NON-AUTONOMOUS DISSIPATIVE SEMIDYNAMICAL SYSTEMS WITH IMPULSES

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Abstract

In this paper, we describe the theory of non-autonomous dissipative semidynamical systems with impulses. We define the concept of a non-autonomous impulsive semidynamical system and we study its topological properties. For dissipativity systems we present results which deal with orbital stability, asymptotic stability and stability in the sense of Lyapunov-Barbashin. Also, we present some results that give conditions for a non-autonomous dissipative impulsive system to admit a global attractor. Finally, we show that systems of impulsive differential equations admit a non-autonomous impulsive semidynamical system associated.

1   Introduction

In the last years, the theory of impulsive systems has been studied and developed intensively. This theory is very important since complex problems can be modeled by systems with impulse conditions. The reader may find some important results and applications in [2, 3, 4], [12, 15] and [21, 22, 23], for instance.

Recently, the theory of dissipative impulsive semidynamical systems has started its study. In [7], we define several types of dissipativity as point, compact, local and bounded. We define the center of Levinson for compact dissipative impulsive semidynamical systems and we study its topological properties. Also, we present results giving necessary and sufficient conditions to obtain dissipativity. In [8], we define some types of attractors and we study some results which relate the concepts of attractors and dissipative systems (point, bounded and compact). The theory of dissipative systems for continuous dynamical systems may be found in [9].

In this paper we develop the theory of non-autonomous dissipative semidynamical systems with impulses as in the sense of Cheban [9]. In the next lines, we describe the organization of the paper and the main results.

In Section 2, we present the basis of the theory of impulsive semidynamical systems as basic definitions and notations.

Section 3 deals with the continuity of a function which describes the times of meeting impulsive sets.

In Section 4, we present additional useful definitions about autonomous dissipative systems.

Section 5 concerns the main results. We divide this section in three subsections. In Subsection 5.1, we create the theory of non-autonomous semidynamical systems with impulses. For these systems, we present the concept of dissipativity and we define some concepts of stability as orbital stability, asymptotic stability and stability in the sense of Lyapunov-Barbashin. Also, we define the center of Levinson for a non-autonomous dissipative compact impulsive semidynamical system and we prove that this center is globally asymptotically stable and globally asymptotically stable in the sense of Lyapunov-Barbashin provided this center satisfies a special condition.

In Subsection 5.2, we show some results which give necessary and sufficient conditions for a non-autonomous impulsive system to admit a global attractor.

Finally, in Subsection 5.3, we consider an impulsive differential equation with impulse effects at variable times,

$$\begin{cases} u' = f(t, u), \\ I : M \to \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $M \subset \mathbb{R}^n$ is an impulsive set, I is an impulse function and we show that system (1.1) admits a non-autonomous impulsive semidynamical system associated.

2   Preliminaries

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The triple $(X, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the function $\pi : X \times \mathbb{R}_+ \to X$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t + s)$ for all $x \in X$ and $t, s \in \mathbb{R}_+$. We denote such system simply by (X, π). For every $x \in X$, we consider the continuous function $\pi_x : \mathbb{R}_+ \to X$ given by $\pi_x(t) = \pi(x, t)$ and we call it the *motion* of x.

Let (X, π) be a semidynamical system. Given $x \in X$, the *positive orbit* of x is given by $\pi^+(x) = \{\pi(x, t) : t \in \mathbb{R}_+\}$. Given $A \subset X$ and $\Delta \subset [0, +\infty)$, we define

$$\pi^+(A) = \bigcup_{x \in A} \pi^+(x) \quad \text{and} \quad \pi(A, \Delta) = \bigcup_{x \in A, \, t \in \Delta} \pi(x, t).$$

For $t \geq 0$ and $x \in X$, we define $F(x, t) = \{y \in X : \pi(y, t) = x\}$ and, for $\Delta \subset [0, +\infty)$ and $D \subset X$, we define $F(D, \Delta) = \cup \{F(x, t) : x \in D \text{ and } t \in \Delta\}$. Then a point $x \in X$ is called an *initial point* if $F(x, t) = \emptyset$ for all $t > 0$.

