac{x}{t} \cdot \frac{1-t}{1-x}$$

we obtain

$$J(x) = \int_o^\infty \frac{2 \cdot \xi^{4/3}}{[x+(1-x).\xi]^{5/3} \cdot (1-\xi^2)} \cdot d\xi$$

$$= \int_o^\infty \frac{\xi^{4/3}}{[x+(1-x).\xi]^{5/3} \cdot (1-\xi)} \cdot d\xi + \int_o^\infty \frac{\xi^{4/3}}{[x+(1-x).\xi]^{5/3} \cdot (1+\xi)} \cdot d\xi$$

Applying Lemma 2 we get

(86)
$$J(x) = T_1(\frac{7}{3}, \frac{5}{3}; x) + T_2(\frac{7}{3}, \frac{5}{3}; x) \quad ,$$

completing the proof of Lemma 3 .

THE BITSADZE-LAVRENTJEV PROBLEM

In 1950 A.V. Bitsadze and M.A. Lavrentjev (Dokl. Akad. Nauk S.S.S.R., 70 , 1950 , 373-376) initiated the work on boundary value problems for partial differential equations of mixed type with discontinuous coefficients.

THE BITSADZE-LAVRENTJEV PROBLEM OR PROBLEM BL :

Consists in finding a function $u = u(x,y)$ which satisfies equation

(BL) $\quad \mathrm{sgn}(y) \cdot u_{xx} + u_{yy} = 0$ (the Bitsadze-Lavrentjev equation)

in a mixed domain D which is simply connected and bounded by a Jordan (non-selfintersecting) "elliptic" arc g_1 (for $y > 0$) with endpoints $0 = (o,o)$ and $A = (1,o)$ and by the "real" characteristics

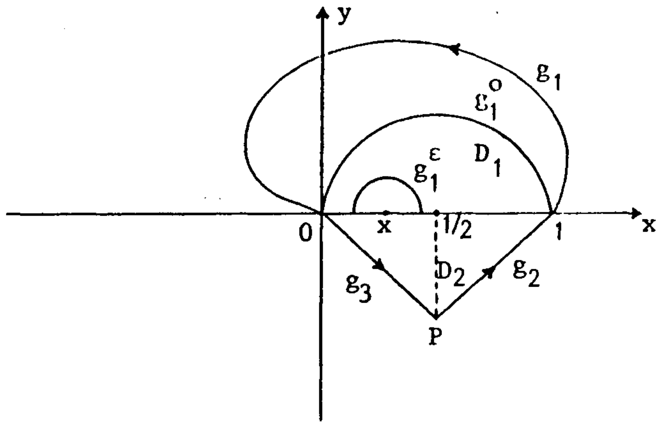
$$g_2 : \quad x + y = 1$$

$$g_3 : \quad x - y = 0$$

of (BL) satisfying the characteristic equation

$$-(dy)^2 + (dx)^2 = 0$$

such that these characteristics meet at a point P (for $y < 0$) , and assumes prescribed continuous boundary values of the (**) form



Consider the normal curve of Bitsadze-Lavrentjev with equation

$$g_1^o : \quad (x-\frac{1}{2})^2 + y^2 = \frac{1}{4} \qquad \text{(upper semi-circle)}$$

or equivalently

$$g_1^o : \quad |z-\frac{1}{2}| = \frac{1}{2} \quad ,$$

where

$$z = x + i \cdot y \quad ,$$

such that g_1 contains g_1^o in its interior.

Definition 2. A function $u = u(x,y)$ is a regular solution of Problem BL if it satisfies conditions 1) , 3) and 5) of Definition 1 and besides if it satisfies the foloowing two conditions:

2)' u_x , u_y are continuous in $\overline{D}$

(except possibly points O , A, where they may have poles of order: < 1 i.e. may go to infinity of order: < 1 as $x \to 0$ and $x \to 1$),

4)' u satisfies equation (BL) at all points of $D \setminus OA$.

IDEA OF BITSADZE AND LAVRENTJEV

The idea of Bitsadze and Lavrentjev of finding solutions for Problem BL it was almost similar to the Fundamental Idea of Tricomi.

We are going to go through the main steps:

Assume in both cases (i.e. Tricomi's or T-case, and Bitzadze-Lavrentjev's or BL-case) :

$$u(0) = u(A) = 0 \tag{87}$$

The general solution of Problem BL in $\overline{D}_2$ (i.e. : $-u_{xx} + u_{yy} = 0$) is given by the familiar formula of D'Alembert:

$$u(x,y) = f_1(x+y) + f_2(x-y) \tag{88}$$

where

$f_1 = f_1(t)$, $f_2 = f_2(t)$: are arbitrary continuous functions on $0 \leq t \leq 1$ twice continuously differentiable on $0 < t < 1$.

The general solution of equation (BL) satisfying condition: $u = \psi$ on g_3 is in D_2 :

(89) $$u(x,y) = f_1(x+y) - f_1(0) + \psi(\frac{x-y}{2}) .$$

Therefore

(90) $$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = \psi'(\frac{x}{2}) , \quad y = 0 , \quad 0 < x < 1$$

or equivalently

$$\tau'(x) - \nu(x) = \psi'(\frac{x}{2}) , \quad 0 < x < 1 .$$

or equivalently

THE FIRST FUNDAMENTAL FUNCTIONAL RELATION FOR PROBLEM BL :

(91) $$\tau'(x) - \nu(x) = \psi'(\frac{x}{2}) , \quad 0 < x < 1$$

THE SECOND FUNDAMENTAL FUNCTIONAL RELATION FOR PROBLEM BL :

Consider the fundamental solution:

(92) $$\ln|\zeta - z| , \quad \zeta = \xi + i\eta , \quad z = x + iy , \quad \eta > 0 , \quad y > 0$$

of equation (BL) in D_1 (i.e.: $u_{xx} + u_{yy} = 0$) .

Denote

$G = G(x\,;\xi\,,\eta)$: the solution of equation (BL) in D_1 with logarithmic singularity at point $\zeta = x \;, \quad 0 < x < 1$:

(93) $$G = -\ln|\zeta - x| + G_o(x\,;\xi\,,\eta)\;,$$

where

(94)
$$\begin{cases} G_o = G_o(x\,;\xi\,,\eta) : & \text{a regular harmonic function of } \xi\,,\eta \text{ in } D_1 \text{ satisfying the conditions:} \\ & G_o(x\,;\xi\,,\eta) - \ln|\zeta - x| = 0 \;, \quad \zeta \in g_1 \;, \\ & \left.\dfrac{\partial G_o}{\partial \eta}\right|_{\eta = o} = 0 \;. \end{cases}$$

Note: The function G is the harmonic Green's function with singularity at $\zeta = x$ in $D_1^* = D_1 \cup D_1^* \cup OA$,

where

D_1^* : is a domain which is the image of D_1 in OA .

LIAPUNOV CONDITION:

Assume that curve g_1 satisfies the Liapunov condition:

i.e. the tangent of the angle formed by the tangent line of g_1

and a constant direction (<u>for example</u>: the positive direction of the Ox-axis) satisfies Hölder's condition.

Isolate point (x,0) from domain D_1 by the curve g_1^{ε} of the semi-circle:

$$g_1^{\varepsilon} : \quad |\zeta - x| = \varepsilon \ , \quad \eta \geq 0$$

and apply in domain D_2 Green's formula

$$(95) \qquad \int (G \cdot \frac{\partial u}{\partial N} - u \cdot \frac{\partial G}{\partial N}) \cdot ds = 0 \ ,$$

where

s : arc length of curve is measured from A in the positive (counter-clockwise) direction,

N : inner normal of the boundary ∂D_2, and $\frac{\partial G}{\partial N} = \frac{\partial G}{\partial \xi} \cdot \frac{\partial \xi}{\partial N} + \frac{\partial G}{\partial \eta} \cdot \frac{\partial \eta}{\partial N}$.

Taking into account the boundary condition

$$(96) \qquad u = \phi \quad \text{on} \quad g_1$$

we shall have from (94) and (96)

$$(97) \quad \int_{g_1} (G \cdot \frac{\partial u}{\partial N} - u \cdot \frac{\partial G}{\partial N}) . ds + \left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^{1} \right) . G . \nu . d\xi = \int_{g_1} \phi \cdot \frac{\partial G}{\partial N} . ds \ .$$

Hence, in the limit, $\varepsilon \to 0$, we get

THE SECOND FUNDAMENTAL FUNCTIONAL RELATION FOR PROBLEM BL :

$$(98) \qquad \tau(x) + \frac{1}{\pi} . \int_o^1 [\ G_o(x\ ;\ \xi\ ,\ 0) - \ell n|\xi - x|\ ] . \nu(\xi) . d\xi = \phi_*(x) \quad ,$$

where

$$(99) \qquad \phi_*(x) = \frac{1}{\pi} . \int_{g_1} \phi . \frac{\partial G}{\partial N} . ds$$

SPECIAL CASE (when g_1 coincides with g_1^o) :

The expression (98) becomes simpler, as follows:

$$(100) \qquad \tau(x) + \frac{1}{\pi} . \int_o^1 [\ \ell n(t+x-2t.x) - \ell n|t-x|\ ] . \nu(t) . dt = \phi_*(x) \quad .$$

Eliminating $\tau = \tau(x)$ from fundamental relations (91) and (100) we get:

$$(101) \qquad \nu(x) + \frac{1}{\pi} . \frac{d}{dx} \int_o^1 [\ \ell n(t+x-2t.x) - \ell n|t-x|\ ] . \nu(t) . dt = F(x) \quad ,$$

where

$$F(x) = \phi_*'(x) - 2 . \frac{d}{dx} \psi(\frac{x}{2}) \quad .$$

Assume that x lies strictly inside the interval $(0,1)$.

It is easy to see that

$$(102) \qquad \frac{d}{dx} \int_o^1 \ell n(t+x-2t.x) . \nu(t) . dt = \int_o^1 \frac{1-2.t}{t+x-2t.x} . \nu(t) . dt \quad .$$

But

$$(103)\qquad \lim_{\varepsilon\to o} I_\varepsilon(x) = \lim_{\varepsilon\to o} \left[\int_o^{x-\varepsilon} \ell n(x-t).\nu(t).dt + \int_{x+\varepsilon}^{1} \ell n(t-x).\nu(t).dt \right]$$

$$= I(x) = \int_o^1 \ell n|t-x|.\nu(t).dt \quad ,$$

where the limit exists uniformly in x .

It is obvious that the uniform limit

$$\lim_{\varepsilon\to o} I'_\varepsilon(x) = \lim_{\varepsilon\to o} [\ \nu(x-\varepsilon) - \nu(x+\varepsilon)\ ] . \ell n\, \varepsilon \ -$$

$$(104)\qquad - \lim_{\varepsilon\to o} \left(\int_o^{x-\varepsilon} \frac{\nu(t)}{t-x} . dt + \int_{x+\varepsilon}^{1} \frac{\nu(t)}{t-x} . dt \right)$$

$$= - \int_o^1 \frac{\nu(t)}{t-x} . dt$$

exists,

where the integral is (in the sense of Cauchy): the principal value.

A WELL-KNOWN FORMULA

$$(105)\qquad \frac{d}{dx} \int_o^1 \ell n|t-x| . \nu(t) . dt = - \int_o^1 \frac{\nu(t)}{t-x} . dt \quad .$$

From expressions (102) and (105) equation (101) takes the form

(106) $$\nu(x) + \frac{1}{\pi} . \int_0^1 \left(\frac{1}{t-x} + \frac{1-2.t}{t+x-2t.x}\right) . \nu(t) . dt = F(x)$$

which is a one-dimensional singular integral equation with respect to ν for Problem T .

This equation (106) is equivalent to Problem BL in the case when g_1 coincides with the normal curve g_1^o (the latter curve gives a simple Green's function).

Note:

$$\frac{1}{t-x} + \frac{1-2.t}{t+x-2t.x} = \frac{t}{x} . \left(\frac{1}{t-x} - \frac{1}{t+x-2t.x}\right)$$

(compare (51) with (106)) .

Similarly as in the Tricomi's case: expression (53) (Tricomi's Theorem) we get here explicitly also that

(107) $$\nu(x) = \frac{1}{2} . \{ F(x) - \frac{1}{\pi} . \int_0^1 \left[\frac{x.(1-t)}{t.(1-x)}\right]^{1/2} . \left(\frac{1}{t-x} + \frac{1-2.t}{t+x-2t.x}\right) . F(t) . dt \} .$$

GENERAL CASE (when g_1 does not coincide with g_1^o) :

Eliminating the function $\tau = \tau(x)$ from expressions (91) and (98) as above we get

the general one-dimensional singular integral equation with respect to ν for Problem BL which is:

$$\nu(x) + \frac{1}{\pi} \cdot \int_0^1 \left(\frac{1}{t-x} + \frac{1-2.t}{t+x-2t.x}\right) . \nu(t) . dt$$

(108)

$$+ \int_0^1 K(x,t) . \nu(t) . dt = F(x) \quad ,$$

where

(109) $$K(x,t) := \frac{1}{\pi} \cdot \frac{\partial}{\partial x} [G_o(x ; t , 0) - \ell n(t+x-2t.x)] \quad .$$

Note:

For $0 < x , t < 1$

$K(x,t)$: continuously differentiable but at the endpoints of these intervals it may become infinite.

In particular, however, whenever g_1 terminates in short portions 00' and AA' of the semi-circle g_1^o , function $K(x,t)$ will have no singularities at the endpoints of the said intervals.

In this case the same can be said about the behavior of the function $F(x)$ as about the right side of equation (106) .

REMARK:

With respect to the kernel $K(x,t)$ (in Tricomi's) in equation (52) the same may be said as above.

THE GELLERSTEDT PROBLEM

In 1935 S. Gellerstedt (Doctoral Thesis, 1935; Jbuch Fortschritte Math. 61 , 1259) generalized Problem T by replacing coefficient y of u_{xx} in equation (*) (Tricomi's equation) by

$$\operatorname{sgn}(y) \,.\, |y|^m \;, \quad m > 0 \;.$$

THE GELLERSTEDT PROBLEM or PROBLEM G :

Consists in finding a function $u = u(x,y)$ which satisfies equation

(G) $\quad \operatorname{sgn}(y) \,.\, |y|^m \,.\, u_{xx} + u_{yy} = 0$ (The Gellerstedt equation)

in a mixed domain D which is simply connected and bounded by a Jordan (non-selfintersecting) "elliptic" arc g_1 (for $y > 0$) with endpoints $0 = (o,o)$ and $A = (1,o)$ and by the "real" characteristics

$$g_2 : \quad x + \frac{2}{m+2} \,.\, (-y)^{\frac{m+2}{2}} = 1$$

$$g_3 : \quad x - \frac{2}{m+2} \,.\, (-y)^{\frac{m+2}{2}} = 0$$

of (*) satysfying the characteristic equation

(110) $$-(-y)^m \,.\, (dy)^2 + (dx)^2 = 0$$

such that these characteristics meet at a point P (for $y < 0$) and assuming prescribed continuous boundary values (**) .

Consider the normal curve of Gellerstedt with equation

$$g_1^o : \quad (x-\frac{1}{2})^2 + \frac{4}{(m+2)^2} \cdot y^{m+2} = \frac{1}{4}$$

or equivalently

$$g_1^o : \quad \left|z-\frac{1}{2}\right| = \frac{1}{2} \quad ,$$

where

$$z = x + i \cdot \frac{2}{m+2} \cdot y^{\frac{m+2}{2}} \quad ,$$

such that g_1 contains g_1^o in its interior.

Note:

The definition for regularity of solutions of Problem G is identical to that one for Problem T .

THE FRANKL PROBLEM

In 1945 F.I. Frankl (Izv. Akad. Nauk SSSR ser. 9 , 2 , 1945 , 121--143) established a generalization of Problem T and Problem G for the Chaplygin equation

(CH) $$K(y) \cdot u_{xx} + u_{yy} = 0 \quad ,$$

$$K(y) \gtreqless 0 \quad \text{whenever} \quad y \gtreqless 0 \quad .$$

THE FRANKL PROBLEM or PROBLEM F :

Consists in finding a function $u = u(x,y)$ which satisfies equation (CH) above in a mixed domain D which is simply connected and bounded by a Jordan (non-selfintersecting) "elliptic" arc g_1 (for $y > 0$) with endpoints $0 = (o,o)$ and $A = (1,o)$, by the "real" characteristic

$$g_2 : \quad x = \int_o^y \sqrt{-K(t)} \,.\, dt + 1 \quad , \quad y < 0$$

of equation (CH) satisfying the characteristic equation

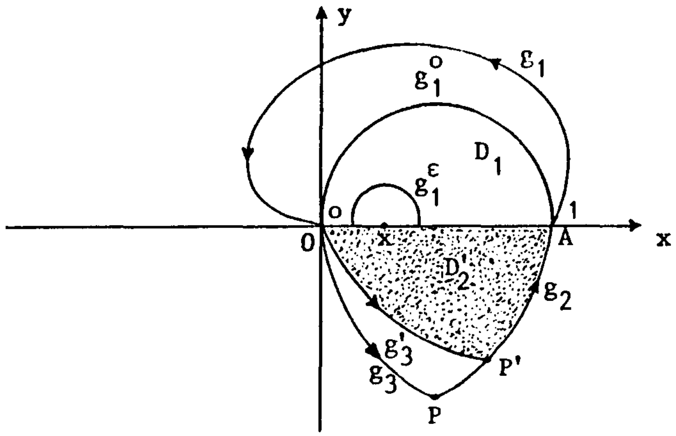
(111) $$-(\sqrt{-K(y)}\,)^2 \,.\, (dy)^2 + (dx)^2 = 0$$

and by the non-characteristic curve g_3' emanating from point 0 lying inside the characteristic tringle OAP and intersecting the characteristic g_2 at most once (g_3' may coincide with the characteristic g_3 of (CH) near point 0) :

$$g_3 : \quad x = -\int_o^y \sqrt{-K(t)} \,.\, dt$$

and assuming prescribed continuous boundary values

$$(F) \quad \begin{cases} u = \phi(s) & \text{on} \quad g_1 , \\ u = \psi(x) & \text{on} \quad g_3' . \end{cases}$$



Note:

On g_3' we must have

$$-(\sqrt{-K(y)})^2 . (dy)^2 + (dx)^2 \geqq 0$$

or

$$g_3' : \quad -\frac{1}{\sqrt{-K}} < \frac{dy}{dx} < 0 \qquad \left(dx\Big|_{g_3'} > 0 , \quad dy\Big|_{g_3'} < 0 \right)$$

or

$$g_3' : \quad dx + \sqrt{-K} . dy > 0$$

or

$$g_3' : \quad \int_o^x dx + \int_o^y \sqrt{-K(t)} . dt > 0$$ (as g_3' emanates from $0 = (o,o)$)

or

$$g_3' : \quad x + \int_0^y \sqrt{-K(t)} \,.\, dt > 0$$

or equivalently

(112) $$g_3' : \quad x > - \int_0^y \sqrt{-K(t)} \,.\, dt$$

REMARKS:

I. The Frankl Problem may be called also the Generalized Tricomi Problem or Problem GT .

II. F.I. Frankl (Izv. Akad. Nauk SSSR ser. 9 , 2 , 1945 , 121 - 143) initiated a new stage in the theory of equations of mixed type . He showed that the problem of a supersonic flow out of a plane--walled vessel (the velocity inside the vessel is subsonic) can be reduced to the Tricomi Problem for the Chaplygin equation (CH) : $K(0) = 0$, $K'(y) > 0$. Besides Frankl studying transonic flows established the uniqueness of the solution of Problem F (or Problem GT) in the case where

(113) $$F(y) = 1 + 2 \,.\, \left(\frac{K}{K'}\right)' > 0 \;, \quad y < 0$$

(Frankl's condition) .

Note:

The positiveness of F is equivalent to the convexity of $(-K)^{-1/2}$.

MORE APPLICATIONS

1. In 1959 I.N. Vekua (Fizmatgiz, Moscow, 1959 ; English transl. , Pergamon Press, Oxford; Addison-Wesley, Reading, Mass. , 1962) pointed out the importance of equations of mixed type for the theory of infinitesimal deformations of surfaces and the zero-moment free theory of shells with curvature of variable sign.

2. In 1961 M.N. Kogan (Prikl. Mat. Mech. 25, 1961, 132-137; J. Appl. Math. Mech. 25, 1961, 180-188) observed that the equations of magnetohydrodynamic flows in the region of sonic and Alfve'n velocities are equations of mixed type.

3. In 1953 F.I. Frankl (Kirgiz. Gos. Univ. Trudy Fiz. - Mat. Fak. 1953, nº 2 , 33-45) noticed that the equations of the flow of water in an open channel at a velocity exceeding the propagation velocity of surface waves are also of mixed type.

4. In 1951 F.I. Frankl (Leningrad Univ. Ser. Mat. Meh. Astronom. 6, 1951, nº 11, 3-7) pointed out two gas-dynamical applications of the Bitsadze-Lavrentjev Problem.

5. In 1959 F.I. Frankl (Izv. Vyss. Ucebn. Zaved. Matematika, 1959, nº 6, 13, 192-201) reduced the problem of flow inside a plane--parallel Laval nozzle of given shape (the direct problem of the theory of the Laval nozzle) to a new boundary value problem for the mixed type equation

$$u_{xx} + \operatorname{sgn}(y) \cdot |y|^m \cdot u_{yy} = 0 \tag{114}$$

with $m = \frac{1}{2}$.

Besides he showed that it is no longer sufficient to specify $u = u(x,y)$ on g_1 and on one of the characteristics in order to ensure the existence and uniqueness of solutions of equations (114) with $0 < m < 1$. Finally instead of the usual continuity condition on the parabolic curve OA :

$$u_y(x, +0) = u_y(x, -0) \tag{115}$$

one requires the Frankl's discontinuity condition:

$$u_y(x, +0) = -u_y(x, -0) \tag{FD}$$

FUNDAMENTAL EXTREMUM PRINCIPLES

1. In 1950 A.V. Bitsadze (Dokl. Akad. Nauk SSSR 70, 1950, 561-564) stated the following extremum principle:

A solution of the Problem BL vanishing on the characteristic g_3 achieves neither a positive maximum nor a negative minimum on an open arc OA of the type degeneracy curve.

2. In 1951 P. Germain and R. Bader (C.R. Acad. Sci. Paris, 232, 1951, 463-465) established the above principle for the Problem T .

3. In 1952 K.I. Babenko (Doctoral dissertation, Steklov Inst. Math., Moscow, 1952) proved the said principle for the Tricomi Problem when the equation (*) (the Tricomi equation) is replaced by equation

$$y.u_{xx} + u_{yy} + \alpha(x,y).u_x + \beta(x,y).u_y + r(x,y).u = 0 \quad , \tag{116}$$

if: $\alpha(x,0) = \beta(x,0) = 0$.

Note:

The above extremum principle is of great importance

FIRST: because it immediately implies uniqueness of the solution, and

SECOND: because it enables us to apply the alternating method of Schwarz to solution of the Tricomi Problem under quite general assumptions on g_1 .

4. In 1953 S. Agmon, L. Nirenberg and M.H. Protter (Comm. Pure and Appl. Math. 6, 1953, 455-470) established an interesting maximum

principle for the equation

(117) $$K(y).u_{xx} + u_{yy} + \alpha(x,y).u_x + \beta(x,y).u_y + r(x,y).u = 0 \quad ,$$

if: $K(y) \gtreqless 0$ whenever $y \gtreqless 0$.

Besides they assumed that a twice continuously differentiable solution $u = u(x,y)$ of equation (117), nondecreasing as a function of y along one of the characteristics, is defined in the characteristic triangle OAP . Finally they supposed that the following conditions held in OAP :

(118) $$\delta(\sqrt{-K}) + \alpha + \beta.\sqrt{-K} \leqq 0 \quad , \quad r \leqq 0 \quad ,$$

(119) $$\delta\left(\frac{\delta(\sqrt{-K})}{\sqrt{-K}} + \frac{\alpha + \beta.\sqrt{-K}}{\sqrt{-K}} \right) - 2.r + \frac{1}{2.K}.[\ \delta(\sqrt{-K}) + \alpha + \beta.\sqrt{-K}\ ].[\delta(\sqrt{-K}) + \alpha - \beta.\sqrt{-K}\ ] \leq 0 \quad ,$$

where

$$\delta := \frac{\partial}{\partial y} + \sqrt{-K}\ .\ \frac{\partial}{\partial x} \quad .$$

Then they proved that: If the maximum of $u = u(x,y)$ is positive, it is attained on the interval OA of the parabolic curve.

