LIMIT CYCLES FOR A CLASS OF GENERALIZED KUKLES DISCONTINUOUS PIECEWISE POLYNOMIAL DIFFERENTIAL SYSTEM

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Abstract. In this paper we divide the plane in $2l$ sectors and define a discontinuous polynomial differential system such that in each sector is defined a smooth generalized Kukles polynomial differential systems. Applying the averaging theory of first order for discontinuous differential systems to get a upper bound to the number of limit cycles that can bifurcates from the linear center $\dot{x} = -y, \dot{y} = x$ when perturbed in the particular class of the generalized Kukles discontinuous piecewise polynomial differential systems when $l = 1, 2, 3$.

1. Introduction and statement of the main theorem

In the qualitative theory of real planar differential system the determination of limit cycles as defined by Poincaré [20] has become one of the main problems. The second part of the 16th Hilbert problem proposes to find a uniform upper bound for the number of limit cycles that a planar polynomial vector field of degree n can have which only depends on the degree of the polynomial differential system.

The investigation of the existence of periodic orbits of differential systems via the averaging methods has a long history, see for instance Marsden and McCracken [19], Sanders and Verhulst [22], Verhulst [23], Buica and Llibre [3], Buica, Francoise and Llibre [4] and the references therein. The averaging methods are useful tools for investigating the number of limit cycles for some differential systems and this method can be applied to obtain the shape, stability and the approximate expression of limit cycles.

A large number of problems from engineer [1], nonlinear oscillations [12], economy [10] among others cannot be described with smooth dynamical systems, so recently many researches are interested to study

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qualitative aspects of the phase space of non-smooth dynamical systems. A representative of this class is the mathematical model

$$\ddot{x} + x + f(x,\dot{x}) = sgn(h(x,\dot{x}))g(x,\dot{x}),$$

which can be found in many applications as for instance in control theory [2, 7]. In order to bring the investigation about the upper bound of limit cycles to the class of non-smooth dynamical systems generalizations of the averaging methods for some particular classes of non-smooth systems were given by Novaes-Llibre-Teixeira [16] and Novaes -Llibre [15].

In this paper we apply the averaging theory of first order for systems in the class of the generalized Kukles discontinuous piecewise polynomial differential systems to get an upper bound to the number of limit cycles for this family of non-smooth differential systems.

The classical Kukles system was introduced by Kukles in [11] which gave necessary and sufficient conditions to system

$$\begin{aligned}&\dot{x} = -y,\\&\dot{y} = x + a_0y + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3,\end{aligned}$$

has a center at the origin. Sadovskii [21] solved the center-focus problem for this system with $a_2a_7 \neq 0$ and proved that it can have seven limit cycles. Zang et al. [24] studied the number and distribuition of limit cycles for a class of reduced Kukles systems under cubic perturbation. Chavarriga et al. [6] studied the maximum numer of small amplitude limit cycles for Kukles system which can coexist with some invariant algebraic curves. Llibre and Mereu [13] studied the maximum number of limit cycles given by averaging methods of first and second orders which can bifurcates form the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$, perturbed inside of the class of generalized Kukles polynomial differential systems

$$\begin{aligned}&\dot{x} = y,\\&\dot{y} = -x - \sum_{k\geq 1}\varepsilon^k(f^k_{n_1}(x) + g^k_{n_2}(x)y + h^k_{n_3}(x)y^2 + d^k_0y^3),\end{aligned} \tag{1}$$

where for every k the polynomials $f^k_{n_1}, g^k_{n_2}, h^k_{n_3}$ have degree n_1, n_2 and n_3 respectively, $d^k_0 \neq 0$ is a real number and ε is a small parameter.

In this work we play with many straight lines of discontinuity through the origin of coordinates and with two different continuous Kukles system (of the form (1)) located alternatively in the sectors defined by the straight lines. This idea was applied to study two distinct classes of discontinuous generalized Lienard polynomial differential equation, see [14] and [18].

Let l a natural number and consider the function $h : \mathbb{R}^2 \to \mathbb{R}$

$$h_l(x,y) = \prod_{k=0}^{l-1} \left(y - \tan\left(\alpha + \frac{k\pi}{l}\right) x \right). \tag{2}$$

The set $h_l^{-1}(0)$ is the product of l straight lines passing through the origin of coordinates dividing the plane in $2l$ sectors with angles π/l when $\alpha \in \left(-\frac{\pi}{l}, \frac{\pi}{l}\right)$.

In this work we investigate an upper bound to the number of limit cycles that can bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ when this is perturbed as following

$$\dot{X} = \begin{cases} X_1(x,y) & \text{if } h(x,y) > 0, \\ X_2(x,y) & \text{if } h(x,y) < 0, \end{cases} \tag{3}$$

where $X_j(x,y) = \begin{pmatrix} y \\ -x - \varepsilon(f^j_{n_1}(x) + g^j_{n_2}(x)y + h^j_{n_3}(x)y^2 + d^j_0 y^3) \end{pmatrix}$ with $f^j_{n_1}(x)$, $g^j_{n_2}(x)$ and $h^j_{n_3}(x)$ polynomials of degrees n_1, n_2 and n_3 respectively, and d^j_0 is a nonzero real number for $j = 1, 2$, when $l \in \{1, 2, 3\}$.

System (3) can be written using the sign function as

$$\dot{X} = Z(x,y) = G_1(x,y) + \operatorname{sign}(h(x,y))G_2(x,y),$$

where $G_1(x,y) = \frac{1}{2}(X_1(x,y) + X_2(x,y))$ and $G_2(x,y) = \frac{1}{2}(X_1(x,y) - X_2(x,y))$.

The main result is the following

Theorem 1. *Assume that $j = 1, 2$, the polynomials $f^j_{n_1}(x)$, $g^j_{n_2}(x)$ and $h^j_{n_3}(x)$ have degree $n_1 \geq 1$, $n_2 \geq 1$ and $n_3 \geq 1$ respectively, d^j_0 is a nonzero constant and $l \in \{1, 2, 3\}$. Then for $|\varepsilon|$ sufficiently small the generalized Kukles discontinuous piecewise polynomial differential system (3) has at most $m(l)$ limit cycles, using the averaging theory, where*

i) $m(1) = \max\left\{2\left[\frac{n_1}{2}\right], n_2 + 1, 2\left[\frac{n_3+2}{2}\right], 3\right\}$;

ii) $m(2) = \max\left\{\left[\frac{n_1-1}{2}\right], \left[\frac{n_2}{2}\right], \left[\frac{n_3+1}{2}\right], 1\right\}$;

iii) $m(3) = \max\left\{2\left[\frac{n_1}{2}\right], n_2 + 1, 2\left[\frac{n_3+2}{2}\right], 3\right\} - 1$.

We recall that in [13] the authors show that smooth generalized Kukles polynomial differential systems have at least $\max\left\{\left[\frac{n_2}{2}, 1\right]\right\}$ limit cycles, using averaging theory. So comparing the obtained results for discontinuous with the results for continuous generalized Kukles polynomial differential systems, this work shows that the discontinuous systms have more or the same number of limit cycles than the continuous systems. In short, we have at least 2 more limit cycles, at least 1 more limit cycle and the same number or more depending on n_1, n_2 and n_3 if $l = 1$, $l = 3$ and $l = 2$, respectively.