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Now we define the concept of semidynamical systems with impulse action. An *impulsive semidynamical system* $(X, \pi; M_X, I_X)$ consists of a semidynamical system, (X, π), a nonempty closed subset M_X of X such that for every $x \in M_X$ there exists $\varepsilon_x > 0$ such that

$$F(x, (0, \varepsilon_x)) \cap M_X = \emptyset \quad \text{and} \quad \pi(x, (0, \varepsilon_x)) \cap M_X = \emptyset,$$

and a continuous function $I_X : M_X \to X$ whose action we explain below in the description of the impulsive trajectory of an impulsive system. The set M_X is called the *impulsive set* and the function I_X is called the *impulse function*. We also define

$$M^+(x) = \left(\bigcup_{t>0} \pi(x, t) \right) \cap M_X.$$

Let $(X, \pi; M_X, I_X)$ be an impulsive semidynamical system. We define the function $\phi_X : X \to (0, +\infty]$ by

$$\phi_X(x) = \begin{cases} s, & \text{if } \pi(x, s) \in M_X \text{ and } \pi(x, t) \notin M_X \text{ for } 0 < t < s, \\ +\infty, & \text{if } M^+(x) = \emptyset. \end{cases}$$

This means that $\phi_X(x)$ is the least positive time for which the trajectory of x meets M_X when $M^+(x) \neq \emptyset$. Thus for each $x \in X$, we call $\pi(x, \phi_X(x))$ the *impulsive point* of x.

The *impulsive trajectory* of x in $(X, \pi; M_X, I_X)$ is an $X-$valued function $\widetilde{\pi}_x$ defined on the subset $[0, s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such trajectory follows inductively as described in the following lines.

If $M^+(x) = \emptyset$, then $\widetilde{\pi}_x(t) = \pi(x, t)$ for all $t \in \mathbb{R}_+$ and $\phi_X(x) = +\infty$. However, if $M^+(x) \neq \emptyset$, there is the smallest positive number s_0 such that $\pi(x, s_0) = x_1 \in M_X$ and $\pi(x, t) \notin M_X$ for $0 < t < s_0$. Then we define $\widetilde{\pi}_x$ on $[0, s_0]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x, t), & 0 \leq t < s_0 \\ x_1^+, & t = s_0, \end{cases}$$

where $x_1^+ = I_X(x_1)$ and $\phi_X(x) = s_0$. Let us denote x by x_0^+.

Since $s_0 < +\infty$, the process now continues from x_1^+ onwards. If $M^+(x_1^+) = \emptyset$, then we define $\widetilde{\pi}_x(t) = \pi(x_1^+, t - s_0)$ for $s_0 \leq t < +\infty$ and $\phi_X(x_1^+) = +\infty$. When $M^+(x_1^+) \neq \emptyset$, there is the smallest positive number s_1 such that $\pi(x_1^+, s_1) = x_2 \in M_X$ and $\pi(x_1^+, t-s_0) \notin M_X$ for $s_0 < t < s_0 + s_1$. Then we define $\widetilde{\pi}_x$ on $[s_0, s_0 + s_1]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - s_0), & s_0 \leq t < s_0 + s_1 \\ x_2^+, & t = s_0 + s_1, \end{cases}$$

where $x_2^+ = I_X(x_2)$ and $\phi_X(x_1^+) = s_1$, and so on. Notice that $\widetilde{\pi}_x$ is defined on each interval $[t_n, t_{n+1}]$, where $t_0 = 0$ and $t_{n+1} = \sum_{i=0}^{n} s_i$ for $n = 0, 1, 2, \dots$. Hence $\widetilde{\pi}_x$ is defined on $[0, t_{n+1}]$.

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The process above ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some n. However, it continues infinitely if $M^+(x_n^+) \neq \emptyset$, $n = 0, 1, 2, \ldots$, and in this case the function $\widetilde{\pi}_x$ is defined in the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$.

Let $(X, \pi; M_X, I_X)$ be an impulsive semidynamical system. Given $x \in X$, the *impulsive positive orbit* of x is defined by the set

$$\widetilde{\pi}^+(x) = \{\widetilde{\pi}(x, t) : t \in [0, T(x))\}.$$

Analogously to the non-impulsive case, an impulsive semidynamical system satisfies the following standard properties: $\widetilde{\pi}(x, 0) = x$ for all $x \in X$ and $\widetilde{\pi}(\widetilde{\pi}(x, t), s) = \widetilde{\pi}(x, t+s)$ for all $x \in X$ and for all $t, s \in [0, T(x))$ such that $t + s \in [0, T(x))$. See [6] for a proof of it.

