A POSTERIORI ERROR ANALYSIS OF CRANK-NICOLSON FINITE ELEMENT METHOD FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

G MURALI MOHAN REDDY* AND RAJEN K. SINHA†

Abstract. We study *a posteriori* error analysis for the space-time finite element discretizations of linear parabolic integro-differential equations in a bounded convex polygonal or polyhedral domain. While the space discretization uses finite element spaces that are allowed to change in time, the time discretization is based on the Crank-Nicolson method. A novel Ritz-Volterra reconstruction operator [2013, IMA J. Numer. Anal., doi:10.1093/imanum/drt059, pp. 1-31], a generalization of elliptic reconstruction operator [2003, SIAM J. Numer. Anal., 41, pp. 1585-1594], is used in a crucial way to obtain optimal rate of convergence in space. In addition, a quadratic (in time) space-time reconstruction operator is introduced to establish second order convergence in time. We use energy method to derive optimal order error estimator in the $L^\infty(L^2)$-norm. Numerical experiment is performed to validate the optimality of the derived error estimators.

Key words. Parabolic integro-differential equations, finite element method, Crank-Nicolson method, *a posteriori* error estimate

AMS subject classifications. 65M15, 65M60

1. Introduction. The main objective of this article is to study *a posteriori* error analysis by means of Crank-Nicolson finite element method for the linear parabolic integro-differential equations (PIDE) of the form

$$\begin{aligned} u_t(x,t) + \mathcal{A}u(x,t) &= \int_0^t \mathcal{B}(t,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega \times (0,T], \\ u(x,t) &= 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) &= u_0(x), \quad x \in \Omega. \end{aligned} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^d, d \geq 1$ is a bounded convex polygonal or polyhedral domain with boundary $\partial\Omega$ and $u_t(x,t) = \frac{\partial u}{\partial t}(x,t)$ with $T < \infty$. Further, $\mathcal{A}$ is a self-adjoint, uniformly positive definite second-order linear elliptic partial differential operator of the form

$$\mathcal{A}u = -\nabla \cdot (A\nabla u)$$

and the operator $\mathcal{B}(t,s)$ is of the form

$$\mathcal{B}(t,s)u = -\nabla \cdot (B(t,s)\nabla u),$$

where "∇" denotes the spatial gradient and and $A = \{a_{ij}(x)\}$ and $B(t,s) = \{b_{ij}(x;t,s)\}$ are two $d \times d$ matrices assumed to be in $L^\infty(\Omega)^{d\times d}$ in space variable. Moreover, the elements of $B(t,s)$ are assumed to be smooth in both t and s. The initial function $u_0(x)$ and the nonhomogeneous term $f(x,t)$ are assumed to be smooth for our purpose.

Such problems and variants of them arise in various applications, such as heat conduction in material with memory [8], the compression of poro-viscoelasticity media [9], nuclear reactor dynamics [13, 14] and the epidemic phenomena in biology [6]. For

*Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati - 781039, India (mmreddy@iitg.ernet.in).

†Corresponding author: Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati - 781039, India (rajen@iitg.ernet.in).

the existence, uniqueness and regularity results of such problems, one may refer to [18] and references therein.

In the recent years, *a posteriori* error analysis for the finite element methods for partial differential equations has been thoroughly studied by several researchers [1, 2, 3, 4, 7, 10, 11, 12, 17]. While much of interest has focussed on elliptic and parabolic problems, relatively less progress has been made in the direction of *a posteriori* error analysis of PIDE [15]. In order to put the results of the paper into proper prospective, we give a brief account of the relevant literature and motivation for the present investigation. In the absence of the memory term i.e., when $\mathcal{B}(t,s) = 0$, *a posteriori* error analysis for linear parabolic problems have been investigated by several authors in [2, 3, 4, 7, 10, 11, 12, 17]. In particular, for the fully discrete Crank-Nicolson method for the heat equation, a continuous, piecewise linear approximation in time is used to obtain suboptimal (with respect to time step) *a posteriori* error bounds using standard energy techniques in [17]. A continuous, piecewise quadratic polynomial so-called Crank-Nicolson reconstruction is then introduced in [2] to restore the second order of convergence for semidiscrete time discretization of a general parabolic problem. Subsequently, the authors of [11] have introduced the reconstruction based on approximations on one time level (two-point reconstruction) as in [2] as well as the reconstructions based on approximations on two time levels (three-point reconstruction) in the $L^2(H^1)$-norm. Recently in [3], for parabolic problems, the elliptic reconstruction in combination with energy techniques are used to derive optimal order *a posteriori* error estimate for Crank-Nicolson method in the $L^\infty(L^2)$-norm.

Since PIDE (1.1) may be thought of as a perturbation of parabolic equation, an attempt has been made to carry over *a posteriori* error analysis of parabolic problems [3] to PIDE (1.1). We wish to remark that such an extension is not straightforward due to the presence of the Volterra integral term in (1.1). To the best of our knowledge no article is available in the literature concerning *a posteriori* error analysis of Crank-Nicolson method for PIDE in $L^\infty(L^2)$-norm. We derive *a posteriori* bounds for PIDE in the $L^\infty(L^2)$-norm of the error for the fully discrete Crank-Nicolson scheme. The optimality in space hinges essentially on this Ritz-Volterra reconstruction operator [15] whereas a quadratic (in time) space-time reconstruction operator is introduced to establish second order *a posteriori* error estimator in time. Here, we emphasis the fact that choice of such a quadratic space-time reconstruction operator is non-trivial and heavily problem dependent. We allow only nested refinement of the space meshes to avoid further complications arise because the Volterra integral term memorizes the jumps over all element edges in all previous space meshes.

The rest of the paper is organized as follows. In Section 2, we introduce some standard notations and preliminary materials to be used in the subsequent sections. Section 3 is devoted to introduce quadratic space-time reconstructions for PIDE. In Section 4, *a posteriori* error estimate for the fully discrete Crank-Nicolson scheme in $L^\infty(L^2)$-norm is derived. Numerical results are presented in Section 5.

2. Notations and preliminaries. Given a Lebesgue measurable set $\omega \subset \mathbb{R}^d$, we denote by $L^p(\omega)$, $1 \leq p \leq \infty$, the Lebesgue spaces with corresponding norms $\|\cdot\|_{L^p(\omega)}$. When $p = 2$, the space $L^2(\omega)$ is equipped with inner product $\langle\cdot,\cdot\rangle_\omega$ and the induced norm $\|\cdot\|_{L^2(\omega)}$. Whenever $\omega = \Omega$, we remove the subscripts of $\|.\|_{L^2(\omega)}$ and $\langle\cdot,\cdot\rangle\omega$. Further, we shall use the standard notation for Sobolev spaces $W^{m,p}(\omega)$ with

$1 \le p \le \infty$. The norm on $W^{m,p}(\omega)$ is defined by

$$\|u\|_{m,p,\omega} = \left(\int_\omega \sum_{|\alpha| \le m} |D^\alpha u|^p dx \right)^{1/p}, \ 1 \le p < \infty$$

with the standard modification for $p = \infty$. When $p = 2$, we write $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and denote the norm by $\|\cdot\|_m$. In particular, $H_0^1(\Omega)$ signifies the space of functions in $H^1(\Omega)$ that vanish on the boundary of Ω (boundary values are taken in the sense of traces).

Let $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be the bilinear form corresponding to the elliptic operator $\mathcal{A}$ defined by

$$a(\phi, \psi) := \langle A\nabla\phi, \nabla\psi \rangle, \quad \forall\, \phi, \psi \in H_0^1(\Omega).$$

Similarly, let $b(t, s; \cdot, \cdot)$ be the bilinear form corresponding to the operator $\mathcal{B}(t, s)$ defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$b(t, s; v(s), \psi) := \langle B(t, s)\nabla v(s), \nabla\psi \rangle, \quad \forall\, v(s), \psi \in H_0^1(\Omega).$$

Let $b_s(t, s; \cdot, \cdot)$ and $b_{ss}(t, s; \cdot, \cdot)$ be the bilinear forms corresponding to the operators $\mathcal{B}_s(t, s)$ and $\mathcal{B}_{ss}(t, s)$ defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$b_s(t, s; \phi(s), \psi) := \langle B_s(t, s)\nabla\phi(s), \nabla\psi \rangle, \quad \forall\, \phi(s), \psi \in H_0^1(\Omega)$$

and

$$b_{ss}(t, s; \phi(s), \psi) := \langle B_{ss}(t, s)\nabla\phi(s), \nabla\psi \rangle, \quad \forall\, \phi(s), \psi \in H_0^1(\Omega),$$

where $B_s(t, s)$ and $B_{ss}(t, s)$ are obtained by differentiating $B(t, s)$ partially with respect to s once and twice respectively. We assume that the bilinear form $a(\cdot, \cdot)$ is coercive and continuous on $H_0^1(\Omega)$ i.e.,

$$a(\phi, \phi) \ge \alpha\|\phi\|_1^2 \quad \text{and} \quad |a(\phi, \psi)| \le \beta\|\phi\|_1\|\psi\|_1, \quad \forall\, \phi, \psi \in H_0^1(\Omega) \tag{2.1}$$

with $\alpha, \beta \in \mathbb{R}^+$.

Further, we assume that the bilinear forms $b(t, s; \cdot, \cdot)$, $b_s(t, s; \cdot, \cdot)$ and $b_{ss}(t, s; \cdot, \cdot)$ are continuous on $H_0^1(\Omega)$ i.e.,

$$|b(t, s; \phi(s), \psi)| \le \gamma\|\phi(s)\|_1\|\psi\|_1, \quad \forall\, \phi(s), \psi \in H_0^1(\Omega), \tag{2.2}$$

$$|b_s(t, s; \phi(s), \psi)| \le \gamma'\|\phi(s)\|_1\|\psi\|_1, \quad \forall\, \phi(s), \psi \in H_0^1(\Omega) \tag{2.3}$$

and

$$|b_{ss}(t, s; \phi(s), \psi)| \le \gamma''\|\phi(s)\|_1\|\psi\|_1, \quad \forall\, \phi(s), \psi \in H_0^1(\Omega) \tag{2.4}$$

with $\gamma, \gamma', \gamma'' \in \mathbb{R}^+$.

The weak formulation of the problem (1.1) may be stated as follows: Find $u : [0, T] \to H_0^1(\Omega)$ such that

$$\begin{aligned} &\int_\Omega u_t\phi dx + a(u, \phi) = \int_0^t b(t, s; u(s), \phi)ds + \int_\Omega f\phi dx, \ \ \forall\phi \in H_0^1(\Omega), \ \ t \in (0, T], \\ &u(., 0) = u_0. \end{aligned} \tag{2.5}$$

Let $0 = t_0 < t_1 < \ldots < t_N = T$ be a partition of $[0, T]$ with $\tau_n := t_n - t_{n-1}$ and $I_n := (t_{n-1}, t_n]$. For $t = t_n$, $n \in [0 : N]$, we set $f^n(\cdot) = f(\cdot, t_n)$. Let $(\mathcal{T}_n)_{n\in[0:N]}$ be a family of conforming triangulations of the domain Ω. Let $h_n(x) = \text{diam}(K)$, where $K \in \mathcal{T}_n$ and $x \in K$ denotes the local mesh-size function corresponds to each given triangulation $\mathcal{T}_n$. Let $\mathcal{S}_n$ denotes the set of internal sides of $\mathcal{T}_n$ representing edges in $d = 2$ or faces in $d = 3$, and $\sum_n$ denotes the union of all internal sides $\cup_{E\in\mathcal{S}_n} E$.