2. Basic definitions and averaging theory

In this section we summarize the main results on the theory of averaging that will be used to prove Theorem 1 as some basic definitions about planar discontinuous piecewise differential system.

Let $D \subset \mathbb{R}^n$ an open subset and $h : \mathbb{R} \times D \to \mathbb{R}$ a C^1 function having 0 as regular value. Consider $F^1, F^2 : \mathbb{R} \times D \to \mathbb{R}^n$ continuous functions and $\Sigma = h^{-1}(0)$. We define the Filippov's system as

$$\dot{x}(t) = F(t,x) = \begin{cases} F^1(t,x) & \text{if } (t,x) \in \Sigma^+, \\ F^2(t,x) & \text{if } (t,x) \in \Sigma^-, \end{cases} \tag{4}$$

where $\Sigma^+ = \{(t,x) \in \mathbb{R} \times D : h(t,x) > 0\}$ and $\Sigma^- = \{(t,x) \in \mathbb{R} \times D : h(t,x) < 0\}$.

The manifold Σ is divided in the closure of two disjoint regions, namely *Crossing region* (Σ^c) and *Sliding region* (Σ^s),

$$\Sigma^c = \{p \in \Sigma : \langle \nabla h(p), (1, F^1(p)) \rangle \cdot \langle \nabla h(p), (1, F^2(p)) \rangle > 0\},$$
$$\Sigma^s = \{p \in \Sigma : \langle \nabla h(p), (1, F^1(p)) \rangle \cdot \langle \nabla h(p), (1, F^2(p)) \rangle < 0\}.$$

Consider the differential system associated to system (4)

$$\dot{x}(t) = F(t,x) = \chi_+(t,x)F^1(t,x) + \chi_-(t,x)F^2(t,x), \tag{5}$$

where χ_+, χ_- are the *characteristic functions* defined as

$$\chi_+(t,x) = \begin{cases} 1 \text{ if } h(t,x) > 0, \\ 0 \text{ if } h(t,x) < 0. \end{cases}$$

and

$$\chi_-(t,x) = \begin{cases} 0 \text{ if } h(t,x) > 0, \\ 1 \text{ if } h(t,x) < 0. \end{cases}$$

Systems (4) and (5) does not coincides in $h(t,x) = 0$, but applying the Fillipov's convention for the solutions of systems (4) and (5) (see

[8]) passing through a point $(t, x) \in \Sigma$ we have that these solutions do not depend on the value of $F(t, x)$, so the solutions are the same.

The averaging methods which will be applied in this paper for a particular discontinuous piecewise differential systems is the following

Theorem 2. [16] *Consider the following system*

$$\dot{x}(t) = \varepsilon F(x,t) + \varepsilon^2 R(t, x, \varepsilon), \tag{6}$$

where $F(t, x) = F_1(t, x) + \text{sign}(h(t, x))F_2(t, x)$ *and* $R(t, x) = R_1(t, x) + \text{sign}(h(t, x))R_2(t, x)$*.Moreover,* $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$*,* $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ *and* $h : \mathbb{R} \times D \to \mathbb{R}$ *are continuous* $T-$*periodic in the first variable and* $D \subset \mathbb{R}^n$ *is an open subset. Suppose that* h *is a* C^1 *function such that* $0 \in \mathbb{R}$ *is a regular value.*
We define the averaged *function* $f : D \to \mathbb{R}^n$ *by*

$$f(x) = \int_0^T F(t, x)dt, \tag{7}$$

and consider valid the following hypotheses:

- i) F_1, F_2, R_1, R_2 *and* h *are locally Lipschitz with respect to* x*;*
- ii) *Exist an bounded open set* $C \subset D$ *such that, for* $|\varepsilon| > 0$ *sufficiently small, every orbit starting in* C *reaches the set of discontinuity only at its crossing regions (crossing hypothesis);*
- iii) *For* $a \in C$*, with* $f(a) = 0$*, exist an neighbourhood* $U \subset C$ *of* a *such that* $f(z) \neq 0$*, for all* $z \in \bar{U}\backslash\{a\}$ *and* $d_B(f, U, 0) \neq 0$*.*

Then, for $|\varepsilon| > 0$ *sufficiently small, exist one* $T-$*periodic solution* $x(t, \varepsilon)$ *of* (6) *such that* $x(0, \varepsilon) \to a$ *if* $\varepsilon \to 0$*.*

Here $d_B(f, U, 0)$ denotes the Brouwer degree for a continuous function(see [5]. For details about the proof of this result, see [17].

Remark 3. *If* $f(x)$ *is a* C^1 *function such that* $f'(a) \neq 0$*, then there exist an neighbourhood* $U \subset C$ *of* a *such that* $f(z) \neq 0$*, for all* $z \in \bar{U}\backslash\{a\}$ *and* $d_B(f, U, 0) \neq 0$*.*

3. Proof of Theorem 1

Writing

$$f^1_{n_1}(x) = \sum_{i=0}^{n_1} a_i x^i, \qquad g^1_{n_2}(x) = \sum_{i=0}^{n_2} b_i x^i, \qquad h^1_{n_3}(x) = \sum_{i=0}^{n_3} c_i x^i,$$

$$f^2_{n_1}(x) = \sum_{i=0}^{n_1} d_i x^i, \qquad g^2_{n_2}(x) = \sum_{i=0}^{n_2} e_i x^i, \qquad h^2_{n_3}(x) = \sum_{i=0}^{n_3} m_i x^i,$$

doing the change of coordinates $x = r\cos\theta$, $y = r\sin\theta$ and taking θ as the new independent variable, system (3) takes the form

$$\frac{dr}{d\theta} = \varepsilon \sin\theta P_j(r,\theta) + \mathcal{O}(\varepsilon^2), \tag{8}$$

where

$$P_1(r,\theta) = \sum_{i=0}^{n_1} a_i r^i \cos^i\theta + \sum_{i=0}^{n_2} b_i r^{i+1}\cos^i\theta\sin\theta + \sum_{i=0}^{n_3} c_i r^{i+2}\cos^i\theta\sin^2\theta + d_0^1 r^3\sin^3\theta,$$

and

$$P_2(r,\theta) = \sum_{i=0}^{n_1} d_i r^i \cos^i\theta + \sum_{i=0}^{n_2} e_i r^{i+1}\cos^i\theta\sin\theta + \sum_{i=0}^{n_3} m_i r^{i+2}\cos^i\theta\sin^2\theta + d_0^2 r^3\sin^3\theta.$$

Observe that system (8) satisfies the condition of Theorem 2 with h given by (2). So, to estimate the limit cycles of system (3), we need to estimate the number of zero of the averaged function (7).