Given $A \subset X$ and $t \geq 0$, we define $\widetilde{\pi}^+(A) = \bigcup_{x \in A} \widetilde{\pi}^+(x)$ and $\widetilde{\pi}(A, t) = \bigcup_{x \in A} \widetilde{\pi}(x, t)$.

If $\widetilde{\pi}^+(A) \subset A$, we say that A is *positively* $\widetilde{\pi}-invariant$. And, we say that A is *minimal* in $(X, \pi; M_X, I_X)$ if $A = \overline{\widetilde{\pi}^+(x)}$ for all $x \in A \setminus M_X$, see [18].

In all this paper, for each $x \in X$, the motion $\widetilde{\pi}(x, t)$ will be defined for every $t \geq 0$, that is, $[0, +\infty)$ denotes the maximal interval of definition of $\widetilde{\pi}_x$.

For details about the structure of these types of impulsive semidynamical systems, the reader may consult [5, 6, 7, 8], [10, 11], [13, 14] and [17, 18, 19].

3 Continuity of ϕ_X

Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing x ($x \in X$) is called a *section* or a λ-*section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that

(a) $F(L, \lambda) = S$;

(b) $F(L, [0, 2\lambda])$ is a neighborhood of x;

(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \leq \mu < \nu \leq 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a λ-*tube* and the set L is called a *bar*. Let $(X, \pi; M_X, I_X)$ be an impulsive semidynamical system. We now present the conditions TC and STC for a tube.

Any tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S \subset M_X \cap F(L, [0, 2\lambda])$ is called $TC-tube$ on x. We say that a point $x \in M_X$ fulfills the *Tube Condition* and we write (TC), if there exists a $TC-$tube $F(L, [0, 2\lambda])$ through x. In particular, if $S = M_X \cap F(L, [0, 2\lambda])$ we have a $STC-tube$ on x and we say that a point $x \in M_X$ fulfills the *Strong Tube Condition* (we write (STC)), if there exists a $STC-$tube $F(L, [0, 2\lambda])$ through x.

The following theorem concerns the continuity of ϕ_X which is accomplished outside M_X for M_X satisfying the condition TC. See [10, Theorem 3.8].

Theorem 3.1. *Consider an impulsive semidynamical system $(X, \pi; M_X, I_X)$. Assume that no initial point in (X, π) belongs to the impulsive set M_X and that each element of M_X satisfies the condition (TC). Then ϕ_X is continuous at x if and only if $x \notin M_X$.*

4   Additional definitions

Let us consider a metric space X with metric ρ_X. By $B_X(x, \delta)$ we mean the open ball with center at $x \in X$ and radius $\delta > 0$. Given $A \subset X$, let $B_X(A, \delta) = \{x \in X : \rho_X(x, A) < \delta\}$ where $\rho_X(x, A) = \inf\{\rho_X(x, y) : y \in A\}$.

Given A and B nonempty bounded subsets of X, we denote by $\beta_X(A, B)$ the semi-deviation of A to B, that is, $\beta_X(A, B) = \sup\{\rho_X(a, B) : a \in A\}$. The Hausdorff distance of A to B is given by

$$d_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\rho_X(a, b), \sup_{b\in B}\inf_{a\in A}\rho_X(a, b)\right\}.$$

Let $(X, \pi; M_X, I_X)$ be an impulsive semidynamical system and $A \subset X$. The *limit set* of $A \subset X$ in $(X, \pi; M_X, I_X)$ is given by

$$\widetilde{L}_X^+(A) = \{y \in X : \text{there exist sequences } \{x_n\}_{n\geq 1} \subset A \text{ and } \{t_n\}_{n\geq 1} \subset \mathbb{R}_+$$

$$\text{such that } t_n \overset{n\to+\infty}{\longrightarrow} +\infty \text{ and } \widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} y\}$$

and the *prolongation set* of $A \subset X$ is defined by

$$\widetilde{D}_X^+(A) = \{y \in X : \text{there are sequences } \{x_n\}_{n\geq 1} \subset X \text{ and } \{t_n\}_{n\geq 1} \subset \mathbb{R}_+ \text{ such that}$$

$$\rho_X(x_n, A) \overset{n\to+\infty}{\longrightarrow} 0 \text{ and } \widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} y\}.$$

If $A = \{x\}$, we set $\widetilde{L}_X^+(x) = \widetilde{L}_X^+(\{x\})$ and $\widetilde{D}_X^+(x) = \widetilde{D}_X^+(\{x\})$.