Each triangulation $(\mathcal{T}_n)$, for $n \in [1 : N]$, is a refinement of a macro-triangulation $\mathcal{M}$ of the domain Ω that satisfies the same conformity and shape-regularity assumptions during refinements (cf. [5]). We assume the following admissible criteria on $\mathcal{T}_n$ (cf. [10]):

1. The refined triangulation is conforming.
2. The shape-regularity of an arbitrary refinement depends only on the shape-regularity of the macro-triangulation $\mathcal{M}$.

We allow only nested refinement of the space meshes at each time level $t = t_n$, $n \in [0 : N]$.

Now, we associate with these triangulations the finite element spaces:

$$\mathbb{V}^n := \{\phi \in H_0^1(\Omega) : \ \phi|_K \in \mathbb{P}_l, \ \forall K \in \mathcal{T}_n\},$$

where $\mathbb{P}_l$ is the space of polynomials in d variables of degree atmost $l \in \mathbb{Z}^+$.

Let σ^n be the quadrature rule used to discretize the Volterra integral term. To be consistent with the Crank-Nicolson scheme, we use the trapezoidal rule given by

$$\begin{aligned} \sigma^n(y) &:= \sum_{j=0}^{n-2} \frac{\tau_{j+1}}{2}\big(y(t_j) + y(t_{j+1})\big) + \frac{\tau_n}{4}\big(y(t_{n-1}) + y(t_{n-1/2})\big) \\ &\approx \int_0^{t_{n-1/2}} y(s)ds. \end{aligned} \tag{2.6}$$

Throughout this paper the following notation will be used for $n = 1, 2, \cdots, N$

$$\partial v^n := \frac{v^n - v^{n-1}}{\tau_n}, \qquad \bar{\partial} v^n = P_0^n \partial v^n := \frac{v^n - P_0^n v^{n-1}}{\tau_n},$$
$$t_{n-1/2} := \frac{t_n + t_{n-1}}{2} \qquad \text{and} \qquad v^{n-1/2} := \frac{v^n + v^{n-1}}{2}.$$

Representation of the bilinear forms. For a function $v \in \mathbb{V}^n$, we can represent the bilinear form $a(\cdot, \cdot)$ as

$$a(v, \phi) = \langle \mathcal{A}_{el} v, \phi \rangle + \langle J_1[v], \phi \rangle_{\sum_n}, \quad \forall \phi \in H_0^1(\Omega), \tag{2.7}$$

where

$$\langle \mathcal{A}_{el} v, \phi \rangle = \sum_{K\in\mathcal{T}_n} \langle -\text{div}(A\nabla v), \phi \rangle, \quad \forall \phi \in H_0^1(\Omega)$$

is the regular part of the distribution $-\text{div}(A\nabla v)$ and

$$J_1[v]|_E(x) = [A\nabla v]_E(x) := \lim_{\varepsilon\to 0}(A\nabla v(x + \varepsilon\nu_E) - A\nabla v(x - \varepsilon\nu_E)) \cdot \nu_E \tag{2.8}$$

is the spatial jump of the field $A\nabla v$ across an element side $E \in \mathcal{S}_n$, where ν_E is a unit normal vector to E at the point x.

Similarly, for all $\phi \in H_0^1(\Omega)$, we represent the bilinear form $b(t_n, s; \cdot, \cdot)$ as

$$\int_0^{t_n} b(t_n, s; v(s), \phi)ds = \langle \int_0^{t_n} \mathcal{B}_{el}(t_n, s)v(s)ds, \phi\rangle + \langle \int_0^{t_n} J_2[v(s)]ds, \phi\rangle_{\sum_n}, \tag{2.9}$$

where $\mathcal{B}_{el}(t_n, s)v(s)$ is the regular part of the distribution $-\text{div}(B(t_n, s)\nabla v(s))$ and is defined as

$$\langle \mathcal{B}_{el}(t_n, s)v(s), \phi\rangle := \sum_{K\in\mathcal{T}_n} \langle -\text{div}(B(t_n, s)\nabla v(s)), \phi\rangle, \quad \forall \phi \in H_0^1(\Omega),$$

and $J_2[v(s)]$ is the spatial jump of the field $-\text{div}(B(t_n, s)\nabla v(s))$ across an element side $E \in \mathcal{S}_n$ as defined in (2.8) with $B(t_n, s)$ replacing A.

We define the fully discrete operators $\mathcal{A}^n : H_0^1(\Omega) \to \mathbb{V}^n$ and $\mathcal{B}^{n-r}(s) : H_0^1(\Omega) \to \mathbb{V}^n$, $0 \le r < 1$ by

$$\langle \mathcal{A}^n w, \chi_n\rangle = a(w, \chi_n) \ \ \forall \chi_n \in \mathbb{V}^n$$

and

$$\langle \mathcal{B}^{n-r}(s)w(s), \chi_n\rangle = b(t_{n-r}, s; w(s)\chi_n) \ \ \forall \chi_n \in \mathbb{V}^n, \quad s \in I_n.$$

Moreover, let $P_0^n : L^2(\Omega) \to \mathbb{V}^n$ denotes the L^2 projection operator and is given by

$$\langle P_0^n w, \chi_n\rangle = \langle w, \chi_n\rangle, \ \ \forall \chi_n \in \mathbb{V}^n.$$

The fully discrete Crank-Nicolson scheme may be stated as follows: Given $U^0 = P_0^0 u(0)$, find $U^n \in \mathbb{V}^n, n \in [1 : N]$ such that

$$\begin{aligned} &\int_\Omega \partial U_h^n \phi_n dx + a(U^{n-1/2}, \phi_n) \\ &\qquad = \sigma^n(b(t_{n-1/2}; U, \phi_n)) + \int_\Omega f^{n-1/2}\phi_n dx, \ \ \forall \phi_n \in \mathbb{V}^n. \end{aligned} \tag{2.10}$$

Here, the quadrature rule σ^n is as defined in (2.6). Let U be a continuous, piecewise linear approximation in time defined for all $t \in I_n$ by

$$U(t) := l_n(t)U^n + l_{n-1}(t)U^{n-1}, \tag{2.11}$$

where

$$l_n(t) := \frac{(t - t_{n-1})}{\tau_n} \quad \text{and} \quad l_{n-1}(t) := \frac{(t_n - t)}{\tau_n}. \tag{2.12}$$

Following [15], we recall the definition of Ritz-Volterra reconstruction operator below.

Definition 2.1 (Ritz-Volterra reconstruction). *We define the Ritz-Volterra reconstruction $\mathcal{W}^n \in H_0^1(\Omega)$ of $v \in H_0^1(\Omega)$ to be a solution of the following elliptic Volterra integral equation in the weak form*

$$a(\mathcal{W}^n, \chi) = \langle g^n, \chi\rangle + \int_0^{t_n} b(t_n, s; \mathcal{W}(s), \chi)ds, \quad \forall \chi \in H_0^1, \tag{2.13}$$

where g^n is given by

$$g^n = \mathcal{A}^n v - \int_0^{t_n} \mathcal{B}^n(s)v(s)ds, \quad v \in H_0^1(\Omega).$$

We denote the Ritz-Volterra reconstruction of a function $v \in H_0^1(\Omega)$ by $\mathcal{R}_w v$.
Remark. The Galerkin orthogonality property holds for the Ritz-Volterra reconstruction

$$a(\mathcal{W}^n - v, \phi_n) - \int_0^{t_n} b(t_n, s; (\mathcal{W} - v)(s), \phi_n) = 0, \;\; \forall \phi_n \in \mathbb{V}^n. \tag{2.14}$$

We use the following definitions in the subsequent error analysis. For $t \in I_n$, we define the Ritz-Volterra reconstructions of $U(t)$ by

$$\mathcal{R}_w U(t) := l_{n-1}(t)\mathcal{R}_w^{n-1} U^{n-1} + l_n(t)\mathcal{R}_w^n U^n, \tag{2.15}$$

where $l_{n-1}(t)$ and $l_n(t)$ are given by (2.12). Now, set

$$\hat{\omega}(t) := \int_0^t B(t,s)\nabla \mathcal{R}_w U(s) ds. \tag{2.16}$$

For $t \in I_n$, define $\hat{\omega}_I(t)$ to be the linear interpolant associated with the integral vectors $\hat{\omega}(t_{n-1})$ and $\hat{\omega}(t_n)$ and is given by

$$\hat{\omega}_I(t) := l_{n-1}(t)\hat{\omega}(t_{n-1}) + l_n(t)\hat{\omega}(t_n). \tag{2.17}$$

Further, let

$$\hat{\mathcal{U}}(t) := \int_0^t \mathcal{B}(t,s)U(s)ds, \;\; t \in I_n. \tag{2.18}$$

For $t \in I_n$, we define $\hat{\mathcal{U}}_{I,1}(t)$ to be the linear interpolant associate with the integrals $\hat{\mathcal{U}}(t_n)$, $\hat{\mathcal{U}}(t_{n-1})$ and $\hat{\mathcal{U}}_{I,2}(t)$ to be the linear interpolant associate with that of the integral $\hat{\mathcal{U}}(t_{n-1/2})$ as follows:

$$\hat{\mathcal{U}}_{I,1}(t) := l_{n-1}(t)\hat{\mathcal{U}}(t_{n-1}) + l_n(t)\hat{\mathcal{U}}(t_n) \tag{2.19}$$

and

$$\begin{aligned} \hat{\mathcal{U}}_{I,2}(t) &:= \hat{\mathcal{U}}(t_{n-1/2}) + (t - t_{n-1/2})\frac{d}{dt}\hat{\mathcal{U}}(t)\big|_{t=t_n}, \\ &:= \hat{\mathcal{U}}(t_{n-1/2}) + (t - t_{n-1/2})\mathcal{Y}_n, \end{aligned} \tag{2.20}$$

where

$$\mathcal{Y}_n = \frac{d}{dt}\hat{\mathcal{U}}(t)\big|_{t=t_n}. \tag{2.21}$$

We define the jump residual, for $n \in [0:N]$, as

$$\begin{aligned} \mathfrak{J}^n[U] &:= J_1[U^n] - \int_0^{t_n} J_2[U(s)]ds, \\ \mathfrak{J}^0[U] &:= J_1[U^0]. \end{aligned} \tag{2.22}$$

Modified Crank-Nicolson scheme: In the case of parabolic problem, the authors in [3] have observed that during refinements the discrete Laplace operator on the finer mesh when applied to coarse grid function leads to the oscillatory behaviour. Since PIDE may be thought of as the perturbation to the parabolic problem, the same oscillatory behaviour is expected concerning the classical Crank-Nicolson scheme for the PIDE (1.1). Therefore, we consider the following modified Crank-Nicolson scheme instead of the classical Crank-Nicolson scheme.