Note:

The uniqueness theorem for the Tricomi Problem for equation

(117) follows easily from this maximum principle.

In particular, if $\alpha = \beta = r = 0$ the uniqueness for Problem T holds if condition

$$\frac{1}{2} + 2 \cdot \left(\frac{K}{K'}\right)' > 0 \quad \text{for} \quad y < 0 \tag{120}$$

(the Agmon-Nirenberg-Protter's condition)

This condition is more restrictive (i.e. stronger) than Frankl's condition (i.e.: $F(y) > 0$ for $y < 0$).

Finally the positiveness in (120) is equivalent to the convexity of $(-K)^{-1/4}$.

FUNDAMENTAL BITSADZE'S RESULT

In 1956 A.V. Bitsadze (Dokl. Akad. Nauk S.S.S.R. 109, 1956, 1091--1094) studied the Frankl Problem in the case that: both characteristics are replaced by two non-characteristics lying inside the characteristic triangle OAP .

FUNDAMENTAL SMIRNOV'S RESULTS

1. In 1963 M.M. Smyrnov (Sibirsk Mat. Z. 4 , 1963 , 1150 - 1161)

investigated the boundary-value problem for Gellerstedt equation (G) with the quantity

$$y^m \cdot \frac{dy}{ds} \cdot \frac{\partial u}{\partial x} - \frac{dx}{ds} \cdot \frac{\partial u}{\partial y}$$

prescribed on g_1 and the values of the unknown function on a characteristic.

2. In 1951, 1957, and 1959 M.M. Smyrnov (Belorusk. Gos. Univ. Ucen Zap. 12, 1951, 3 - 9 ; Vestnik Leningrad Univ. 12, 1957, nº 1, 80--96, 209-210; Vestnik Leningrad Univ. 14, 1959, nº 1 , 130-133) investigated the fourth-order equation

$$\text{(S)} \qquad \frac{\partial^4 u}{\partial x^4} + 2.\operatorname{sgn}(y).\frac{\partial^4 u}{\partial x^2\, \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0 \quad \left(\text{ or: } \left(\frac{\partial^2}{\partial x^2} + \operatorname{sgn}(y).\frac{\partial^2}{\partial y^2}\right)^2 u = 0\right)$$

in a domain D bounded by a smooth curve g_1 in the upper half--plane with endpoints $0 = (o,o)$ and $A = (1,o)$, and by the characteristics

$$g_2 : \quad y = x - 1$$

$$g_3 : \quad y = -x$$

of equation (S) . Then he proved the existence and uniqueness of solutions of (S) satisfying the boundary conditions

$$[S] \quad \begin{cases} u\big|_{g_1} = \phi_1(s) \;, & \dfrac{\partial u}{\partial n}\Big|_{g_1} = \phi_2(s) \;, \\ \dfrac{\partial u}{\partial n}\Big|_{g_3} = \psi_1(x) \;, & 0 \leq x \leq \dfrac{1}{2} \;, \\ \dfrac{\partial u}{\partial n}\Big|_{g_2} = \psi_2(x) \;, & \dfrac{1}{2} \leq x \leq 1 \;. \end{cases}$$

He also proved uniqueness theorems for boundary-value problems in the case that, besides: $u\big|_{g_1} = \phi_1(s)$, $\frac{\partial u}{\partial n}\Big|_{g_1} = \phi_2(s)$, the unknown function $u = u(x,y)$ is prescribed on both characteristics, or the case that the unknown function is prescribed on one of the characteristics and the normal derivative $\frac{\partial u}{\partial n}$ on the other.

MORE RESULTS

1. In 1957 L.I. Chibrikova (Gos. Univ. Ucen. Zap. 117, 1957, nº 9, 44-47) obtained a solution of the Tricomi problem for equation (BL) in explicit form by using the properties of simple automorphic functions, doing this without use of conformal mappings when g_1 is half the boundary of one of the fundamental domains of an elementary or Fuchsian group of Möbius transformations.

2. In 1945 F.I. Frankl (Izv. Akad. Nauk S.S.S.R. Ser. Mat. 9, 1945, 121-143; English transl. techn. memos Nat. Adv. Comm. Aeronaut.

nº 1155 , 1947) considered the problem of sufficiently wide supersonic jet flow impinging on a wedge, when a zone of subsonic velocities is formed ahead of the wedge; he reduced this to the boundary-value problem for the Chaplygin equation (CH) , with : $u = 0$ on part of g_1 near 0 and on the characteristic g_3 , and on the other part of g_1 a homogeneous equation

(121) $$P(x,y) \cdot u_x + Q(x,y) \cdot u_y = 0 ,$$

where $P = P(x,y)$, $Q = Q(x,y)$ are given functions.

This problem is homogeneous and therefore determines the solution up to a constant factor.

Frankl proved a uniqueness theorem for this problem, assuming condition: $F(y) = 1 + 2 \cdot (\frac{K}{K'})' > 0 , \quad y < 0 .$

3. In 1962 B.V. Melentev (Dokl. Akad. Nauk S.S.S.R., 143, 1962, 38--41 ; Soviet Math. Dokl., 3 , 1962 , 338-342) proved the uniqueness of the solution to the Frankl problem for equations (*) and (BL), and also for a more general problem, in which (121) is replaced by

(122) $$P(x,y) \cdot u_x + Q(x,y) \cdot u_y + R(x,y) \cdot u = 0 ,$$

where $P = P(x,y)$, $Q = Q(x,y)$, $R = R(x,y)$ are given functions, and on the assumption that

$$[P.\cos(n.x) + Q.\cos(n.y)] \cdot R < 0 .$$

Melentev proved an existence theorem for the Frankl Problem for e quation (BL) in D , wheen the curve g_1 has a special form.

4. In 1964 B.V. Melentev (Dokl. Akad. Nauk S.S.S.R. 154, 1964, 1262–1265 ; Soviet Math. Dokl. 5 , 1964 , 270-273) also studied the Hilbert-Poincaré Problem for the Tricomi equation.

5. In 1954, 1958 V.G. Karmanov (Dokl. Akad. Nauk S.S.S.R. 95 , 1954, 439-442; Izv. Akad. Nauk S.S.S.R. Ser. Mat. 22 , 1958 , 117-134) used finite differences to prove the existence of a solution to the Tricomi Problem for equation (BL) under very weak restrictions on g_1 .

6. In 1965 L.I. Kovalenko (Dokl. Akad. Nauk S.S.S.R. 162, 1965, 751–754 ≡ Soviet Math. Dokl. 6 , 1965 , 747-751 ; Dokl. Akad. Nauk S.S.S.R. 162 , 1965 , 988-991 ≡ Soviet Math. Dokl. 6 , 1965 , 789–793) used finite differences to prove the uniqueness of solution of the Tricomi Problem for the equation

$$(123)\quad \operatorname{sgn}(y).|y|^m.h(y).u_{xx} + u_{yy} + \alpha(x,y).u_x + \beta(x,y).u_y + r(x,y).u = f(x,y),$$

$$m > 0 , \quad h(y) > 0 ,$$

under additional restrictions on g_1 in the neighborhoods of 0 and A and on the coefficients α , β , r and K :

$$K(y) = \operatorname{sgn}(y) . |y|^m . h(y) .$$

REDUCTION FORMULAS

Consider the Chaplygin equation

$$\text{(CH)} \qquad K(y) \,.\, u_{xx} + u_{yy} = 0 \;, \quad K(0) = 0 \;, \quad K'(y) > 0$$

and characteristics

$$g_2 \;:\; x = \int_o^y \sqrt{-K(t)} \,.\, dt + 1 \quad ,$$

$$g_3 \;:\; x = -\int_o^y \sqrt{-K(t)} \,.\, dt \quad .$$

Introduce new variables

$$\xi = x \quad , \quad \eta^{3/2} = \frac{3}{2} \,.\, \int_o^y \sqrt{K(t)} \,.\, dt \;.$$

Then equation (CH) becomes

$$\eta \,.\, u_{\xi\xi} + u_{\eta\eta} + \beta(\eta) \,.\, u_\eta = 0 \quad ,$$

where

$$\beta(\eta) := \frac{d^2\eta}{dy^2} \,/\, (\frac{d\eta}{dy})^2 \quad .$$

Set

$$u = z \,.\, \exp(-\frac{1}{2} \,.\, \int_o^\eta \beta(t) \,.\, dt)$$

Then

$$\eta \,.\, z_{\xi\xi} + z_{\eta\eta} + r(\eta) \,.\, z = 0 \quad ,$$

where

$$r(\eta) := -\frac{1}{4} \,.\, \beta^2 - \frac{1}{2} \,.\, \beta' \quad .$$

QUASI-REGULARITY OF SOLUTIONS

Definition 2. A function $u = u(x,y)$ is a quasi-regular solution of Problem T for Chaplygin equation (in the sense of Protter) if

1) $u \in C^2(D) \cap C(\bar{D})$,

2) the integrals

$$\int_0^1 u(x,0) \,.\, u_y(x,0) \,.\, dx \quad , \quad \iint_{D_1} (K.u_x^2 + u_y^2) \,.\, dx\, dy$$

exist,

3) Green's theorem is applicable to the integrals

$$\iint_D u \,.\, Lu \,.\, dx \,.\, dy \quad , \quad \iint_D u_x \,.\, Lu \,.\, dx\, dy \quad , \quad \iint_D u_y \,.\, Lu \,.\, dx\, dy \;.$$

4) the boundary integrals which arise exist in the sense that : the limits taken over corresponding interior curves exist as these interior curves approach the boundary,

5) u satisfies Chaplygin equation (CH) in D ,

6) u satisfies boundary conditions

$$(P) \qquad \begin{cases} u = \phi(s) & \text{on} \quad g_1 \\ u = \psi(x) & \text{on} \quad g_2 \end{cases}$$

(Protter's condition or Adjoint Tricomi's conditions)

THE a , b , c ENERGY INTEGRAL METHOD AND NEW UNIQUENESS RESULTS FOR QUASI-REGULAR SOLUTIONS OF MIXED TYPE BOUNDARY VALUE PROBLEMS

In 1953 M.H. Protter (J. Rat. Mech. & Anal., 2, nº 1, 1953, 107--114) used the so-called a , b , c - method, based on an idea of Friedrichs in order to establish uniqueness results for quasi-regular solutions of Chaplygin equation (CH) in a domain D with boundary conditions (P) .

The idea is the following:

Consider $u = u(x,y)$ be a quasi-regular solution of (CH) defined in D . Besides consider the integral

$$\iint_D (a.u + b.u_x + c.u_y) . (K . u_{xx} + u_{yy}) . dx\,dy$$

where a , b , c : sufficiently smooth functions of (x,y) .

By virtue of (CH) this integral vanishes. The functions a,b,c : are chosen in such a way that, after a transformation of the integral by Green's formula, one obtains a positive (or non-negative) definite expression which vanishes only if $u \equiv 0$ in D .

APPLICATION:

Take

$$(124) \qquad Lu \equiv K(y) . u_{xx} + u_{yy} + r(x,y) . u = f(x,y)$$

and boundary conditions (P) ,

where

$$K \in C^2(\cdot) \ , \quad r \in C^1(\cdot) \ , \quad f \in C^o(\cdot) \ .$$

Take

(125) $$J = 2 \,.\, \langle Mu, Lu \rangle_D := 2 \,.\, \iint_D Mu \,.\, Lu \,.\, dx\, dy \quad ,$$

where

(126) $$Mu := a(x,y) \,.\, u + b(x,y) \,.\, u_x + c(x,y) \,.\, u_y \quad \text{in} \quad D \ .$$

Consider identities

$$2.a.r.u^2 = 2.a.r.u^2$$

$$2.b.r.u.u_x = (b.r.u^2)_x - (b.r)_x \,.\, u^2 \quad ,$$

$$2.c.r.u.u_y = (c.r.u^2)_y - (c.r)_y \,.\, u^2 \quad ,$$

$$2.a.K.u.u_{xx} = (2.a.K.u.u_x)_x - 2.a.K.u_x^2 - (a_x.K.u^2)_x + a_{xx}.K.u^2 \quad ,$$

$$2.a.u.u_{yy} = (2.a.u.u_y)_y - 2.a.u_y^2 - (a_y.u^2)_y + a_{yy}.u^2 \quad ,$$

$$2.b.K.u_x.u_{xx} = (b.K.u_x^2)_x - b_x.K.u_x^2 \quad ,$$

$$2.b.u_x.u_{yy} = (2.b.u_x.u_y)_y - 2.b_y.u_x.u_y - (b.u_y^2)_x + b_x.u_y^2 \quad ,$$

$$2.c.K.u_y\, u_{xx} = (2.c.K.u_x\, u_y)_x - (c.K.u_x^2)_y + (c.K)_y.u_x^2 - 2.c_x.K.u_x\, u_y \ ,$$

$$2.c.u_y.u_{yy} = (c.u_y^2)_y - c_y.u_y^2 \quad .$$

Besides employ Green's theorem:

(127) $$\iint_D P(x,y).dx + Q(x,y).dy = \oint_{\partial D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) . dx.dy$$

or

(128) $$\begin{cases} \displaystyle\iint_D \frac{\partial Q}{\partial x} . dx.dy = \oint_{\partial D} Q.dy := \oint_{\partial D} Q.\nu_1.ds \quad , \\ \displaystyle\iint_D \frac{\partial P}{\partial y} . dx.dy = -\oint_{\partial D} P.dx := \oint_{\partial D} P.\nu_2.ds \quad , \end{cases}$$

where

$\nu = (\nu_1 , \nu_2)$: the outer unit normal vector on ∂D ,

s : arc length ,

$$\nu_1 = \frac{dy}{ds} , \quad \nu_2 = - \frac{dx}{ds} \quad (|\nu| = \sqrt{\nu_1^2 + \nu_2^2} = 1) \quad .$$

<u>REMEMBER THE RULE:</u>

(129) $$\begin{cases} (\ )_x \longrightarrow dy \longrightarrow \nu_1 \\ (\ )_y \longrightarrow -dx \longrightarrow \nu_2 \end{cases}$$

Then employing above identities and applying Green's theorem into relation (125) we get:

$$J = \iint_D [\,2.a.r.u^2 - (b.r)_x.u^2 - (c.r)_y.u^2 - 2.a.K.u_x^2 + a_{xx}.K.u^2 - 2.a.u_y^2$$

$$+ a_{yy}.u^2 - b_x.K.u_x^2 - 2.b_y.u_x.u_y + b_x.u_y^2 + (c.K)_y.u_x^2 - 2.c_x.K.u_x.u_y$$

$$- c_y.u_y^2\,].dx.dy$$

(130)

$$+ \int_{\partial D} [\,b.r.u^2.\nu_1 + c.r.u^2.\nu_2 + 2.a.K.u.u_x.\nu_1 - a_x.K.u^2.\nu_1 + 2.a.u.u_y.\nu_2$$

$$- a_y.u^2.\nu_2 + b.K.u_x^2.\nu_1 + 2.b.u_x.u_y.\nu_2 - b.u_y^2.\nu_1$$

$$+ 2.c.K.u_x.u_y.\nu_1 - c.K.u_x^2.\nu_2 + c.u_y^2.\nu_2\,].ds \quad .$$

Therefore

$$J = \iint_D [\,2.a.r - (b.r)_x - (c.r)_y + K.a_{xx} + a_{yy}\,].u^2.dx.dy$$

$$+ \iint_D [\,(-2a.K - b_x.K + (c.K)_y).u_x^2 - 2.(b_y + c_x.K).u_x.u_y + (-2a + b_x - c_y).u_y^2\,].dx.dy$$

$$+ \oint_{\partial D} [\,(b.\nu_1 + c.\nu_1).r\,].u^2.ds$$

(131)

$$+ \oint_{\partial D} [\,2.a.u.(K.u_x.\nu_1 + u_y.\nu_2) - (K.a_x.\nu_1 + a_y.\nu_2).u^2\,].ds$$

$$+ \oint_{\partial D} [\,(b.\nu_1 - c.\nu_2).K.u_x^2 + 2.(b.\nu_2 + c.K.\nu_1).u_x.u_y + (-b.\nu_1 + c.\nu_2).u_y^2\,].ds$$

$$= I_1 + I_2 + J_1 + J_2 + J_3 \quad .$$

CASE 1 : UNIQUENESS (The Adjoint Tricomi Case) :

Assume u_1 , u_2 : two solutions of Adjoint Problem T for equation (124) and adjoint boundary conditions (P) . Then take

(132) $$u = u_1 - u_2 \quad .$$

Clain that

(133) $$u = 0 \quad \text{in} \quad D \quad .$$

In fact

(134) $$L\, u \equiv K(y) \cdot u_{xx} + u_{yy} + r(x,y) \cdot u = 0 \quad ,$$

and

[P] : $$\begin{cases} u = 0 & \text{on} \quad g_1 \\ u = 0 & \text{on} \quad g_2 \quad . \end{cases}$$

Therefore it is enough to show that

(135) $$u = 0 \quad \text{on} \quad g_3 \quad .$$

Then by a standard maximum principle we'll get (133) .

From (125) and (134) we get

(136) $$J = 0 \quad .$$

From (131) and (136) to prove (135) it is enough to show that <u>all</u>

<u>integrals</u> I_1, I_2, J_1, J_2, J_3 <u>are non-negative</u>.

<u>FIRST</u>: The integrals I_1 , I_2 are non-negative (≥ 0) if the following two conditions hold in D :

(c_1) $$(2.a - b_x - c_y).r - (b.r_x + c.r_y) + K.a_{xx} + a_{yy} \geq 0 ,$$

(c_2) $$\begin{cases} A = -2\,a.K - b_x.K + (c.K)_y \geq 0 , \\ B = -2a + b_x - c_y \geq 0 , \\ C = A.B - (b_y + c_x.K)^2 \geq 0 . \end{cases}$$

<u>SECOND</u>: The integrals J_1, J_2, J_3 are non-negative (≥ 0) if the following four conditions hold on $\partial D := g_1 \cup g_2 \cup g_3$:

(c_3) $$(b + c.\sqrt{-K}).r \leq 0 \quad \text{on} \quad g_3$$

(c_4) $$b.\nu_1 + c.\nu_2 \geq 0 \quad \text{on} \quad g_1 ,$$

where: $\nu = (\nu_1, \nu_2)$ the outer unit normal vector on g_3 ,

(c_5) $$b - c.\sqrt{-K} \geq 0 \quad \text{on} \quad g_3 ,$$

(<u>equivalently</u>: $b.\nu_1 - c.\nu_2 \leq 0$ on g_3)

(c_6) $$a_x.\sqrt{-K} - a_y + \frac{a.K'}{4.(-K)} \leq 0 \quad \text{on} \quad g_3$$

<u>JUSTIFICATION:</u>

Condition (c_1) and (c_2) hold obviously.

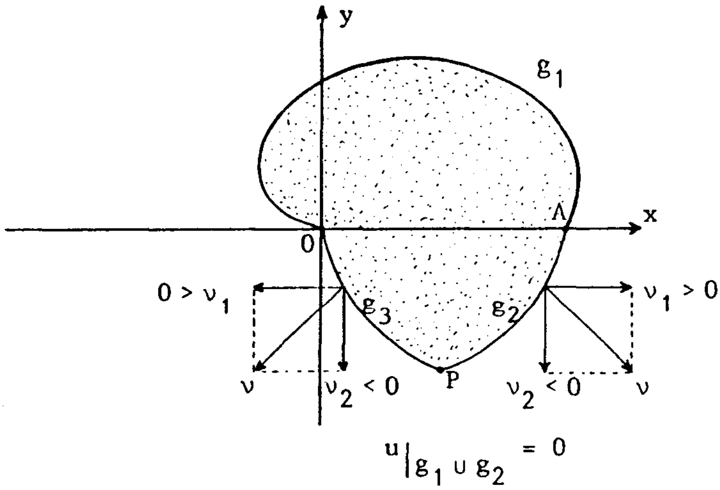
Condition (c_3) : From [P] and the fact that

$$dx|_{g_3} = -\sqrt{-K}\,.\,dy \quad \text{or} \quad -\nu_2.ds|_{g_3} = -\sqrt{-K}\,.\,\nu_1.ds \quad \text{or}$$

(137) $$\nu_2|_{g_3} = \nu_1\,.\,\sqrt{-K}$$

we get

$$J_1 = \int_{g_3} [\ (b + c.\sqrt{-K}).r\ ]\,.\,\nu_1.u^2.ds \quad .$$



But

$$\nu_1 < 0 \quad \text{on} \quad g_3 \quad .$$

Therefore condition (c_3) holds.

Conditions (c_4) and (c_5) :

(138) $$J_3 = \int_{g_1 \cup g_2} Q.ds + \int_{g_3} Q.ds \ := J_3^{(1)} + J_3^{(2)} \quad ,$$

where

$$Q := Q(u_x,u_y) := (b.\nu_1-c.\nu_2).K.u_x^2+2.(b.\nu_2+c.K.\nu_1).u_x.u_y+(-b.\nu_1+c.\nu_2).u_y^2$$

is a quadratic form with respect to u_x , u_y .

From [P] we get

$$d\,u = 0 \quad \text{on} \quad g_1 \cup g_2 ,$$

or

$$0 = u_x.d_x + u_y.d_y = (u_y.\nu_1 - u_x.\nu_2).ds \quad \text{on} \quad g_1 \cup g_2$$

or

$$u_x . \nu_2 = u_y . \nu_1 \quad \text{on} \quad g_1 \cup g_2$$

or

(139) $$u_x = N . \nu_1 , \quad u_y = N . \nu_2 \quad \text{on} \quad g_1 \cup g_2 ,$$

where

N := normalizing factor.

Therefore from (138) : $J_3^{(1)}$ and (139) we get:

$$Q|_{g_1 \cup g_2} := [(b.\nu_1-c.\nu_2).K.\nu_1^2+2.(b.\nu_2+c.K.\nu_1).\nu_1.\nu_2+(-b.\nu_1+c.\nu_2).\nu_2^2] . N^2$$

or

(140) $$Q|_{g_1 \cup g_2} := (b.\nu_1 + c.\nu_2) . (K.\nu_1^2 + \nu_2^2) . N^2 .$$

But

(141) $$K.\nu_1^2 + \nu_2^2 > 0 \quad \text{on} \quad g_1 , \quad \text{and} \quad K.\nu_1^2 + \nu_2^2 = 0 \quad \text{on} \quad g_2$$

because $K > 0$ on g_1 and g_2 is characteristic.

Therefore from (140)-(141) we obtain

$$Q\big|_{g_1 \cup g_2} := Q\big|_{g_1} := (b.\nu_1 + c.\nu_2).(K.\nu_1^2 + \nu_2^2) . N^2 \geq 0 \quad ,$$

if (c_4) holds. Therefore $J_3^{(1)} \geqq 0$ if (c_4) holds.

Also from (138) we have $J_3^{(2)} \geqq 0$ if

(142) $$Q\big|_{g_3} \geqq 0 \quad .$$

But on g_3 :

$$\begin{vmatrix} (b.\nu_1 - c.\nu_2).K & b.\nu_2 + c.K.\nu_1 \\ b.\nu_2 + c.K.\nu_1 & -b.\nu_1 + c.\nu_2 \end{vmatrix}$$

$$= -(b^2 + c^2.K) . (K.\nu_1^2 + \nu_2^2) = 0 \quad ,$$

because

$$K.\nu_1^2 + \nu_2^2 = 0 \quad \text{on} \quad g_3 \quad \text{(as characteristic)} .$$

Therefore (142) holds if

$$(b.\nu_1 - c.\nu_2).K \geqq 0 \quad , \quad \text{and} \quad -b.\nu_1 + c.\nu_2 \geqq 0 \quad \text{on} \quad g_3 \quad ,$$

or if

$$b.\nu_1 - c.\nu_2 \leqq 0 \quad \text{on} \quad g_3 \qquad (\text{as} \ K < 0 \ \text{on} \ g_3) \quad ,$$

or if

$$(b - c.\sqrt{-K}).\nu_1 \leqq 0 \quad \text{on} \quad g_3 \qquad (\text{as} \quad \nu_2 = \nu_1.\sqrt{-K} \quad \text{on} \quad g_3) \quad ,$$

or if condition (c_5) holds (as $\nu_1 < 0$ on g_3) .