Denoting

$$\varphi_{ijl}(\alpha) = \sum_{k=1}^{l}\int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{(2k-1)\pi}{l}} \cos^i\theta\sin^j\theta d\theta,$$

and

$$\overline{\varphi}_{ijl}(\alpha) = \sum_{k=1}^{l}\int_{\alpha+\frac{(2k-1)\pi}{l}}^{\alpha+\frac{2k\pi}{l}} \cos^i\theta\sin^j\theta d\theta,$$

the averaged function (7) becomes

$$f(r) = \sum_{i=0}^{n_1} r^i\Big[a_i\varphi_{i1l}(\alpha) + d_i\overline{\varphi}_{i1l}(\alpha)\Big] + \sum_{i=0}^{n_2} r^{i+1}\Big[b_i\varphi_{i2l}(\alpha) + e_i\overline{\varphi}_{i2l}(\alpha)\Big] + \sum_{i=0}^{n_3} r^{i+2}\Big[c_i\varphi_{i3l}(\alpha) + m_i\overline{\varphi}_{i3l}(\alpha)\Big] + r^3\Big[d_0^1\varphi_{04l}(\alpha) + d_0^2\overline{\varphi}_{04l}(\alpha)\Big]. \tag{9}$$

The next lemma exhibit the relation between some integrals in the expression of the averaged function.

Lemma 4. *For each $i \in \mathbb{N}$ and $l \neq 0$ the following holds:*

i) $\varphi_{i1l} = -\overline{\varphi}_{i1l}$,
ii) $\varphi_{i3l} = -\overline{\varphi}_{i3l}$.
Moreover, if i is odd then
iii) $\varphi_{i2l} = -\overline{\varphi}_{i2l}$.

Proof. Note that for each $l \neq 0$

$$(\varphi_{i1l} + \overline{\varphi}_{i1l})(\alpha) = \sum_{k=1}^{l} \left(\int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{(2k-1)\pi}{l}} \cos^i\theta \sin\theta d\theta + \int_{\alpha+\frac{(2k-1)\pi}{l}}^{\alpha+\frac{2k\pi}{l}} \cos^i\theta \sin\theta d\theta \right)$$
$$= \sum_{k=1}^{l} \int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{2k\pi}{l}} \cos^i\theta \sin\theta d\theta = \int_{\alpha}^{\alpha+2\pi} \cos^i\theta \sin\theta d\theta = 0.$$

Analogously

$$(\varphi_{i3l} + \overline{\varphi}_{i3l})(\alpha) = \sum_{k=1}^{l} \left(\int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{(2k-1)\pi}{l}} \cos^i\theta \sin^3\theta d\theta + \int_{\alpha+\frac{(2k-1)\pi}{l}}^{\alpha+\frac{2k\pi}{l}} \cos^i\theta \sin^3\theta d\theta \right)$$
$$= \sum_{k=1}^{l} \int_{\alpha+\frac{2(k-1)\pi}{l}}^{\alpha+\frac{2k\pi}{l}} \cos^i\theta \sin^3\theta d\theta = \int_{\alpha}^{\alpha+2\pi} \cos^i\theta \sin^3\theta d\theta = 0.$$

These computations prove items i) and ii).

As $(\varphi_{i2l} + \overline{\varphi}_{i2l})(\alpha) = \int_{\alpha}^{\alpha+2\pi} \cos^i\theta \sin^2\theta d\theta$, assuming $i = 2m+1$ the item iii) follows directly from the formulae 2.511-4 of [9]

(10)
$$\int \cos^{2m+1}\theta \sin^2\theta d\theta =$$
$$= \frac{\sin^3\theta}{2m+3}\left(\sum_{j=1}^{m} \frac{2^j m(m-1)...(m-j+1)}{(2m+1)(2m-1)...(2m-2j+3)} \cos^{2m-2j}\theta + \cos^{2m}\theta \right).$$

□

Remark 5. *Observe that*

1) *if i is even $\varphi_{i2l}(\alpha), \overline{\varphi}_{i2l}(\alpha) > 0$ for all $l \neq 0$ and $\alpha \in \mathbb{R}$, in fact if i is even then $\cos^i\theta \sin^2\theta \geq 0$.*
2) *$\varphi_{04l}(\alpha), \overline{\varphi}_{04l}(\alpha) > 0$ for all $\alpha \in \mathbb{R}$ because $\sin^4\theta \geq 0$.*

In order to estimate the number of zeros of the averaged function remains to study three functions $\varphi_{i1l}, \varphi_{i3l}$ and φ_{i2l}, the first and second for each $i \in \mathbb{N}$ and $l \neq 0$ and the third one, for i odd. For these functions we must to fix the number $l \neq 0$.

3.1. **Proof of item (i).** Let $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The case $\alpha = \frac{\pi}{2}$ will be considered separately. If $l = 1$ then $h_1(x, y) = y - (\tan\alpha)x$ and $h_1^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : y = (\tan\alpha)x\}$.

The next lemmas will be used to estimate the maximum number of zeros of the averaged function (9) when $l = 1$.

Lemma 6. *If i is odd then $\varphi_{i11} \equiv 0$ and φ_{i11} does not vanish for $\alpha \neq \pm\frac{\pi}{2}$ and i even.*

Proof. Now we note that

$$\varphi_{i11}(\alpha) = \int_{\alpha}^{\alpha+\pi} \cos^i\theta \sin\theta d\theta = -\frac{(-1)^{i+1}\cos^{i+1}\alpha - \cos^{i+1}\alpha}{i+1}$$
$$= \begin{cases} 0 & \text{if } i \text{ is odd,} \\ 2\dfrac{\cos^{i+1}\alpha}{i+1} & \text{if } i \text{ is even.} \end{cases}$$

Then if i is even the function $\varphi_{i11}(\alpha)$ vanishes if and only if $\alpha = \pm\pi/2$. □

Lemma 7. *If i is odd and $\alpha \neq 0$ the functions φ_{i21} do not vanish.*

Proof. If $i = 1$ then

$$\varphi_{121}(\alpha) = \int_{\alpha}^{\alpha+\pi} \sin^2\theta \sin\theta d\theta = -\frac{2}{3}\sin^3\alpha.$$

So if $\alpha \neq 0$, the function φ_{121} is not zero.