For $n, 1 \le n \le N$, find $U^n \in \mathbb{V}^n$ such that

$$\bar{\partial} U^n + \frac{1}{2}\mathcal{A}^n U^n + \frac{1}{2} P_0^n \mathcal{A}^{n-1} U^{n-1} - P_0^n(\sigma^n(\mathcal{B}^{n-1/2}U)) - P_0^n f^{n-1/2} = 0. \tag{2.23}$$

3. Quadratic (in time) space-time reconstructions. In this section, we start with some notations necessary to introduce quadratic space-time reconstruction. Moreover, these notations will be fruitful in the error analysis in Section 4. Let $\Theta : [0,T] \to H_0^1(\Omega)$ be defined by

$$\Theta(t) := l_{n-1}(t) P_0^n \mathcal{A}^{n-1} U^{n-1} + l_n(t) \mathcal{A}^n U^n - P_0^n(\sigma^n(\mathcal{B}^{n-1/2}U)), \ t \in I_n. \tag{3.1}$$

Define $\hat{F} : [0,T] \to H_0^1(\Omega)$ by

$$\hat{F}(t) := \Theta(t) - P_0^n \varphi(t), \ t \in I_n, \tag{3.2}$$

where $\varphi(t) := \hat{I} f(t)$. Here, $\hat{I}$ is a piecewise linear interpolant chosen such that

$$\hat{I}(\phi) \in \mathbb{P}_1(I_n), \ \ \hat{I}(\phi)(t_{n-1}) = \phi^{n-1} \ \text{ and } \ \hat{I}(\phi)(t_{n-1/2}) = \phi^{n-1/2}. \tag{3.3}$$

We now define the quadratic space-time reconstruction $\hat{U} : [0,T] \to H_0^1(\Omega)$ as follows

$$\begin{aligned} \hat{U}(t) := {} & \mathcal{R}_w^{n-1} U^{n-1} - \mathcal{R}_w^n \int_{t_{n-1}}^t \hat{F}(s) ds \\ & + (t - t_{n-1}) \frac{\mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^{n-1} U^{n-1}}{\tau_n}, \ \ t \in I_n. \end{aligned} \tag{3.4}$$

This definition is motivated by the fact that $\hat{U}(t)$ satisfies the following relation:

$$\hat{U}_t(t) + \mathcal{R}_w^n \hat{F}(t) = \frac{\mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^{n-1} U^{n-1}}{\tau_n}. \tag{3.5}$$

Observe that

$$\hat{U}(t_{n-1}) = \mathcal{R}_w^{n-1} U^{n-1}$$

and

$$\begin{aligned} \hat{U}(t_n) = {} & \mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^n \tau_n \Big[\frac{P_0^n \mathcal{A}^{n-1} U^{n-1} + \mathcal{A}^n U^n}{2} - P_0^n f^{n-1/2} \\ & - P_0^n \sigma^n(\mathcal{B}^{n-1/2} U)) \Big] = \mathcal{R}_w^n U^n, \end{aligned}$$

where we have used (2.23) and the integral is evaluated using the mid-point rule. Equivalently, the modified Crank-Nicolson scheme can be rewritten as

$$\bar{\partial} U^n + \Theta^{n-1/2} = P_0^n f^{n-1/2}. \tag{3.6}$$

In view of (3.2)

$$\bar{\partial} U^n + F^{n-1/2} = 0, \tag{3.7}$$

where $\hat{F}(t_{n-1/2}) := F^{n-1/2}$.

4. Error analysis. In this section, we shall derive *a posteriori* error estimate for the error $e(t) := u(t) - U(t)$, where u is the exact solution of the PIDE (1.1) and U is defined in (2.11). We now decompose the total error $e(t)$ as $e(t) := \hat{\rho}(t) - \varepsilon(t)$, where $\hat{\rho}(t) := u(t) - \hat{U}(t)$ denotes the parabolic error and $\varepsilon(t) := U(t) - \hat{U}(t)$ denotes the reconstruction error. Further, we decompose the reconstruction error as $\varepsilon(t) := \epsilon(t) - \sigma(t)$, where $\epsilon(t) := U(t) - \mathcal{R}_w U(t)$ is the Ritz-Volterra reconstruction error and $\sigma(t) := \hat{U}(t) - \mathcal{R}_w U(t)$ is the time reconstruction error. With the above decompositions, the error $e(t)$ may be expressed as

$$e(t) := \hat{\rho}(t) + \sigma(t) - \epsilon(t). \tag{4.1}$$

The idea behind the above error decomposition is as follows: (i) optimal order *a posteriori* error estimates for Ritz-Volterra reconstruction error $\epsilon(t)$ in standard norms like L^2 and H^1 can be obtained; (ii) the parabolic error $\hat{\rho}(t)$ satisfies a variant of the original PIDE (1.1) with a right hand side that can be controlled *a posteriori* in an optimal way; (iii) the time reconstruction $\hat{U}$ is chosen in such a way that the difference $\hat{U}(t) - \mathcal{R}_w U(t)$ can be estimated *a posteriori* and will be of $O(\tau^2)$.

We now recall from [16] the following interpolation error estimates.

PROPOSITION 4.1. *Let $\Pi^n : H_0^1(\Omega) \to \mathbb{V}^n$ be the Clément-type interpolation operator. Then, for sufficiently smooth ψ and finite element polynomial space of degree l, there exist constants $C_{1,j}$ and $C_{2,j}$ depending only upon the shape-regularity of the family of triangulations such that for $j \leq l+1$*

$$\|h_n^{-j}(\psi - \Pi^n \psi)\| \leq C_{1,j} \|\psi\|_j,$$

and

$$\|h_n^{1/2-j}(\psi - \Pi^n \psi)\|_{\sum_n} \leq C_{2,j} \|\psi\|_j.$$

Below, we now state the following *a posteriori* error estimates for the Ritz-Volterra reconstruction error. For a proof, we refer to Lemma 4.2 of [15].

LEMMA 4.2 (Ritz-Volterra reconstruction error estimates).
For any $v \in \mathbb{V}^n$, the following estimates hold:

$$\|\mathcal{R}_w^n v - v\|_1 \leq \alpha_n(v)$$

and

$$\|\mathcal{R}_w^n v - v\| \leq \beta_n(v),$$

where

$$\begin{aligned} \alpha_n(v) := C_1 h_n \|\mathcal{A}^n v - \mathcal{A}_{el} v - \int_0^{t_n} \mathcal{B}^n(s) v(s) ds \\ + \int_0^{t_n} \mathcal{B}_{el}(t_n, s) v(s) ds\| + C_2 h_n^{1/2} \|\mathfrak{J}^n[v]\|_{\sum_n} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \beta_n(v) := C_3 h_n^2 \|\mathcal{A}^n v - \mathcal{A}_{el} v - \int_0^{t_n} \mathcal{B}^n(s) v(s) ds \\ + \int_0^{t_n} \mathcal{B}_{el}(t_n, s) v(s) ds\| + C_4 h_n^{3/2} \|\mathfrak{J}^n[v]\|_{\sum_n} \end{aligned} \tag{4.3}$$

are the Ritz-Volterra reconstruction error estimators. Moreover, the constants appeared in the estimators are positive constants depend upon the interpolation constants and the final time T.

We state the main result of this section concerning fully discrete Crank-Nicolson *a posteriori* error estimate in the $L^\infty(L^2)$-norm.

THEOREM 4.3 ($L^\infty(L^2)$ *a posteriori* error estimate). *Let $u(t)$ be the exact solution of (1.1) and $U(t)$ be as defined in (2.11). Then, for each $m \in [1:N]$, the following error estimate hold:*

$$\max_{t\in[0,t_m]} \|u(t) - U(t)\| \leq \Big[\|\hat{\rho}(t_0)\|^2 + C_7 \sum_{n=1}^{m} \tau_n \Lambda_n^2\Big]^{1/2} + \left(\Xi_{1,m}^2 + \Xi_{2,m}^2\right)^{1/2} + \max_{0\leq n\leq m} \beta_n(U^n) + \max_{0\leq n\leq m} \nu_n,$$

where ν_n and Λ_n are the time reconstruction error estimators and are defined by

$$\nu_n := \tau_n^2\Big[\beta_n(\mathcal{W}_n) + \|\mathcal{W}_n\|\Big] \tag{4.4}$$

and

$$\Lambda_n := \frac{\beta\tau_n^2}{\sqrt{30}\alpha}\Big[\alpha_n(\mathcal{W}_n) + \|\mathcal{W}_n\|_1\Big]. \tag{4.5}$$

Here, $\alpha_n(\mathcal{W}_n)$ and $\beta_n(\mathcal{W}_n)$ are given by (4.2) and (4.3) respectively. $\mathcal{W}_n$ is an a posteriori quantity given by

$$\mathcal{W}_n := \left[\frac{1}{2}\bar{\partial}\mathcal{A}^n U^n - \frac{P_0^n\left(f^{n-1/2} - f^{n-1}\right)}{\tau_n}\right]. \tag{4.6}$$

$\Xi_{1,m}^2$ and $\Xi_{2,m}^2$ are the total estimators corresponding to parabolic error $\hat{\rho}(t)$ and are defined by

$$\Xi_{1,m}^2 := \left(C_7 \sum_{n=1}^{m} \tau_n \Big[\lambda_n + \eta_n + \zeta_n + \vartheta_{n,1}\Big]\right)^2 \tag{4.7}$$

and

$$\Xi_{2,m}^2 := \frac{4(C_7)^2}{\alpha} \sum \tau_n \Big[\mu_n + \xi_n + \vartheta_{n,2}\Big]^2. \tag{4.8}$$

Here, λ_n is the time estimator which captures quadrature error and linear approximation errors and, is defined by

$$\lambda_n := C_8\left[\theta_n + \|\hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t)\| + \|\hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t)\|\right], \tag{4.9}$$

where $\hat{\mathcal{U}}(t)$, $\hat{\mathcal{U}}_{I,1}(t)$ and $\hat{\mathcal{U}}_{I,2}(t)$ are given by (2.18), (2.19) and (2.20), respectively and θ_n is given by

$$\theta_n := \sum_{j=0}^{n} \tau_j^2 \left[\tau_j\|\Delta^n U^j\| + \tau_j\|\Delta^n \partial U^j\|\right]. \tag{4.10}$$

$$
\begin{aligned}
(4.11)\quad \zeta_n := C_9 C_\Omega\left(\frac{\hat{\tau}_n}{\tau_n}\right)\Bigg[h_n^2\|\tau_n^{-1}\Big(\mathcal{A}^n U^n - \mathcal{A}^{n-1}U^{n-1} + \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s)U(s)ds \\
- \int_0^{t_n} \mathcal{B}^n(s)U(s)ds + \mathcal{A}_{el}U^{n-1} - \mathcal{A}_{el}U^n + \int_0^{t_{n-1}} \mathcal{B}_{el}(t_{n-1},s)U(s)ds \\
- \int_0^{t_n} \mathcal{B}_{el}(t_n,s)U(s)ds\Big)\| + h_n^{3/2}\|\partial\mathfrak{J}^n[U]\|_{\Sigma_n} + \sum_{j=0}^{n-1}\beta_j(U^j)\Bigg]
\end{aligned}
$$

and

$$
(4.12)\qquad \eta_n := \frac{\tau_n}{2}\beta_n(\mathcal{W}_n)
$$

are the spatial error estimators, where $\beta_n(U^n)$ and $\beta_n(\mathcal{W}_n)$ are given by (4.3).

$$
(4.13)\qquad \xi_n := \left(\frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} \|\hat{\omega}(t) - \hat{\omega}_I(t)\|^2 dt\right)^{1/2},
$$

is the linear interpolation error estimator for the Volterra integral term where $\hat{\omega}(t)$ and $\hat{\omega}_I(t)$ are given by (2.16) and (2.17), respectively.

$$
\begin{aligned}
(4.14)\quad \mu_n := C_{1,1}h_n\Bigg[\frac{1}{\sqrt{3}}\|(P_0^n - I)\mathcal{A}^{n-1}U^{n-1}\| + \|(P_0^n - I)\sigma^n(\mathcal{B}^{n-1/2}U)\| \\
+\tau_n^{-1}\|(P_0^n - I)U^{n-1}\|\Bigg],
\end{aligned}
$$

is the mesh change estimator.

$$
(4.15)\qquad \vartheta_{n,1} := \frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} \|f(t) - \varphi(t)\| dt
$$

and

$$
(4.16)\qquad \vartheta_{n,2} := 2C_{1,1}h_n \max\Big\{\|(I - P_0^n)f^{n-1}\|, \|(I - P_0^n)f^{n-1/2}\|\Big\}
$$

are the data approximation error estimators, where $\varphi(t) := \hat{I}f(t)$ and $\hat{I}$ is given by (3.3). Moreover, the constants appeared in the estimators are positive constants independent of the discretization parameters but depend upon the interpolation constants and the final time T.

The proof of Theorem 4.3 needs some preparations. We first proceed to estimate $\hat{\rho}(t)$ which is a cumbersome task.