Condition (c_6) : From [P] we get

$$(143) \qquad \begin{aligned} J_2 &= \int_{g_3} 2.a.u.(K.u_x.\nu_1 + u_y.\nu_2).ds - \int_{g_3} (K.a_x.\nu_1 + a_y.\nu_2).u^2.ds \\ &= J_2^{(1)} + J_2^{(2)} \quad . \end{aligned}$$

But

$$\begin{aligned} du|_{g_3} &= u_x.dx + u_y.dy = u_x.(-\nu_2.ds) + u_y.(\nu_1.ds) \\ &= (-u_x.\sqrt{-K} + u_y).\nu_1.ds \qquad (\text{as} \quad \nu_2 = \nu_1.\sqrt{-K} \quad \text{on} \quad g_3) \end{aligned}$$

or

$$du|_{g_3} = \frac{(K.u_x + u_y.\sqrt{-K}).\nu_1.ds}{\sqrt{-K}} = \frac{(K.u_x.\nu_1 + u_y.\nu_2).ds}{\sqrt{-K}}$$

Therefore

$$J_2^{(1)} = \int_{g_3} 2.a.\sqrt{-K}.u.du = \int_{g_3} a.\sqrt{-K}.d(u^2) \quad .$$

By integration by parts we get

$$J_2^{(1)} = a.\sqrt{-K}\,.\,u^2\Big|_{g_3} - \int_{g_3} u^2\,.\,d(a\,.\,\sqrt{-K}\,) \quad .$$

But

$$u = 0 \quad \text{on the upper and lower limits of} \quad g_3$$

$$(\text{as} \quad u(0) = 0 \;, \quad \text{and} \quad u|_{g_2} = 0 \quad \text{from} \quad [P])$$

Therefore

(144) $$J_2^{(1)} = -\int_{g_3} u^2 \,.\, d(a.\sqrt{-K}) \;.$$

Besides

$$da|_{g_3} = ax.dx + a_y.dy = a_x \,.\, (-\nu_2.ds) + a_y \,.\, (\nu_1.ds)$$

$$= (-a_x.\sqrt{-K} + a_y).\nu_1.ds \qquad (\text{as} \quad \nu_2 = \nu_1.\sqrt{-K} \quad \text{on} \quad g_3)$$

or

$$da|_{g_3} = \frac{(K.a_x + a_y.\sqrt{-K}).\nu_1.ds}{\sqrt{-K}} = \frac{(K.a_x.\nu_1 + a_y.\nu_2).ds}{\sqrt{-K}} \;.$$

Therefore

$$(K.a_x.\nu_1 + a_y.\nu_2) \,.\, ds|_{g_3} = \sqrt{-K} \,.\, da \;.$$

Thus

(145) $$J_2^{(2)} = \int_{g_3} (\sqrt{-K} \,.\, da).u^2$$

From expressions (143)-(145) we get

$$J_2 = -\int_{g_3} [\, d(a.\sqrt{-K}) + \sqrt{-K}.da \,] \,.\, u^2 \;.$$

Then $J_2 \geqq 0$ if

(146) $$d(a.\sqrt{-K}) + \sqrt{-K}\,.\,da \leq 0 \quad \text{on} \quad g_3 \;.$$

But

$$d(a.\sqrt{-K})\big|_{g_3} = (a.\sqrt{-K})_x.dx + (a.\sqrt{-K})_y\,.\,dy$$

$$= (a_x.\sqrt{-K}).dx + (a_y.\sqrt{-K} + a.\frac{-K'}{2\sqrt{-K}}).dy$$

$$= \sqrt{-K}.\,[\,a_x.dx + (a_y + \frac{a.K'}{2.K}).dy\,]$$

$$= \sqrt{-K}.\,[\,a_x.(-\sqrt{-K}) + (a_y + \frac{a.K'}{2.K})\,].dy$$

$$(\text{as} \quad dx\big|_{g_3} = -\sqrt{-K}.dy\,)\,, \quad \text{or}$$

(147) $$d(a.\sqrt{-K})\big|_{g_3} = \sqrt{-K}\,.\,(a_x.\sqrt{-K} - a_y + \frac{a.K'}{-2K}).(-\,dy)$$

Besides

$$\sqrt{-K}.da\big|_{g_3} = \sqrt{-K}\,.\,(a_x.dx + a_y.dy)$$

$$= \sqrt{-K}\,.\,[\,a_x.(-\sqrt{-K}) + a_y\,].dy \qquad \text{or}$$

(148) $$\sqrt{-K}.da\big|_{g_3} = \sqrt{-K}\,[\,a_x.\sqrt{-K} - a_y\,]\,.\,(-\,dy) \;.$$

From (147)-(148) we get

(149) $$d(a.\sqrt{-K}) + \sqrt{-K}\,.\,da\big|_{g_3} = \sqrt{-K}.\,(\,2.a_x.\sqrt{-K} - 2.a_y + \frac{a.K'}{-2K}\,).(-\,dy)$$

But

(150) $$(-\,dy)\big|_{g_3} = -\,\nu_1\,.\,ds\big|_{g_3} > 0 \qquad (\text{as} \quad \nu_1\big|_{g_3} < 0\,)\;.$$

Expressions (146) and (149)-(150) yield condition (c_6).

Therefore all the integrals: $I_1, I_2, J_1, J_2, J_3 :\geqq 0$ if conditions (c_i), $i = 1,2,\dots,6$ hold.

Finally we must choose: "nice functions"

$$a = a(x,y) \ , \quad b = b(x,y) \ , \quad c = c(x,y) \quad \text{in} \quad D$$

so that all these conditions hold. If this occurs then uniqueness follows immediately.

REMARK:

Choose $a := \text{const}$.

Then (146) is equivalent to

$$a \,.\, d(\sqrt{-K}) \leqq 0 \quad \text{on} \quad g_3 \ ,$$

or

$$a \,.\, (\sqrt{-K})' \,.\, dy \leqq 0 \quad \text{on} \quad g_3 \ ,$$

or

$$a \,.\, \frac{-K'}{2\sqrt{-K}} \,.\, dy \leqq 0 \quad \text{on} \quad g_3 \ ,$$

or to :

$$a := \text{const} :\leqq 0 \quad \text{on} \quad g_3 \ .$$

WELL-KNOWN CHOICES:

1. (Frankl's choice) :

$$a = -\frac{1}{2} \quad \text{in } D \;, \qquad b = c = 0 \quad \text{for} \quad y \geq 0$$

and $$b = c.\sqrt{-K} \;, \qquad c = \frac{4a.K}{K'} \quad \text{for} \quad y \leq 0 \;.$$

Assume Frankl's condition:

$$F(y) = 1 + 2.\left(\frac{K}{K'}\right)' > 0 \;, \qquad y < 0 \;.$$

2. (Protter's choice) :

$$a = -e^{\beta.x}.\cos(\gamma.y) \;, \quad b = 0 \;, \quad c = 0 \;, \qquad \text{for} \quad y \geq 0$$

$$a = -e^{\beta.x} \;, \quad b = c.\sqrt{-K} \;, \quad c = \frac{4.a.K}{K'} \;, \qquad \text{for} \quad y \leq 0$$

Assume Generalized Frankl's Condition:

$$F(y) \geq 2.\frac{K.\sqrt{-K}}{K'}.\beta \;,$$

$$\beta = \frac{\gamma}{\sqrt{K(y_m)}} \;, \qquad \gamma = \frac{\pi}{2.y_m} \;,$$

y_m := max. of the ordinates of points on g_1

Note: Both choices (i.e. Frankl's and Protter's) work with

$$r = 0 \;.$$

CASE 2 : UNIQUENESS (The Ordinary Tricomi Case) :

Assume u_1, u_2 : two solutions of Problem T for equation (124) and Tricomi boundary conditions

$$(T) \qquad \begin{cases} u = \phi(s) & \text{on} \quad g_1 \\ u = \psi(x) & \text{on} \quad g_3 \end{cases} .$$

Then based on above (Case 1) discussion

(151) $\quad [T] \qquad u = 0 \quad \text{on} \quad g_1 \cup g_3$.

Therefore it is enough to show that

(152) $\qquad u = 0 \quad \text{on} \quad g_2$.

FIRST: (c_1) and (c_2) are same as those in Case 1 . Therefore integrals I_1, I_2 : $\geqq 0$ in D .

SECOND: The integrals J_1, J_2, J_3 are : $\geqq 0$ if the following conditions hold on $\partial D := g_1 \cup g_2 \cup g_3$:

(c_3) $\qquad (b - c.\sqrt{-K}).r \geqq 0 \quad \text{on} \quad g_2$,

(c_4) $\qquad$ the same as in the Case 1 ,

(c_5) $\qquad b + c.\sqrt{-K} \leqq 0 \quad \text{on} \quad g_2$,

(c_6) $\qquad a_x.\sqrt{-K} + a_y + \frac{a.K'}{4.K} \geqq 0 \quad \text{on} \quad g_2$.

JUSTIFICATION:

Condition (c_3) : From [T] and the fact that

$$dx|_{g_2} = \sqrt{-K} \,.\, dy \quad \text{or} \quad -\nu_2 \,.\, ds|_{g_2} = \sqrt{-K} \,.\, \nu_1 .ds \quad \text{or}$$

(153) $$\nu_2|_{g_2} = -\nu_1 \,.\, \sqrt{-K}$$

we get

$$J_1 = \int_{g_2} [\,(b - c.\sqrt{-K}).r\,] \,.\, \nu_1 \,.\, u^2 \,.\, ds \quad ,$$

But

$$\nu_1 > 0 \quad \text{on} \quad g_2 \,.$$

Therefore condition (c_3) holds.

Conditions (c_4) and (c_5) :

(154) $$J_3 = \int_{g_1 \cup g_3} Q.ds + \int_{g_2} Q.ds \;:=\; J_3^{(1)} + J_3^{(2)} \quad ,$$

where Q : the same as in (138).

From [T] we get as above (in (139)) :

(155) $$u_x = N.\nu_1 \,, \quad u_y = N.\nu_2 \quad \text{on} \quad g_1 \cup g_3 \quad ,$$

where

N : = normalizing factor.

Therefore from (154) : $J_3^{(1)}$ and (155) we get:

(156) $$Q\big|_{g_1 \cup g_3} := (b.\nu_1 + c.\nu_2).(K.\nu_1^2 + \nu_2^2).N^2$$

(similar to that of (140))

But

(157) $K.\nu_1^2 + \nu_2^2 > 0$ on g_1 , and $K.\nu_1^2 + \nu_2^2 = 0$ on g_3

because $K > 0$ on g_1 and g_3 is characteristic.

Therefore

$$Q\big|_{g_1 \cup g_3} \geqq 0 \quad \text{if } (c_4) \text{ holds.}$$

Besides from (154) we have $J_3^{(2)} \geqq 0$ if

(158) $$Q\big|_{g_2} \geqq 0 \quad .$$

But on g_2 :

$$\begin{vmatrix} (b.\nu_1 - c.\nu_2).K & b.\nu_2 + c.K.\nu_1 \\ b.\nu_2 + c.K.\nu_1 & -b.\nu_1 + c.\nu_2 \end{vmatrix}$$

$$= -(b^2 + c^2.K).(K.\nu_1^2 + \nu_2^2) = 0 \quad ,$$

because

$$K.\nu_1^2 + \nu_2^2 = 0 \quad \text{on} \quad g_2 \quad \text{(as characteristic)} \; .$$

Therefore (158) holds if

$$(b.\nu_1 - c.\nu_2).K \geqq 0 \quad , \quad -b.\nu_1 + c.\nu_2 \geqq 0 \quad \text{on} \quad g_2 \quad ,$$

or if

$$b.\nu_1 - c.\nu_2 \leqq 0 \quad \text{on} \quad g_2 \quad (\text{as} \quad K < 0 \quad \text{on} \quad g_2) \quad ,$$

or if

$$(b + c.\sqrt{-K}).\nu_1 \leqq 0 \quad \text{on} \quad g_2 \quad (\text{as} \quad \nu_2 = -\nu_1.\sqrt{-K} \quad \text{on} \quad g_2)$$

or if condition (c_5) holds (as $\nu_1 > 0$ on g_2) .

Condition (c_6) : From [T] we get

(159)
$$J_2 = \int_{g_2} 2.a.u.(K.u_x.\nu_1 + u_y.\nu_2).ds - \int_{g_2} (K.a_x.\nu_1 + a_y.\nu_2).u^2.ds$$

$$= J_2^{(1)} + J_2^{(2)} \quad .$$

But

$$du|_{g_2} = u_x.dx + u_y.dy = u_x.(-\nu_2.ds) + u_y.(\nu_1.ds)$$

$$= (u_x.\sqrt{-K} + u_y).\nu_1.ds \quad (\text{as} \quad \nu_2 = -\nu_1.\sqrt{-K} \quad \text{on} \quad g_2)$$

$$du|_{g_2} = \frac{(-K.u_x + u_y.\sqrt{-K}).\nu_1.ds}{\sqrt{-K}} = -\frac{(K.u_x.\nu_1 + u_y.\nu_2).ds}{\sqrt{-K}}$$

Therefore

$$J_2^{(1)} = -\int_{g_2} 2.a.\sqrt{-K}.u.du = -\int_{g_2} a.\sqrt{-K}.d(u^2) \quad .$$

By integration by parts we get

$$J_2^{(1)} = - a.\sqrt{-K} . u^2 \Big|_{g_2} + \int_{g_2} u^2.d(a.\sqrt{-K}) \quad .$$

Then similar to the Case 1 we get:

(160) $$J_2^{(1)} = \int_{g_2} u^2.d(a.\sqrt{-K})$$

Besides

$$da\Big|_{g_2} = a_x.dx + a_y.dy = a_x.(-\nu_2.ds) + a_y. \ (\nu_1.ds)$$

$$= (a_x.\sqrt{-K} + a_y).\nu_1.ds \qquad (\text{ as } \ \nu_2 = -\nu_1.\sqrt{-K} \ \text{ on } \ g_2 \ )$$

or

$$da\Big|_{g_2} = \frac{(-K.a_x + a_y.\sqrt{-K}).\nu_1.ds}{\sqrt{-K}} = - \frac{(K.a_x.\nu_1 + a_y.\nu_2).ds}{\sqrt{-K}}$$

Therefore

$$(K.a_x.\nu_1 + a_y.\nu_2).ds\Big|_{g_3} = - \sqrt{-K} . da$$

Thus

(161) $$J_2^{(2)} = - \int_{g_2} (\sqrt{-K}.da) . u^2$$

From expressions (159)-(161) we get

$$J_2 = \int_{g_2} [\ d(a.\sqrt{-K}) + \sqrt{-K}.da \ ] . u^2$$

Then $J_2 \geqq 0$ if

(162) $$d(a.\sqrt{-K}) + \sqrt{-K}\,.\,da \geqq 0 \quad \text{on} \quad g_2 \,.$$

But

$$d(a.\sqrt{-K})|_{g_2} = (a.\sqrt{-K})_x\,.\,dx + (a.\sqrt{-K})_y\,.\,dy$$

$$= (a_x.\sqrt{-K}).dx + (a_y.\sqrt{-K} + a\,.\,\frac{-K'}{2.\sqrt{-K}})\,.\,dy$$

$$= \sqrt{-K}\,.[\,a_x.dx + (a_y + \frac{a.K'}{2.K})\,.\,dy\,]$$

or

$$d(a.\sqrt{-K})|_{g_2} = \sqrt{-K}\,.[\,a_x.(\sqrt{-K}) + (a_y + \frac{a.K'}{2.K})\,].dy$$

$$(\text{as} \quad dx|_{g_2} = \sqrt{-K}\,.\,dy\,)\,.$$

Besides

(164) $$\sqrt{-K}.da|_{g_2} = \sqrt{-K}\,.\,(a_x.dx + a_y.dy)$$

$$= \sqrt{-K}\,.[\,a_x.(\sqrt{-K}) + a_y\,].dy\,.$$

From (163)-(164) we get

(165) $$d(a.\sqrt{-K}) + \sqrt{-K}.da|_{g_2} = \sqrt{-K}.(2.a_x.\sqrt{-K} + 2.a_y + \frac{a.K'}{2.K}).dy$$

But

(166) $$dy|_{g_2} = \nu_1.ds|_{g_2} > 0 \qquad (\text{as} \quad \nu_1|_{g_2} > 0\,)$$

Expressions (162) and (165)-(166) yield condition (c_6) .

SPECIAL CASE:

Choose $a := \text{const.}$

Then (162) is equivalent to

$$a \,.\, d(\sqrt{-K}) \geq 0 \quad \text{on} \quad g_2 \ ,$$

or

$$a \,.\, (\sqrt{-K})' \,.\, dy \geq 0 \quad \text{on} \quad g_2$$

or

$$a \,.\, \frac{-K'}{2\sqrt{-K}} \,.\, dy \geq 0 \quad \text{on} \quad g_2$$

or to:

$$a := \text{const.} : \leq 0 \quad \text{on} \quad g_2 \ .$$

REMARK:

In both cases if

$$a := \text{constant in } D$$

Then

$$a \leq 0 \quad \text{in} \quad D \ .$$

WELL-KNOWN CHOICES:

1. (Frankl's new choice) :

$$a = -\frac{1}{2} \quad \text{in} \quad D \ , \quad b = c = 0 \quad \text{for} \quad y \geqq 0 \ ,$$

and $$b = -c \, . \sqrt{-K} \ , \quad c = \frac{4 \, . \, a \, . \, K}{K'} \quad \text{for} \quad y \leqq 0 \ .$$

Assume Frankl's condition

$$F > 0 \ , \quad y < 0 \ .$$

2. (Protter's new choice):

$$a = -e^{\beta . x} . \cos(\gamma . y) \ , \quad b = 0 \ , \quad c = 0 \ , \quad \text{for} \quad y \geqq 0$$

$$a = -e^{\beta . x} \ , \quad b = -c \, . \sqrt{-K} \ , \quad c = \frac{4 \, . \, a \, . \, K}{K'} \quad \text{for} \quad y \leqq 0$$

Assume Generalized Frankl's condition

$$F \geqq 2 \, . \frac{K \, . \sqrt{-K}}{K'} . \beta \ ,$$

β , γ :> 0 as above.

Note: Also here both choices work with

$$r = 0 \ .$$

ON THE EXTERIOR TRICOMI AND FRANKL PROBLEM

F.G. Tricomi (1923 -), S. Gellerstedt (1935 -), F.I. Frankl (1945 -), A.V. Bitsadze and M.A. Lavrentiev (1950 -), M.H. Protter (1953 -) and most of the recent workers in the field of mixed type boundary value problems have considered only one parabolic line of degeneracy. The problem with more than one parabolic line of degeneracy becomes more complicated. The above researchers and many others have restricted their attention to the Chaplygin equation: $K(y) . u_{xx} + u_{yy} = f(x , y)$ and not considered the "generalized Chaplygin equation" : $Lu \equiv K(y).u_{xx} + u_{yy} + r(x,y).u = f(x,y)$ because of the difficulties that arise when $r := $ non-trivial $(: \neq 0)$. Also it is unusual for anyone to study such problems in a doubly connected region. In this paper I consider a case of this type with two parabolic lines of degeneracy, $r :=$ non-trivial $(: \neq 0)$, in a doubly connected region, and such that boundary conditions are prescribed only on the "exterior boundary" of the mixed domain, and I obtain uniqueness results for quasi-regular solutions of the characteristic and non-characteristic Problem by applying the b,c - energy integral method in the mixed domain.

THE EXTERIOR TRICOMI PROBLEM

Consider

$$(+) \qquad Lu \equiv K(y).u_{xx} + u_{yy} + r(x,y).u = f(x,y) \quad ,$$

$$K \in C^2(\cdot) \ , \quad r \in C^1(\cdot) \ , \quad f \in C^0(\cdot) \quad ,$$

and such that

$$K = K(y) > 0 \quad \text{for} \quad y < 0 \quad \text{and} \quad y > 1 \ ,$$

$$: \quad = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad y = 1 \ , \quad \text{and}$$

$$: \quad < 0 \quad \text{for} \quad 0 < y < 1 \ .$$

Consider a mixed domain D which is doubly connected contains the two parabolic arcs: $A_1 B_1$, $A_2 B_2$, with end points: $A_1 = (-1,1)$, $B_1 = (1,1)$, $A_2 = (-1,0)$, $B_2 = (1,0)$, and has boundary

$$\partial D = \text{Ext}(D) \cup \text{Int}(D) \ ,$$

Ex(D) : exterior boundary of $D := \Gamma_o \cup \Gamma'_o \cup \Gamma_2 \cup \Gamma'_2 \cup \Delta_1 \cup \Delta'_1$, and

Int(D) : interior boundary of $D := \Gamma_1 \cup \Gamma'_1 \cup \Delta_2 \cup \Delta'_2$,

with boundary curves:

Γ_o : "elliptic arc" for $y > 1$ connecting points: A_1 , B_1 ,

Γ'_o : "elliptic arc" for $y < 0$ connecting points: A_2 , B_2 ,

Γ_1 : characteristic for $0 < y < 1$, $0 < x < 1$ emanating from point $O_1 = (o,1)$:

$$: \int_o^x dx = - \int_1^y \sqrt{-K} \, . \, dy \ , \quad \text{or} \quad \Gamma_1 : \ x = - \int_1^y \sqrt{-K(t)} \, . \, dt \ ,$$

Γ'_1 : characteristic for $0 < y < 1$, $0 < x < 1$ emanating from point $O_2 = (o,o)$:

$: \int_o^x dx = \int_o^y \sqrt{-K} \, . \, dy$, or $\Gamma_1' : x = \int_o^y \sqrt{-K(t)} \, . \, dt$,

Γ_2 : characteristic for $0 < y < 1$, $0 < x < 1$ emanating from point: $B_1 = (1,1)$:

$: \int_1^x dx = \int_1^y \sqrt{-K} \, . \, dy$, or $\Gamma_2 : x = \int_1^y \sqrt{-K(t)} \, . \, dt + 1$,

Γ_2' : characteristic for $0 < y < 1$, $0 < x < 1$ emanating from point: $B_2 = (1,o)$:

$: \int_1^x dx = - \int_o^y \sqrt{-K} \, . \, dy$, or $\Gamma_2' : x = - \int_o^y \sqrt{-K(t)} \, . \, dt + 1$,

Δ_1 : characteristic for $0 < y < 1$, $-1 < x < 0$ emanating from point: $A_1 = (-1,1)$:

$: \int_{-1}^x dx = - \int_1^y \sqrt{-K} \, . \, dy$, or $\Delta_1 : x = - \int_1^y \sqrt{-K(t)} \, . \, dt - 1$,

Δ_1' : characteristic for $0 < y < 1$, $-1 < x < 0$ emanating from point: $A_2 = (-1,o)$:

$: \int_{-1}^x dx = \int_o^y \sqrt{-K} \, . \, dy$, or $\Delta_1' : x = \int_o^y \sqrt{-K(t)} \, . \, dt - 1$,

Δ_2 : characteristic for $0 < y < 1$, $-1 < x < 0$ emanating from point: $O_1 = (o,1)$:

$: \int_o^x dx = \int_1^y \sqrt{-K} \, . \, dy$, or $\Delta_2 : x = \int_1^y \sqrt{-K(t)} \, . \, dt$,

Δ_2' : characteristic for $0 < y < 1$, $-1 < x < 0$ emanating from point: $O_2 = (o,o)$

$$: \int_o^x dx = - \int_o^y \sqrt{-K} \,.\, dy \quad , \quad \text{or} \quad \Lambda_2' : \; x = - \int_o^y \sqrt{-K(t)} \,.\, dt \quad .$$

Besides

$$D = G_1 \cup G_1' \cup G_2 \cup G_2' \cup (A_1 B_1) \cup (A_2 B_2) \quad ,$$

where

G_1 : upper elliptic region $:= \{ (x,y) \in D \,,\; |x| < 1 \,,\; y > 1 \}$

G_1' : lower elliptic region $:= \{ (x,y) \in D \,,\; |x| < 1 \,,\; y < 0 \}$

G_2 : right-hand side hyperbolic region $:= \{(x,y) \in D \,,\; 0 < x < 1 \,,\; 0 < y < 1\}$

G_2' : left-hand side hyperbolic region $:= \{(x,y) \in D \,,\; -1 < x < 0 \,,\; 0 < y < 1\}$

with boundary

$$\partial G_1 := \Gamma_o \cup (A_1 B_1) \quad , \quad \partial G_1' := \Gamma_o' \cup (B_2 A_2) \quad ,$$

$$\partial G_2 := \Gamma_1 \cup \Gamma_1' \cup \Gamma_2 \cup \Gamma_2' \cup (B_1 O_1) \cup (O_2 B_2) \quad ,$$

$$\partial G_2' := \Delta_1 \cup \Delta_1' \cup \Delta_2 \cup \Delta_2' \cup (O_1 A_1) \cup (A_2 O_2) \quad .$$

The above characteristic curves intersect at the following points:

$\Gamma_1 \cap \Gamma_1' = P_1$, $\Gamma_2 \cap \Gamma_2' = P_2$ for $0 < y < 1$ and $0 < x < 1$, and

$\Delta_1 \cap \Delta_1' = P_1'$, $\Delta_2 \cap \Delta_2' = P_2'$ for $0 < y < 1$ and $-1 < x < 0$.