The *stable manifold* of a set $A \subset X$ in $(X, \pi; M_X, I_X)$ is defined by

$$\widetilde{W}_X^s(A) = \{x \in X : \lim_{t\to+\infty} \rho_X(\widetilde{\pi}(x, t), A) = 0\}.$$

A set A in $(X, \pi; M_X, I_X)$ is said to be:

1. *orbitally $\widetilde{\pi}-$stable*, if given $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that $\rho_X(x, A) < \delta$ implies $\rho_X(\widetilde{\pi}(x, t), A) < \epsilon$ for all $t \geq 0$;

2. *$\widetilde{\pi}-$attracting*, if there exists $\gamma > 0$ such that $B_X(A, \gamma) \subset \widetilde{W}_X^s(A)$;

3. *asymptotically $\widetilde{\pi}-$stable*, if it is orbitally $\widetilde{\pi}-$stable and $\widetilde{\pi}-$attracting;

4. *globally asymptotically $\widetilde{\pi}-$stable*, if it is asymptotically $\widetilde{\pi}-$stable and $\widetilde{W}_X^s(A) = X$;

5. *uniform $\widetilde{\pi}-$attracting*, if there is $\gamma > 0$ such that $\lim\limits_{t \to +\infty} \sup\limits_{x \in B_X(A,\gamma)} \rho_X(\widetilde{\pi}(x,t), A) = 0$.

An impulsive system $(X, \pi; M_X, I_X)$ is said to be:

6. *point dissipative*, if there exists a bounded subset $K \subset X \setminus M_X$ such that for every $x \in X$ the convergence

$$\lim\limits_{t \to +\infty} \rho_X(\widetilde{\pi}(x,t), K) = 0 \tag{4.1}$$

holds;

7. *compact dissipative*, if the convergence (4.1) takes place uniformly with respect to x on the compact subsets from X;

8. *locally dissipative*, if for any point $x \in X$ there exists $\delta_x > 0$ such that the convergence (4.1) takes place uniformly with respect to $y \in B_X(x, \delta_x)$;

9. *bounded dissipative*, if the convergence (4.1) takes place uniformly with respect to x on every bounded subset from X.

Remark 4.1. If K is compact in the above definition, the impulsive system $(X, \pi; M_X, I_X)$ will be called $k-dissipative$.

Let $(X, \pi; M_X, I_X)$ be compact $k-$dissipative and K be a nonempty compact set such that $K \cap M_X = \emptyset$ and it is an attractor for all compact subsets of X. The set

$$J := \widetilde{L}_X^+(K) = \cap\{\widetilde{\pi}(K,t) : t \geq 0\}$$

is called the *center of Levinson* of the compact $k-$dissipative system $(X, \pi; M_X, I_X)$. The reader may find properties of the center of Levinson in [7], see Theorem 3.1 for instance.

The next definition presents several concepts established in [8].

Definition 4.1. An impulsive semidynamical system $(X, \pi; M_X, I_X)$:

a) satisfies the condition of Ladyzhenskaya, if for every bounded set $\mathcal{B} \subset X$ there exists a nonempty compact set $K_{\mathcal{B}} \subset X$, $K_{\mathcal{B}} \cap M_X = \emptyset$, such that $\lim\limits_{t \to +\infty} \beta_X(\widetilde{\pi}(\mathcal{B}, t), K_{\mathcal{B}}) = 0$;

b) is called $\widetilde{\pi}-$asymptotically compact, if for every bounded positively $\widetilde{\pi}-$invariant set $\mathcal{B} \subset X$ there exists a nonempty compact $K_{\mathcal{B}} \subset X$ with $K_{\mathcal{B}} \cap M_X = \emptyset$ such that $\lim\limits_{t \to +\infty} \beta_X(\widetilde{\pi}(\mathcal{B}, t), K_{\mathcal{B}}) = 0$;