Lemma 4.4 (*A posteriori* error estimate for the parabolic error). *For each $m \in [1 : N]$, the following estimate holds for $\hat{\rho}(t)$:*

$$
\begin{aligned}
\left(\max_{t\in[0,t_m]}\|\hat{\rho}(t)\|^2 + \frac{\alpha}{4}\int_0^{t_m}\|\hat{\rho}(t)\|_1^2 dt\right)^{1/2} \\
\leq \Big[\|\hat{\rho}(t_0)\|^2 + C_7\sum_{n=1}^m \tau_n\Lambda_n^2\Big]^{1/2} + \left(\Xi_{1,m}^2 + \Xi_{2,m}^2\right)^{1/2},
\end{aligned}
$$

where Λ_n, $\Xi^2_{1,m}$ *and* $\Xi^2_{2,m}$ *are given by (4.5), (4.7) and (4.8), respectively and* C_7 *is a positive constant independent of the discretization parameters but depends upon the interpolation constants and the final time* T.

The proof of the above lemma in turn hinges essentially on several auxiliary results which we shall discuss in detail below. We shall use the notation $\rho(t)$ for the error $u(t) - \mathcal{R}_w U(t)$ in the subsequent error analysis. We begin with the following error equation for $\hat{\rho}(t)$.

LEMMA 4.5. *For* $t \in I_n, n \in [1:N]$ *and for each* $\phi \in H^1_0(\Omega)$, *we have the following error equation for* $\hat{\rho}(t)$:

$$\langle \hat{\rho}_t, \phi\rangle + a(\rho, \phi) - \int_0^t b(t, s; \rho(s), \phi)ds = \langle G, \phi\rangle, \tag{4.17}$$

where G *is defined by*

$$\langle G, \phi\rangle := \langle G_1, \phi\rangle + (t - t_{n-1/2})\langle \mathcal{Y}_n, \phi\rangle$$

with

$$\begin{aligned}\langle G_1, \phi\rangle := &\langle (P^n_0 - I)\Big\{l_{n-1}(t)\mathcal{A}^{n-1}U^{n-1} - \sigma^n(\mathcal{B}^{n-1/2}U) - \tau_n^{-1}U^{n-1}\Big\}, \phi\rangle\\ &+\langle (\mathcal{R}^n_w - I)(\hat{F}(t) - F^{n-1/2}), \phi\rangle + \langle \hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t), \phi\rangle + \langle \hat{\omega}(t) - \hat{\omega}_I(t), \nabla\phi\rangle\\ &+\langle \hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t), \phi\rangle + \langle \int_0^{t_{n-1/2}} \mathcal{B}^{n-1/2}U(s)ds - \sigma^n(\mathcal{B}^{n-1/2}U), \phi\rangle\\ &+\langle \hat{F}(t) - \Theta(t) + f(t), \phi\rangle - \langle \tau_n^{-1}\Big[(\mathcal{R}^n_w - I)U^n - (\mathcal{R}^{n-1}_w - I)U^{n-1}\Big], \phi\rangle\end{aligned}$$

and $\mathcal{Y}_n$ *is given by (2.21).*

Proof. For $t \in I_n$ and $\forall\phi \in H^1_0(\Omega)$, we first multiply (3.5) by ϕ and integrate over Ω. Then, subtract the resulting equation from (2.5) to obtain

$$\begin{aligned}&\langle \hat{\rho}_t(t), \phi\rangle + a(u(t), \phi) - \int_0^t b(t, s; u(s), \phi)ds\\ &\qquad = \langle f, \phi\rangle + \langle \mathcal{R}^n_w \hat{F}(t), \phi\rangle - \tau_n^{-1}\langle \mathcal{R}^n_w P^n_0 U^{n-1} - \mathcal{R}^{n-1}_w U^{n-1}, \phi\rangle.\end{aligned}$$

Using (2.15)-(2.17) and (2.13), we obtain

$$\begin{aligned}&\langle \hat{\rho}_t(t), \phi\rangle + a(\rho(t), \phi) - \int_0^t b(t, s; \rho(s), \phi)ds\\ &= -l_{n-1}(t)\Big[\langle \mathcal{A}^{n-1}U^{n-1}, \phi\rangle - \langle \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s)U(s)ds, \phi\rangle\Big]\\ &\quad -l_n(t)\Big[\langle \mathcal{A}^n U^n, \phi\rangle - \langle \int_0^{t_n} \mathcal{B}^n(s)U(s)ds, \phi\rangle\Big] + \langle f(t), \phi\rangle + \langle \mathcal{R}^n_w \hat{F}(t), \phi\rangle\\ &\quad -\tau_n^{-1}\langle \mathcal{R}^n_w P^n_0 U^{n-1} - \mathcal{R}^{n-1}_w U^{n-1}, \phi\rangle + \langle \hat{\omega}(t) - \hat{\omega}_I(t), \nabla\phi\rangle.\end{aligned}$$

Using (3.1) and (2.18)-(2.20), we get

$$\langle \hat{\rho}_t(t), \phi\rangle + a(\rho(t), \phi) - \int_0^t b(t, s; \rho(s), \phi)ds \tag{4.18}$$

$$
\begin{aligned}
&= \langle (P_0^n - I)\Big\{ l_{n-1}(t)\mathcal{A}^{n-1}U^{n-1} - \sigma^n(\mathcal{B}^{n-1/2}U)\Big\}, \phi\rangle + \langle \hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t), \phi\rangle \\
&\quad + \langle \hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t), \phi\rangle + \langle \int_0^{t_{n-1/2}} \mathcal{B}^{n-1/2}U(s)ds - \sigma^n(\mathcal{B}^{n-1/2}U), \phi\rangle \\
&\quad + (t - t_{n-1/2})\langle \mathcal{Y}_n, \phi\rangle + \langle f(t), \phi\rangle + \langle \hat{\omega}(t) - \hat{\omega}_I(t), \nabla\phi\rangle \\
&\quad + \langle \mathcal{R}_w^n \hat{F}(t) - \Theta(t), \phi\rangle - \tau_n^{-1}\langle \mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^{n-1}U^{n-1}, \phi\rangle.
\end{aligned}
$$

For the last two terms on the right hand side of (4.18), an application of (2.23) yields

$$
\begin{aligned}
(4.19)\quad & \langle \mathcal{R}_w^n \hat{F}(t) - \Theta(t), \phi\rangle - \tau_n^{-1}\langle \mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^{n-1}U^{n-1}, \phi\rangle \\
&= \langle (\mathcal{R}_w^n - I)(\hat{F}(t) - F^{n-1/2}), \phi\rangle + \langle \hat{F}(t) - \Theta(t), \phi\rangle \\
&\quad - \tau_n^{-1}\langle (\mathcal{R}_w^n - I)U^n - (\mathcal{R}_w^{n-1} - I)U^{n-1}, \phi\rangle - \tau_n^{-1}\langle P_0^n U^{n-1} - U^{n-1}, \phi\rangle.
\end{aligned}
$$

Thus, the error equation (4.17) for $\hat{\rho}(t)$ now follows from (4.18) and (4.19). ∎

The next lemma presents a clear picture of the terms to be estimated in order to obtain a bound on $\hat{\rho}(t)$.

LEMMA 4.6. *The following estimate holds for* $\hat{\rho}(t)$

$$
\max_{t\in[0,t_m]} \|\hat{\rho}(t)\|^2 + \frac{\alpha}{2}\int_0^{t_m} \Big[2\|\rho(t)\|_1^2 + \|\hat{\rho}(t)\|_1^2\Big]dt \le \|\hat{\rho}(0)\|^2 + C_7\mathcal{I}_m,
$$

where

$$
\begin{aligned}
\mathcal{I}_m &:= \sum_{n=1}^{m} \Big(\mathcal{I}_n^{T,1} + \mathcal{I}_n^{T,2} + \mathcal{I}_n^{M,3} + \mathcal{I}_n^{S,4} + \mathcal{I}_n^{S,5} + \mathcal{I}_n^{D,6}\Big) \\
&:= \mathcal{I}_m^1 + \mathcal{I}_m^2 + \mathcal{I}_m^3 + \mathcal{I}_m^4 + \mathcal{I}_m^5 + \mathcal{I}_m^6
\end{aligned}
$$

with

$$
(4.20)\qquad \mathcal{I}_n^{T,1} := \int_{t_{n-1}}^{t_n} \|\hat{\rho}(t) - \rho(t)\|_1^2 dt,
$$

$$
\begin{aligned}
(4.21)\qquad \mathcal{I}_n^{T,2} := \int_{t_{n-1}}^{t_n} \Bigg[& \Big|\langle \int_0^{t_{n-1/2}} \mathcal{B}^{n-1/2}U(s)ds - \sigma^n(\mathcal{B}^{n-1/2}U), \hat{\rho}(t)\rangle\Big| \\
& + \Big|\langle \hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t), \hat{\rho}(t)\rangle\Big| + \Big|\langle \hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t), \hat{\rho}(t)\rangle\Big| \\
& + |\langle \hat{\omega}(t) - \hat{\omega}_I(t), \nabla\hat{\rho}(t)\rangle| \Bigg] dt,
\end{aligned}
$$

$$
\begin{aligned}
(4.22)\qquad \mathcal{I}_n^{M,3} := \int_{t_{n-1}}^{t_n} \Bigg[& \Big|\Big\langle (P_0^n - I)\Big\{ l_{n-1}(t)\mathcal{A}^{n-1}U^{n-1} - \sigma^n(\mathcal{B}^{n-1/2}U) \\
& - \tau_n^{-1}U^{n-1}\Big\}, \hat{\rho}(t)\Big\rangle\Big| \Bigg] dt,
\end{aligned}
$$

$$
(4.23)\qquad \mathcal{I}_n^{S,4} := \int_{t_{n-1}}^{t_n} \Big|\Big\langle (\mathcal{R}_w^n - I)(\hat{F}(t) - F^{n-1/2}), \hat{\rho}(t)\Big\rangle\Big| dt,
$$

$$
(4.24)\qquad \mathcal{I}_n^{S,5} := \tau_n^{-1}\int_{t_{n-1}}^{t_n} \Big|\Big\langle (\mathcal{R}_w^n - I)U^n - (\mathcal{R}_w^{n-1} - I)U^{n-1}, \hat{\rho}(t)\Big\rangle\Big| dt
$$

and

$$\mathcal{I}_n^{D,6} := \int_{t_{n-1}}^{t_n} \Big|\Big\langle \hat{F}(t) - \Theta(t) + f(t), \hat{\rho}(t)\Big\rangle\Big| dt. \tag{4.25}$$

Proof. Set $\phi = \hat{\rho}(t)$ in (4.17) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\hat{\rho}(t)\|^2 + a(\rho(t), \hat{\rho}(t)) = \int_0^t b(t, s; \rho(s), \hat{\rho}(t))ds + \langle G, \hat{\rho}(t)\rangle.$$

We integrate from t_{n-1} to t_n and use the fact

$$a(\rho(t), \hat{\rho}(t)) = \frac{1}{2}a(\rho(t), \rho(t)) + \frac{1}{2}a(\hat{\rho}(t), \hat{\rho}(t)) - \frac{1}{2}a(\hat{\rho}(t) - \rho(t), \hat{\rho}(t) - \rho(t))$$

to obtain

$$\begin{aligned}
&\frac{1}{2}\Big\{\|\hat{\rho}(t_n)\|^2 - \|\hat{\rho}(t_{n-1})\|^2\Big\} + \frac{1}{2}\int_{t_{n-1}}^{t_n} a(\rho(t), \rho(t))dt + \frac{1}{2}\int_{t_{n-1}}^{t_n} a(\hat{\rho}(t), \hat{\rho}(t))dt \\
&\qquad = \frac{1}{2}\int_{t_{n-1}}^{t_n} a(\hat{\rho}(t) - \rho(t), \hat{\rho}(t) - \rho(t))dt + \int_{t_{n-1}}^{t_n}\int_0^t b(t, s; \rho(s), \hat{\rho}(t))dsdt \\
&\qquad\quad + \int_{t_{n-1}}^{t_n} \langle G, \hat{\rho}(t)\rangle dt.
\end{aligned}$$

Using the coercivity of $a(\cdot,\cdot)$, and continuity of $a(\cdot,\cdot)$, $b(t, s; \cdot,\cdot)$ together with a standard kickback argument, we obtain

$$\begin{aligned}
&\frac{1}{2}\Big\{\|\hat{\rho}(t_n)\|^2 - \|\hat{\rho}(t_{n-1})\|^2\Big\} + \frac{\alpha}{2}\int_{t_{n-1}}^{t_n} \|\rho(t)\|_1^2 dt + \frac{\alpha}{4}\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt \\
&\le \frac{\beta}{2}\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t) - \rho(t)\|_1^2 dt + C_5(T)\int_{t_{n-1}}^{t_n}\int_0^t \|\rho(s)\|_1^2 dsdt + \int_{t_{n-1}}^{t_n} |\langle G_1, \hat{\rho}(t)\rangle| dt,
\end{aligned}$$

where we have used the fact that

$$\int_{t_{n-1}}^{t_n} (t - t_{n-1/2})dt = 0.$$

Summing from $n = 1 : m$ with an application of Gronwall's lemma gives the desired result with $C_7 = \max\{\beta C_6(T), 2C_6(T)\}$, where $C_6(T)$ is a Gronwall's constant. ◻

Now, we proceed to estimate the terms appeared in Lemma 4.6. We start with providing *a posteriori* error bounds on the time discretization error.