Besides assume <u>boundary conditions</u>

$$(++)\qquad \begin{cases} u = \phi_1(s) \quad \text{on} \quad \Gamma_o \;, & u = \phi_2(s) \quad \text{on} \quad \Gamma'_o \\ u = \psi_1(x) \quad \text{on} \quad \Gamma_2 \;, & u = \psi_2(x) \quad \text{on} \quad \Gamma'_2 \\ u = \psi_3(x) \quad \text{on} \quad \Delta_1 \;, & u = \psi_4(x) \quad \text{on} \quad \Delta'_1 \end{cases}$$

(i.e. : u := continuous prescribed values on Ext(D)) .

<u>THE EXTERIOR TRICOMI PROBLEM</u>, or <u>PROBLEM (ET)</u> :

Consists in finding a function $u = u(x,y)$ which satisfies equation (+) and boundary conditions (++) .

<u>A NEW UNIQUENESS THEOREM</u>

<u>Assume the above-mentioned domain</u> $D \subset \mathbb{R}^2$, <u>and the conditions</u>

(R_1) : $\quad r \leqq 0 \quad$ <u>on</u> $\quad \text{Int}(D)$

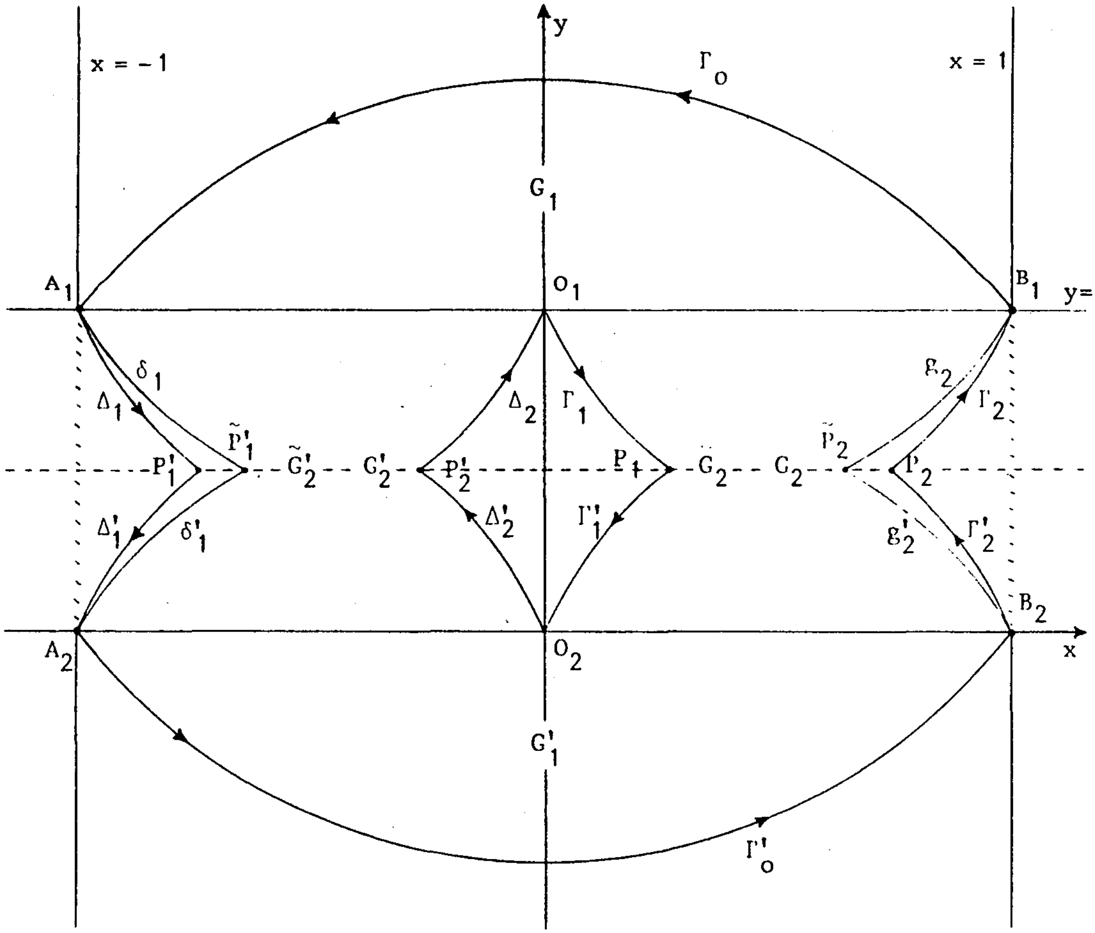
$$(R_2)\ :\qquad \begin{cases} x.dy - (y-1).dx \geqq 0 & \text{on} \quad \Gamma_o \\ x.dy - y.dx \geqq 0 & \text{on} \quad \Gamma'_o \end{cases}$$

" <u>star-likedness</u> "

$$(R_3) : \quad \begin{cases} 2.r + x.r_x + (y-1).r_y \leqq 0 & \underline{\text{in}} \quad G_1 \\ r + x.r_x \leqq 0 & \underline{\text{in}} \quad G_2 \cup G_2' \\ 2.r + x.r_x + y.r_y \leqq 0 & \underline{\text{in}} \quad G_1' \end{cases}$$

$$(R_4) : \quad K' > 0 \ \underline{\text{in}} \ G_1 \ , \ \text{and} \ K' < 0 \ \underline{\text{in}} \ G_1' \ .$$

<u>Then Problem</u> (ET) <u>has at most one quasi-regular solution in the mixed domain</u> D .



Proof. We apply the b, c energy integral method ($a = 0$ in D) and use (++).

First, we assume u_1, u_2: two quasi-regular solutions satisfying equation (+) and boundary conditions (++). Then claim that

$$u = u_1 - u_2 = 0 \quad \text{in} \quad D .$$

It is clear now that

[+] $$Lu \equiv K(y).u_{xx} + u_{yy} + r(x,y).u = 0 , \quad \text{and}$$

[++] $$u = 0 \quad \text{on} \quad \mathrm{Ext}(D) .$$

It is enough to show that

$$u = u_1 - u_2 = 0 \quad \text{on} \quad \mathrm{Int}(D) .$$

Second, investigate

$$0 = J = 2 . \iint_D (b.u_x + c.u_y).Lu.dx\,dy \quad ,$$

where

(c) : $$\begin{cases} b = x \ , & c = y-1 \quad \text{in} \quad G_1 \\ b = x \ , & c = 0 \quad \text{in} \quad G_2 \cup G_2' \\ b = x \ , & c = y \quad \text{in} \quad G_1' \ . \end{cases}$$

Then consider the identities

$$2.b.r.u.u_x = (b.r.u^2)_x - (b.r)_x . u^2 ,$$

$$2.c.r.u.u_y = (c.r.u^2)_y - (c.r)_y . u^2 ,$$

$$2.b.K.u_x.u_{yy} = (b.K.u_x^2)_x - b_x.K.u_x^2 ,$$

$$2.b.u_x.u_{yy} = (2.b.u_x.u_y)_y - (b.u_y^2)_x + b_x.u_y^2 ,$$

$$2.c.K.u_y.u_{xx} = (2.c.K.u_x.u_y)_x - (c.K.u_x^2)_y + (c.K)_y . u_x^2 ,$$

$$2.c.u_y.u_{yy} = (c.u_y^2)_y - c_y.u_y^2 .$$

Then employing above identities and applying Green's theorem we obtain:

$$0 = J = \iint_D [-(b.r)_x.u^2 - (c.r)_y.u^2 - b_x.K.u_x^2 + b_x.u_y^2 + (c.K)_y.u_x^2 - c_y . u_y^2].dxdy$$
$$+ \oint_{\partial D} [b.r.u^2.\nu_1 + c.r.u^2.\nu_2 + b.K.u_x^2.\nu_1 + 2.b.u_x.u_y.\nu_2 - b.u_y^2.\nu_1 + 2.c.K.u_x.u_y.\nu_1 - c.K.u_x^2.\nu_2 + c.u_y^2.\nu_2].ds ,$$

where $\nu = (\nu_1 , \nu_2) := (\frac{dy}{ds}, - \frac{dx}{ds})$: outer unit normal vector on ∂D .

Therefore

$$0 = - \iint_D [(b.r)_x + (c.r)_y].u^2.dx\,dy$$
$$+ \iint_D [(-b_x.K + (c.K)_y).u_x^2 + (b_x - c_y).u_y^2].dx\,dy$$

$$+ \oint_{\partial D} [(b.\nu_1 + c.\nu_2).r] . u^2 . ds$$

$$+ \oint_{\partial D} [(b.\nu_1 - c.\nu_2).K.u_x^2 + 2.(b.\nu_2 + c.K.\nu_1).u_x.u_y + (-b.\nu_1 + c.\nu_2).u_y^2] . ds$$

$$= I_1 + I_2 + J_1 + J_3 .$$

Claim that all integrals: I_1, I_2, J_1, and J_3 are non-negative.

FIRST: The integrals I_1, I_2 are non-negative if the following two conditions hold in D :

(c_1) : $$(b_x + c_y).r + (b.r_x + c.r_y) \leqq 0$$

(c_2) : $$\begin{cases} A := -b_x.K + (c.K)_y \geqq 0 \\ B := b_x - c_y \geqq 0 \end{cases} .$$

SECOND: The integrals J_1, and J_3 are non-negative if the following conditions hold on ∂D :

(c_3) : $(b.\nu_1).r \geqq 0$ on $Int(D)$,

(c_4) : $b.\nu_1 + c.\nu_2 \geqq 0$ or $\Gamma_o \cup \Gamma_o'$,

(c_5) : $b.\nu_1 \leqq 0$ on $Int(D)$.

JUSTIFICATION:

Condition (c_3) : From [++] and (c) we get

$$J_1 = \int_{Int(D)} [\,(b.\nu_1).r\,].u^2.ds$$

Therefore condition (c_3) holds.

<u>Conditions (c_4) and (c_5) :</u>

$$J_3 = \int_{Ext(D)} Q_1.ds + \int_{Int(D)} Q_2.ds \;:=\; J_3^{(1)} + J_3^{(2)} \quad ,$$

where

$$Q_1 := Q_1(u_x,u_y) := (b.\nu_1-c.\nu_2).K.u_x^2+2.(b.\nu_2+c.K.\nu_1).u_x.u_y+(-b.\nu_1+c.\nu_2).u_y^2 \;,$$

$$Q_2 := Q_1(u_x,u_y) := (b.\nu_1).K.u_x^2 + 2.(b.\nu_2).u_x.u_y + (-b.\nu_1)\,.\,u_y^2$$

are two quadratic forms with respect to u_x , u_y on Ext(D), and Int(D) , respectively.

From [++] we get

$$u_x = N.\nu_1 \;,\quad u_y = N.\nu_2 \quad \text{on} \quad Ext(D) \quad ,$$

where

N : = normalizing factor.

Therefore

$$Q_1 \;:=\; (b.\nu_1 + c.\nu_2).(K.\nu_1^2 + \nu_2^2).N^2 \quad .$$

But

$K.\nu_1^2 + \nu_2^2 > 0$ on $\Gamma_o \cup \Gamma_o'$ (as $K > 0$ in $G_1 \cup G_1'$) ,

$K.\nu_1^2 + \nu_2^2 = 0$ on $Ext(D) \setminus \Gamma_o \cup \Gamma_o'$ (as $\Gamma_2, \Gamma_2', \Delta_1, \Delta_1'$ are characteristics)

Therefore

$$Q_1 = Q_1|_{Ext(D)} := Q_1|_{\Gamma_o \cup \Gamma_o'} \geq 0$$

if (c_4) holds. Therefore $J_3^{(1)} \geq 0$ if (c_4) holds.

Also $J_3^{(2)} \geq 0$ if

$$Q_2 := Q_2|_{Int(D)} \geq 0 \quad .$$

But on $Int(D)$:

$$\begin{vmatrix} (b.\nu_1).K & b.\nu_2 \\ b.\nu_2 & -b.\nu_1 \end{vmatrix} := -b^2.(K.\nu_1^2 + \nu_2^2) := 0 \quad ,$$

because

$K.\nu_1^2 + \nu_2^2 = 0$ on $Int(D)$ (as $\Gamma_1, \Gamma_1', \Delta_2, \Delta_2'$ are characteristics)

From (c) therefore

$$Q_2 \geq 0 \quad \text{holds if}$$

$$(b.\nu_1).K \geq 0 \quad \text{and} \quad -b.\nu_1 \geq 0 \quad \text{on} \quad Int(D) \quad ,$$

or if condition (c_5) holds (as $K < 0$ in $G_2 \cup G_2'$), and the justification is complete.

REDUCTION OF CONDITIONS $(c_1) - (c_5)$ (by using choices (c)) :

Conditions (c_3) and (c_5) are reduced to condition:

$$(R)_1 : \qquad r \leq 0 \quad \text{on} \quad Int(D) ,$$

because

$$x.\nu_1 \leq 0 \quad \text{on} \quad Int(D) .$$

Also condition (c_4) is reduced to condition:

$$(R)_2 : \quad \begin{cases} x.dy - (y-1).dx \geq 0 & \text{on} \quad \Gamma_o , \\ x.dy - y.dx \geq 0 & \text{on} \quad \Gamma_o' . \end{cases}$$

Besides condition (c_1) is reduced to condition:

$$(R)_3 : \quad \begin{cases} 2.r + x.r_x + (y-1).r_y \leq 0 & \text{in} \quad G_1 \\ r + x.r_x \leq 0 & \text{in} \quad G_2 \cup G_2' \\ 2.r + x.r_x + y.r_y \leq 0 & \text{in} \quad G_1' . \end{cases}$$

Finally condition (c_2) is reduced to condition:

$(R)_4$: $K' > 0$ in G_1 , $: < 0$ in G_1' .

because

$$-b_x.K + (c.K)_y = \begin{cases} -K + (K + (y-1).K') = (y-1).K' > 0 \text{ in } G_1 \\ \qquad\qquad \text{if } K' > 0 \text{ in } G_1 \\ -K > 0 \text{ in } G_2 \\ -K + (K + y.K') = y.K' > 0 \text{ in } G_1' \\ \qquad\qquad \text{if } K' < 0 \text{ in } G_1' \end{cases}$$

and $b_x - c_y = 0$ in $G_1 \cup G_1'$, and $b_x - c_y = 1$ in G_2 .

SPECIAL CASE:

$$(S) : K = \operatorname{sgn}(y.(y-1)).|y|^{\alpha}.|y-1|^{\beta}.h(y) \quad \text{in} \quad D ,$$

$$\alpha , \beta > 0 \quad , \quad \text{and}$$

$$h = h(y) > 0 \quad \text{for all} \quad y ,$$

where

$$\operatorname{sgn}(y.(y-1)) := \begin{cases} 1 , & y > 1 \\ -1 , & 0 < y < 1 \\ 1 , & y < 0 \end{cases}$$

and $:= 0$ for $y = 0$ and $y = 1$.

Therefore

$$K(y) = \begin{cases} K_1(y) = y^{\alpha}.(y-1)^{\beta}.h(y) > 0 \quad , \quad y > 1 \\ K_2(y) = -y^{\alpha}\ (1-y)^{\beta}\ h(y) < 0 \quad , \quad 0 < y < 1 \\ K_3(y) = (-y)^{\alpha}.(1-y)^{\beta}.h(y) > 0 \quad , \quad y < 0 \end{cases}$$

and : = 0 for $y = 0$ and $y = 1$.

COROLLARY

If $K = K(y)$ is of the form (S) in D , if conditions (R_1)-(R_3) of Theorem hold, and if

$$R = R(y;\alpha,\beta) = [\,\alpha.(y-1) + \beta.y\,]\,.\,h(y) + y(y-1).h'(y)$$

is such that the following condition

(B) : $R > 0$ in G_1 , and $R < 0$ in G_1'

holds, then Problem (ET) has at most one quasi-regular solution in the mixed domain $D \subset \mathbb{R}^2$.

REMARKS:

1). It is clear than on the parabolic lines of degeneracy : $y = 1$ and $y = 0$:

$\lim_{y\to 1^+} R(y;\alpha,\beta) = \beta.h(1) > 0$, and $\lim_{y\to o^-} R(y;\alpha,\beta) = -\alpha.h(0) < 0$ hold, because $\alpha,\beta > 0$, and $h(y) > 0$ for all y in D .

2). If $r :=$ constant, then conditions (R_1) and (R_3) are replaced by only condition (R_1) .

3). If

$$\alpha = \beta = 1 \quad , \quad h = 1$$

in (S) , then

$$K(y) = \text{sgn}(y.(y-1)).|y|.|y-1| := y.(y-1)$$

and condition (B) in Corollary or condition (R_4) in Theorem is <u>not</u> needed.

THE EXTERIOR FRANKL PROBLEM

Replace characteristics $\Gamma_2 , \Gamma_2' , \Delta_1 , \Delta_1'$ by smooth non-characteristics: $g_2 , g_2' , \delta_1 , \delta_1'$ so that:

(NC) : $\quad H := K.\nu_1^2 + \nu_2^2 > 0$ on $g_2 \cup g_2' \cup \delta_1 \cup \delta_1'$, and

i). g_2 emanating from point B_1 lying inside the characteristic truncated triangle $O_1 P_1 P_2 B_1$ and intersecting Γ_1 at most once. This curve g_2 may coincide with Γ_2 near point B_1 , ii). g_2' emanating from point B_2 lying inside the characteristic truncated triangle $O_2\ B_2\ P_2\ P_1$ and intersecting Γ_1' at most once. This curve g_2' may coincide with Γ_2' near point B_2,

iii). δ_1 emanating from point A_1 lying inside the characteristic truncated triangle $A_1\ P_1'\ P_2'\ O_1$ and intersecting Δ_2 at most once. This curve δ_1 may coincide with Δ_1 near point A_1, and

iv). δ_1' emanating from point A_2 lying inside the characteristic truncated triangle $A_2\ O_2\ P_2'\ P_1'$ and intersecting Δ_2' at most once. This curve δ_1' may coincide with Δ_1' near point A_2.

Besides assume <u>boundary conditions</u>

$$
(F): \quad \begin{cases} u = \phi_1(s) & \text{on} \quad \Gamma_o \ , \quad u = \phi_2(s) \quad \text{on} \quad \Gamma_o' \\ u = \psi_1(x) & \text{on} \quad g_2 \ , \quad u = \psi_2(x) \quad \text{on} \quad g_2' \\ u = \psi_3(x) & \text{on} \quad \delta_1 \ , \quad u = \psi_4(x) \quad \text{on} \quad \delta_1' \end{cases}
$$

<u>The new mixed domain</u> D' is such that:

$$
\partial D' = Ext(D') \cup Int(D') \ ,
$$

$$
Ext(D') = \Gamma_o \cup \Gamma_o' \cup N\,ch(D') \ , \qquad Int(D') = Int(D) \ ,
$$

$$
N\,ch(D') = g_2 \cup g_2' \cup \delta_1 \cup \delta_1' : \text{ the non-characteristic part of } D'.
$$

Besides

$$D' = G_1 \cup G_1' \cup \tilde{G}_2 \cup \tilde{G}_2' \cup (A_1 B_1) \cup (A_2 B_2) ,$$

where

$$\tilde{G}_2 \ (\subset G_2) := \{ (x,y) \in D' , \ 0 < x < 1 , \ 0 < y < 1 \}$$

$$\tilde{G}_2' \ (\subset G_2') := \{ (x,y) \in D' , \ -1 < x < 0 , \ 0 < y < 1 \}$$

with boundary

$$\partial\tilde{G}_2 := \Gamma_1 \cup \Gamma_1' \cup g_2 \cup g_2' \cup (B_1 O_1) \cup (O_2 B_2) ,$$

$$\partial\tilde{G}_2' := \delta_1 \cup \delta_1' \cup \Delta_2 \cup \Delta_2' \cup (O_1 A_1) \cup (A_2 O_2) .$$

The above non-characteristic curves intersect as follows:

$$g_2 \cap g_2' = \tilde{P}_2 , \quad \delta_1 \cap \delta_1' = \tilde{P}_1' .$$

THE EXTERIOR FRANKL PROBLEM, or PROBLEM (EF):

Consists in finding a function $u = u(x,y)$ which satisfies equation (+) and boundary conditions (F) in the mixed domain D'.

Then it is clear that a corresponding new uniqueness theorem and a corollary hold in the new domain D' under the same conditions as those of the above proved theorem (and the corollary).

The only difference in statement is that we must change $G_2 \cup G_2'$ with $\tilde{G}_2 \cup \tilde{G}_2'$ in (R_3)

(c)' : $$(b.\nu_1).H \geq 0 \quad \text{on} \quad N\,ch(D')$$

In fact, from (NC) : $H > 0$ on $N\,ch(D')$, the fact that $b = x$ in $\tilde{G}_2 \cup \tilde{G}'_2$ (analogous to that one of (c)), and the fact that

$$x.\nu_1 > 0 \quad \text{on} \quad N\,ch(D')$$

we obtain the validity of condition (c)'. This yields that we don't need to assume finally an additional condition in our new theorem (and corollary) in D'.

WEAK SOLUTIONS

Let

$$p = (x,y) \in \mathbb{R}^2 \quad , \quad \alpha = (\alpha_1 , \alpha_2) \quad ,$$

$$\alpha_1 , \alpha_2 : > 0 \quad , \quad |\alpha| = \alpha_1 + \alpha_2 \quad , \quad p^\alpha = x^{\alpha_1} . y^{\alpha_2} \quad ,$$

$$< p,q > = x.\tilde{x} + y.\tilde{y} \quad , \quad |p| = (< p,p >)^{1/2} \quad ,$$

$$q = (\tilde{x},\tilde{y}) \in \mathbb{R}^2 \quad .$$

Also let

$$(D^\alpha u)(p) := (D_1^{\alpha_1} D_2^{\alpha_2} u)(p) \quad :$$

$$D_1 = \frac{\partial}{\partial x} \quad , \quad D_2 = \frac{\partial}{\partial y}$$

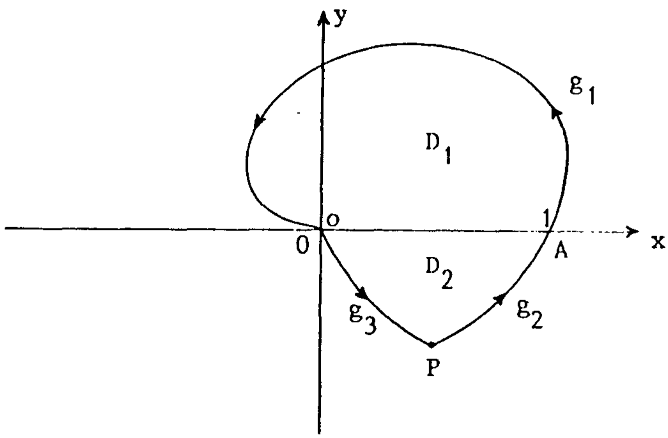
for sufficiently smooth functions: $u = u(p)$.

Consider the generalized Chaplygin equation

$$Lu \equiv K(y).u_{xx} + u_{yy} + r(x,y).u = f(x,y) \ , \quad K(y) \gtreqless 0 \quad \text{if} \quad y \gtreqless 0 \ ,$$

(EQ) :

$$K \in C^2(.) \ , \quad r \in C^1(.) \ , \quad f \in C^o(.) \ , \quad (\text{or} \quad f \in L^2(.))$$



g_2 : may be either characteristic (Tricomi case):

$$\int_1^x dx = \int_o^y \sqrt{-K(t)} \ . \ dt$$

g_2 : $$x = \int_o^y \sqrt{-K(t)} \ . \ dt + 1 \quad , \quad \text{or}$$

a smooth non-characteristic (Frankl case) .

The domain $G := \partial D := g_1 \cup g_2 \cup g_3$ is a piecewise smooth boundary of the mixed domain $D \subset \mathbb{R}^2$.