If $i \geq 1$ and odd we write $i = 2m + 1$ for some $m \in \mathbb{N}\backslash\{0\}$ and from (10) we get

$$\varphi_{(2m+1)21}(\alpha) = \int_{\alpha}^{\alpha+\pi} \sin^2\theta \cos^{2m+1}\theta d\theta =$$
$$= -2\frac{\sin^3\alpha}{2m+3}\left(\sum_{j=1}^{m} \frac{2^j m(m-1)...(m-j+1)}{(2m+1)(2m-1)...(2m-2j+3)}\cos^{2m-2j}\alpha\right.$$
$$\left. + \cos^{2m}\alpha\right).$$

So $\varphi_{(2m+1)20}(\alpha) = 0$ if, and only if $\alpha = 0$. □

Lemma 8. *Consider i even and $\alpha \neq \pm\frac{\pi}{2}$ then the function φ_{i31} does not vanish. Moreover $\varphi_{i31} \equiv 0$ if i is odd.*

Proof. We have that

$$\begin{aligned}\varphi_{i31}(\alpha) &= \int_{\alpha}^{\alpha+\pi} \cos^i\theta \sin^3\theta d\theta \\ &= \frac{(-1)^{i+3}\cos^{i+3}\alpha - \cos^{i+3}\alpha}{i+3} - \frac{(-1)^{i+1}\cos^{i+1}\alpha - \cos^{i+1}\alpha}{i+1} \\ &= \begin{cases} 0 & \text{if } i \text{ is odd,} \\ 2\left(\dfrac{\cos^{i+1}\alpha}{i+1} - \dfrac{\cos^{i+3}\alpha}{i+3}\right) \neq 0 & \text{if } i \text{ is even.} \end{cases}\end{aligned}$$

So the function $\varphi_{i31}(\alpha)$ vanishes if and only if i odd or $\alpha = \pm\frac{\pi}{2}$. □

Proof of Theorem 1 (i). By the previous lemmas the averaged function is given by

$$\begin{aligned} f(r) = &\sum_{\substack{i=0\\ i \text{ even}}}^{n_1} A_i(\alpha)(a_i - d_i)r^i + \sum_{\substack{i=0\\ i \text{ even}}}^{n_2} B_i(\alpha)(b_i + e_i)r^{i+1} \\ &+ \sum_{\substack{i=1\\ i \text{ odd}}}^{n_2} C_i(\alpha)(b_i - e_i)r^{i+1} + \sum_{\substack{i=0\\ i \text{ even}}}^{n_3} D_i(\alpha)(c_i - m_i)r^{i+2} \\ &+ (E^1(\alpha)d_0^1 + E^2(\alpha)d_0^2)r^3, \end{aligned}$$

with $A_i(\alpha)$ and $D_i(\alpha)$ does not vanish if $\alpha \neq \pm\frac{\pi}{2}$, $B_i(\alpha), E^1(\alpha), E^2(\alpha) > 0$, and $C_i(\alpha) \neq 0$ if $\alpha \neq 0$.

Then the degree of the averaged function is m given by

(i) $\max\{n_2 + 1, 3\}$, if $\alpha \in \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$;

(ii) $\max\left\{2\left[\frac{n_1}{2}\right], 2\left[\frac{n_2}{2}\right]+1, 2\left[\frac{n_3+2}{2}\right], 3\right\}$, if $\alpha = 0$;

(iii) $\max\left\{2\left[\frac{n_1}{2}\right], n_2 + 1, 2\left[\frac{n_3+2}{2}\right], 3\right\}$, if $\alpha \notin \{0, \pm\frac{\pi}{2}\}$.

So the maximum number of zeros of the averaged function will be m in each case.

Moreover it is possible to choose the coefficients $a_i, b_i, c_i, d_i, e_i, m_i$ and d_0^j and α in the expression of the averaged function such that $f(r)$ has exactly $m(1) = \max\left\{2\left[\frac{n_1}{2}\right], n_2 + 1, 2\left[\frac{n_3+2}{2}\right], 3\right\}$ simple positive roots, what guarantee that the maximum number of zeros is reached. Then making this choose we conclude by Theorem 2 that system (3) has at most $m(1)$ limit cycles, each one with period near to 2π. This proofs the item (i) from Theorem 1. □

3.2. **Proof of item (ii).** For $l = 2$ in (2) we have $h_2(x,y) = (y - (\tan\alpha)x)(y + \tan(\alpha + \frac{\pi}{2})x)$. The degree of (9) with $l = 2$ will be determinate using the next lemmas.

Lemma 9. *The function φ_{i12} do not vanish for $\alpha \neq \pm\frac{\pi}{4}$ and i odd. If i is even then $\varphi_{i12} \equiv 0$.*

Proof. We have

$$\varphi_{i12}(\alpha) = \int_{\alpha}^{\alpha+\frac{\pi}{2}} \cos^i\theta \sin\theta d\theta + \int_{\alpha+\pi}^{\alpha+\frac{3\pi}{2}} \cos^i\theta\sin\theta d\theta$$
$$= \begin{cases} 0 & \text{if } i \text{ is even,} \\ 2\dfrac{\cos^{i+1}\alpha - \sin^{i+1}\alpha}{i+1} & \text{if } i \text{ is odd.} \end{cases}$$

Obviously $\varphi_{112}(\alpha) = 0$ if and only if $\cos^2\alpha = \sin^2\alpha$, i.e., if and only if $\alpha = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$.

If $i \geq 1$ and it is odd then $i + 1 = 2n$ for some $n \in \mathbb{N}$ and

$$\left(\cos^{2n-2}\alpha + \sum_{k+j=2(n-1)} \cos^k\alpha\sin^j\alpha + \sin^{2n-2}\alpha\right)(\cos^2\alpha - \sin^2\alpha)$$
$$= \cos^{2n}\alpha - \sin^{2n}\alpha.$$

So the zeros of the function $\varphi_{i12}(\alpha)$ are $\alpha = \frac{\pi}{4}, \frac{3\pi}{4}$ in the established interval. □

Lemma 10. *If i is odd the function φ_{i22} is identically null.*

Proof. If $i = 1$ then

$$\varphi_{122}(\alpha) = \frac{1}{3}\Big(\cos^3\alpha - \sin^3\alpha\Big) + \frac{1}{3}\Big(-\cos^3\alpha + \sin^3\alpha\Big) = 0.$$

If $i \geq 1$ and odd then we apply (10) to write

$$\varphi_{i22}(\alpha) = \frac{\sin^3\left(\alpha + \frac{\pi}{2}\right)}{2m+3}\left(\cos^{2m}\left(\alpha + \frac{\pi}{2}\right) + \sum_{j=1}^{m} A(m,j)\cos^{2m-2j}\left(\alpha + \frac{\pi}{2}\right)\right)$$
$$- \frac{\sin^3\alpha}{2m+3}\left(\cos^{2m}\alpha + \sum_{j=1}^{m} A(m,j)\cos^{2m-2j}\alpha\right)$$
$$+ \frac{\sin^3(\alpha + \frac{3\pi}{2})}{2m+3}\left(\cos^{2m}\left(\alpha + \frac{3\pi}{2}\right) + \sum_{j=1}^{m} A(m,j)\cos^{2m-2j}\left(\alpha + \frac{3\pi}{2}\right)\right)$$
$$- \frac{\sin^3(\alpha+\pi)}{2m+3}\left(\cos^{2m}(\alpha+\pi) + \sum_{j=1}^{m} A(m,j)\cos^{2m-2j}(\alpha+\pi)\right),$$

where $A(m,j) = \dfrac{2^j m(m-1)...(m-j+1)}{(2m+1)(2m-1)...(2m-2j+3)}$.