LEMMA 4.7 (Time error estimators). *The following a posteriori bounds hold for the time discretization error terms $\mathcal{I}_m^1$ and $\mathcal{I}_m^2$:*

$$\mathcal{I}_m^1 \le \sum_{n=1}^m \tau_n \Lambda_n^2 \tag{4.26}$$

and

$$\mathcal{I}_m^2 \le \sum_{n=1}^m \tau_n \max_{[0,t_m]} \|\hat{\rho}(t)\|\lambda_n + \sum_{n=1}^m \tau_n^{1/2}\xi_n\left(\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt\right)^{1/2}, \tag{4.27}$$

where Λ_n, λ_n and ξ_n are given by (4.5), (4.9) and (4.13), respectively.

Proof. We know that

$$\hat{\rho}(t) - \rho(t) = -(\hat{U}(t) - \mathcal{R}_w U(t)). \tag{4.28}$$

Thus, to estimate the term $I_n^{T,1}$, we have to first estimate $\hat{U}(t) - \mathcal{R}_w U(t)$.
Using (2.15), (3.5) and (3.7), we have

$$\hat{U}_t(t) - (\mathcal{R}_w U)_t(t) = \mathcal{R}_w^n \Big[F^{n-1/2} - \hat{F}(t) \Big].$$

We integrate from t_{n-1} to t and use the fact that $\hat{U}(t)$ interpolates with $\mathcal{R}_w U(t)$ at t_{n-1} to obtain

$$\hat{U}(t) - \mathcal{R}_w U(t) = -\mathcal{R}_w^n \int_{t_{n-1}}^{t} \Big\{ \hat{F}(s) - F^{n-1/2} \Big\} ds. \tag{4.29}$$

Using (3.1), (3.2) and the identity $l_{n-1}(t) + l_n(t) = 1$, $t \in I_n$, we have

$$\begin{aligned} \hat{F}(t) - F^{n-1/2} &= \Theta(t) - \Theta(t_{n-1/2}) - P_0^n[\varphi(t) - \varphi(t_{n-1/2})] \\ &= 2(t - t_{n-1/2})\mathcal{W}_n, \end{aligned} \tag{4.30}$$

where $\mathcal{W}_n$ is given by (4.6).

Substituting (4.30) in (4.29), we get

$$\hat{U}(t) - \mathcal{R}_w U(t) = (t_n - t)(t - t_{n-1})\mathcal{R}_w^n \mathcal{W}_n. \tag{4.31}$$

Using the coercivity and the continuity of the bilinear form $a(\cdot,\cdot)$, it follows that

$$\begin{aligned} \alpha\|\hat{\rho}(t) - \rho(t)\|_1^2 &\le a(\hat{\rho}(t) - \rho(t), \hat{\rho}(t) - \rho(t)) \\ &= (t_n - t)(t - t_{n-1}) a(\mathcal{R}_w^n \mathcal{W}_n, \hat{\rho}(t) - \rho(t)) \\ &\le (t_n - t)(t - t_{n-1})\beta\|\mathcal{R}_w^n \mathcal{W}_n\|_1 \|\hat{\rho}(t) - \rho(t)\|_1, \end{aligned}$$

where we have used (4.28) and (4.31).

Thus, using (4.2) we deduce that

$$\|\hat{\rho}(t) - \rho(t)\|_1 \le \frac{\beta(t_n - t)(t - t_{n-1})}{\alpha} \Big[\alpha_n(\mathcal{W}_n) + \|\mathcal{W}_n\|_1 \Big]. \tag{4.32}$$

Finally, with an aid of (4.32), we obtain

$$\mathcal{I}_n^{T,1} := \int_{t_{n-1}}^{t_n} \|\hat{\rho}(s) - \rho(s)\|_1^2 ds \le \frac{\beta^2 \tau_n^5}{30\alpha^2} \Big[\alpha_n(\mathcal{W}_n) + \|\mathcal{W}_n\|_1 \Big]^2. \tag{4.33}$$

Thus, the first inequality (4.26) follows by taking summation over n and using (4.5).

Next, to prove the inequality (4.27), we first note that

$$\begin{aligned} \mathcal{I}_n^{T,2} := \int_{t_{n-1}}^{t_n} \Bigg[&\Big| \langle \int_0^{t_{n-1/2}} \mathcal{B}^{n-1/2} U(s) ds - \sigma^n(\mathcal{B}^{n-1/2} U), \hat{\rho}(t) \rangle \Big| + \Big| \langle \hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t), \hat{\rho}(t) \rangle \Big| \\ &+ \Big| \langle \hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t), \hat{\rho}(t) \rangle \Big| + |\langle \hat{\omega}(t) - \hat{\omega}_I(t), \nabla \hat{\rho}(t) \rangle| \Bigg] dt, \\ := \int_{t_{n-1}}^{t_n} &\Big[|\mathcal{J}_1| + |\mathcal{J}_2| + |\mathcal{J}_3| + |\mathcal{J}_4| \Big] dt. \end{aligned}$$

We start with estimating the term $\mathcal{J}_1$. A standard Trapezoidal rule argument for a sufficiently smooth function $g(s)$ yields

$$\int_a^b g(s)ds - \frac{(b-a)}{2}(g(a)+g(b)) = \frac{1}{2}\int_a^b (s-a)(s-b)g''(s)ds.$$

If we define

$$\psi_{2j}(s) := \begin{cases} (s-t_{j-1})(s-t_j) & \text{for } s\in[t_{j-1},t_j] \text{ and } 1\le j\le n-1, \\ (s-t_{j-1})(s-t_{j-1/2}) & \text{for } s\in[t_{j-1},t_{j-1/2}] \text{ and } j=n, \end{cases}$$

then

$$\int_{t_{j-1}}^{t_j} g(s)ds - \frac{\tau_j}{2}[g(t_j)+g(t_{j-1})] = \frac{1}{2}\int_{t_{j-1}}^{t_j}\psi_{2j}(s)g''(s)ds \tag{4.34}$$

and

$$\int_{t_{n-1}}^{t_{n-1/2}} g(s)ds - \frac{\tau_n}{4}[g(t_{n-1})+g(t_{n-1/2})] = \frac{1}{2}\int_{t_{n-1}}^{t_{n-1/2}}\psi_{2n}(s)g''(s)ds. \tag{4.35}$$

Using (2.6), (2.11), (4.34) and (4.35), we obtain

$$\begin{aligned}
&\int_0^{t_{n-1/2}} \langle \mathcal{B}^{n-1/2}(s)U(s)ds, \hat\rho(t)\rangle - \langle \sigma^n(\mathcal{B}^{n-1/2}U), \hat\rho(t)\rangle \qquad (4.36)\\
&= \frac{1}{2}\langle \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\psi_{2j}(s)\frac{d^2}{ds^2}\big\{\mathcal{B}^{n-1/2}(s)U(s)\big\}ds \\
&\quad + \int_{t_{n-1}}^{t_{n-1/2}}\psi_{2n}(s)\frac{d^2}{ds^2}\big\{\mathcal{B}^{n-1/2}(s)U(s)\big\}ds, \hat\rho(t)\rangle \\
&= \frac{1}{2}\langle \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\psi_{2j}(s)\left\{\frac{d^2(\mathcal{B}^{n-1/2}(s))}{ds^2}U(s) + \frac{d(\mathcal{B}^{n-1/2}(s))}{ds}\frac{d(U(s))}{ds}\right\}ds \\
&\quad + \int_{t_{n-1}}^{t_{n-1/2}}\psi_{2n}(s)\left\{\frac{d^2(\mathcal{B}^{n-1/2}(s))}{ds^2}U(s) + \frac{d(\mathcal{B}^{n-1/2}(s))}{ds}\frac{d(U(s))}{ds}\right\}ds, \hat\rho(t)\rangle \\
&\le \frac{1}{2}\left(\sum_{j=1}^{n-1}\tau_j^2\left[\frac{\gamma''}{2}\tau_j\Big(\|\Delta^n U^{j-1}\| + \|\Delta^n U^j\|\Big) + \gamma'\tau_j\|\Delta^n\partial U^j\|\right]\right. \\
&\quad \left. +\tau_n^2\left[\frac{\gamma''}{2}\tau_j\Big(\|\Delta^n U^{n-1}\| + \|\Delta^n U^n\|\Big) + \gamma'\tau_j\|\Delta^n\partial U^n\|\right]\right)\|\hat\rho(t)\| \\
&\le \bar\gamma\theta_n\|\hat\rho(t)\|,
\end{aligned}$$

where θ_n is given by (4.10) and $\bar\gamma = \max\left\{\frac{\gamma''}{2}, \frac{\gamma'}{2}\right\}$.

Thus, in view of (4.36) we have the following bound on $\mathcal{J}_1$

$$|\mathcal{J}_1| \le \bar\gamma\theta_n\|\hat\rho(t)\|.$$

Moreover, an application of Cauchy-Schwarz inequality gives

$$\begin{aligned}
|\mathcal{J}_2| &\le \|\hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t)\|\|\hat\rho(t)\|, \\
|\mathcal{J}_3| &\le \|\hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t)\|\|\hat\rho(t)\|
\end{aligned}$$

and

$$|\mathcal{J}_4| \leq \|\hat{\omega}(t) - \hat{\omega}_I(t)\|\|\nabla\hat{\rho}(t)\|.$$

Combine the bounds on $|\mathcal{J}_1|$, $|\mathcal{J}_2|$, $|\mathcal{J}_3|$ and $|\mathcal{J}_4|$ to obtain

$$\mathcal{I}_n^{T,2} \leq \int_{t_{n-1}}^{t_n} \left[\bar{\gamma}\theta_n + \|\hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t)\| + \|\hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t)\| \right] \|\hat{\rho}(t)\| dt$$
$$+ \left(\int_{t_{n-1}}^{t_n} \|\hat{\omega}(t) - \hat{\omega}_I(t)\|^2 dt \right)^{1/2} \left(\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt \right)^{1/2}.$$

Thus, we obtain

$$\mathcal{I}_n^{T,2} \leq \tau_n \max_{[0,t_m]} \|\hat{\rho}(t)\| \lambda_n + \tau_n^{1/2} \xi_n \left(\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt \right)^{1/2},$$

where λ_n, ξ_n are given by (4.9), (4.13), respectively and $C_8 := \max\{\bar{\gamma}, 1\}$. The desired estimate now follows by taking summation over n. □

The next lemma gives information on the *a posteriori* contributions due to mesh change.