Assume boundary conditions

(B) : $$u = 0 \quad \text{on} \quad g_1 \cup g_2 .$$

Also consider adjoint equation

(AQ) : $$L^*w \equiv K(y).w_{xx} + w_{yy} + r(x,y).w = f(x,y) ,$$

where L^* (: = L) is the formal adjoint operator of the formal operator L .

Assume now adjoint boundary condition

(AT) : $w = 0$ on $g_1 \cup g_3$ (Tricomi case)

if g_2 : characteristic , and

(AF) : $w = 0$ on G (Frankl case)

if g_2 : smooth non-characteristic .

Denote

$$C^m(\bar{D}) := \{ u(p) \mid p \in \bar{D} : \ u = u(p) \ \text{is m-times continuously differentiable in } \bar{D} \} .$$

Note: This space is a complete normed space with the following norm:

$$\|u\|_{C^m(\bar{D})} := \max \{ |(D^\alpha u)(p)| \ | p \in \bar{D} : \ |\alpha| \leq m \} .$$

Also denote

$L^2(D) : \{ u(p) \mid p \in D \}$ with inner product $\langle u,w\rangle_{L^2(D)} = \int_D u(p).w(p).dp$,

and norm $\|u\|_{L^2(D)} = \left(\int_D |u(p)|^2 .dp \right)^{1/2}$.

SOBOLEV SPACE:

$W_2^m(D) = W^{m,2}(D) := \{ u(p) \mid p \in D ,\ u(p) \in L^2(D) ,\ D^\alpha u(p) \in L^2(D) ,\ |\alpha| \leq m \}$:

is the Sobolev space with norm

$$\|u\|_m := \|u\|_{W_2^m(D)} := \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^2(D)} ,$$

or equivalently

$$\|u\|_m := \|u\|_{W_2^m(D)} := \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(D)} \right)^{1/2} ,$$

or also equivalently

$$\|u\|_m := \|u\|_{W_2^m(D)} := \left(\|u\|^2_{L^2(D)} + \sum_{|\alpha| = m} \|D^\alpha u\|^2_{L^2(D)} \right)^{1/2} ,$$

and inner product

$$\langle u,w\rangle_m := \langle u,w\rangle_{W_2^m(D)} := \left(\sum_{|\alpha| \leq m} \langle D^\alpha u , D^\alpha w\rangle^{1/2}_{L^2(D)} \right)^2 ;$$

or equivalently

$$\langle u,w\rangle_m := \langle u,w\rangle_{W_2^m(D)} := \sum_{|\alpha| \leq m} \langle D^\alpha u , D^\alpha w\rangle_{L^2(D)} ,$$

or also equivalently

$$< u,w >_m := < u,w >_{W_2^m(D)} := < u,w >_{L^2(D)} + \sum_{|\alpha| = m} < D^\alpha u, D^\alpha w >_{L^2(D)}$$

REMARKS:

i). In general if

$$Lu \equiv \sum_{i,j=1}^{2} \alpha_{ij}(p).D_i D_j u + \sum_{i=1}^{2} \alpha_i(p).D_i u + \alpha(p).u$$

then

$$L^*w = \sum_{i,j=1}^{2} D_i D_j(\alpha_{ij}(p).w) - \sum_{i=1}^{2} D_i(\alpha_i(p).w) + \alpha(p).w$$

ii) The above space $W_2^m(D)$ is a complete normed space.

iii) $W_2^m(D) \subset \ldots \subset W_2^1(D) \subset W_2^o(D) = L^2(D)$

$$\|\cdot\|_m \geq \ldots \geq \|\cdot\|_1 \geq \|\cdot\|_o$$

SPECIAL CASE (: m = 2) :

Let

$D(L)$: domain of the formal operador L

$:= \{ u \in C^2(\bar{D}) : u = 0$ on $g_1 \cup g_2 \}$,

$W_2^2(D,bd)$: closure of the function space $D(L)$ with respect to the norm $\|\cdot\|_2$

$$:= \overline{D(L)}\Big|_{\|\cdot\|_2} .$$

See: Ju. M. Berezanskii (Transl. Math. Mon., 17, AMS, Providence, R.I., 1968, p. 79-80).

Similarly,

$D(L^*)$: domain of the formal adjoint operator L^*

$$:= \{ w \in C^2(\overline{D}) : w = 0 \text{ on } g_1 \cup g_3 \} \text{ (Tricomi case)}$$

or

$$:= \{ w \in C^2(\overline{D}) : w = 0 \text{ on } G \} \text{ (Frankl case)}$$

$W_2^2(D,b\overset{*}{d})$:= closure of the function space $D(L^*)$ with respect to the norm $\|\cdot\|_2$

$$:= \overline{D(L^*)}\Big|_{\|\cdot\|_2} ,$$

or equivalently

$$:= \{ w \mid w \in W_2^2(D) , \langle Lu,w \rangle_o = \langle u,L^*w \rangle_o , \text{ for all } u \in W_2^2(D,bd) \} .$$

<u>Definition 3</u>. A function $u \in L^2(D)$ is a <u>weak solution</u> of Problem (EQ) & (B) if

$$\langle f,w \rangle_o = \langle u,L^*w \rangle_o \quad \text{for all} \quad w \in W_2^2(D,b\overset{*}{d}) .$$

EXISTENCE

CRITERION. A necessary and sufficient condition for the existence of a weak solution of Problem (EQ) and (B) is that

$$(AP) : \quad \|w\|_o \leq C.\|L^* w\|_o \qquad (\text{ a - priori - estimate })$$

holds for all $w \in W_2^2(D, \overset{*}{bd})$, $C = \text{const.} > 0$.

See: Ju. M. Berezanskii (Transl. Math., 17, AMS, Providence, R.I., 1968, p. 79-80 ; and J.M. Rassias ("Mathematics - Space Technology" , Athens, Greece, 1981 ; "Partial Differential Equations of Mixed Type", manuscript at I.C.M.S.C. / S.P. , Brasil, 1988, p. 12-27).

Definition 4. A function $u \in L^2(D)$ is a strong solution of Problem (EQ) & (B) if there is a sequence $\{u_n\} : u_n \in C^2(\bar{D})$ such that

$$\|u_n - u\| \to 0 \quad \text{and} \quad \|Lu_n - f\| \to 0 , \quad n \to \infty$$

in the L^2-norm in D .

REMARKS:

i). { strong solution } $\subset$ { weak solution } .

i.e. a strong solution is a weak solution but a weak solution is not always a strong solution.

ii). In 1958 K.O. Friedrichs (Comm. Pure Appl. Math., 11, 1958, 333-418) worked extensively on symmetric positive linear differential equations.

iii). In 1960 P.D. Lax and R.S. Phillips (Comm. Pure Appl. Math., 13, 1960, 427-455) proved that a weak solution is also a strong solution in the above classical sense or equivalently in the sense of Friedrichs (1958) by assuming local boundary conditions for dissipative symmetric linear differential operators.

iv). In 1965 Ju. M. Berezanskii (Naukova Dumka, Kiev, 1965; Transl. Math. Mon., 17, AMS, Providence, R.I., 1968) developed a functional-analytic approach to existence proofs for weak solutions of the Tricomi and Frankl Problems.

v). In 1966 N.G. Sorokina (Ukrain. Mat. Z., 18, 1966, 65-77) proved the uniqueness of the weak solution of the Tricomi Problem and showed that this solution coincided with the strong solution.

vi). In 1980 J.M. Rassias (Bull. Soc. Roy. Sci. Liège, 5-8, 1980, 278-280) established a new existence theorem for weak solutions of a mixed type boundary value problem with prescribed boundary values on a piece of the boundary of the hyperbolic region in the three- dimensional euclidean space. Uniqueness results for quasi--regular solutions of the above problem were established in 1977 via the doctoral dissertation of the same author (Doctoral Dissertation, U.C.-Berkeley, 1977). The generalization of these results in $\mathbb{R}^{n+1}$ $(n \geq 2)$ was established by J. M. Rassias in 1988 (Comp. Rend. Acad. Bulg. Sci., 41, 1988, 35-37; Comp. Rend. Acad. Bulg. Sci., to appear).

vii). We get an analogous Criterion <u>for the existence of a weak solution</u> if (AP) is replaced by <u>the new a-priori estimate</u>

[AP] : $$\|w\|_1 \leq C.\|L^* w\|_o ,$$

because

$$\|w\|_1 \geq \|w\|_o .$$

<u>Note</u>: <u>In the 2-dimensional case</u>:

$$\|w\|_1^2 = \iint_D (w^2 + w_x^2 + w_y^2).dx.dy \geq \iint_D w^2.dx.dy = \|w\|_o^2 .$$

THE HAHN-BANACH THEOREM AND THE RIESZ REPRESENTATION THEOREM FOR EXISTENCE OF WEAK SOLUTIONS

<u>Note</u>: The following discussion is <u>not</u> necessary because the above-mentioned Criterion is enough, but it was chosen as it gives a better understanding of the subject.

Prove: If $u \in L^2(\bar{D})$ and (AP), or [AP] a-priori estimate holds, then u is a weak solution of Problem (EQ) & (B).

Proof: In fact

$$f \in L^2(D) \quad , \quad f : D \to \mathbb{R}$$

FIRST: Define the linear functional F in $D(L^*)$

$$F : \; U = L^*(D(L^*)) \to \mathbb{R} \quad ,$$

$(R)_1$:
$$F(L^*w) = \langle f, w \rangle_0$$

for all $w \in D(L^*)$.

Then

$|F(L^*w)| = |\langle f,w \rangle_0|$

$\leq \|f\|_0 . \|w\|_0$ (by Cauchy-Schwarz-Buniakowski inequality)

$\leq \|f\|_0 . \|w\|_1$ (by inequality $\|w\|_1 \geq \|w\|_0$ if [AP] a-priori estimate appears through, otherwise we apply (AP) a-priori estimate and don't consider this step at all ; go straight to the following step)

$\leq \|f\|_0 . C . \|L^*w\|_0$ (by (AP) a-priori estimate)

or

$$|F(L^*w)| \leq C \cdot \|f\|_o \cdot \|L^*w\|_o$$

yielding that F : is bounded.

Note: U is a linear subspace of $L^2(D)$.

SECOND: Employ the Hahn-Banach theorem to extend F from U onto the whole space $L^2(D)$ with preservation of the norm.

In fact, exists a linear functional

$$\tilde{F} : \tilde{U} = L^2(D) \to \mathbb{R}$$

as an extension of F (i.e. $\tilde{F}|_U = F$) with preservation of the norm (i.e. $\|\tilde{F}\|_{\tilde{U}} = \|F\|_U$) .

THIRD: Apply the Riesz representation theorem to find $u \in L^2(D)$ such that

$$\tilde{F}(\tilde{u}) = \int_D \tilde{u}.u$$

and

$$\|\tilde{F}\| = \|u\|_{L^2(D)}$$

for all $\tilde{u} \in \tilde{U}$.

Therefore

$$\tilde{F}(\tilde{u}) = \langle \tilde{u}, u \rangle_o \qquad \text{(by the definition of the inner product } \langle \cdot \rangle_o \text{) .}$$

FOURTH: Use the Hahn-Banach extension (above-mentioned).

Thus

$$\tilde{F}(\tilde{u}) = F(\tilde{u}) \qquad \text{for all} \qquad \tilde{u} \in U \ (\subset \tilde{U})$$

Therefore

$$F(\tilde{u}) = \langle \tilde{u}, u \rangle_o \qquad \text{for all} \qquad \tilde{u} \in U$$

FINALLY:

Choose

$$\tilde{u} = L^* w \qquad \text{for all} \qquad w \in D(L^*) \ .$$

Hence

$(R)_2$:

$$F(L^* w) = \langle L^* w, u \rangle_o \qquad \text{for all} \qquad w \in D(L^*) \ .$$

Therefore relations $(R)_1$ - $(R)_2$ yield

$$\langle f, w \rangle_o = \langle L^* w, u \rangle_o \qquad \text{for all} \qquad w \in D(L^*) \ ,$$

completing the proof that u is a weak solution of Problem (EQ) & (B) .

JUSTIFICATION OF THE DEFINITION OF WEAK SOLUTION

Assume $u \in C^2(\overline{D})$ and

$$< f,w >_o = < u,L^*w >_o$$

for all $w \in D(L^*)$.

Claim that:

(i). $Lu = f$ in D

(ii). $u = 0$ on $g_1 \cup g_2$.

In fact,

$$< f,w >_o = < u,L^*w >_o$$

for all $w \in D(L^*)$, by assumption.

FIRST: By applying Green's theorem we get $<Lu,w>_o = <u,L^*w>_o + (u,w)_G$,

or $<f,w>_o = <u,L^*w>_o = <Lu,w>_o - (u,w)_G$ (we do not know yet if $Lu = f$ in D)

for all $w \in D(L^*)$,

where

$$(u,w)_G = \int_G [w . (K . u_x . \nu_1 + u_y . \nu_2)$$

$$- u . (k . w_x . \nu_1 + w_y . \nu_2)] . ds ,$$

where $\nu = (\nu_1, \nu_2)$ is the outer normal unit vector on G .

SECOND: Consider also the functional spaces:

$$C_o^m(\bar{D}) \; (= \{ w \mid w \in C^m(\bar{D}) \text{ with compact support: } \operatorname{supp} w \subset \bar{D} \} \, ,$$

$$\operatorname{supp} w = \overline{\{ x \mid w(x) \neq 0 \}} \;) \, ,$$

and

$$C_o^\infty(\bar{D}) = \bigcap_{m \in \mathbb{N}} C_o^m(\bar{D}) \quad (: \subset D(L^*) \subset C^2(\bar{D}) \;)$$

which is dense in

$$L^2(D) \quad (\supset L^*(D(L^*)) \;) \; .$$

THIRD: In particular, choose

$$w \in C_o^\infty(\bar{D}) \quad (\subset D(L^*) \;) \; :$$

such that

$$w = \nabla w = 0 \quad \text{on} \quad G$$

and

$$(u,w)_G = 0 \quad \text{for all} \quad w \in C_o^\infty(\bar{D}) \; .$$

Therefore

$$< f,w >_o \; = \; < Lu,w >_o \quad \text{for all} \quad w \in C_o^\infty(\bar{D})$$

or

$$< f - Lu , w >_o = 0 \quad \text{for all} \quad w \in C_o^\infty(\bar{D}) :$$

such that

$$w = \nabla w = 0 \quad \text{on} \quad G .$$

FOURTH: Density of functions w ($\in C_o^\infty(\bar{D})$) in $L^2(\bar{D})$ implies

$$f = Lu \quad \text{in the } L^2\text{-sense} .$$

Continuity of Lu implies

$$Lu = f \quad \text{in } D \quad (\text{ that is the validity of (i) })$$

FIFTH: From (i) we get that:

$$(u,w)_G = 0 \quad \text{for all} \quad w \in D(L^*) .$$

But since

$$w \in D(L^*) \quad (\text{ i.e. } w = 0 \text{ on } G : \underline{\text{Frankl case}})$$

then

$$0 = (u,w)_G = -\int_G u \cdot (K \cdot w_x \cdot \nu_1 + w_y \cdot \nu_2) \cdot ds$$

$$= -\int_G u \cdot (K \cdot \nu_1^2 + \nu_2^2) \cdot N \cdot ds$$

(as $w = 0$ on G yields $\nu = \frac{\nabla w}{|\nabla w|}$: outer unit normal vector, or $\nabla w = N.\nu$, N : normalizing factor $:= |\nabla w|$ or $w_x = N.\nu_1$, $w_y = N.\nu_2$) ,

or

$$0 = (u,w)_G = - \int_{g_1 \cup g_2} u \, . \, (K \, . \, \nu_1^2 + \nu_2^2) \, . \, N \, . \, ds$$

(as g_3 : characteristic : $K \, . \, \nu_1^2 + \nu_2^2 = 0$ on g_3) .

But N is an arbitrary function.

Therefore

$$u = 0 \quad \text{on} \quad g_1 \cup g_2 \ ,$$

completing the proof of (ii) .

Similarly the definition of weak solution is justified for the Tricomi Case (i.e. $w = 0$ on $g_1 \cup g_3$) .

a - PRIORI ESTIMATE

We apply the a^*, b^*, c^* energy integral method:

$$J^* = 2 \, . < M^*w \, , L^*w >_o = \iint_D 2 \, . \, M^*w \, . \, L^*w \, . \, dx \, dy \ ,$$

where

$$M^*w := a^*(x,y) \, . \, w + b^*(x,y) \, . \, w_x + c^*(x,y) \, . \, w_y \quad \text{in} \quad D \ ,$$

and $\qquad a^* \in C^2(\bar{D})$, $b^* \in C^1(\bar{D})$, $c^* \in C^1(\bar{D})$.

Employing Green's theorem we get

$$J^* = I_1^* + I_2^* + J_1^* + J_2^* + J_3^* \quad ,$$

where

$$I_1^* = \iint_D A_1^* \, . \, w^2 \, . \, dx\, dy \quad ,$$

$$I_2^* = \iint_D (A_2^* \, . \, w_x^2 - 2 \, . \, B^* \, . \, w_x \, . \, w_y + A_3^* \, . \, w_y^2) \, . \, dx\, dy \quad ,$$

$$J_1^* = \oint_G B_1^* \, . \, w^2 \, . \, ds \quad ,$$

$$J_2^* = \oint_G B_2^* \, . \, ds \quad , \quad J_3^* = \oint_G Q^* \, . \, ds \quad ,$$

$$A_1^* \; := \; (2a^* - b_x^* - c_y^*) . r - (b^* . r_x + c^* . r_y) + K \, . \, a_{xx}^* + a_{yy}^* \quad ,$$

$$A_2^* \; := \; -2 . a^* \, . \, K - b_x^* \, . \, K + (c^* \, . \, K)_y \quad ,$$

$$A_3^* \; := \; -2 . a^* + b_x^* - c_y^* \quad ,$$

$$B^* \; := \; b_y^* + c_x^* \, . \, K \quad ,$$

$$B_1^* \; := \; (b^* \, . \, \nu_1 + c^* \, . \, \nu_2) \, . \, r \quad ,$$

$$B_2^* \; := \; 2 \, . \, a^* \, . \, w \, . \, (K \, . \, w_x \, . \, \nu_1 + w_y \, . \, \nu_2) - (K \, . \, a_x^* \, . \, \nu_1 + a_y^* \, . \, \nu_2) \, . \, w^2$$

and the quadratic form

$$Q^* \; := \; Q^*(w_x, w_y)$$

with respect to w_x , w_y :

$$Q^* := (b^*.\nu_1 - c^*.\nu_2) . K . w_x^2 + 2 . (b^*.\nu_2 + c^*.K.\nu_1) . w_x . w_y + (-b^*.\nu_1 + c^*.\nu_2) . w_y^2 .$$

FRANKL CASE:

$$L^*w = f \quad , \quad w\big|_G = 0 .$$

FIRST: On the boundary G :

$$J_1^* = 0 \quad , \quad J_2^* = 0 \quad , \quad \text{and}$$

$$J_3^* = \oint_G (b^*.\nu_1 + c^*.\nu_2) . (K.\nu_1^2 + \nu_2^2) . N^2 . ds$$

(as $w_x = N.\nu_1$, $w_y = N.\nu_2$ on G , N : normalizing factor)

or

$$J_3^* = \int_{g_1 \cup g_2} (b^*.\nu_1 + c^*.\nu_2) . (K.\nu_1^2 + \nu_2^2) . N^2 . ds$$

(as g_3 : characteristic : $K.\nu_1^2 + \nu_2^2 = 0$ on g_3)

or

$$J_3^* \geqq 0$$

if conditions

$b^*.\nu_1 + c^*.\nu_2 \geqq 0$ on $g_1 \cup g_2$, and

$K.\nu_1^2 + \nu_2^2 > 0$ on g_2 (: non-characteristic) hold, as $K.\nu_1^2 + \nu_2^2 > 0$ on g_1 (as $K > 0$ on g_1) , or equivalently if conditions:

(R_1F) : $$b^*.dy - c^*.dx \geq 0 \quad \text{on} \quad g_1$$

" star-likedness "

(R_2F) : $$\begin{cases} b^*.dy - c^*.dx \geq 0 & \text{on} \quad g_2 \quad (\text{ : non-characteristic }) \\ 0 < \dfrac{dy}{dx} < \dfrac{1}{\sqrt{-K}} & \text{on} \quad g_2 \end{cases}$$

hold.

SECOND: In the domain D :

Denote

$$D_1 = D \cap \{y > 0\} , \quad D_2 = D \cap \{y < 0\}$$

$$(OA) := D \cap \{y = 0\}$$

Then

$$D = D_1 \cup D_2 \cup (OA)$$

Note: These D_1 , D_2 are different from the D_1 , D_2 ($:= \frac{\partial}{\partial x}$, $:= \frac{\partial}{\partial y}$, respectively) .

From

$$\left(\sqrt{\mu} . |a| - \frac{1}{\sqrt{\mu}} . |b| \right)^2 \geq 0 , \quad \mu > 0$$

we get

$$\mu . a^2 + \frac{1}{\mu} . b^2 \geq 2 . |a . b| \ , \quad \mu > 0 \ .$$

But

$$2 . M^* w . L^* w = 2 . (a^* . w + b^* . w_x + c^* . w_y) . L^* w$$

$$= 2 . (a^* . w) . (L^* w) + 2 . (b^* . w_x) . (L^* w) + 2 . (c^* . w_y) . (L^* w) \ .$$

Therefore

$$J^* \leq \iint_D 2 . |M^* w \ . \ L^* w| \ . \ dx\, dy$$

$$\leq \iint_D [\ 2 . |a^* . w| . |L^* w| + 2 . |b^* . w_x| . |L^* w| + 2 . |c^* . w_y| . |L^* w|\ ] \ . \ dx\, dy$$

$$\leq \iint_D \{\ [\ \mu_1 \ . \ (a^* . w)^2 + \frac{1}{\mu_1} \ . \ (L^* w)^2\ ] + [\ \mu_2 . (b^* . w_x)^2 + \frac{1}{\mu_2} \ . \ (L^* w)^2\ ]$$

$$+ [\ \mu_3 \ . \ (c^* . w_y)^2 + \frac{1}{\mu_3} \ . \ (L^* w)^2\ ]\ \} \ . \ dx\, dy$$

or

$$J^* \leq \iint_D [\ \mu_1 \ . \ (a^*)^2 \ . \ w^2 + \mu_2 \ . \ (b^*)^2 \ . \ (w_x)^2 + \mu_3 \ . \ (c^*)^2 \ . \ (w_y)^2\ ] \ . \ dx\, dy$$

$$+ \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}\right) . \ \|L^* w\|_o^2 \quad ; \quad \mu_1 , \mu_2 , \mu_3 :> 0 \ .$$

But

$$\boxed{J^* \geq I_1^* + I_2^*}$$

(as $J_1^* = J_2^* = 0$, and $J_3^* \geq 0$ from conditions $(R_1F) - (R_2F)$)

Therefore:

$$I_1^* + I_2^* \leq \iint_D [\, \mu_1.(a^*)^2.w^2 + \mu_2.(b^*)^2.w_x^2 + \mu_3.(c^*)^2.w_y^2 \,] \,.\, dx\, dy + C_1^2.\|L^*w\|_o^2 \,,$$

where

$$C_1 = \sqrt{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}} \; := \; \text{const.} \; :> 0$$

Thus

$$(I_1^* - \iint_D \mu_1.(a^*)^2.w^2.dx\, dy) + [\, I_2^* - \iint_D (\mu_2.(b^*)^2.w_x^2 + \mu_3.(c^*)^2.w_y^2) \,.\, dx\, dy \,]$$

$$\leq C_1^2 \,.\, \|L^*w\|_o^2 \quad .$$

<u>Therefore</u>:

$$\iint_D \beta_1.w^2.dx\, dy \; + \; \iint_D (\beta_2.w_x^2 - 2.B^*.w_x w_y + \beta_3.w_y^2).dx\, dy \leq C_1^2.\|L^*w\|_o^2 \;,$$

where:

$$\beta_1 \; := \; A_1^* - \mu_1 \,.\, (a^*)^2 \quad ,$$

$$\beta_2 \; := \; A_2^* - \mu_2 \,.\, (b^*)^2 \quad ,$$

$$\beta_3 \; := \; A_3^* - \mu_3 \,.\, (c^*)^2 \quad .$$

<u>LEMMA</u>.