c) is called completely continuous if for every bounded subset $\mathcal{B} \subset X$ there exist $l = l(\mathcal{B}) > 0$ such that $\widetilde{\pi}(\mathcal{B}, l)$ is relatively compact and $\overline{\widetilde{\pi}(\mathcal{B}, l)} \cap M_X = \emptyset$;

d) is called weakly $b-$dissipative if there exist a nonempty bounded set $B_0 \subset X$ such that $\widetilde{\pi}^+(x) \cap B_0 \neq \emptyset$ for every point x from X. In this case we will call B_0 the bounded weak $b-$attractor of $(X, \pi; M_X, I_X)$;

e) is called weakly $k-$dissipative if there exist a nonempty compact set $K_0 \subset X$ such that for every $\epsilon > 0$ and $x \in X$ there is $\tau = \tau(x, \epsilon) > 0$ for which $\widetilde{\pi}(x, \tau) \in B_X(K_0, \epsilon)$. In this case we will call K_0 the compact weak $k-$attractor of $(X, \pi; M_X, I_X)$.

The theory of dissipative systems in the continuous case can be found in [9].

5 The main results

In this section we present the main results of the paper. We divide this section in three subsections. In Subsection 5.1, we construct the theory of non-autonomous dissipative semidynamical systems with impulses and we study the theory of stability for these systems. In Subsection 5.2, we define the concept of global attractor for a non-autonomous dissipative impulsive semidynamical system and we present conditions for this kind of system to admit a global attractor. In the last subsection, we show that non-autonomous systems of impulsive differential equations admit a non-autonomous semidynamical system with impulses associated.

Throughout this section we shall consider the following conditions for the impulsive semidynamical system $(X, \pi; M_X, I_X)$:

(H1) No initial point in (X, π) belongs to the impulsive set M_X and each element of M_X satisfies the condition (STC), consequently ϕ_X is continuous on $X \setminus M_X$ (see Theorem 3.1).

(H2) $M_X \cap I_X(M_X) = \emptyset$.

Conditions (H1)-(H2) are motivated by several results in the theory of impulsive systems which can be found, in particular, in [7, 8].

5.1 Non-autonomous semidynamical systems with impulses

In [9], the author defines the concept of non-autonomous dissipative dynamical systems and he presents a systematic study of topological properties for these systems. In this section, we construct this theory for impulsive semidynamical systems.

Let $(X, \pi; M_X, I_X)$ and $(Y, \sigma; M_Y, I_Y)$ be impulsive semidynamical systems, where (X, ρ_X) and (Y, ρ_Y) are metric spaces. In the next definition we establish the concept of a homomorphism between two impulsive semidynamical systems.

Definition 5.1. A mapping $h : X \to Y$ is called a homomorphism from the impulsive semidynamical system $(X, \pi; M_X, I_X)$ taking values in $(Y, \sigma; M_Y, I_Y)$, if the following conditions hold:

a) h is continuous in X;

b) h is surjective;

c) $h(\widetilde{\pi}(x,t)) = \widetilde{\sigma}(h(x),t)$ for all $x \in X$ and for all $t \geq 0$.

We cannot assure that h takes impulsive points in impulsive points. If h is a homeomorphism, for instance, then h takes impulsive points in impulsive points, see [5].

In the sequel, we define the concept of a non-autonomous impulsive semidynamical system with impulses.

Definition 5.2. The triple $\langle (X, \pi; M_X, I_X), (Y, \sigma; M_Y, I_Y), h \rangle$, where h is a homomorphism from $(X, \pi; M_X, I_X)$ to $(Y, \sigma; M_Y, I_Y)$, is called a non-autonomous impulsive semidynamical system. The impulsive system $(Y, \sigma; M_Y, I_Y)$ is called a factor of the impulsive system $(X, \pi; M_X, I_X)$ by the homomorphism h.

Next we define the concepts of dissipativity and center of Levinson for a non-autonomous systems with impulses.