LEMMA 4.8 (Mesh change estimate). *We have the following bound on the mesh change error term* $\mathcal{I}_m^3$*:*

$$\mathcal{I}_m^3 \leq \sum_{n=1}^{m} \tau_n^{1/2} \mu_n \left(\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt \right)^{1/2}, \tag{4.37}$$

where μ_n *is given by (4.14).*

Proof. The orthogonality property of P_0^n now leads to

$$\int_{t_{n-1}}^{t_n} \left[\left\langle (P_0^n - I) \left\{ l_{n-1}(t) \mathcal{A}^{n-1} U^{n-1} - \sigma^n (\mathcal{B}^{n-1/2} U) - \tau_n^{-1} U^{n-1} \right\}, \hat{\rho}(t) \right\rangle \right] dt$$
$$= \int_{t_{n-1}}^{t_n} \left[\left\langle (P_0^n - I) \left\{ l_{n-1}(t) \mathcal{A}^{n-1} U^{n-1} - \sigma^n (\mathcal{B}^{n-1/2} U) \right.\right.\right.$$
$$\left.\left.\left. - \tau_n^{-1} U^{n-1} \right\}, \hat{\rho}(t) - \Pi^n \hat{\rho}(t) \right\rangle \right] dt.$$

An application of the Cauchy-Schwarz inequality yields

$$\mathcal{I}_n^{M,3} \leq C_{1,1} h_n \left[\left\{ \int_{t_{n-1}}^{t_n} l_{n-1}^2(t) dt \right\}^{1/2} \|(P_0^n - I)\mathcal{A}^{n-1} U^{n-1}\| \right.$$
$$\left. + \tau_n^{1/2} \|(P_0^n - I)\sigma^n(\mathcal{B}^{n-1/2} U)\| + \tau_n^{-1/2} \|(P_0^n - I) U^{n-1}\| \right] \left\{ \int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt \right\}^{1/2}$$
$$\leq \tau_n^{1/2} \mu_n \left\{ \int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt \right\}^{1/2},$$

where μ_n is given by (4.14). Taking summation over n completes the proof. □

The next lemma captures contributions due to the spatial discretizations.

LEMMA 4.9 (Spatial error estimates). *The following a posteriori error bound holds on the spatial discretization error term $\mathcal{I}_m^4$:*

$$\mathcal{I}_m^4 \leq \sum_{n=1}^{m} \tau_n \max_{[0,t_m]} \|\hat{\rho}(t)\| \eta_n, \tag{4.38}$$

where η_n is given by (4.12). Moreover, the error bound holds for $\mathcal{I}_m^5$ corresponds to the spatial discretization error due to mesh change:

$$\mathcal{I}_m^5 \leq \max_{t\in[0,t_m]} \|\hat{\rho}(t)\| \sum_{n=1}^{m} \tau_n \zeta_n, \tag{4.39}$$

where ζ_n is given by (4.11).

Proof. Using (4.30) and (4.3), we have

$$\int_{t_{n-1}}^{t_n} \left| \left\langle (\mathcal{R}_w^n - I)(\hat{F}(t) - F^{n-1/2}), \hat{\rho}(t) \right\rangle \right| dt \leq \frac{\tau_n^2}{2} \max_{[0,t_m]} \|\hat{\rho}(t)\| \beta_n(\mathcal{W}_n).$$

Hence, we obtain

$$\mathcal{I}_n^{S,4} \leq \tau_n \max_{[0,t_m]} \|\hat{\rho}(t)\| \eta_n, \tag{4.40}$$

where η_n is given by (4.12) and the estimate (4.38) follows by taking the summation over n.

Next, to estimate $\mathcal{I}_n^{S,5}$ as given in (4.24), we exploit the orthogonality property of the Ritz-Volterra reconstructions. We use the standard duality technique here.

For $t \in (0,T)$, let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the following elliptic problem in the weak form

$$a(\chi, \psi(t)) = \langle \chi, \hat{\rho}(t) \rangle, \quad \forall \chi \in H_0^1(\Omega) \tag{4.41}$$

satisfying the following regularity estimate:

$$\|\psi(t)\|_2 \leq C_\Omega \|\hat{\rho}(t)\|, \tag{4.42}$$

where the constant C_Ω depends on the domain Ω.

Setting $\chi = \mathcal{R}_w^n U^n - \mathcal{R}_w^{n-1} U^{n-1} - U^n + U^{n-1}$ in (4.41) and using (2.13), (2.7) and, (2.9), we obtain

$$\begin{aligned}
&\langle \mathcal{R}_w^n U^n - \mathcal{R}_w^{n-1} U^{n-1} - U^n + U^{n-1}, \hat{\rho}(t) \rangle \\
&= a(\mathcal{R}_w^n U^n - \mathcal{R}_w^{n-1} U^{n-1} - U^n + U^{n-1}, \psi(t) - \Pi^n \psi(t)) \\
&\quad - \int_0^{t_n} b(t_n, s; (\mathcal{R}_w U - U)(s), \psi(t) - \Pi^n \psi(t)) ds \\
&\quad + \int_0^{t_{n-1}} b(t_{n-1}, s; (\mathcal{R}_w U - U)(s), \psi(t) - \Pi^n \psi(t)) ds \\
&\quad + \int_0^{t_n} b(t_n, s; (\mathcal{R}_w U - U)(s), \psi(t)) ds - \int_0^{t_{n-1}} b(t_{n-1}, s; (\mathcal{R}_w U - U)(s), \psi(t)) ds
\end{aligned}$$

$$
\begin{aligned}
&= \langle \mathcal{A}^n U^n - \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \mathcal{A}_{el}U^n + \int_0^{t_n} \mathcal{B}_{el}(t_n,s)U(s)ds, \psi(t) - \Pi^n\psi(t)\rangle \\
&\quad -\langle \mathcal{A}^{n-1}U^{n-1} - \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s)U(s)ds - \mathcal{A}_{el}U^{n-1} \\
&\quad + \int_0^{t_{n-1}} \mathcal{B}_{el}(t_{n-1},s)U(s)ds, \psi(t) - \Pi^n\psi(t)\rangle \\
&\quad +\langle \int_0^{t_n} J_2[U(s)]ds - J_1[U^n] - \int_0^{t_{n-1}} J_2[U(s)]ds + J_1[U^{n-1}], \psi(t) - \Pi^n\psi(t)\rangle_{\Sigma_n} \\
&\quad + \int_0^{t_n} b(t_n,s;(\mathcal{R}_wU-U)(s),\psi(t))ds - \int_0^{t_{n-1}} b(t_{n-1},s;(\mathcal{R}_wU-U)(s),\psi(t))ds.
\end{aligned}
$$

We now use (2.22) together with $\mathfrak{J}^n[U] - \mathfrak{J}^{n-1}[U] = \tau_n \partial \mathfrak{J}^n[U]$ to obtain

$$
\begin{aligned}
(4.43)\ &|\langle \mathcal{R}_w^n U^n - \mathcal{R}_w^{n-1}U^{n-1} - U^n + U^{n-1}, \hat{\rho}(t)\rangle| \\
&\le \|\mathcal{A}^n U^n - \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \mathcal{A}_{el}U^n + \int_0^{t_n} \mathcal{B}_{el}(t_n,s)U(s)ds \\
&\quad -\mathcal{A}^{n-1}U^{n-1} + \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s)U(s)ds + \mathcal{A}_{el}U^{n-1} - \int_0^{t_{n-1}} \mathcal{B}_{el}(t_{n-1},s)U(s)ds\| \\
&\quad \|\psi(t) - \Pi^n\psi(t)\| + \tau_n\|\partial\mathfrak{J}^n[U]\|_{\Sigma_n}\|\psi(t) - \Pi^n\psi(t)\|_{\Sigma_n} \\
&\quad +|\int_0^{t_n} b(t_n,s;(\mathcal{R}_wU-U)(s),\psi(t))ds - \int_0^{t_{n-1}} b(t_{n-1},s;(\mathcal{R}_wU-U)(s),\psi(t))ds|.
\end{aligned}
$$

To handle the last term above, we use the fact

$$
b(t_n,s;(\mathcal{R}_wU-U)(s),\psi(t)) := \langle(\mathcal{R}_wU-U)(s), \mathcal{B}^*(t_n,s)\psi(t)\rangle, \tag{4.44}
$$

where $\mathcal{B}^*(t_n,s)$ is the formal adjoint of the operator $\mathcal{B}(t_n,s)$. Now, we apply Cauchy-Schwarz inequality together with $\|\mathcal{B}^*(t_n,s)\psi(t)\| \le C_{\mathcal{B}_1^*}\|\psi(t)\|_2$ to obtain

$$
\begin{aligned}
(4.45)\ &|\int_0^{t_n} b(t_n,s;(\mathcal{R}_wU-U)(s),\psi(t))ds - \int_0^{t_{n-1}} b(t_{n-1},s;(\mathcal{R}_wU-U)(s),\psi(t))ds| \\
&\le |\int_0^{t_n} \langle(\mathcal{R}_wU-U)(s),\mathcal{B}^*(t_n,s)\psi(t)\rangle ds - \int_0^{t_{n-1}} \langle(\mathcal{R}_wU-U)(s),\mathcal{B}^*(t_{n-1},s)\psi(t)\rangle ds| \\
&\le \|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(\mathcal{R}_wU-U)(s)\|\|\mathcal{B}^*(t_n,s)\psi(t)\|ds \\
&\quad +\|\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}(\mathcal{R}_wU-U)(s)\|\|\mathcal{B}^*(t_{n-1},s)\psi(t)\|ds \\
&\le 2C_{\mathcal{B}_1^*}\Big[\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\Big\{l_{j-1}(s)(\mathcal{R}_w^{j-1}U^{j-1}-U^{j-1}) + l_j(s)(\mathcal{R}_w^jU^j - U^j)\Big\}ds\|\Big]\|\psi(t)\|_2 \\
&\le C_{\mathcal{B}_1^*}\Big[\hat{\tau}_n\sum_{j=1}^{n}\beta_j[U] + \hat{\tau}_{n-1}\sum_{j=0}^{n-1}\beta_j[U]\Big]\|\psi(t)\|_2 \le 2C_{\mathcal{B}_1^*}\hat{\tau}_n\Big[\sum_{j=0}^{n}\beta_j[U]\Big]\|\psi(t)\|_2.
\end{aligned}
$$

Using (4.45) in (4.43) and applying Proposition 4.1 with $C_9 = \max\Big(2C_{\mathcal{B}_1^*}, 1\Big)$, we obtain

$$
\begin{aligned}
(4.46)\ &|\langle \mathcal{R}_w^n U^n - \mathcal{R}_w^{n-1} U^{n-1} - U^n + U^{n-1}, \hat{\rho}(t)\rangle| \\
&\le C_9 \|\psi\|_2 \Bigg(\hat{\tau}_n \Bigg[h_n^2 \|\tau_n^{-1} (\mathcal{A}^n U^n - \int_0^{t_n} \mathcal{B}^n(s) U(s) ds - \mathcal{A}_{el} U^n \\
&+ \int_0^{t_n} \mathcal{B}_{el}(t_n, s) U(s) ds - \mathcal{A}^{n-1} U^{n-1} + \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s) U(s) ds \\
&+ \mathcal{A}_{el} U^{n-1} - \int_0^{t_{n-1}} \mathcal{B}_{el}(t_{n-1}, s) U(s) ds)\| + h_n^{3/2} \|\partial \mathfrak{J}^n [U]\|_{\Sigma_n} + \sum_{j=0}^{n} \beta_j[U] \Bigg] \Bigg).
\end{aligned}
$$

Combining (4.24) and (4.46), we arrive at

$$
\begin{aligned}
\mathcal{I}_n^{S,5} &\le C_9 \tau_n^{-1} \int_{t_{n-1}}^{t_n} \|\psi(t)\|_2 dt \Bigg(\hat{\tau}_n \Bigg[h_n^2 \|\tau_n^{-1} (\mathcal{A}^n U^n - \int_0^{t_n} \mathcal{B}^n(s) U(s) ds - \mathcal{A}_{el} U^n \\
&+ \int_0^{t_n} \mathcal{B}_{el}(t_n, s) U(s) ds - \mathcal{A}^{n-1} U^{n-1} + \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s) U(s) ds \\
&+ \mathcal{A}_{el} U^{n-1} - \int_0^{t_{n-1}} \mathcal{B}_{el}(t_{n-1}, s) U(s) ds)\| + h_n^{3/2} \|\partial \mathfrak{J}^n [U]\|_{\Sigma_n} + \sum_{j=0}^{n} \beta_j[U] \Bigg] \Bigg) \\
&\le \max_{t \in I_n} \|\hat{\rho}(t)\| \tau_n \zeta_n,
\end{aligned}
$$

where we have used (4.11) and the regularity result (4.42). Summing from $n = 1 : m$, the desired result is obtained. □

The data approximation error is estimated in the following lemma.