<u>Denote</u>

$$Q_k \; := \; k_1 \,.\, w_x^2 + 2 \,.\, k_2 \,.\, w_x \,.\, w_y + k_3 \,.\, w_y^2 \quad \underline{\text{in}} \quad D \;.$$

Assume

$$\mathcal{D} := k_1 . k_3 - k_2^2 \geqq \delta > 0 \quad \text{in} \quad D$$

where

$$k_1 := k_1(x,y) \; ,$$

$$k_2 := k_2(x,y) \; ,$$

$$k_3 := k_3(x,y)$$

given functions of x,y in D .

Besides assume

$$2 . k := k_1 + k_3 \geqq \varepsilon > 0$$

$$k_1 \geqq 0$$

$$k_3 \geqq 0$$

in D .

Then

$$Q_k \geqq k . [1 - \sqrt{1 - \frac{\mathcal{D}}{k^2}}] . (w_x^2 + w_y^2) \quad \text{in} \quad D \; .$$

APPLICATION:

Take:

$$k_1 := \beta_2$$

$k_2 \; := \; - B^*$

$k_3 \; := \; \beta_3$

$\mathcal{D} \; := \; \beta_2 \cdot \beta_3 - (B^*)^2$

Assume <u>conditions in D</u> :

[R_1F] : $$\beta_1 \geqq \delta_1 > 0$$

[R_2F] : $$\mathcal{D} \geqq \delta > 0$$

[R_3F] : $$\begin{cases} \beta_2 \geqq 0 \;, \quad \beta_3 \geqq 0 \\ \beta_2 + \beta_3 \geqq \varepsilon > 0 \end{cases}$$

Then

$$\iint_D \beta_1 . w^2 . dx . dy + \iint_D (\beta_2 . w_x^2 - 2 . B^* . w_x . w_y + \beta_3 . w_y^2) \, . \, dx\, dy \geqq C_2^2 \, . \, \|w\|_1^2 \quad ,$$

$$C_2 \; := \; \text{const} \, :> 0 \quad ,$$

$$\|w\|_1^2 = \iint_D (w^2 + w_x^2 + w_y^2) \, . \, dx\, dy \quad .$$

Therefore

[AP] : $$\|w\|_1 \leqq C \, . \, \|L^* w\|_o \quad ,$$

$$C = c_2 / c_1 \; := \; \text{const.} > 0 \quad .$$

REMARKS:

1). If

$$\beta_2 = \beta_3 = B^* := 0$$

Then

$$\mathcal{D} = 0 \quad ,$$

and a-priori estimete [AP] is replaced by a-priori estimate

(AP) : $$\|w\|_o \leq C \cdot \|L^* w\|_o \quad .$$

2). If

$$B^* := 0$$

$$\beta_2 , \beta_3 : > 0$$

Then an a-priori estimate of the form [AP] holds immediately (without employing above Lemma) .

In fact,

$$B^* := 0$$

if we choose:

$$b^* := b^*(x) \quad , \quad c^* := c^*(y)$$

3). If

$$r := \text{const.} \quad , \quad a^* := a^*(y)$$

then

$$A_1^* = (2a^* - b_x^* - c_y^*) \, . \, r + (a^*)''$$

PROOF OF LEMMA:

$$\begin{vmatrix} k_1 - \lambda & k_2 \\ k_2 & k_3 - \lambda \end{vmatrix} = 0 \quad ,$$

where

λ : eigenvalues of matrix

$$[\,M\,] \; : = \begin{pmatrix} k_1 & k_2 \\ k_2 & k_3 \end{pmatrix}$$

of the quadratic form Q_K .

Then

$$\lambda^2 - (k_1 + k_3) \, . \, \lambda + \mathcal{D} = 0$$

or

$$\lambda_{1,2} \; : = \frac{2.k \pm \sqrt{(2.k)^2 - 4.\mathcal{D}}}{2}$$

$$: = k \pm \sqrt{k^2 - \mathcal{D}}$$

$$: = k \, . \, [\; 1 \pm \sqrt{1 - \frac{\mathcal{D}}{k^2}} \;]$$

But

$$\lambda_{min} : = \min.(\lambda_1,\lambda_2) : = k . [1 - \sqrt{1 - \frac{\mathcal{D}}{k^2}}] .$$

But

$$Q_k \geqq \lambda_{min} . (w_x^2 + w_y^2) ,$$

completing the proof of Lemma.

<u>REMARK</u>:

If

$$\mathcal{D} = 0$$

then

$$\lambda_{min} : = 0 ,$$

and

$$Q_k \geqq 0 .$$

In this case we have an a-priori estimate of the form (AP) :

$$\|w\|_o \leqq C . \|L^* w\|_o .$$

In fact, in this case

$$\lambda_1 = 0 , \lambda_2 = 2.k$$

and

$$Q_k : = (\sqrt{k_1} . w_x + \sqrt{k_3} . w_y)^2 : \geq 0 \qquad (: = \text{min. eigenvalue}) ,$$

where

$$\sqrt{k_1} \cdot \sqrt{k_3} = k_2$$

as

$$\mathcal{D} \ := k_1 \cdot k_3 - k_2^2 = 0$$

TRICOMI CASE:

$$L^* w = f \quad , \qquad w \big|_{g_1 \cup g_3} = 0$$

There are two main differences here from Frankl case:

FIRST: g_2 is a characteristic.

SECOND: $w = 0$ only on $g_1 \cup g_3$.

The existence of a weak solution of Problem (EQ) & (B) can be found if we assume

CONDITIONS ON BOUNDARY G :

$(R_1 T)$: $$b^* \cdot dy - c^* \cdot dx \geqq 0 \qquad \text{on} \qquad g_1$$

" star-likedness "

(R_1F) on g_1 , (R_2F) on g_3 (: non-characteristic) ,

and

$[R_1F]$ - $[R_3F]$ in D .

Note: Here g_2 is characteristic.

On the other hand, the adjoint boundary value Problem (Tricomi case) :

(AQ) : the same as above, and

$$[AT] : \quad w = 0 \quad \text{on} \quad g_1 \cup g_2 .$$

In this case we assume conditions exactly the same as the conditions $[R_1F]$ - $[R_3F]$ in D , but on boundary G we assume here the following new conditions instead :

$[R_1T]$: $b^*.dy - c^*.dx \geq 0$ on g_1 (this is the same as (R_1T)) " star-likedness "

$$[R_2T] : \begin{cases} (b^* + c^* . \sqrt{-K}.r \leq 0 & \text{on} \quad g_3 \\ b^* - c^* . \sqrt{-K} \geq 0 & \text{on} \quad g_3 \\ a^*_x . \sqrt{-K} - a_y + \dfrac{a^* . K'}{4 . (-K)} \leq 0 & \text{on} \quad g_3 . \end{cases}$$

Note: Here both g_2 , g_3 : are characteristics.

APPLICATION OF THE ENERGY INTEGRAL METHOD SEPARATELY IN D_1 AND D_2 :

Denote

$$\overline{D}_1 = \overline{D} \cap \{y > 0\} \quad ,$$

$$\overline{D}_2 = \overline{D} \cap \{y < 0\} \quad .$$

Assume

$$a^* \in C^2(\overline{D}_1) \cap C^2(\overline{D}_2) \quad ,$$

$$b^* \in C^1(\overline{D}_1) \cap C^1(\overline{D}_2) \quad ,$$

$$c^* \in C^1(\overline{D}_1) \cap C^1(\overline{D}_2) \quad .$$

Applying the energy integral method separately in D_1 and D_2 we get:

$$J^*_{D_1} = 2.\langle M^*w \,,\, L^*w \rangle_{oD_1} \;:=\; \iint_{D_1} 2.M^*w.L^*w.dx\,dy \quad ,$$

$$J^*_{D_2} = 2.\langle M^*w \,,\, L^*w \rangle_{oD_2} \;:=\; \iint_{D_2} 2.M^*w.L^*w.dx\,dy \quad .$$

Employing Green's theorem in each case and then adding side by side we get:

$$J^*_{D_1} + J^*_{D_2} \;:=\; \iint_{D_1 \cup D_2} A^*_1.w^2.dx\,dy$$

$$+ \iint_{D_1 \cup D_2} (\, A^*_2.w_x^2 - 2.B^*.w_x.w_y + A^*_3.w_y^2).dx\,dy$$

$$+ J_1^* + J_2^* + J_3^*$$

$$+ \int_0^1 [((a^*_{y+} - a^*_{y-}) - (c^*_+ - c^*_-) . r).w^2$$

$$- 2.(a^*_+ - a^*_-).w.w_y - 2.(b^*_+ - b^*_-).w_x.w_y$$

$$- (c^*_+ - c^*_-).w_y^2] . dx \quad .$$

<u>REMARKS</u>:

1). Because of the last integral (: $\int_0^1$) we have to assume the following <u>additional condition</u> (to all the above cases concerning uniqueness of quasi-regular solutions or existence of weak solutions) :

$$\text{(KM)} \quad : \quad \begin{cases} (a^*_{y_+} - a^*_{y_-}) - (c^*_+ - c^*_-).r \geqq 0 \quad , \\ c^*_+ - c^*_- \leqq 0 \quad , \\ a^*_+ - a^*_- = 0 \quad , \\ b^*_+ - b^*_- = 0 \quad , \end{cases}$$

for all $x : 0 \leqq x \leqq 1$,

where

$$(\ )_+ = \lim_{\to o^+} (\ ) ,$$

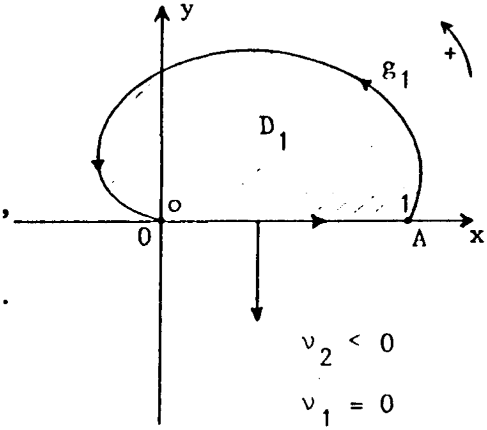
$$(\ )_- = \lim_{\to o^-} (\ ) .$$

2). On $OA \subset \partial D_1$:

$B^*_{1+} := (c^*_+ . \nu_2).r$,

$B^*_{2+} := 2.a^*_+.\nu_2.w.w_y - a^*_{y_+}.\nu_2.w^2$,

$Q^*_+ := 2.b^*_+.\nu_2.w_x.w_y + c^*_+.\nu_2.w_y^2$.



$\nu_2 < 0$

$\nu_1 = 0$

$K(0) = 0$

$\nu_2.ds = -dx$

$G_1 = \partial D_1$

Therefore

$$\text{(i)} : \int_{OA} [\ (c^*_+.r.w^2) + (2.a^*_+.w.w_y - a^*_{y_+}.w^2) + (2.b^*_+.w_x.w_y + c^*_+.w_y^2)\ ].\nu_2.ds := \int_o^1 (\cdot) .$$

Similarly On $AO \subset \partial D_2$:

$B^*_{1-} := (c^*_- . \nu_2).r$,

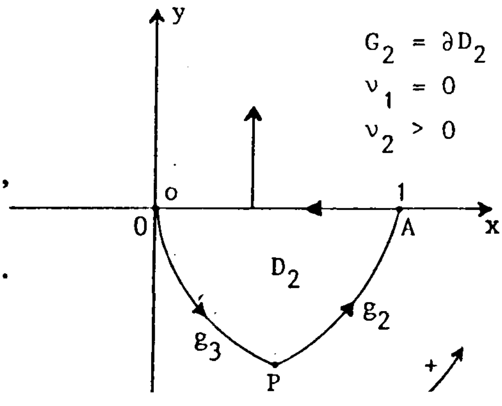
$B^*_{2-} := 2.a^*_-.\nu_2.w.w_y - a^*_{y_-}.\nu_2.w^2$,

$Q^*_- := 2.b^*_-.\nu_2.w_x.w_y + c^*_-.\nu_2.w_y^2$.

$G_2 = \partial D_2$

$\nu_1 = 0$

$\nu_2 > 0$



Therefore

$$ii) : \int_{AO} [(c_-^*.r.w^2) + (2.a_-^*.w.w_y - a_{y_-}^*.w^2) + (2.b_-^*.w_x.w_y + c_-^*.w_y^2)].\nu_2.ds := \int_1^o (\cdot) = -\int_o^1 (\cdot).$$

Adding (i) and (ii) replacing $\nu_2.ds = -dx$

$$\int_o^1 [((a_{y_+}^* - a_{y_-}^*) - (c_+^* - c_-^*).r).w^2 - 2.(a_+^* - a_-^*).w.w_y - 2.(b_+^* - b_-^*).w_x.w_y - (c_+^* - c_-^*).w_y^2] . dx .$$

3). Choose:

$$a^* = \begin{cases} -y & \text{if} \quad y \geq 0 \\ y & \text{if} \quad y \leq 0 \end{cases} := -|y| \quad \text{in} \quad D .$$

and

b^* , c^* so that :

$$b_+^* = b_-^* , \quad c_+^* = c_-^* .$$

Then we see that

$$a_+^* = a_-^* := 0 ,$$

but

$$a^*_y = \begin{cases} -1 & \text{if} \quad y \geq 0 \\ 1 & \text{if} \quad y \leq 0 \end{cases}$$

so that

$$a^*_{y_+} - a^*_{y_-} \; := \; -2 \; (: \; 0)$$

and condition (KM) fails to hold.

But a^* has to be chosen so that it is a $C^2(\cdot)$ function. In our choice above a^* is not $C^2(\cdot)$.

4). The above additional condition (KM) is very important especially if the considered equation has discontinuous coefficients. In particular, in this case (with discontinuity) the energy integral method must be applied separately in D_1 and D_2.

A UNIQUENESS THEOREM IN A THREE DIMENSIONAL REGION

In 1986 J.M. Rassias (Comp. Rend. Acad. Bulg. Sci., 39, 1986, 29-32) imposed the Bi-hyperbolic Bitsadze-Lavrentiev-Rassias equation

$$(*) \qquad Lu = \mathrm{sgn}(z).(u_{xx} - u_{yy}) + u_{zz} + r(x,y,z).u = f(x,y,z) ,$$

and established uniqueness results for quasi-regular-solutions. In particular, he considered the domain G in $\mathbb{R}^3$, bounded by the surfaces:

$S_3^y : \; y + 1 = (x^2+z^2)^{1/2}$, $\Sigma_4^y : \; y - 1 = -(x^2+z^2)^{1/2}$ for $z > 0$, and

$S_4^x :\ x - 1 = -(y^2+z^2)^{1/2}$, $\overset{x}{\underset{3}{\Sigma}} :\ x + 1 = (y^2+z^2)^{1/2}$ for $z < 0$, such that the boundary ∂G of G is given by

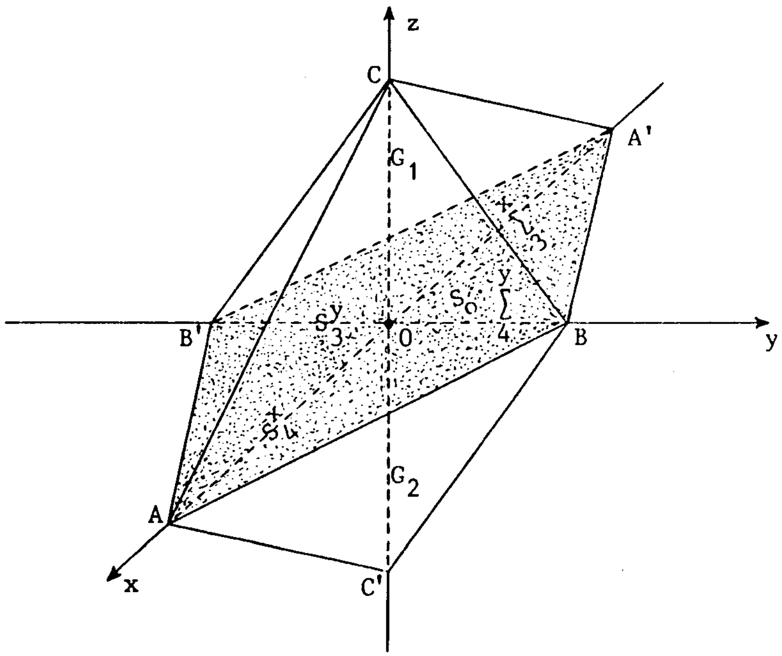
$$\partial G = S_4^x \cup \overset{x}{\underset{3}{\Sigma}} \cup \overset{y}{\underset{4}{\Sigma}} \cup S_3^y \ .$$

Note

$$S_3^y \cap \overset{y}{\underset{4}{\Sigma}} = (AA'C) \ , \quad S_4^x \cap \overset{x}{\underset{3}{\Sigma}} = (BB'C')$$

and all the above surfaces intersect the $\{x,y\}$-plane at (ABA'B'). Besides, the surface $S_o = (ABA'B') := \{ (x,y) \in \mathbb{R}^2 : |x| + |y| \leq 1 \}$ is a parabolic degenerate surface for equation $(\ast)$. Finally,

G_1 : denotes that part of G above S_o (for $z > 0$) : $= G \cap \{z > 0\}$, and
G_2 : denotes that part of G below S_o (for $z < 0$) : $= G \cap \{z < 0\}$.



Assume conditions

$$r \in C^1(G_i) , \quad f \in C^o(G_i) , \quad i = 1,2,$$

$$r_x - r_y \geq 0 \text{ in } G_i , \text{ and } r \leq 0 \text{ on } S_4^x \cup S_3^y .$$

In addition, assume boundary condition

$$(**) \qquad u = 0 \quad \text{on} \quad \sum_3^x \cup \sum_4^y .$$

Finally Rassias proved: Assume the above domain $G \subset \mathbb{R}^3$ and conditions. Then Problem (*) and (**) has at most one quasi-regular solution u in G.

Note: That the case:

$$Lu = K(z).(u_{xx} - u_{yy}) + u_{zz} + r(x,y,z).u = f(x,y,z) ,$$

$$K(z) \gtreqless 0 \quad \text{whenever} \quad z \gtreqless 0 , \quad K \in C^2(\cdot) ,$$

was investigated through the doctoral dissertation of J.M. Rassias (U.C. - Berkeley, 1977).

MIXED TYPE EQUATIONS (COLLECTION OF RESULTS)

In the same year J.M. Rassias (BSB B.G. Teubner Verlagsgesellschaft, " Teubner-Texte zur Mathematik ", Leipzig, 90, 1986) collected most of the results on mixed type equations with applications in fluid dynamics. This collection contains significant results by the expert researchers in the field of partial differential equations of mixed type: E. Ammicht & R.J. Weinacht (Newark), K.I. Babenko (Moscow), R.G. Barantsev (Leningrad), Chiu-Chun Chang (Taiwan), I.A. Chernov (Saratov), L.I. Chibrikova & N. B. Pleshchinskii (Kazan), G.C. Dong Hangchow), M.Y. Chi (Wuhan), T.V. Gramchev (Sofia), Chaohao Gu & Jiaxing Hong (Shanghai), O. Jokhadze (Tbilisi), A.I. Kozhanov (Novosibirsk), M. Kracht (Düsseldorf), E. Kreyszig (Ottawa), A.G. Kuz'min (Leningrad), S.G. Mikhlin (Leningrad), A.M. Nakhushev (Nal'chik), S. Nocilla (Torino), I.E. Pleshchikskaya (Kazan), A.G. Podgaev (Novosibirsk), Ji Xinhua & Chen Dequan (Beijing), V. I. Zhegalov (Kazan), and the Editor (Athens). The contributions to this collection of works follow along the lines of the important work by F.G. Tricomi on boundary value problems of mixed type. Their originality and contact with many problems in fluid mechanics make this collection a most useful source of information about equations of mixed type. The topics covered include axially symmetric bodies, the Tricomi equation, the, Bitsadze-Lavrentiev equation, nonlinear problems, Galerkin's method, maximum principle, geometry, and gas dynamics.

Through the above-mentioned collection of works J.M. Rassias (269--279) established uniqueness results for <u>regular solutions</u>.

THE CHAPLYGIN EQUATION

In 1988 J.M. Rassias (Comp. Rend. Acad. Bulg. Sci., 1989: to appear) considered

$$(E) \quad : \quad Lu \equiv K(x_{n+1}) \cdot \sum_{i=1}^{n} u_{x_i x_i} + u_{x_{n+1} x_{n+1}} + r(x).u = f(x) \quad ,$$

$$x = (x_1, x_2, \ldots, x_n, x_{n+1})$$

$$K(x_{n+1}) \gtreqless 0 \quad \text{for} \quad x_{n+1} \gtreqless 0$$

$$K'(x_{n+1}) > 0 \quad .$$

$$K \in C^2(\cdot) \ , \quad r \in C^1(\cdot) \ , \quad f \in C^o(\cdot) \quad .$$

Take as domain $D \subset \mathbb{R}^{n+1}$, $n \geq 2$ a simply connected multi-dimensional region bounded for $x_{n+1} > 0$ by a smooth hypersurface S_1 intersecting the hyperplane $\pi : x_{n+1} = 0$ at

$$S_o : \sum_{i=1}^{n} x_i^2 = 1 \quad ,$$

and for $x_{n+1} < 0$ by two hypersurfaces S_3 , S_4 , so that S_3 is a smooth non-characteristic conic hypersurface intersecting the hyperplane π at S_o with vertex on the x_{n+1} - axis, and S_4 is a characteristic conic hypersurface intersecting S_3 at $S_o'(: x_{n+1} = t_{n+1}^o < 0)$ with vertex at the origin 0 , so that:

$$S_4 \; : \; -\Phi = \Big(\sum_{i=1}^{n} x_i^2 \Big)^{1/2} + \int_0^{x_{n+1}} \sqrt{-K(s)} \, . \, ds = 0 \; ,$$

where the " $-\Phi$ " is used because $\nabla\Phi$ has to be outward on S_4 ; $\nu = (\nu_1, \nu_2, \dots, \nu_n, \nu_{n+1})$ is an outer normal vector on the boundary $G = \partial D := S_1 \cup S_4 \cup S_3$ such that on S_4 :

$$\nu = \frac{\nabla\Phi}{2 . \sqrt{-K}} \; .$$

It is clear that on S_4 :