So

$$\begin{aligned}
\varphi_{i22}(\alpha) &= \frac{\cos^3\alpha}{2m+3}\left(\sin^{2m}\alpha + \sum_{j=1}^{m} A(m,j)\sin^{2m-2j}\alpha\right)\\
&- \frac{\sin^3\alpha}{2m+3}\left(\cos^{2m}\alpha + \sum_{j=1}^{m} A(m,j)\cos^{2m-2j}\alpha\right)\\
&- \frac{\cos^3\alpha}{2m+3}\left(\sin^{2m}\alpha + \sum_{j=1}^{m} A(m,j)\sin^{2m-2j}\alpha\right)\\
&+ \frac{\sin^3\alpha}{2m+3}\left(\cos^{2m}\alpha + \sum_{j=1}^{m} A(m,j)\cos^{2m-2j}\alpha\right)\\
&= 0.
\end{aligned}$$

□

Lemma 11. *If i is odd the only zeros of the function φ_{i32} are $\alpha = \pm\frac{\pi}{4}$. When i is even the function is identically null.*

Proof. From straightforward calculations we get that $\varphi_{i32}(\alpha) \equiv 0$, when i is even. Otherwise

$$\varphi_{i32}(\alpha) = \frac{2}{i+3}(\sin^{i+3}\alpha - \cos^{i+3}\alpha) + \frac{2}{i+1}(-\sin^{i+1}\alpha + \cos^{i+1}\alpha).$$

Besides $\alpha = \pm\frac{\pi}{4}$ are roots of the function $\varphi_{i32}(\alpha)$ when i is odd. To show that they are the unique roots in the established interval we study the sign of its derivative

$$\varphi'_{i32}(\alpha) = -2(\cos^i\alpha\sin^3\alpha + \cos^3\alpha\sin^i\alpha).$$

For $\alpha \in (-\frac{\pi}{2}, 0)$, $\varphi'_{i32}(\alpha) > 0$ and hence $\varphi_{i32}(\alpha)$ is strictly increasing. For $\alpha \in (0, \frac{\pi}{2})$, $\varphi'_{i32}(\alpha) < 0$ and hence $\varphi_{i32}(\alpha)$ is strictly decreasing. So the unique roots of $\varphi_{i32}(\alpha)$ in the established interval are $\alpha = \pm\frac{\pi}{4}$. □

Proof of Theorem 1 (ii). It follows from Lemmas 9 – 11 that the averaged function is given by

$$\begin{aligned}
f(r) &= \sum_{\substack{i=1\\ i\text{odd}}}^{n_1} A_i(\alpha)(a_i - d_i)r^i + \sum_{\substack{i=0\\ i\text{even}}}^{n_2} (B_i(\alpha)b_i + \overline{B}_i(\alpha)e_i)r^{i+1}\\
&+ \sum_{\substack{i=1\\ i\text{odd}}}^{n_3} C_i(\alpha)(c_i - m_i)r^{i+2} + (E^1(\alpha)d_0^1 + E^2(\alpha)d_0^2)r^3,
\end{aligned}$$

where $A_i(\alpha)$ and $C_i(\alpha)$ are not zero except by $\alpha = \pm\frac{\pi}{4}$ and $B_i(\alpha), \overline{B}_i(\alpha)$ are not zero.

Then the averaged function $f(r)$ is a polynomial of degree m where m is given by

1) $\max\left\{2\left[\frac{n_1-1}{2}\right]+1, 2\left[\frac{n_2}{2}\right]+1, 2\left[\frac{n_3+1}{2}\right]+1, 3\right\}$, if $\alpha \notin \{-\frac{\pi}{4}, \frac{\pi}{4}\}$;
2) $\max\left\{2\left[\frac{n_2}{2}\right]+1, 3\right\}$ otherwise.

Moreover, if $\alpha \neq \pm\frac{\pi}{4}$ the averaged function is an odd function hence it has at most $\frac{m-1}{2}$ positive roots. From the averaging theory it follows the proof of Theorem 1 (ii).

Besides it is not difficult to verify that the polynomials, whose roots provide the number of limit cycles which bifurcates from the linear center, have independent coefficients as functions of the coefficients of the perturbed system. Therefore the upper bound provides in statements (ii) can be reached. □

3.3. **Proof of item (iii).** When $l = 3$ in (2) the function h_3 becomes $h_3(x, y) = (y - (\tan\alpha)x)(y - (\tan(\alpha + \frac{2\pi}{3}))x)(y - (\tan(\alpha + \frac{4\pi}{3}))x)$.

Lemma 12. *About the function φ_{i13} the following hold*

(i) $\varphi'_{i13}(\alpha) < 0$ *in* $(0, \frac{\pi}{3})$.
(ii) $\varphi_{i13}(\alpha)$ *is a even function.*
(iii) $\varphi_{i13}(\alpha)$ *is* $2\pi/3$*–periodic.*
(iv) $\varphi_{i13}(\alpha) \neq 0$, *if* $\alpha \neq \pm\frac{\pi}{6}$ *and* $i \neq 0$ *is even.*

Proof. We have

$$\varphi_{i13}(\alpha) = \int_{\alpha}^{\alpha+\frac{\pi}{3}} \cos^i\theta \sin\theta d\theta + \int_{\alpha+\frac{2\pi}{3}}^{\alpha+\pi} \cos^i\theta \sin\theta d\theta + \int_{\alpha+\frac{4\pi}{3}}^{\alpha+\frac{5\pi}{3}} \cos^i\theta \sin\theta d\theta$$
$$= \frac{(-1)^i \cos^{i+1}\alpha + \cos^{i+1}\alpha - (-1)^{i+1}\sin^{i+1}(\alpha - \frac{\pi}{6}) + \sin^{i+1}(\alpha - \frac{\pi}{6})}{i+1}$$
$$+ \frac{(-1)^{i+1}\sin^{i+1}(\alpha + \frac{\pi}{6}) - \sin^{i+1}(\alpha + \frac{\pi}{6})}{i+1}.$$

So $\varphi_{i13}(\alpha)$ vanishes if i odd. If i is even then

$$\varphi_{i13}(\alpha) = \frac{2}{i+1}(\cos^{i+1}\alpha + \sin^{i+1}(\alpha - \frac{\pi}{6}) - \sin^{i+1}(\alpha + \frac{\pi}{6})).$$

Therefore $\varphi_{013}(\alpha) = 0$ if and only if $\alpha = \pm\pi/2$. To consider the case i even and $i \neq 0$ we take the derivative of φ_{i13} with respect to α. Using computational tools like Mathematica we get

$$\varphi'_{i13}(\alpha)=\Big(-2\cos^i\alpha\sin\alpha$$
$$-(\sqrt{3}\cos\alpha+\sin\alpha)\left((\sqrt{3}\sin\alpha+\cos\alpha)^i-(\sqrt{3}\sin\alpha-\cos\alpha)^i\right)/2^i\Big).$$

In the case of 3 straight lines, without loss of generality we can make all the computations in the interval $(0,\frac{\pi}{3})$. and in this interval we have
(a) $\sin\alpha\cos^i\alpha>0$;
(b) $\sqrt{3}\cos\alpha+\sin\alpha>0$ and
(c)$(\cos\alpha+\sqrt{3}\sin\alpha)^i-(-\cos\alpha+\sqrt{3}\sin\alpha)^i=$

$$=\sum_{\substack{k=1\\ kimpar}}^{i-1}\binom{i}{k}(\sqrt{3})^k\cos^{i-k}\alpha\sin^k\alpha+\sum_{\substack{k=2\\ kpar}}^{i-2}\binom{i}{k}(\sqrt{3})^k\cos^{i-k}\alpha\sin^k\alpha$$
$$+\sum_{\substack{k=1\\ kimpar}}^{i-1}\binom{i}{k}(\sqrt{3})^k\cos^{i-k}\alpha\sin^k\alpha-\sum_{\substack{k=2\\ kpar}}^{i-2}\binom{i}{k}(\sqrt{3})^k\cos^{i-k}\alpha\sin^k\alpha$$
$$=2\sum_{\substack{k=1\\ kimpar}}^{i-1}\binom{i}{k}(\sqrt{3})^k\cos^{i-k}\alpha\sin^k\alpha>0.$$

Hence $\varphi'_{i13}(\alpha)<0$ if $\alpha\in(0,\frac{\pi}{3})$, so $\varphi_{i13}(\alpha)$ is strictly decreasing in this interval. Item (i) is proved.