Lemma 4.10 (Data approximation error estimate). *The following bound holds on the data approximation error term* $\mathcal{I}_m^6$

$$
(4.47) \qquad \mathcal{I}_m^6 \le \sum_{n=1}^{m} \left[\tau_n \vartheta_{n,1} \max_{t \in [0, t_m]} \|\hat{\rho}(t)\| + \tau_n^{1/2} \vartheta_{n,2} \left(\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt \right)^{1/2} \right],
$$

where $\vartheta_{n,1}$ *and* $\vartheta_{n,2}$ *are given by (4.15) and (4.16).*

Proof. Using (3.1) and (3.2), we have

$$
\begin{aligned}
\mathcal{I}_n^{D,6} &:= \int_{t_{n-1}}^{t_n} \left| \left\langle \hat{F}(t) - \Theta(t) + f(t), \hat{\rho}(t) \right\rangle \right| dt = \int_{t_{n-1}}^{t_n} \left| \left\langle f(t) - P_0^n \varphi(t), \hat{\rho}(t) \right\rangle \right| dt \\
&\le \int_{t_{n-1}}^{t_n} \left| \left\langle f(t) - \varphi(t), \hat{\rho}(t) \right\rangle \right| dt + \int_{t_{n-1}}^{t_n} \left| \left\langle (I - P_0^n) \varphi(t), \hat{\rho}(t) \right\rangle \right| dt \\
&:= \mathfrak{J}_1 + \mathfrak{J}_2.
\end{aligned}
$$

Using Cauchy-Schwarz inequality, we obtain

$$
\mathfrak{J}_1 \le \max_{t \in [0, t_m]} \|\hat{\rho}(t)\| \int_{t_{n-1}}^{t_n} \|f(t) - \varphi(t)\| dt.
$$

For $\mathfrak{J}_2$, we use orthogonality property of P_0^n to have

$$\begin{aligned}\mathfrak{J}_2 &= \int_{t_{n-1}}^{t_n} \Big|\Big\langle (I-P_0^n)\varphi(t), \hat{\rho}(t)\Big\rangle\Big| dt = \int_{t_{n-1}}^{t_n} \Big|\Big\langle (I-P_0^n)\varphi(t), \hat{\rho}(t) - \Pi^n\hat{\rho}(t)\Big\rangle\Big| dt \\ &\leq C_{1,1}h_n \int_{t_{n-1}}^{t_n} \|(I-P_0^n)\varphi(t)\|\|\hat{\rho}(t)\|_1 dt \\ &\leq 2C_{1,1}h_n\tau_n^{1/2}\max\Big\{\|(I-P_0^n)f^{n-1}\|, \|(I-P_0^n)f^{n-1/2}\|\Big\}\Bigg(\int_{t_{n-1}}^{t_n}\|\hat{\rho}(t)\|_1^2 dt\Bigg)^{1/2}.\end{aligned}$$

Therefore,

$$\mathcal{I}_n^{D,6} \leq \tau_n\vartheta_{n,1}\max_{t\in[0,t_m]}\|\hat{\rho(t)}\| + \tau_n^{1/2}\vartheta_{n,2}\Bigg(\int_{t_{n-1}}^{t_n}\|\hat{\rho}(t)\|_1^2 dt\Bigg)^{1/2},$$

where $\vartheta_{n,1}$ and $\vartheta_{n,2}$ are given by (4.15) and (4.16), respectively. Now, taking summation over n, we obtain the desired result. □

Proof of Lemma 4.4. Application of Lemmas 4.7-4.10 in Lemma 4.6 yields

$$\begin{aligned}&\max_{t\in[0,t_m]}\|\hat{\rho}(t)\|^2 + \frac{\alpha}{2}\int_0^{t_m}\Big[2\|\rho(t)\|_1^2 + \|\hat{\rho}(t)\|_1^2\Big] dt \\ &\qquad \leq \|\hat{\rho}(0)\|^2 + C_7\Big[\sum_{n=1}^m \tau_n\Lambda_n^2 + \max_{t\in[0,t_m]}\|\hat{\rho}(t)\|\sum_{n=1}^m\tau_n\Big[\lambda_n+\eta_n+\zeta_n+\vartheta_{n,1}\Big] \\ &\qquad\quad + \sum_{n=1}^m \tau_n^{1/2}\Big[\mu_n+\vartheta_{n,2}+\xi_n\Big]\Big(\int_{t_{n-1}}^{t_n}\|\hat{\rho}(t)\|_1^2 dt\Big)^{1/2}\Big],\end{aligned}$$

We now use the following elementary fact to complete the proof. For $a = (a_0, a_1, \ldots, a_m)$, $b = (b_0, b_1, \ldots, b_m) \in \mathbb{R}^{m+1}$ and $c \in \mathbb{R}$, if $|a|^2 \leq c^2 + a.b$, then $|a| \leq |c| + |b|$.

In particular for $n = [1:m]$, taking

$$a_0 = \max_{t\in[0,t_m]}\|\hat{\rho}(t)\|,\quad a_n = \Bigg(\frac{\alpha}{2}\int_{t_{n-1}}^{t_n}\|\hat{\rho}(t)\|_1^2 dt\Bigg)^{1/2},\quad c = \Big[\|\hat{\rho}(t_0)\|^2 + C_7\sum_{n=1}^m\tau_n\Lambda_n^2\Big]^{1/2},$$

$$b_0 = C_7\sum_{n=1}^m\tau_n\Big[\lambda_n+\eta_n+\zeta_n+\vartheta_{n,1}\Big],\quad b_n = C_7(2\tau_n/\alpha)^{1/2}\Big[\mu_n+\xi_n+\vartheta_{n,2}\Big],$$

we obtain the desired result. □

Proof of Theorem 4.3. In view of (4.1), we apply triangle inequality to have

$$\|u(t) - U(t)\| \leq \|\hat{\rho}(t)\| + \|\sigma(t)\| + \|\epsilon(t)\|. \tag{4.48}$$

For $t \in I_n$,

$$\|\epsilon(t)\| = \|l_{n-1}(t)\epsilon^{n-1} + l_n(t)\epsilon^n\| \leq \max\Big(\|\epsilon^{n-1}\|, \|\epsilon^n\|\Big).$$

Therefore, for $t \in [0, t_m]$, using Lemma 4.2, we have

$$\|\epsilon(t)\| \leq \max_{n\in[0,m]}\Big(\|\epsilon^{n-1}\|, \|\epsilon^n\|\Big) \leq \max_{n\in[0,m]}\beta_n[U]. \tag{4.49}$$

Also,

$$\|\hat{U}(t) - \mathcal{R}_w U(t)\| \leq (t - t_{n-1})(t_n - t)\|\mathcal{R}_w^n \mathcal{W}_n\| \leq (t - t_{n-1})(t_n - t)\Big[\|(\mathcal{R}_w^n - I)\mathcal{W}_n\| + \|\mathcal{W}_n\|\Big] \leq \nu_n, \tag{4.50}$$

where ν_n is given by (4.4).

Finally, we use (4.48)-(4.50) and Lemma 4.4 to obtain the desired result. □

Remarks. (*i*) The estimator appeared in Theorem 4.3 is formally of optimal order. Moreover, in the absence of the memory term (i.e., $\mathcal{B}(t, s) = 0$), the error estimator obtained in Theorem 4.3 is similar to that for the parabolic problems [3]. Further, we note that the estimator λ_n, the contribution to the error from the approximation of the integral term, is of $O(\tau^2)$. Thus, the *a posteriori* error bound in Theorem 4.3 generalizes the results of [3] to PIDE.

(*ii*) The mesh change error term $\mathcal{I}_m^3$ can alternatively be estimated as

$$\mathcal{I}_m^3 \leq \sum_{n=1}^{m} \tau_n \max_{t\in[0,t_m]} \|\hat{\rho}(t)\|\mu_n',$$

where μ_n' is given by

$$\mu_n' = C_{1,1} h_n \Big[\frac{1}{2}\|(P_0^n - I)\mathcal{A}^{n-1}U^{n-1}\| + \|(P_0^n - I)\sigma^n(\mathcal{B}^{n-1/2}U)\| + \tau_n^{-1}\|(P_0^n - I)U^{n-1}\|\Big].$$

This estimate for mesh change error will lead to an alternative *a posteriori* error estimate for the main error e. In particular, the terms $\Xi_{1,m}^2$ and $\Xi_{2,m}^2$ in Lemma 4.4 take the form

$$\Xi_{1,m}^2 := \left(C_7 \sum_{n=1}^{m} \tau_n \Big[\lambda_n + \mu_n' + \eta_n + \zeta_n + \vartheta_{n,1}\Big]\right)^2$$

and

$$\Xi_{2,m}^2 := \frac{4C_7^2}{\alpha} \sum \tau_n (\xi_n + \vartheta_{n,2})^2.$$

The corresponding changes take place in the Theorem 4.3.

(*iii*) The term

$$\left(\frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} \|\hat{\omega}(t) - \hat{\omega}_I(t)\|^2 dt\right)^{1/2} \tag{4.51}$$

appeared in Theorem 4.3 (see (4.13)) is not a traditional *a posteriori* quantity, where $\hat{\omega}(t)$ and $\hat{\omega}_I(t)$ are given by (2.16) and (2.17), respectively. Since, the error in linear interpolation is bounded as

$$\|\hat{\omega}(t) - \hat{\omega}_I(t)\| \leq C\tau_n^2 \max_{t\in I_n} \|\frac{d^2}{dt^2}(\hat{\omega}(t))\|, \quad t \in I_n,$$

where $\frac{d^2}{dt^2}(\hat{\omega}(t))$ depends upon the quantities $\nabla\omega_t(t)$ and $\nabla\omega(t)$. The term $\|\nabla\omega_t(t)\|$ can be estimated as

$$\|\nabla\omega_t(t)\| = \|\nabla\epsilon_t(t) - \nabla U_t(t)\| \leq \|\nabla\epsilon_t(t)\| + \|\nabla U_t(t)\| \leq \frac{1}{\tau_n}\Big(\|\nabla\epsilon^n\| + \|\nabla\epsilon^{n-1}\|\Big) + \|\nabla\partial U^n\|,$$

and for the term $\|\nabla\omega(t)\|$, we have

$$\begin{aligned}\|\nabla\omega(t)\| &\le \|\nabla\epsilon(t)\| + \|\nabla U(t)\| \\ &\le \|l_{n-1}(t)\nabla\epsilon^{n-1} + l_n(t)\nabla\epsilon^n\| + \|l_{n-1}(t)\nabla U^{n-1} + l_n(t)\nabla U^n\| \\ &\le \max\Big(\|\nabla\epsilon^{n-1}\|, \|\nabla\epsilon^n\|\Big) + \max\Big(\|\nabla U^{n-1}\|, \|\nabla U^n\|\Big).\end{aligned}$$

This shows that (4.51) is now a meaningful *a posteriori* quantity by noting the fact that $\|\nabla\epsilon^n\|$ is bounded and is of $O(h)$ (see Lemma 4.2). Taking $\tau \approx h$, it is easy to see that the term (4.51) is of optimal order.