Definition 5.3. The non-autonomous impulsive semidynamical system

$$\langle (X, \pi; M_X, I_X), (Y, \sigma; M_Y, I_Y), h \rangle \tag{5.1}$$

is said to be point-wise (compact, local, bounded) dissipative if the autonomous impulsive semidynamical system $(X, \pi; M_X, I_X)$ possesses this property. Analogously, the system (5.1) is said to be $k-$dissipative if the system $(X, \pi; M_X, I_X)$ is $k-$dissipative.

Definition 5.4. Let system (5.1) be compact $k-$dissipative and the set J be the center of Levinson of $(X, \pi; M_X, I_X)$. The set J is said to be the Levinson's center of the non-autonomous system (5.1).

Let $B_c(X)$ be the family of all bounded closed subsets from X equipped with the Hausdorff distance. In the sequel, we present a special condition defined in [9].

Definition 5.5. A nonempty compact subset $K \subset X$ satisfies the condition (C) in the system (5.1) if the mapping $F : Y \to B_c(X)$ defined by

$$F(y) = K_y = \{x \in K : h(x) = y\},$$

is continuous in $h(K)$.

Given a subset $A \subset X$ and $y \in Y$, we define $A_y = \{x \in A : h(x) = y\}$.
Example 5.1 shows a simple impulsive system which satisfies the condition (C).

Example 5.1. Consider the impulsive semidynamical system $(X, \pi; M_X, I_X)$, where $X = \mathbb{R}^2$, $M_X = \{(x_1, 3) : x_1 \in \mathbb{R}\} \cup \{(x_1, 1) : x_1 \in \mathbb{R}\}$, $I_X(x_1, x_2) = \left(\dfrac{x_1}{2}, \dfrac{x_2}{2}\right)$ for $(x_1, x_2) \in M_X$ and $\pi((x_1, x_2), t) = (e^{-t}x_1, e^{-t}x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2$ and $t \geq 0$. Now, we consider the system $(Y, \sigma; M_Y, I_Y)$, where $Y = \mathbb{R}$, $M_Y = \{1, 3\}$, $I_Y(1) = \frac{1}{2}$, $I_Y(3) = \frac{3}{2}$ and $\sigma(y, t) = e^{-t}y$ for all $y \in Y$ and $t \geq 0$. Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\}$ and h be the projection in the second coordinate. Then $\langle (X, \pi; M_X, I_X), (Y, \sigma; M_Y, I_Y), h \rangle$ is a non-autonomous impulsive system and A satisfies the condition (C) in this system.

Next, we define some concepts of stability for non-autonomous systems with impulses.

Definition 5.6. A nonempty set $K \subset X$ is said to be orbitally stable with respect to the non-autonomous impulsive system (5.1) if for all $y \in Y$ with $K_y \neq \emptyset$ we have:

a) $K_{\widetilde{\sigma}(y,t)} \neq \emptyset$ for all $t \geq 0$, and

b) for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $\rho_X(x, K_y) < \delta$ ($x \in X$ and $y = h(x)$) then $\rho_X(\widetilde{\pi}(x,t), K_{\widetilde{\sigma}(y,t)}) < \epsilon$ for all $t \geq 0$.

Definition 5.7. If $K \subset X$ is orbitally stable with respect to the non-autonomous impulsive system (5.1) and

a) there is $\gamma > 0$ such that for all $y \in Y$ with $K_y \neq \emptyset$ satisfying $\rho_X(x, K_y) < \gamma$, $x \in X_y$, we have $\rho_X(\widetilde{\pi}(x,t), K_{\widetilde{\sigma}(y,t)}) \overset{t \to +\infty}{\longrightarrow} 0$, then K is said to be asymptotically orbitally stable with respect to the system (5.1);

b) for all $y \in Y$ such that $K_y \neq \emptyset$ implies $\rho_X(\widetilde{\pi}(x,t), K_{\widetilde{\sigma}(y,t)}) \overset{t \to +\infty}{\longrightarrow} 0$ for all $x \in X_y$, then K is said to be globally asymptotically stable with respect to the system (5.1).

If system (5.1) is compact $k-$dissipative then its center of Levinson is orbitally stable as shown in the next result.