Moreover,

$$\begin{aligned}\varphi_{i13}(-\alpha)&=\frac{2}{i+1}(\cos^{i+1}(-\alpha)+\sin^{i+1}(-\alpha-\frac{\pi}{6})-\sin^{i+1}(-\alpha+\frac{\pi}{6}))\\&=\frac{2}{i+1}(\cos^{i+1}(\alpha)-\sin^{i+1}(\alpha+\frac{\pi}{6})+\sin^{i+1}(\alpha-\frac{\pi}{6}))\\&=\varphi_{i13}(\alpha),\end{aligned}$$

from what we conclude the proof of item (ii), i.e., $\varphi_{i13}(\alpha)$ is an even function.

Because

$$\begin{aligned}\varphi_{i13}(\alpha+\frac{2\pi}{3})&=\frac{2}{i+1}(\cos^{i+1}(\alpha+\frac{2\pi}{3})-\sin^{i+1}(\alpha+\frac{5\pi}{6})+\sin^{i+1}(\alpha-\frac{\pi}{2}))\\&=\frac{2}{i+1}(-\sin^{i+1}(\alpha+\frac{\pi}{6})+\sin^{i+1}(\alpha-\frac{\pi}{2})-\sin^{i+1}(\alpha+\frac{5\pi}{6})\\&=\frac{2}{i+1}(\cos^{i+1}(\alpha)-\sin^{i+1}(\alpha+\frac{\pi}{6})+\sin^{i+1}(\alpha-\frac{\pi}{6}))\\&=\varphi_{i13}(\alpha),\end{aligned}$$

we get that $\varphi_{i13}(\alpha)$ is $\frac{2\pi}{3}$–periodic.

Finally if $i \neq 0$

$$\varphi_{i13}(0) = \frac{2^{1-i}}{i+1}(2^i - 1) \neq 0.$$

See the graphic of $\varphi_{i13}(\alpha)$, for i even not zero in Figure 1.

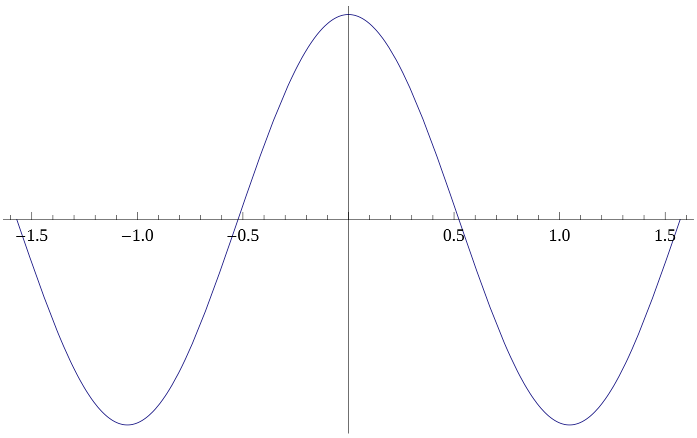

FIGURE 1. Graphic of $\varphi_{i13}(\alpha)$, for i even

As φ_{i13} is strictly decreasing in $(0, \frac{\pi}{3})$ and $\varphi_{i13}(\frac{\pi}{6}) = 0$ it follows that $\frac{\pi}{6}$ is the unique root of φ_{i13} in $(0, \frac{\pi}{3})$. Analogously, as φ_{i13} is strictly increasing in $(-\frac{\pi}{3}, 0)$, $-\frac{\pi}{6}$ is the unique root in this interval.

Now suppose that there exist an $\alpha_0 \in (\frac{\pi}{3}, \frac{\pi}{2})$ such that $\varphi_{i13}(\alpha_0) = 0$. So $\varphi_{i13}(\alpha_0 - \frac{2\pi}{3}) = 0$. But if $\alpha_0 \in (\frac{\pi}{3}, \frac{\pi}{2})$, then $\alpha_0 - \frac{2\pi}{3} \in (-\frac{\pi}{3}, -\frac{\pi}{6})$ will be a root of φ_{i13}, contradiction.

□

Lemma 13. *If i is odd and $\alpha \notin \{0, \pm\frac{\pi}{3}\}$ then the function φ_{i23} do not vanish.*

Proof. If $i = 1$ then $\varphi_{123}(\alpha) = \frac{1}{2}\sin(3\alpha)$. Otherwise we can write $i = 2m+1$ for some $m \in \mathbb{N}\backslash\{0\}$ and use (10) to get

$$\begin{aligned}
&\varphi_{i23}(\alpha) = \\
&= 2\Bigg[\frac{\sin^3(\alpha+\frac{\pi}{3})}{2m+3}\Bigg(\cos^{2m}(\alpha+\frac{\pi}{3}) + \sum_{j=1}^{m} A(m,j)\cos^{2m-2j}(\alpha+\frac{\pi}{3})\Bigg) \\
&+ \frac{\sin^3(\alpha+\frac{5\pi}{3})}{2m+3}\Bigg(\cos^{2m}(\alpha+\frac{5\pi}{3}) + \sum_{j=1}^{m} A(m,j)\cos^{2m-2j}(\alpha+\frac{5\pi}{3})\Bigg) \\
&+ \frac{\sin^3(\alpha+\pi)}{2m+3}\Bigg(\cos^{2m}(\alpha+\pi) + \sum_{j=1}^{m} A(m,j)\cos^{2m-2j}(\alpha+\pi)\Bigg)\Bigg],
\end{aligned}$$

where $A(m,j) = \dfrac{2^j m(m-1)...(m-j+1)}{(2m+1)(2m-1)...(2m-2j+3)}$.