(iv) The *a posteriori* error analysis of the classical Crank-Nicolson scheme leads to one additional term $\frac{1}{2}\|(P_0^n - I)\mathcal{A}^{n-1}U^{n-1}\|$ in the error bounds. However, it does not affect the optimality of the main estimator.

5. Numerical assessment. In this section, we study the behaviour of the *a posteriori* error estimator presented in Theorem 4.3 for a two dimensional test problem. We solve the PIDE (1.1) in a square domain $\Omega = (0,\pi)^2 \subset \mathbb{R}^2$ with homogeneous Dirichlet boundary conditions. We select the coefficient matrices to be $A = I$ and $B(t,s) = I$, where I denotes the identity matrix. Then, the forcing term f is chosen such that the exact solution is given by

$$u(x,y,t) = \exp(-t/2)\sin(x)\sin(y).$$

Our main emphasis here is to understand the asymptotic behaviour of the estimators following which we perform numerical test on uniform meshes with uniform time-step. All computations have been carried out using MATLAB-7.8. We choose a sequence of mesh-sizes $(h(i) : i \in [1:l])$, to which we couple a sequence of step-sizes $(\tau(i) : i \in [1:l])$ i.e., $\tau(i) = c_0 h(i)$, where c_0 is taken to be $\frac{0.4}{\pi}$. Here, l denotes the number of runs. The initial mesh size $h(1)$ and the time step $\tau(1)$ are chosen to be $\frac{\pi}{8}$ and 0.05. Bisection algorithm is used to generate the refined triangulations. For each run $i \in [1:l]$, we compute the following quantities of interest:

- the Ritz-Volterra reconstruction error estimator: $\max_{n\in[0:m]} \beta_n$
- the space error estimator: $\sum_{n=1}^m \tau_n\zeta_n$
- the time reconstruction error estimators: $(\sum_{n=1}^m \tau_n\Lambda_n^2)^{1/2}$ and $\max_{n\in[0:m]} \nu_n$

for each time $t_m \in [0 = t_0 : \tau(i) : t_N = .1]$. We dropped the linear interpolation estimator, mesh change estimator and data approximation estimator from study. The experiment is carried out with $\mathbb{P}_1$ elements.

For each quantities of interest we observe their experimental order of convergence (EOC). The EOC is defined as follows: For a given finite sequence of successive runs (indexed by i), the EOC of the corresponding sequence of quantities of interest $E(i)$ (estimator or part of an estimator) itself is a sequence defined by

$$EOC(E(i)) = \frac{log(E(i+1)/E(i))}{log(h(i+1)/h(i))},$$

where $h(i)$ denotes the mesh size of the run i.

In order to measure the quality of our estimator the estimated error is compared to the true error so-called effectivity index (EI). The effectivity index is defined by

$$\left(\max_{n\in[0:m]}\beta_n + \max_{n\in[0:m]}\nu_n + \sum_{n=1}^m \tau_n\zeta_n + \left(\sum_{n=1}^m \tau_n\Lambda_n^2\right)^{1/2}\right) \Big/ \max_{n\in[0:m]}\|e(t_n)\|.$$

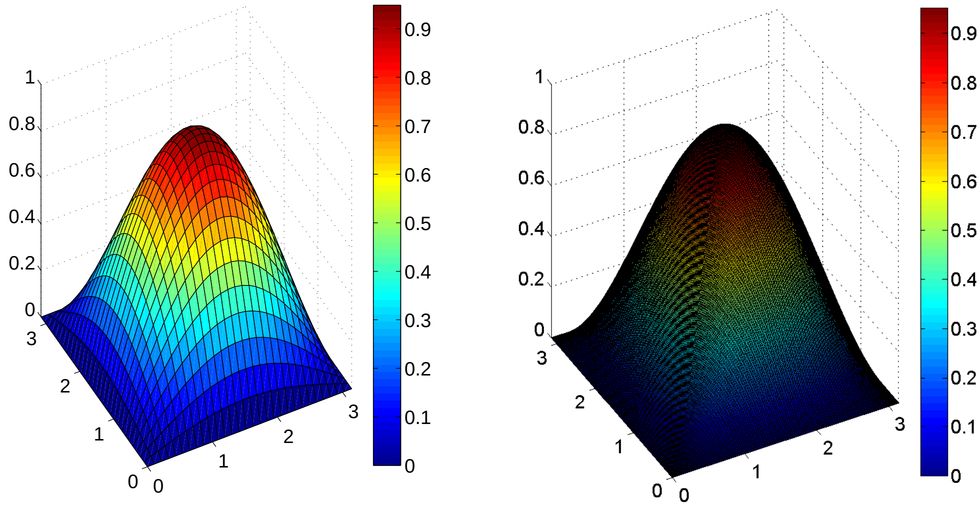

FIG. 5.1. ***The first plot shows the exact solution and the second one corresponds to Crank-Nicolson FEM solution. Crank-Nicolson FEM solution is computed using $\mathbb{P}_1$ elements with 33025 free nodes at T = 0.1 corresponding to*** $\tau = .003125$

All the constants involved in the estimators are taken to be equal to 1 except Gronwall's constant which is taken to be $\exp(T)$. The effectivity index is to be understood only qualitatively in this paper as the main emphasis is on observing asymptotic behaviour of the estimator.

In FIG. 5.2, the abscissa represents time which ranges in [0,0.1]. Each curve in each of the plots in first and third rows of FIG. 5.2 shows the estimator behaviour corresponds to a given run. The most coarse grid corresponds to curve with the largest error value and the finest grid corresponds to curve with the smallest error value. The value of EOC of a given estimator indicates its order.

From the FIG. 5.2, it is apparent that the estimators have the optimal rate of convergence.

Concluding remarks. In this paper, we have derived optimal order residual based *a posteriori* error estimator for PIDE (1.1) in the $L^\infty(L^2)$-norm for the fully discrete Crank-Nicolson method. Moreover, computational results are provided to illustrate that the estimator exhibit optimal rate of convergence which support our theoretical findings. Despite the importance of PIDE, and their variants in the modeling of several physical phenomena, the topic of *a posteriori* analysis for such kind of equations remains unexplored. We believe the work presented here could be a first step towards the development of various space-time adaptive algorithms for PIDE. The Ritz-Volterra reconstruction operator [15] unifies *a posteriori* approach from parabolic problems to PIDE. Moreover, for the optimality of the estimator, the linear approximation of the Volterra integral term is found to be crucial.

It is challenging to study the problem of obtaining *a posteriori* error estimates with the constants appeared in the bounds be independent of the final time T and hence they can serve as long-time estimates. Moreover, *a posteriori* error analysis for hyperbolic integro-differential equations in the $L^\infty(L^2)$-norm is an interesting research problem which will be reported elsewhere.

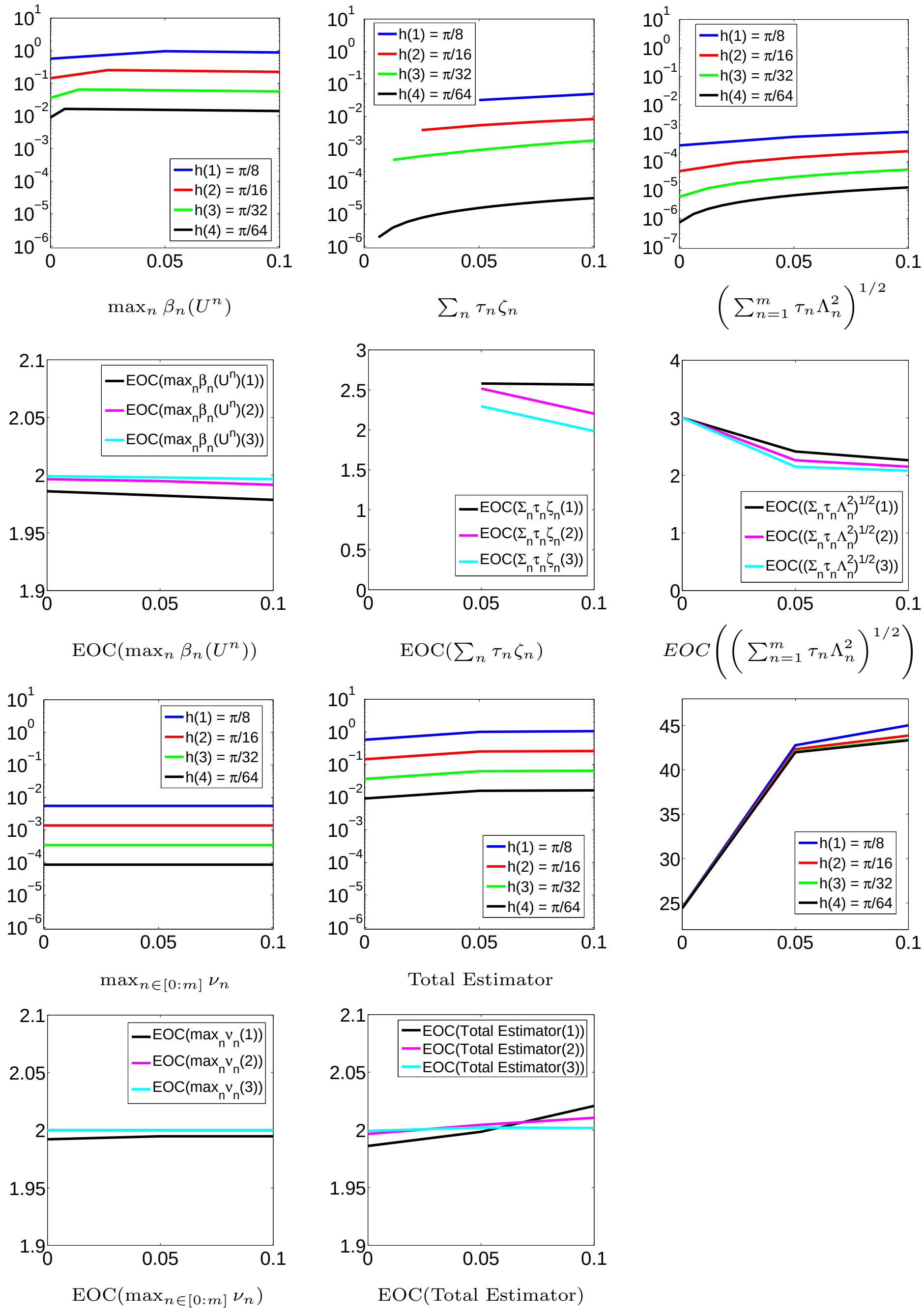

FIG. 5.2. ***Simulation with $\mathbb{P}_1$ elements for the PIDE with exact solution $u(x, y, t) = \exp(-t/2)\sin(x)\sin(y)$. On first and third rows we show the behaviour of the different estimators and below them the corresponding EOC. At the end of third row, we plot the effectivity index of the total estimator. We observe that the estimators decrease with at least second order with respect to the mesh parameters. Moreover, the Ritz-Volterra reconstruction estimator i.e.*** $\max_n \beta_n(U^n)$ ***dominates all other error estimators.***

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