Theorem 5.1. *Let system (5.1) be compact $k-$dissipative and J be its center of Levinson satisfying the condition (C). Then J is orbitally stable with respect to the system (5.1).*

Proof. Suppose to the contrary that J is not orbitally stable with respect to the system (5.1). Since J is positively $\widetilde{\pi}-$invariant the condition $a)$ from the Definition 5.6 holds. Then there are $\epsilon_0 > 0$, a sequence $\{\delta_n\}_{n \geq 1} \subset \mathbb{R}_+$ with $\delta_n \overset{n \to +\infty}{\longrightarrow} 0$, a sequence $\{x_n\}_{n \geq 1} \subset X$ with $h(x_n) = y_n$, $J_{y_n} \neq \emptyset$, $\rho_X(x_n, J_{h(x_n)}) < \delta_n$ and $t_n \geq 0$ such that

$$\rho_X(\widetilde{\pi}(x_n, t_n), J_{\widetilde{\sigma}(h(x_n), t_n)}) \geq \epsilon_0 \tag{5.2}$$

for all $n = 1, 2, 3 \ldots$.

Since $\langle (X, \pi; M_X, I_X), (Y, \sigma; M_Y, I_Y), h \rangle$ is compact $k-$dissipative, we may assume that

$$\widetilde{\pi}(x_n, t_n) \overset{n \to +\infty}{\longrightarrow} a \quad \text{and} \quad \widetilde{\sigma}(h(x_n), t_n) \overset{n \to +\infty}{\longrightarrow} b,$$

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with $b = h(a)$.

Now, since $\widetilde{\pi}(x_n, t_n) \stackrel{n \to +\infty}{\longrightarrow} a$ and $\rho_X(x_n, J) \stackrel{n \to +\infty}{\longrightarrow} 0$ it follows that $a \in \widetilde{D}_X^+(J)$. But J is orbitally $\widetilde{\pi}$−stable (see Theorem 3.1, [7]), thus $\widetilde{D}_X^+(J) = J$ and $a \in J$. Hence, $a \in J_b$ because $h(a) = b$.

On the other hand, since J satisfies the condition (C), when n approaches to $+\infty$ in (5.2) we get $\rho_X(a, J_b) \geq \epsilon_0$ which is a contradiction as $a \in J_b$.

Therefore, J is orbitally stable with respect to $\langle (X, \pi; M_X, I_X), (Y, \sigma; M_Y, I_Y), h \rangle$. $\square$

Next, we define a concept of stability in the sense of Lyapunov-Barbashin.

Definition 5.8. A nonempty set $K \subset X$ is called globally asymptotically stable in the sense of Lyapunov-Barbashin if $h(K) = Y$ and the limit

$$\lim_{t \to +\infty} \rho_X(\widetilde{\pi}(x, t), K_{\widetilde{\sigma}(h(x), t)}) = 0$$

holds uniformly with respect to x on the compact subsets of X.

In Theorem 5.2, we present a result concerning the global asymptotic stability of the Levinson's center of system (5.1).

Theorem 5.2. *Let system (5.1) be compact k−dissipative and assume that its Levinson's center J satisfy the condition (C). Then the following conditions hold:*

  a) *the set J is globally asymptotically stable with respect to system (5.1);*

  b) *the set J is globally asymptotically stable in the sense of Lyapunov-Barbashin provided $h(J) = Y$.*

Proof. Let us prove item b) because the proof of item a) is similar. Suppose to the contrary that there are a compact set $K \subset X$, $\epsilon_0 > 0$, $\{x_n\}_{n \geq 1} \subset K$, $\{t_n\}_{n \geq 1} \subset \mathbb{R}_+$ with $t_n \stackrel{n \to +\infty}{\longrightarrow} +\infty$ such that

$$\rho_X(\widetilde{\pi}(x_n, t_n), J_{\widetilde{\sigma}(h(x_n), t_n)}) \geq \epsilon_0 \tag{5.3}$$

for all $n = 1, 2, \ldots$.