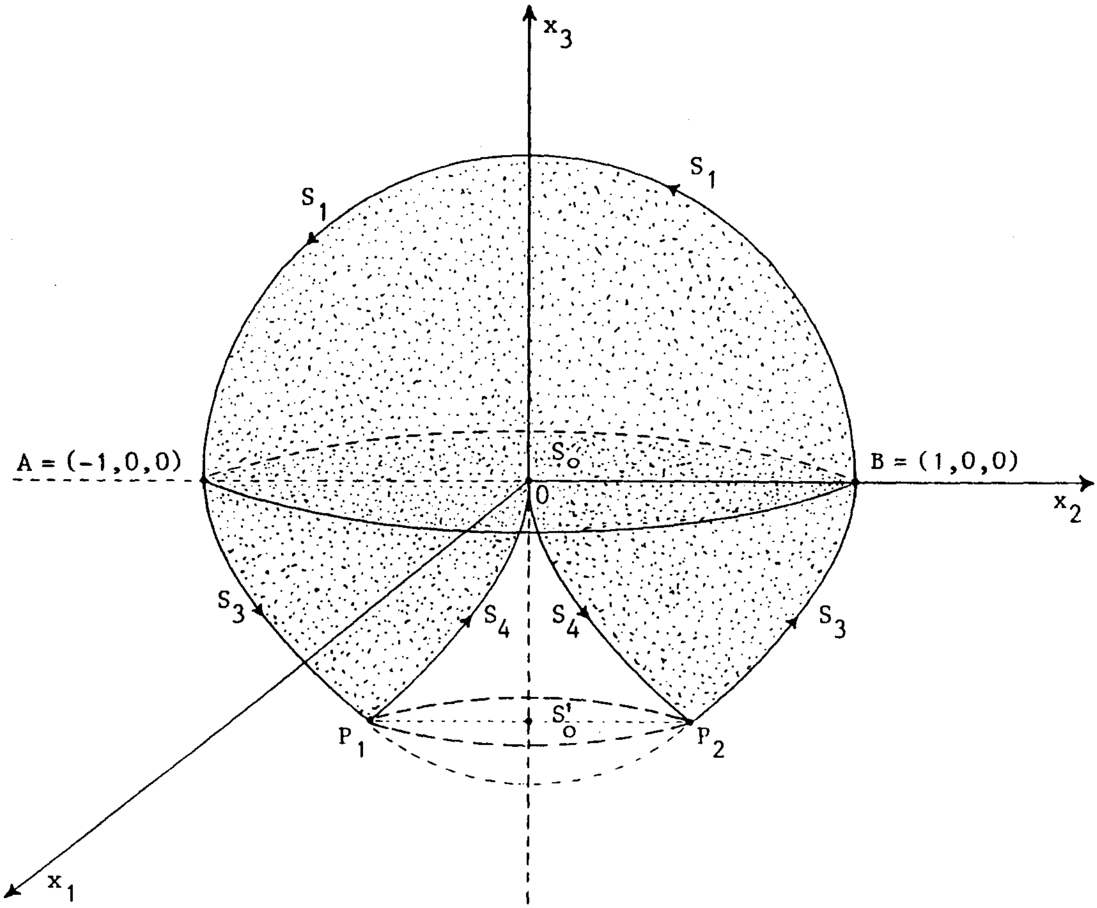
$$\nu_i = \frac{1}{2 \, . \, \sqrt{-K} \, . \, \int_0^{x_{n+1}} \sqrt{-K(s)} \, . \, ds} \, . \, x_i \; , \qquad i = 1, 2, \dots, n \; ,$$

$$\nu_{n+1} = -\frac{1}{2} \; .$$

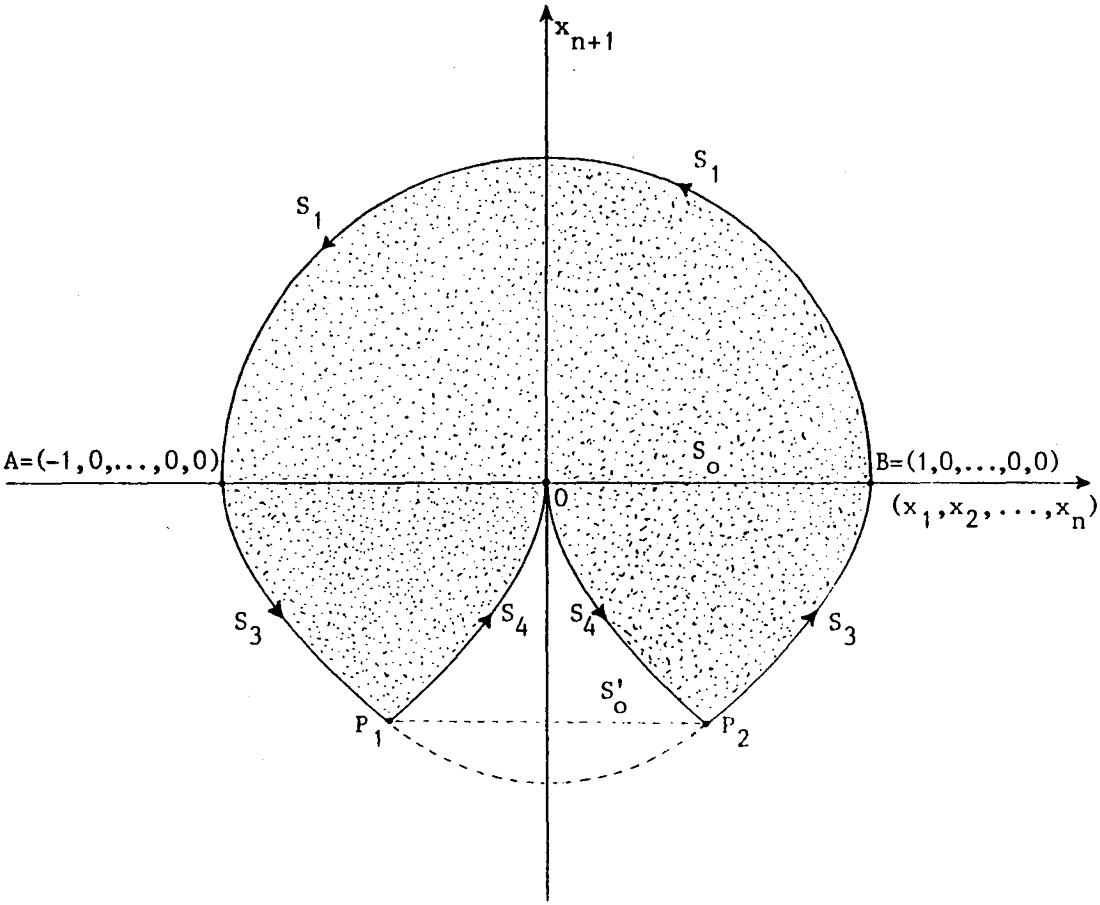
Assume <u>boundary conditions</u>

(BC) : $\qquad u = 0 \quad \text{on} \quad S_1 \cup S_3 \; .$

GRAPH FOR 3-DIMENSIONAL CASE (: n = 2) :



GRAPH FOR (n+1) - DIMENSIONAL CASE (: n ≥ 2) :



MIXED TYPE PROBLEM or PROBLEM (MF) :

Consists in finding a functión $u = u(x)$ which satisfies equation (E) and boundary consitions (BC) in D .

UNIQUENESS OF QUASI-REGULAR SOLUTIONS

Consider operator M :

$$Mu = a.u + \sum_{n=1}^{n} b_i . u_{x_i} + c . u_{x_{n+1}} \quad \text{in} \quad D ,$$

where

$$(C) \quad : \quad \begin{cases} a = a(x_{n+1}) \; : = \dfrac{n-1}{2} + \dfrac{1}{4} . \dfrac{K'}{K} . c(x_{n+1}) \; , \\ b_i = b_i(x_i) \; : = x_i \; , \quad i = 1,2 , \ldots , n \; , \\ c = c(x_{n+1}) \; : = \dfrac{\int_0^{x_{n+1}} \sqrt{|K(t)|} \; . \; dt}{\sqrt{|K(x_{n+1})|}} \end{cases}$$

in D .

Assume conditions:

$$(R_1) \quad : \quad \sum_{i=1}^{n} x_i \, \nu_i + c \, . \, \nu_{n+1} \geq 0 \quad \text{on} \quad S_1 \cup S_3$$

Note: If S_3 is characteristic then (R_1) is assumed only on S_1 .

$$(R_2) \quad : \quad a'' - (r.c)_{x_{n+1}} + 2.r.(a - \tfrac{n}{2}) - \sum_{i=1}^{n} x_i . r_{x_i} \geq 0 \quad \text{in} \quad D \ ,$$

$$(R_3) \quad : \quad a' \geq 0 \quad \text{on} \quad S_4 \ ,$$

$$(R_4) \quad : \quad K . \sum_{i=1}^{n} \nu_i^2 + \nu_{n+1}^2 > 0 \ , \quad \nu_{n+1} < 0 \quad \text{on} \quad S_3 \ ,$$

where (') means differentiation with respect to x_{n+1} .

Denote <u>in D</u> :

$$A_o = a'' + 2.r.a - [\sum_{i=1}^{n} (r.b_i)_{x_i} + (r.c)_{x_{n+1}}] \ ,$$

$$A_i = -2a.K - (b_i)_{x_i}.K + \sum_{j \neq i}^{n} (b_j)_{x_j}.K + (c.K)_{x_{n+1}} \ , \quad i = 1,2, \ldots, n \ ,$$

$$A_{n+1} = -2a + \sum_{i=1}^{n} (b_i)_{x_i} - c' \ , \quad \text{and}$$

<u>on G (: = ∂D)</u> :

$$B_o = r.(\sum_{i=1}^{n} b_i . \nu_i + c . \nu_{n+1}) \ ,$$

$$B_j = K. (b_j . \nu_j - \sum_{i \neq j}^{n} b_i . \nu_i - c . \nu_{n+1}) \ , \quad j = 1,2, \ldots, n$$

$$B_{n+1} = c . \nu_{n+1} - \sum_{i=1}^{n} b_i . \nu_i \ , \quad B_{ij} = b_j . K . \nu_i + b_i . K . \nu_j \ , \ i \neq j := 1,2, \ldots, n \ ,$$

$$B^i = c.K.\nu_i + b_i . \nu_{n+1} \ , \quad i = 1,2, \ldots, n \ .$$

Assume two quasi-regular solutions u_1, u_2 exist for Problem (MT).

Then claim that

$$u = u_1 - u_2 = 0 \quad \text{in} \quad D \ .$$

Therefore

$$[E] : \quad Lu = K(x_{n+1}) . \sum_{i=1}^{n} u_{x_i x_i} + u_{x_{n+1} x_{n+1}} + r.u = 0 ,$$

and

$$[BC] : \quad u = 0 \quad \text{on} \quad S_1 \cup S_3 \ .$$

It is enough to show that

$$u = 0 \quad \text{on} \quad S_4 \ .$$

To prove this we apply Green's theorem in

$$0 = J = 2. \langle Mu, Lu \rangle_D = \int_D Mu.Lu.dx$$

and get

$$\begin{aligned} 0 = & \int_D (A_o.u^2 + \sum_{i=1}^{n} A_i.u_{x_i}^2 + A_{n+1}.u_{x_{n+1}}^2) . dx \\ & + \int_G [K . \sum_{i=1}^{n} 2a.u.u_{x_i}.\nu_i + (2a.u.u_{x_{n+1}} - a'.u^2).\nu_{n+1}] . dS \\ & + \int_C (B_o.u^2 + \sum_{j=1}^{n} B_j.u_{x_j}^2 + B_{n+1} u_{x_{n+1}}^2 + \sum_{i \neq j}^{n} B_{ij}.u_{x_i}.u_{x_j} \\ & \qquad + 2 . \sum_{i=1}^{n} B^i.u_{x_i x_{n+1}}) . dS \\ = & J_1 + J_2 + J_3 \ . \end{aligned}$$

Then it is clear that all integrals J_i , $i = 1,2,3$ are non-negative, completing the proof for uniqueness of quasi-regular solutions.

EXISTENCE OF WEAK SOLUTIONS

It remains to show the existence of a weak solution of Problem (MF). Assume additional conditions

$$\operatorname{sgn}(x_{n+1}) \cdot (K' - M_o) \geqq d_o \quad ,$$

$$M_o = x_{n+1}^2 - c_1 \cdot x_{n+1} + K'(0) \quad ,$$

$$d_o , c_1 := \text{const.} > 0 \quad ,$$

and

$$A_1^+ \geqq d_1 \; , \quad A_2^+ > d_2 \; , \quad A_3^+ > d_3 \quad ,$$

$$A_1^+ - R_o \cdot (a^+)^2 \geqq 0 \; , \quad A_2^+ - R_{n+1} \cdot (c^+)^2 > 0 \; , \quad A_3^+ - R_i \cdot (c_2 . x_i)^2 > 0 \; ,$$

$$i = 1,2, \ldots, n \quad ,$$

$$d_j := \text{const.} > 0 \; , \quad j = 1,2,3,$$

$$R_i := \text{const.} > 0 \; , \quad i = 1,2,3, \ldots, n \quad ,$$

$$c_2 := \text{const.} > 0 \quad ,$$

$$a^+ = M_o + (n-2) / 2 \cdot c_2 \; , \quad c^+ = d + K \; , \quad d := \text{const.} > 0 \quad ,$$

$$A_1^+ = 2 - (r.c^+)_{x_{n+1}} + 2r \,.\, (a^+ - \frac{n}{2} . c_2) - c_2 \,.\, \sum_{i=1}^{n} r_{x_i} . x_i \quad ,$$

$$A_2^+ = 2.K.(K' - M_o) + d.K' \quad , \quad A_3^+ = 2.(c_2 - M_o) - K' \qquad \text{in} \qquad D \quad .$$

Assume adjoint boundary condition

$$w = 0 \qquad \text{on} \qquad G \quad .$$

Besides assume condition

$$c_2 \,.\, \sum_{i=1}^{n} x_i . v_i + c^+ . v_{n+1} \geq 0 \qquad \text{on} \qquad S_1 \cup S_3 \quad .$$

It is enough then to show that the following a-priori estimate holds

$$\|w\|_1 \leq C \,.\, \|L^+ w\| \quad , \quad C := \text{const.} > 0 \quad ,$$

for all $w \in D(L^+) := \{ w \in C^2(D) \,,\ w = 0 \text{ on } G \}$.

Note: $\|\cdot\| = \|\cdot\|_o \,,\quad L^+ = L$.

To prove it he applied Green's theorem, Hahn-Banach theorem and Riesz representation theorem or a Criterion (necessary and sufficient conditions for existence of weak solutions). See: Ju.M. Berezanskii (Transl. Math. Mon., A.M.S., 1968) and the corresponding 2-dimensional case in this book for further techniques and for the statement of the said Criterion. Then

$$M^+w = a^+.w + c_2 . \sum_{i=1}^{n} x_i.w_{x_i} + c^+.w_{x_{n+1}} ,$$

$$2.< M^+w, L^+w >_D = I_1 + I_2 ,$$

where

$$I_1 = \int_D (\cdot) . dx \geq C_1 . \|w\|_1^2 , \qquad C_1 := \text{const.} > 0 ,$$

for all $w \in D(L^+)$, and

$$I_2 = \int_G (\cdot) . dS \geq 0 .$$

Thus the a-priori estimate holds and the proof for the existence of a weak solution of Problem (GM) is complete.

Therefore Rassias proved: Assume above domain D and conditions. Then Problem (MF) is well-posed in the sense that: there is at most one quasi-regular solution and a weak solution exists.

Note: That the uniqueness part was carried out at U.C. Berkeley (1977) through the doctoral dissertation of J.M. Rassias.

THE EXTENDED CHAPLYGIN EQUATION

In the same year J.M. Rassias (Comp. Rend. Acad. Bulg. Sci., 41 , 1988, 35-37) considered the extended Chaplygin equation

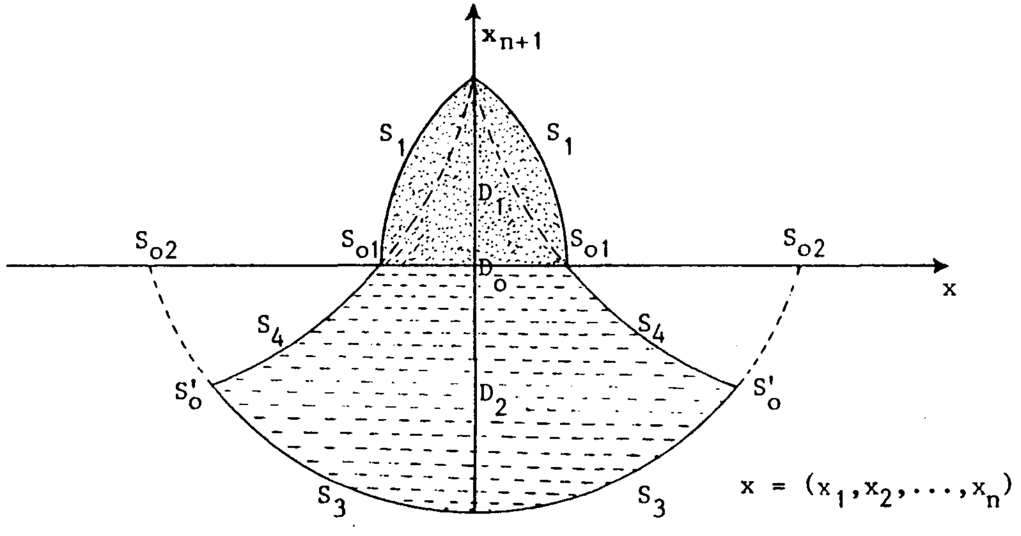
$$(*) \qquad Lu = \sum_{i=1}^{n} K_i(x_{n+1}).u_{x_i x_i} + u_{x_{n+1} x_{n+1}} + r(x).u = f(x) ,$$

where

$$x = (x_1, x_2, \ldots, x_n, x_{n+1}) , \quad K_i = K_i(x_{n+1}) \gtreqless 0 \quad \text{for} \quad x_{n+1} \gtreqless 0 ,$$

$$K_i'(x_{n+1}) > 0 , \quad i = 1, 2, \ldots, n .$$

Then he assumed the simply connected multi-dimensional "bell-shaped" region $D \in \mathbb{R}^{n+1}$, bounded for $x_{n+1} > 0$ by a smooth hypersurface S_1 intersecting the hyperplane: $x_{n+1} = 0$ at S_{o1}, and for $x_{n+1} < 0$ by two hypersurfaces S_3, S_4, so that S_3 is a smooth noncharacteristic conic hypersurface intersecting the hyperplane: $x_{n+1} = 0$ at S_{o2} with vertex on the x_{n+1}-axis, and S_4 is a truncated characteristic conic hypersurface intersecting S_3 at S_o' with vertex at the positive x_{n+1}-axis (only the truncated part of S_4 for $x_{n+1} < 0$ is considered).



The outer normal vector $v = (v_1, v_2, \dots, v_n, v_{n+1})$ on the boundary $G = \partial D = S_1 \cup S_4 \cup S_3$ is such that

$$v_{n+1} > 0 \quad \text{on} \quad S_1 \cup S_4 , \quad \text{and} \quad v_{n+1} < 0 \quad \text{on} \quad S_3 .$$

Denote

$$D_o = D \cap \{ x : x_{n+1} = 0 \} , \quad D_1 = D \cap \{ x : x_{n+1} \geq 0 \}, \quad D_2 = D \cap \{ x : x_{n+1} \leq 0 \}.$$

Take

$S_{o1} : \sum_{i=1}^{n} x_i^2 = m_1^2$, $\quad S_{o2} : \sum_{i=1}^{n} x_i^2 = m_2^2$, $\quad (m_1, m_2 := \text{const.} > 0 : m_1 < m_2)$, and truncated (for $x_{n+1} < 0$) characteristic

$$S_4 : \tilde{r} = \tilde{r}(\theta, \phi, x_{n+1}) = \{ P_1/\rho_1, P_2/\rho_2, \dots, P_n/\rho_n , x_{n+1} \} ,$$

where

$$P_1 = \cos\theta . \cos\phi_1 . \cos\phi_2 \dots \cos\phi_{n-2} , \quad P_2 = \sin\theta . \cos\phi_1 . \cos\phi_2 \dots \cos\phi_{n-2}$$

$$P_3 = \sin\theta . \cos\phi_1 \dots \cos\phi_{n-2} , \quad \dots , \quad P_n = \sin\phi_{n-2} ,$$

$$\phi = (\phi_1, \phi_2, \dots, \phi_{n-2}) , \quad \phi_j \in [0,\pi] , \quad j = 1,2, \dots, n-2 , \quad \theta \in [0,2\pi],$$

$$M_o = M_o(s,\theta,\phi) = \Big(- \sum_{i=1}^{n} K_i(s) . P_i^2 \Big)^{1/2} ,$$

and

$$\rho_i = 1 / \Big[m_1 + \int_o^{x_{n+1}} K_i(s) / M_o(s,\theta,\phi) . ds \Big] , \quad i = 1,2,\dots,n , \quad \rho_i > 0 .$$

Assume boundary condition

$$(**) \qquad u = 0 \quad \text{on} \quad S_1 \cup S_3 \ .$$

The Extended Mixed Type Boundary Value Problem, or Problem (EF): consists in finding a function $u = u(x)$ which satisfies equation $(*)$ and boundary condition $(**)$.

Assume conditions

$$K_i \in C^2(D) \ , \quad r \in C^1(D) \ , \quad \text{and} \quad f \in C^o(D) \ ,$$

$$r \geq 0 \quad \text{on} \quad S_4 \ , \quad r_{x_{n+1}} \leq 0 \quad \text{in} \quad D \ , \quad \text{and}$$

$$H = \sum_{i=1}^{n} K_i \cdot v_i^2 + v_{n+1}^2 < 0 \quad \text{on} \quad S_3 \ .$$

Therefore he proved the uniqueness of a quasi-regular solution of Problem (EM). In fact, denote $Mu = u_{x_{n+1}}$ in $D : A_o = -r_{x_{n+1}}$, $A_i = K_i'$, $i = 1,2,\ldots,n$, and on $G : B_o = r.v_{n+1}$, $B_j = -K_j.v_{n+1}$, $j = 1,2,\ldots,n$, $B_{n+1} = v_{n+1}$, $B^i = K_i.v_i$, $i = 1,2,\ldots,n$. Assume u_1, u_2 two quasi-regular solutions, then claim that $u = u_1 - u_2 = 0$ in D. To prove this fact he applied the energy integral method in D, where Mu is defined above and $Lu = 0$. Therefore, Green's theorem implies

$$0 = 2.\langle Mu,Lu\rangle_D = \int_D \left(A_o.u^2 + \sum_{i=1}^{n} A_i.u_{x_i}^2 \right).dx + \int_G \left(B_o.u^2 + \sum_{j=1}^{n} B_j.u_{x_j}^2 + B_{n+1}.u_{x_{n+1}}^2 + 2.\sum_{i=1}^{n} B^i.u_{x_i}.u_{x_{n+1}} \right).dS = J_1 + J_2 \ .$$

It is clear that both integrals J_i, $i = 1,2$, are positive. This completes the proof for the said uniqueness of u in D.

It remains to show the existence of a weak solution of Problem (EM). Assume additional conditions

$$c_1.K_i + c^+.K_i' > 0 , \quad i = 1,2,\ldots,n , \quad \text{in} \quad D_1 .$$

$$c^+ = c_1.x_{n+1} + c_2 \; (>0) , \quad c_1 := \text{const.} < 0 , \quad c_2 := \text{const.} > 0,$$

$$c_1.r + c^+.r_{x_{n+1}} \leqq 0 \quad \text{in} \quad D , \quad \text{and}$$

$$A_o^+ = -(r.c^+)_{x_{n+1}} \geqq 0 , \quad A_i^+ = (c^+.K_i)_{x_{n+1}} > 0 , \quad i = 1,2,\ldots,n ,$$

$$c_1 + R_{n+1}.(c_{n+1}^+)^2 < 0 , \quad R_{n+1} := \text{const.} > 0 , \quad \text{in} \quad D .$$

Assume adjoint boundary condition

$$w = 0 \quad \text{on} \quad G .$$

It is enough then to show that: A sufficient condition for the existence of a weak solution $L^2(D)$ of Problem (EM) is the a-priori estimate holds

$$\|w\|_1 \leqq C.\|L^+w\| , \quad \|w\|_1^2 = \int_D (w^2 + \sum_{i=1}^{n+1} w_{x_i}^2).dx , \quad \|w\|^2 = \|w\|_o^2 = \int_D w^2.dx$$

for all $w \in D(L^+)$, $C := \text{const.} > 0$.

In fact, $M^+w = c^+.w_{x_{n+1}}$, and

$$2.\langle M^+w, L^+w\rangle_D = I_1 + I_2 \quad ,$$

where

$$I_1 = \int_D (A_o^+ . w^2 + \sum_{i=1}^{n} A_1^+ . w_{x_i}^2 - c_1 . w_{x_{n+1}}^2).dx \quad ,$$

$$I_2 = \int_G (\sum_{j=1}^{n} B_j^+ . w_{x_j}^2 + B_{n+1}^+ . w_{x_{n+1}}^2 + 2 . \sum_{i=1}^{n} B^{+\,i} . w_{x_i} . w_{x_{n+1}}).dS \quad ,$$

where

$$B_j^+ = K_j . c^+ . v_{n+1} \; , \quad j = 1,2,\ldots,n \; , \qquad \text{and}$$

$$B_{n+1}^+ = c^+ . v_{n+1} \; , \quad B^{+\,i} = c^+ . K_i . v_i \; , \quad i = 1,2,\ldots,n \quad .$$

It is clear that

$$I_1 \geq C_1 . \|w\|_1^2 \; , \quad (\|w\| \leq \|w_1\|) \quad ,$$

for all $w \in D(L^+)$, $C_1 =:$ const. > 0 , and

$$I_2 = 0 \quad .$$

Thus the above a-priori estimate holds. Therefore the proof for the existence of a weak solution of Problem (EF) is complete.

Therefore Rassias proved: Assume above domain D and conditions. Then Problem (EF) is well-posed in the sense that: there is at most one quasi-regular solution and a weak solution exists.