Now, using the software Mathematica we can evaluate the expressions of $\varphi_{i23}(\alpha)$ with $i = 2m+1$, for some values of m, below we give some of them ($m = 1, 2, ..., 7$),

(i) $\varphi_{323}(\alpha) = 1/8\sin(3\alpha)$,
(ii) $\varphi_{523}(\alpha) = -61/1.120\sin(3\alpha)$,
(iii) $\varphi_{723}(\alpha) = -31/630\sin(3\alpha) + 1/384\sin(9\alpha)$,
(iv) $\varphi_{923}(\alpha) = -26.123/887.040\sin(3\alpha) + 7/1.536\sin(9\alpha)$,
(v) $\varphi_{11,23}(\alpha) = (1/92.252.160)(-1.612.309\sin(3\alpha) + 389.550\sin(9\alpha))$,
(vi) $\varphi_{13,23}(\alpha) = (1/5.535.129.600)(-60.534.421\sin(3\alpha) + 17.493.735\sin(9\alpha) + 135.135\sin(15\alpha))$,
(vi) $\varphi_{15,23}(\alpha) = (1/376.388.812.800)(-2.715.648.724\sin(3\alpha) + 811.055.280\sin(9\alpha) + 29.864.835\sin(15\alpha))$.

From the expressions of $\varphi_{i23}(\alpha)$ for $i = 1, 2, ...10, ..$ we conclude that

$$\varphi_{i23}(\alpha) = a_m\sin(3\alpha) + b_m\sin(9\alpha) + c_m\sin(15\alpha) + ...,$$

with $a_m, b_m, c_m \in \mathbb{R}$. Therefore the roots of these functions are $0, -\frac{\pi}{3}, \frac{\pi}{3}$. □

Working as in the case $l = 2$ we can prove the following lemma

Lemma 14. *The function φ_{i33} do not vanish for $\alpha \neq \pm\frac{\pi}{6}$ and i even. If i is odd we have $\varphi_{i33} \equiv 0$.*

Proof. We have

$$\begin{aligned}
\varphi_{i33}(\alpha) &= \sum_{k=1}^{3} \int_{\alpha+\frac{2(k-1)\pi}{3}}^{\alpha+\frac{(2k-1)\pi}{3}} \cos^i\theta \sin^3\theta d\theta \\
&= \frac{1}{i+1}\Big(2\cos^{i+1}\alpha + 2\sin^{i+1}\Big(\alpha-\frac{\pi}{6}\Big) - 2\sin^{i+1}\Big(\alpha+\frac{\pi}{6}\Big)\Big) \\
&+ \frac{1}{i+3}\Big(-2\cos^{i+3}\alpha - 2\sin^{i+3}\Big(\alpha-\frac{\pi}{6}\Big) + 2\sin^{i+3}\Big(\alpha+\frac{\pi}{6}\Big)\Big).
\end{aligned}$$

So $\varphi_{i33}(\alpha) \equiv 0$ if i is odd. If i is even the function $\varphi_{i33}(\alpha)$ vanishes only at $\alpha = \pm\frac{\pi}{6}$. The prove of this fact follows from similar arguments used in the study of function φ_{i13}. □

Proof of Theorem 1 (iii). From Lemmas 12 – 14 we conclude that, in the case $l = 3$ the averaged function becomes

$$\begin{aligned}
f(r) = &\sum_{\substack{i=2\\ i\text{even}}}^{n_1} r^i A_i(\alpha)(a_i - d_i) + \sum_{\substack{i=0\\ i\text{even}}}^{n_2} r^{i+1}(b_i B_i(\alpha) + e_i\overline{B}_i(\alpha)) \\
&+ \sum_{\substack{i=1\\ i\text{odd}}}^{n_2} r^{i+1} C_i(\alpha)(b_i - e_i) + \sum_{\substack{i=2\\ i\text{even}}}^{n_3} r^{i+2} D_i(\alpha)(c_i - m_i) \\
&+ (E^1(\alpha)d_0^1 + E^2(\alpha)d_0^2)r^3,
\end{aligned}$$

where $B_i(\alpha), \overline{B}_i(\alpha) > 0$ for all α, $C_i(\alpha)$ do not vanish if $\alpha \notin \{0, \pm\frac{\pi}{3}\}$ and $A_i(\alpha), D_i(\alpha)$ are different from zero if $\alpha \neq \pm\frac{\pi}{6}$.

Then $f(r)$ is a polynomial function of degree m where

1) $m = \max\left\{2\left[\frac{\mathrm{n}_1}{2}\right], 2\left[\frac{\mathrm{n}_2}{2}\right]+1, 2\left[\frac{\mathrm{n}_3+2}{2}\right], 3\right\}$ if $\alpha \in \left\{0, -\frac{\pi}{3}, \frac{\pi}{3}\right\}$;
2) $m = \max\{\mathrm{n}_2 + 1, 3\}$ if $\alpha \in \left\{-\frac{\pi}{6}, \frac{\pi}{6}\right\}$;
3) $m = \max\left\{2\left[\frac{\mathrm{n}_1}{2}\right], \mathrm{n}_2 + 1, 2\left[\frac{\mathrm{n}_3+2}{2}\right], 3\right\}$ if $\alpha \notin \left\{0, \pm\frac{\pi}{6}, \pm\frac{\pi}{3}\right\}$.

Applying the averaging theory it follows that an upper bound to the number of limit cycles of system (3) when $l = 3$ is given by $m(3) = \max\left\{2\left[\frac{\mathrm{n}_1}{2}\right], \mathrm{n}_2 + 1, 2\left[\frac{\mathrm{n}_3+2}{2}\right], 3\right\}$.

We remark that can choose the coefficients of the perturbation such that the averaged function $f(r)$ has exactly $m(3)$ positive roots. Then as discussed previously the proof of Theorem 1 is concluded. □

4. Examples

In this section we illustrate Theorem 1 studying the existence of periodic solutions for three generalized Kukles discontinuous piecewise polynomial differential systems.

Example 15 (One line of discontinuity)**.** *Consider $l = 1$, $\alpha = 0$ and the functions*

$$\begin{array}{llll} f^1_{n_1}(x) = & 24 + x + x^2, & f^2_{n_1}(x) = & 12 + 2x - \frac{103}{2}x^2, \\ g^1_{n_2}(x) = & 2x + x^2, & g^2_{n_2}(x) = & -\frac{100}{\pi} + x - \frac{(80+\pi)}{\pi}x^2, \\ h^1_{n_3}(x) = & 1 + 3x + 15x^2, & h^2_{n_3}(x) = & 1 - 3x - \frac{45}{4}x^2. \end{array}$$

Take the constants $d^1_0 = 1$ and $d^2_0 = -1$. Under this conditions we have $n_1 = n_2 = n_3 = 2$, $m = 4$ and the discontinuous piecewise system

$$Z(x,y) = \begin{cases} X_1(x,y) & \text{if } h(x,y) > 0, \\ X_2(x,y) & \text{if } h(x,y) < 0. \end{cases} \tag{11}$$

where $h(x,y) = y$,

$$X_1(x,y) = \begin{pmatrix} y \\ -x - \varepsilon F_1(x,y) \end{pmatrix} \text{ and } X_2(x,y) = \begin{pmatrix} y \\ -x - \varepsilon F_2(x,y) \end{pmatrix},$$

with

$$F_1(x,y) = 24 + x + x^2 + (2x + x^2)y + (1 + 3x + 15x^2)y^2 + y^3,$$

and

$$F_2(x,y) = 12+2x-\frac{103}{2}x^2+(-\frac{100}{\pi}+x-\frac{(80+\pi)}{\pi}x^2)y+(1-3x-\frac{45}{4}x^2)y^2-y^3.$$