THE NEW EXTENDED CHAPLYGIN EQUATION

In 1988 J.M. Rassias obtained a new result for well-posedness of a boundary value problem (in the sense that: there is at most one quasi-regular solution and a weak solution exists) for the new extended Chaplygin equation:

$$(*) \qquad Lu = \sum_{i=1}^{n} K_i(x_{n+1}).u_{x_i x_i} + u_{x_{n+1} x_{n+1}} + r(x).u = f(x) \quad ,$$

$$x = (x_1, x_2, \ldots, x_n, x_{n+1}) \quad ,$$

$$K_1(x_{n+1}) \gtreqless 0 \quad \text{for} \quad x_{n+1} \gtreqless 0 \quad ,$$

$$K_1'(x_{n+1}) > 0 \quad ,$$

$$K_j(x_{n+1}) > 0 \quad ,$$

$$K_j'(x_{n+1}) \geq 0 \quad , \quad j = 2,3,\ldots,n \quad ,$$

for any $x_{n+1} \in D$ (: = given domain) ,

$$K_1(m) \geq -1 \;, \quad \text{where} \quad m := \inf\{ x_{n+1} : x_{n+1} \in D \} \;.$$

Then he considered the simply connected domain $D \subset \mathbb{R}^{n+1}$, bounded for $x_{n+1} > 0$ by a smooth hypersurface S_1 intersecting the hyperplane: $x_{n+1} = 0$ at S_{o1} : $y = (x_2, x_3, \ldots, x_n) = -(x_2^o, x_3^o, \ldots, x_n^o) = -y^o$,

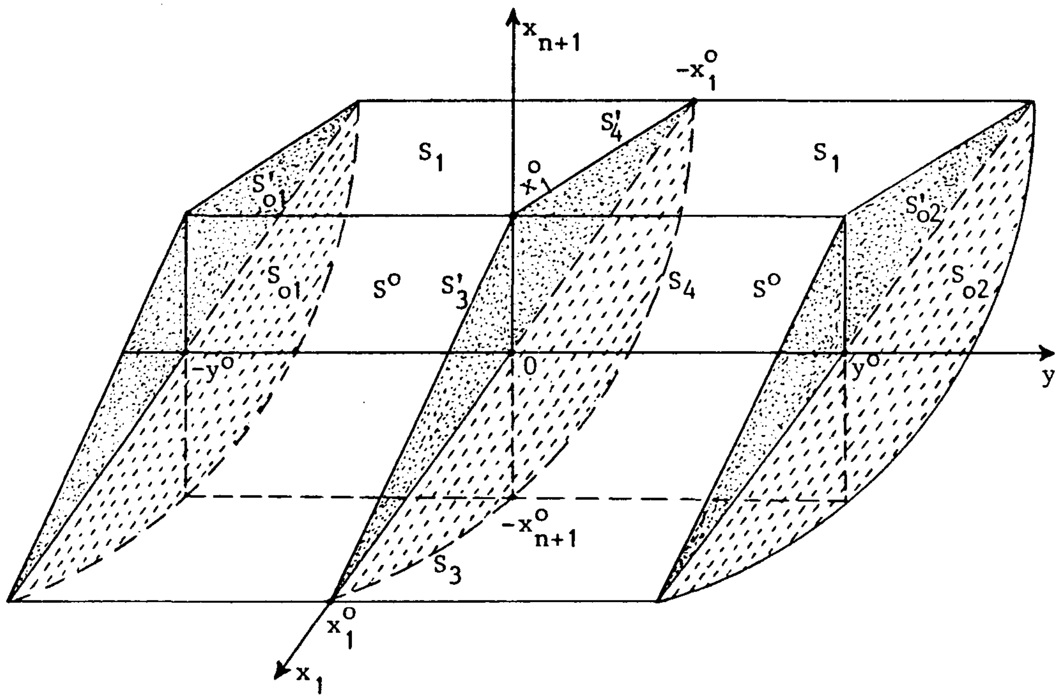
S_{o2} : $y = y^o$, x_j^o : = const. > 0 , $j = 2,3,\dots,n$, and two charac teristic hypersurfaces

$$S_3 : \Psi = x_1 - x_1^o - \int_o^{x_{n+1}} \sqrt{-K_1(s)} \,.\, ds = 0 \quad ,$$

$$S_4 : -\Phi = x_1 + x_1^o + \int_o^{x_{n+1}} \sqrt{-K_1(s)} \,.\, ds = 0$$

of equation (107), x_1^o : = const. > 0 , both intersecting the negative x_{n+1}-axis at

$$x_{n+1} = -x_{n+1}^o : \; x_1^o + \int_o^{-x_{n+1}^o} \sqrt{-K_1(s)} \,.ds = 0 \quad , \quad x_{n+1}^o \; : = \text{const.} > 0 \;.$$



The outer normal vector $v = (v_1, v_2, \ldots, v_n, v_{n+1})$ on the boundary $G = \partial D = S_1 \cup S_3 \cup S_4 \cup S_{o1} \cup S_{o2}$ is determined, as follows

$$v = \nabla\Psi / 2.\sqrt{-K_1} = (1/2.\sqrt{-K_1}, 0,0, \ldots, 0, -1/2) \quad \text{on} \quad S_3,$$

$$v = \nabla\Phi / 2.\sqrt{-K_1} = (-1/2.\sqrt{-K_1}, 0,0, \ldots, 0, -1/2) \quad \text{on} \quad S_4,$$

$$v = (0, -1/2.\sqrt{-K_1}, -1/2.\sqrt{-K_1}, \ldots, -1/2.\sqrt{-K_1}, 0) \quad \text{on} \quad S_{o1},$$

$$v = (0, 1/2.\sqrt{-K_1}, 1/2.\sqrt{-K_1}, \ldots, 1/2.\sqrt{-K_1}, 0) \quad \text{on} \quad S_{o2},$$

$$v = \nabla X / 2.\sqrt{K_1} \quad \text{on} \quad S_1 : X = X(x) = 0, \quad v_{n+1} \geq 0, \quad x_1.v_1 \geq 0 .$$

Then he considered

$$S_1 = S_3' \cup S_4' \cup S_{o1}' \cup S_{o2}',$$

$$S_{o1}' : Y = -y^o, \quad S_{o2}' : y = y^o,$$

$$S_3' : \Psi' = x_1 - x_1^o + x_{n+1} = 0, \quad S_4' : -\Phi' = x_1 + x_1^o - x_{n+1} = 0,$$

and S_1 intersects the positive x_{n+1} - axis at $x_{n+1} = x_1^o$.

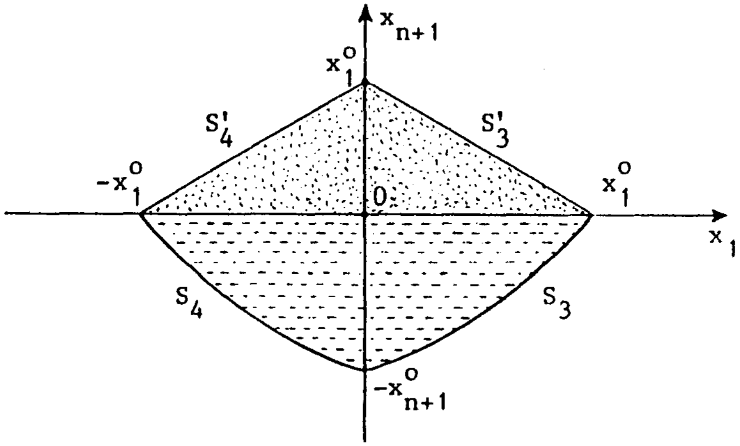
Therefore the outer normal vector on S_1 is:

$$v = \nabla\Psi' / 2.\sqrt{K_1} = (1/2.\sqrt{K_1}, 0,0, \ldots, 0, 1/2.\sqrt{K_1}) \quad \text{on} \quad S_3',$$

$$v = \nabla\Phi' / 2.\sqrt{K_1} = (-1/2.\sqrt{K_1}, 0, 0, \ldots, 0, 1/2.\sqrt{K_1}) \quad \text{on} \quad S'_4 ,$$

$$v = (0, -1/2.\sqrt{K_1}, -1/2.\sqrt{K_1}, \ldots, -1/2.\sqrt{K_1}, 0) \quad \text{on} \quad S'_{o1} ,$$

$$v = (0, 1/2.\sqrt{K_1}, 1/2.\sqrt{K_1}, \ldots, 1/2.\sqrt{K_1}, 0) \quad \text{on} \quad S'_{o2} .$$



Assume boundary condition

$$(**) \qquad u = 0 \quad \text{on} \quad S_1 \cup S_3 \cup S_{o1} \cup S_{o2}$$

<u>The New Extended Mixed Type Boundary Value Problem</u>, or <u>Problem (NT)</u> consists in finding a function $u = u(x)$ which satisfies equation $(*)$ and boundary condition $(**)$.

Assume conditions

$$K_i \in C^2(D) , \quad r \in C^1(D) , \quad f \in C^o(D) ,$$

$$r < 0 \quad \text{on} \quad S_4 , \quad 2r + (x_1 + x_1^o).r_{x_1} + x_{n+1}.r_{x_{n+1}} < 0 \quad \text{if} \quad x_{n+1} \geq 0 ,$$

$$r + (x_1 + x_1^o) \, . \, r_{x_1} < 0 \quad \text{if} \quad x_{n+1} \leq 0 \ .$$

Note: That above conditions can be replaced by conditions

$$r < 0 \ , \quad r_{x_1} \geq 0 \ , \quad r_{x_{n+1}} \leq 0 \quad \text{in} \quad D \ .$$

It is clear now by applying Green's theorem and conditions above that there exists at most one quasi-regular solution for the Problem (NT) .

To prove the existence of a weak solution of Problem (NT) assume conditions

$$r < 0 \quad \text{on} \quad S_3 \ , \quad 3r + (x_1 - \lambda).r_{x_1} + (x_{n+1} + \mu).r_{x_{n+1}} < 0 \quad \text{if} \quad x_{n+1} \geqq 0 \ ,$$

$$2r + (x_1 - \lambda).r_{x_1} + \mu.r_{x_{n+1}} < 0 \quad \text{if} \quad x_{n+1} \leq 0 \ ,$$

where

$$\lambda = x_1^o + \lambda_o \ , \quad \lambda_o := \text{const.} > 0 \ , \quad \mu := \text{const.} > 0 \ ,$$

$$-\lambda_o + \mu.\sqrt{-K_1} + \int_o^{x_{n+1}} \sqrt{-K_1(s)} \, . \, ds \leq 0 \quad \text{on} \quad S_3 \ ,$$

and

$$\lambda_o \leq \mu \quad \text{on} \quad S_3' \ ,$$

$$A_o^+ - R_o.(a^+)^2 \geq 0 \ , \quad A_1^+ - R_1.(b_1^+)^2 > 0 \ ,$$

$$A_j^+ > 0 \ , \quad j = 2,3,\dots,n \ , \quad A_{n+1}^+ - R_{n+1} \, . \, (c^+)^2 > 0 \ ,$$

where

$$R_i \ := \ \text{const.} > 0 \ , \quad i = 0,1,2,\ldots,n,n+1 \ ,$$

$$a^+ = -1/2 \ , \quad b_1^+ = x_1 - \lambda \quad \text{in} \quad D \ ,$$

$$c^+ = x_{n+1} + \mu \quad \text{if} \quad x_{n+1} \geq 0 \ ,$$

$$c^+ = \mu \quad \text{if} \quad x_{n+1} \leq 0 \ ,$$

$$A_o^+ = 2r \cdot a^+ - [\ (r.b_1^+)_{x_1} + (r.c^+)_{x_{n+1}}\ ] \ ,$$

$$A_1^+ = -2a^+.K_1 - (b_1^+)_{x_1} \cdot K_1 + (c^+ \cdot K_1)_{x_{n+1}} \ ,$$

$$A_j^+ = -2a^+.K_j + (b_1^+)_{x_1} \cdot K_j + (c^+ \cdot K_j)_{x_{n+1}} \ , \quad j = 2,3,\ldots,n \ ,$$

$$A_{n+1}^+ = -2a^+ + (b_1^+)_{x_1} - (c^+)_{x_{n+1}} \quad \text{in} \quad D \ ,$$

and adjoint boundary condition

[**] $$w = 0 \quad \text{on} \quad S_1 \cup S_4 \cup S_{o1} \cup S_{o2} \ .$$

Then it is easy to show the following a-priori estimate

$$\|w\|_1 \leq C \cdot \|L^+ w\| \ , \quad C \ := \ \text{const.} > 0 \ ,$$

for all $w \in D(L^+) \ := \{ w \in C^2(D) \ , \ w$ satisfies [**] } .

Therefore the following result holds: Assume above domain D and conditions.

Then Problem (NT) is well-posed in D in the sense that: there is at most one quasi-regular solution and a weak solution exists.

REMARKS:

i. The Problem (NT) is the Tricomi Problem, or the Characteristic Problem. To investigate the Frankl Problem, or the Noncharacteristic Problem is not difficult and in this case someone must assume adjoint boundary condition on the whole boundary G .

ii. If someone considers an arbitrary hypersurface S_1 such that

$$(x_1 + x_1^o) \cdot v_1 + x_{n+1} \cdot v_{n+1} \geq 0 \ , \qquad (x_1 - \lambda) \cdot v_1 + (x_{n+1} + \mu) \cdot v_{n+1} \geq 0 \ ,$$

on S_1 , in addition, then a new result holds.

THE EXTENDED BITSADZE-LAVRENTJEV EQUATION

In the same year J.M. Rassias obtained another new result for well-posedness of a boundary value problem for the extended Bitsadze--Lavrentjev equation

$$(*) \qquad Lu = \operatorname{sgn}(x_{n+1}).u_{x_1 x_1} + \sum_{j=2}^{n+1} u_{x_j x_j} + r(x).u = f(x) \quad ,$$

$$x = (x_1, x_2, \ldots, x_n, x_{n+1}) \ , \quad \operatorname{sgn}(x_{n+1}) \ := 1 \ \text{ if } \ x_{n+1} > 0 \ ,$$

$$:= 0 \ \text{ if } \ x_{n+1} = 0 \ , \quad := -1 \ \text{ if } \ x_{n+1} < 0 \ .$$

Then he considered the simply connected domain $D \subset \mathbb{R}^{n+1}$, bounded for $x_{n+1} > 0$ by a smooth hypersurface S_1 intersecting the hyperplane: $x_{n+1} = 0$ at S^o and for $x_{n+1} < 0$ by two characteristic hypersurfaces

$$S_3 : \ \Psi = x_1 - x_1^o + \Big(\sum_{j=2}^{n+1} x_j^2\Big)^{1/2} = 0 \ , \qquad S_4 := -\Phi = x_1 + x_1^o - \Big(\sum_{j=2}^{n+1} x_j^2\Big)^{1/2}$$

of equation $(*)$, $x_1^o := \text{const.} > 0$, both intersecting the negative x_{n+1}-axis at $x_{n+1} = -x_1^o$. Denote: $y = (x_2, x_3, \ldots, x_n)$.

The outer normal vector $v = (v_1, v_2, \ldots, v_n, v_{n+1})$ on $G = \partial D$ is:

$$v = \nabla\Psi/2 = (1/2, x_2/2.(x_1^o - x_1), x_3/2.(x_1^o - x_1), \ldots, x_n/2.(x_1^o - x_1), x_{n+1}/2.(x_1^o - x_1))$$

on S_3 ,

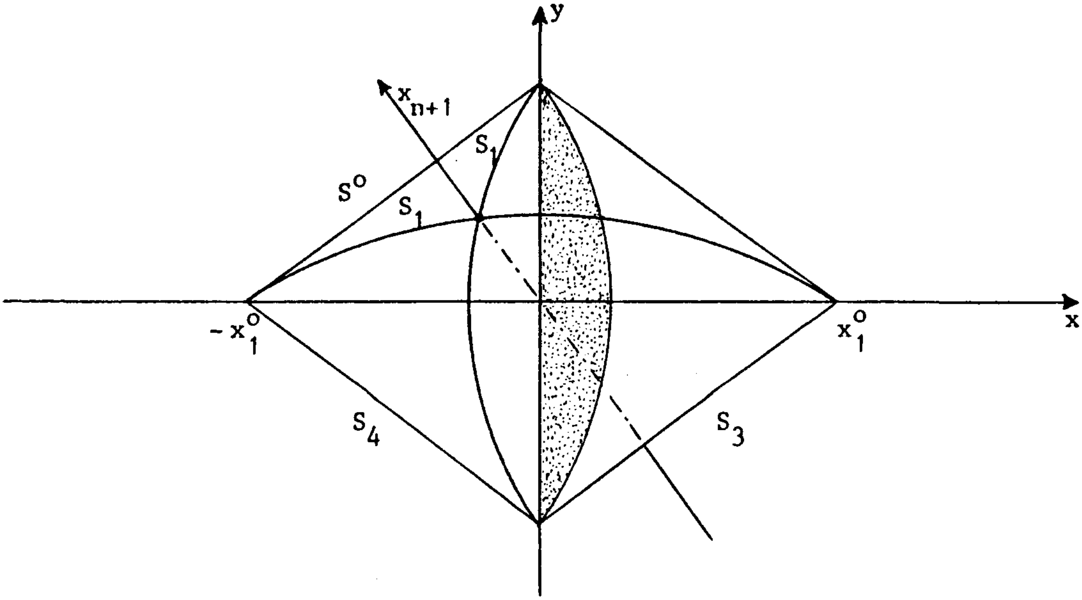
$$v = \nabla\Phi/2 = (-1/2, x_2/2.(x_1 + x_1^o), x_3/2.(x_1 + x_1^o), \ldots, x_n/2.(x_1 + x_1^o), x_{n+1}/2.(x_1 + x_1^o))$$

on S_4 ,

$$v = \nabla X/2 \qquad \text{on} \qquad S_1 : \ X = X(x) = 0 \ .$$

Denote

$$D_1 = D \cap \{x : x_{n+1} > 0\}, \quad D_2 = D \cap \{x : x_{n+1} < 0\},$$



Assume boundary condition

$$(**) \qquad u = 0 \quad \text{on} \quad S_1 \cup S_4 ,$$

and conditions

$$r \in C^1(D), \quad f \in C^o(D) ,$$

$$r < 0 \text{ on } S_3, \quad 2r + (x_1 - x_1^o).r_{x_1} + x_{n+1}.r_{x_{n+1}} < 0 \text{ in } \overline{D}_1 (= D_1 \cup \partial D_1)$$

$$r + (x_1 - x_1^o).r_{x_1} < 0 \quad \text{in} \quad \overline{D}_2 (= D_2 \cup \partial D_2) ,$$

$$(\text{ or } r < 0, \quad r_{x_1} \geq 0, \quad r_{x_{n+1}} \leq 0 \quad \text{in} \quad D),$$

$$(x_1-x_1^o).\nu_1 + x_{n+1}.\nu_{n+1} \geq 0 \quad \text{on} \quad S_1 \quad .$$

If someone applies Green's theorem in D_i , $j = 1,2$ separately (: because of the discontinuous coefficient $\text{sgn}(x_{n+1})$ of $u_{x_1 x_1}$) and employs boundary condition (**) and above additional conditions he gets the uniqueness of a quasi-regular solution of the characteristic Extended Bitsadze-Lavrentjev Problem (BL) : (*) and (**) .

To prove the existence of a weak solution of Problem (BL) assume the adjoint boundary condition

[**] $$w = 0 \quad \text{on} \quad S_1 \cup S_3 \quad ,$$

and additional conditions

$$(x_1+x_1^o).\nu_1 + x_{n+1}.\nu_{n+1} \geq 0 \quad \text{on} \quad S_1 \quad ,$$

$$r < 0 \quad \text{on} \quad S_4 \; , \quad 5r + 2.(x_1+x_1^o).r_{x_1} + 2.x_{n+1}.r_{x_{n+1}} < 0 \quad \text{on} \quad \overline{D}_1$$

$$3r + 2.(x_1+x_1^o).r_{x_1} < 0 \quad \text{in} \quad D_2 \quad ,$$

$$(\text{ or } \quad r < 0 \; , \quad r_{x_1} \geq 0 \; , \quad r_{x_{n+1}} \leq 0 \quad \text{in} \quad D \;) \quad ,$$

$$A_o^+ - R_o.(a^+)^2 \geq 0 \; , \quad A_1^+ - R_1.(b_1^+)^2 > 0 \quad ,$$

$$A_j^+ > 0 \; , \quad j = 2,3,\ldots,n \; , \quad A_{n+1}^+ - R_{n+1}.(c^+)^2 > 0 \quad ,$$

where R_i : = const. > 0 , $i = 0,1,2,\dots,n,n+1$.

$$a^+ = -1/4 \ , \quad b_1^+ = x_1 + x_1^o \quad \text{in } D \ , \quad \text{and}$$

$$c^+ = x_{n+1} \quad \text{in } \overline{D}_1 \ , \quad c^+ = 0 \quad \text{in } \overline{D}_2 \ ,$$

$$A_o^+ = 2r.a^+ - [\,(r.b_1^+)_{x_1} + (r.c^+)_{x_{n+1}}\,] \ ,$$

$$A_1^+ = \operatorname{sgn}(x_{n+1}) \,.\, [\,-2a^+ - (b_1^+)_{x_1} + (c^+)_{x_{n+1}}\,] \ ,$$

$$A_j^+ = -2a^+ + (b_1^+)_{x_1} + (c^+)_{x_{n+1}} \ , \quad j = 2,3,\dots,n \ ,$$

$$A_{n+1}^+ = -2a^+ + (b_1^+)_{x_1} - (c^+)_{x_{n+1}} \quad \text{in} \quad D \ .$$

Then it is easy to prove the a-priori estimate

$$\|w\|_1 \leq C \,.\, \|L^+w\| \ , \quad C := \text{const.} > 0 \ ,$$

for all $w \in D(L^+) = \{\, w \in C^2(D) \ , \ w$ satisfies [**] } .

Therefore the following result: Assume above domain D and conditions, then the characteristic Problem (BL) is well-posed in D .

OPEN PROBLEMS

1. An open question concerns the regularity of solutions for the bound-

ary value problems of mixed type discussed above.

2. In connection with Problem F of equation (++) the main question remains of proving the existence and uniqueness of regular solutions without restrictions on $K = K(y)$ or the size or shape of the domain of (++).

3. No serious work is known on nonlinear boundary value problems of mixed type in three and more dimensions.

4. The problem concerning the solution of elliptic systems in a domain on the boundary of which the type degenerates has not been investigated.

5. Little is known about the Cauchy problem for hyperbolic equations of order higher than 2 with boundary conditions on the curve of parabolic degeneracy.

6. The difficulty of the correct statement of the problem for equations of mixed type in higher dimensions still remains.

7. The study of higher order equations and systems of equations of mixed type requires more attention.

8. It would be interesting to clear up the question whether there is an extremal principle for the boundary value problems of mixed type.

9. One of the most important problems of mathematical physics is to study the properties of solutions of partial differential equations of mixed type with boundary conditions.

10. To consider regions of mixed type (elliptic-parabolic-hyperbolic): multi-connected with parabolic lines of degeneracy replaced by arbitrary curves (different from straight lines) is a question of high demand and difficult to handle.

11. The existence problem for regular transonic flow around given general profiles with given velocity at ∞ is difficult to be solved completely.

12. Regarding the work of J.M. Rassias (Bull. Acad. Polonaise Sci. Ser. Math., 28, 1980, 311-313) there is still open question if n : any odd > 2.

BASIC REFERENCES

BASIC BOOKS

1. Ju.M. BEREZANSKII, "Expansion in eigenfunctions of selfadjoint operators", Amer. Math. Soc., Providence, R.I. (1968).

2. A.V. BITSADZE, "Equations of the Mixed Type", Acad. Sci. U.S.S.R., Moscow (1959); English Transl. Mac Millan Co., New York (1964).

3. J.M. RASSIAS, "Mixed Type Equations", BSB Teubner, Leipzig, 90, (1986).

4. ————— , "Mathematics - Space Technology", Athens, Greece (1981).

5. ————— , "Partial Differential Equations of Mixed Type", manuscript at I.C.M.S.C. / S.P., Brasil, (1988) and, "Boundary Value Problems of Mixed Type", manuscript at I.C.M.S.C. / S.P., Brasil, (1988).

6. M.M. SMYRNOV, "Equations of Mixed Type", Moscow (1970) ; English Transl. Amer. Math. Soc., Providence, R.I. (1978).

MORE APPLIED BOOKS

1. R.G. BARANCEV, "Lectures on transonic gas dynamics", Izdat. Leningrad. Univ., Leningrad (1965).

2. L. BERS, "Mathematical aspects of subsonic and transonic gas dynamics", Wiley, New York (1958).

3. C. FERRARI & F.G. TRICOMI, "Transonic Aerodynamics", Acad. Press (1968), English. Transl. of "Aerodinamica Transonica", Cremonese (1962).