Therefore the averaged *function is given by*

$$\begin{aligned} f(r) = {} & 24\int_0^{\pi} \sin\theta d\theta + 12\int_{\pi}^{2\pi} \sin\theta d\theta - \frac{100}{\pi}r\int_{\pi}^{2\pi} \sin^2\theta d\theta + \\ & + r^2\Big(\int_0^{\pi} \cos^2\theta\sin\theta d\theta + \int_0^{\pi}\sin^3\theta d\theta + \int_{\pi}^{2\pi}\sin^3\theta d\theta + \\ & - \frac{103}{2}\int_{\pi}^{2\pi}\sin\theta\cos^2\theta d\theta\Big) + r^3\Big(\int_0^{\pi}\cos^2\theta\sin^2\theta d\theta + \\ & + \int_0^{\pi}\sin^4\theta d\theta - \frac{(80+\pi)}{\pi}\int_{\pi}^{2\pi}\sin^2\theta\cos^2\theta d\theta\Big) + \\ & + r^4\Big(15\int_0^{\pi}\cos^2\theta\sin^3\theta d\theta + \frac{45}{4}\int_{\pi}^{2\pi}\sin^3\theta\cos^2\theta d\theta\Big) = \\ & = r^4 - 10r^3 + 35r^2 - 50r + 24, \end{aligned}$$

whose roots are $r = 1, 2, 3, 4$. *Hence by Theorem* (2) *it follows that for* $\varepsilon \neq 0$ *sufficiently small the discontinuous differential system* (11) *has at least four limit cycles.*

Example 16 (Two lines of discontinuity). *Consider* $l = 2$, $\alpha = 0$, *the functions*

$$\begin{array}{llll} f^1_{n_1}(x) = & 4 + 2x - 3x^2 + 2x^3, & f^2_{n_1}(x) = & 7 + 3x + 9x^2 + 2x^3, \\ g^1_{n_2}(x) = & 3 + 5x, & g^2_{n_2}(x) = & -3 + 2x, \\ h^1_{n_3}(x) = & 9 + x, & h^2_{n_3}(x) = & 5 - x. \end{array}$$

Taking the constants $d^1_0 = 1$ *and* $d^2_0 = -1$ *we have* $n_1 = 3, n_2 = n_3 = 1$ *and* $m = 3$. *Therefore the get the discontinuous system*

$$Z(x,y) = \begin{cases} X_1(x,y) & \text{if } h(x,y) > 0, \\ X_2(x,y) & \text{if } h(x,y) < 0, \end{cases} \tag{12}$$

where $h(x,y) = y$,

$$X_1(x,y) = \begin{pmatrix} y \\ -x - \varepsilon F_1(x,y) \end{pmatrix} \text{ and } X_2(x,y) = \begin{pmatrix} y \\ -x - \varepsilon F_2(x,y) \end{pmatrix},$$

with

$$F_1(x,y) = 4 + 2x - 3x^2 + 2x^3 + (3 + 5x)y + (9 + x)y^2 + y^3,$$

and

$$F_2(x,y) = 7 + 3x + 9x^2 + 2x^3 + (-3 + 2x)y + (5 - x)y^2 - y^3.$$

The averaged *function for this system is given by*

$$f(r) = r^3 - r,$$

whose roots are $r = 0, 1, -1$. *Hence by Theorem 2 it follows that for* $\varepsilon \neq 0$ *sufficiently small the discontinuous differential system* (12) *has at least one limit cycle.*

Example 17 (Three lines of discontinuity). *Consider* $l = 3$, $\alpha = \frac{\pi}{4}$, *the functions*

$$\begin{array}{llll} f^1_{n_1}(x) = & 3 + 2x + x^2, & f^2_{n_1}(x) = & -2 + 7x + x^2, \\ g^1_{n_2}(x) = & 2x, & g^2_{n_2}(x) = & -\frac{12}{\pi} + (2 - 22\sqrt{2})x, \\ h^1_{n_3}(x) = & 1 + 3x + 2x^2, & h^2_{n_3}(x) = & 1 + 2x + (2 + 8\sqrt{2})x^2, \end{array}$$

and the constants $d^1_0 = d^2_0 = -\frac{8}{\pi}$. *In this case* $n_1 = n_3 = 2, n_2 = 1$, $m = 4$ *and the discontinuous system is*

$$Z(x,y) = \begin{cases} X_1(x,y) & \text{if } h(x,y) > 0, \\ X_2(x,y) & \text{if } h(x,y) < 0. \end{cases} \tag{13}$$

where $h(x, y) = y$,

$$X_1(x, y) = \begin{pmatrix} y \\ -x - \varepsilon F_1(x, y) \end{pmatrix} \text{ and } X_2(x, y) = \begin{pmatrix} y \\ -x - \varepsilon F_2(x, y) \end{pmatrix},$$

with

$$F_1(x, y) = 3 + 2x + x^2 + 2xy + (1 + 3x + 2x^2)y^2 - \frac{8}{\pi}y^3,$$

and

$$\begin{aligned} F_2(x, y) = & -2 + 7x + x^2 + (-\frac{12}{\pi} + (2 - 22\sqrt{2})x)y + \\ & + (1 + 2x + (2 + 8\sqrt{2})x^2)y^2 - \frac{8}{\pi}y^3. \end{aligned}$$

The averaged *function is given by*

$$f(r) = r^4 - 6r^3 + 11r^2 - 6r,$$

whose roots are $r = 0, 1, 2, 3$. *Hence by Theorem 2 it follows that for* $\varepsilon \neq 0$ *sufficiently small the discontinuous differential system* (13) *has at least three limit cycles.*

5. Conjecture

From the study of the system (3) with 1, 2 or 3 lines of discontinuity we observe some particularities but because the computations are becoming increasingly complicated as we increase the number of lines we cannot established a general expression for the averaged function (9) with many lines of discontinuity. Based on our study we establish the following conjecture.

Conjecture 18. *Assume that* $j = 1, 2$, *the polynomials* $f^j_{n_1}(x)$, $g^j_{n_2}(x)$ *and* $h^j_{n_3}(x)$ *have degree* $n_1 \geq 1$, $n_2 \geq 1$ *and* $n_3 \geq 1$ *respectively,* d^j_0 *is a nonzero constant and* $l \in \mathbb{N}$. *Then for* $|\varepsilon|$ *sufficiently small the generalized Kukles discontinuous piecewise polynomial differential system* (3) *has at most* $m(l)$ *limit cycles, where*

i) $m(l) = \max\left\{2\left[\frac{n_1}{2}\right], n_2 + 1, 2\left[\frac{n_3 + 2}{2}\right], 3\right\}$ *if* l *is odd;*

ii) $m(l) = \max\left\{\left[\frac{n_1 - 1}{2}\right], \left[\frac{n_2}{2}\right], \left[\frac{n_3 + 1}{2}\right], 1\right\}$ *if* l *is even;*

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