Since $(X, \pi; M_X, I_X)$ is compact k−dissipative, we may assume that

$$\widetilde{\pi}(x_n, t_n) \stackrel{n \to +\infty}{\longrightarrow} a \quad \text{and} \quad \widetilde{\sigma}(h(x_n), t_n) \stackrel{n \to +\infty}{\longrightarrow} h(a).$$

Note that $a \in \widetilde{L}_X^+(K) \subset J$. By the condition (C) we may pass the limit in (5.3) as $n \to +\infty$ and we obtain $\rho_X(a, J_{h(a)}) \geq \epsilon_0$ which is a contradiction because $a \in J \cap h^{-1}(h(a)) = J_{h(a)}$. Hence the result follows. $\square$

In the next result, we give sufficient conditions for an asymptotically orbitally stable compact set to be asymptotically $\widetilde{\pi}$−stable with respect to the system $(X, \pi; M_X, I_X)$. But before that, we present an auxiliary result whose proof is similar to the proof of Lemma 2.3 from [9].

Lemma 5.1. *Let $A \subset X$ be a nonempty compact set. Assume that A satisfies the condition (C) and $h(A) = Y$. Then for all $\delta > 0$ there is $\gamma = \gamma(\delta) > 0$ such that the following inclusion*

$$B_X(A, \gamma) \subset \bigcup_{y \in Y} B_y(A, \delta)$$

holds, where $B_y(A, \delta) = \{x \in X : h(x) = y \text{ and } \rho_X(x, A_y) < \delta\}$.

Theorem 5.3. *Let $A \subset X$ be nonempty, compact and asymptotically orbitally stable with respect to the system (5.1). Assume that A satisfies the condition (C) and $h(A) = Y$. Then A is asymptotically $\widetilde{\pi}-$stable with respect to the autonomous system $(X, \pi; M_X, I_X)$.*

Proof. Let $\epsilon > 0$ be given. Then there exists $\delta_1 = \delta_1(\epsilon) > 0$ such that if $\rho_X(x, A_y) < \delta_1$ $(x \in X, \; y = h(x))$ then $\rho_X(\widetilde{\pi}(x, t), A_{\widetilde{\sigma}(y,t)}) < \epsilon$ for all $t \geq 0$. By Lemma 5.1, there is $\delta_2 = \delta_2(\delta_1) > 0$ such that $B_X(A, \delta_2) \subset \bigcup_{y \in Y} B_y(A, \delta_1)$.

Let us show that A is orbitally $\widetilde{\pi}-$stable. Let $x \in X$ such that $\rho_X(x, A) < \delta_2$. Then $x \in B_{h(x)}(A, \delta_1)$ and therefore

$$\rho_X(\widetilde{\pi}(x, t), A_{\widetilde{\sigma}(h(x),t)}) < \epsilon \quad \text{for all} \quad t \geq 0.$$

Thus,

$$\rho_X(\widetilde{\pi}(x, t), A) \leq \rho_X(\widetilde{\pi}(x, t), A_{\widetilde{\sigma}(h(x),t)}) < \epsilon \quad \text{for all} \quad t \geq 0.$$

Hence, A is orbitally $\widetilde{\pi}-$stable.

Now, let us show that A is $\widetilde{\pi}-$attracting. Since A is asymptotically orbitally stable with respect to the system (5.1), there is $\gamma > 0$ such that if $x \in X_y$ and $\rho_X(x, A_y) < \gamma$ then $\rho_X(\widetilde{\pi}(x, t), A_{\widetilde{\sigma}(y,t)}) \xrightarrow{t \to +\infty} 0$. By Lemma 5.1, there is $\delta = \delta(\gamma) > 0$ such that $B_X(A, \delta) \subset \bigcup_{y \in Y} B_y(A, \gamma)$. Thus, if $x \in B_X(A, \delta)$ then $\rho_X(\widetilde{\pi}(x, t), A) \xrightarrow{t \to +\infty} 0$. The result is proved. $\square$

In the sequel, we define the concept of a fiber of a stable manifold.

Definition 5.9. Let $A \subset X$ be a nonempty set in the non-autonomous system (5.1) such that $h(A) = Y$. The set

$$\widetilde{W}_y^s(A) = \left\{ x \in X : h(x) = y \;\; \text{and} \;\; \lim_{t \to +\infty} \rho_X(\widetilde{\pi}(x, t), A_{\widetilde{\sigma}(y,t)}) = 0 \right\}$$

is called a fiber of the stable manifold $\widetilde{W}_X^s(A)$.

Lemma 5.2. *Let $A \subset X$ be a nonempty compact set, satisfying the condition (C) and $h(A) = Y$. Then $\widetilde{W}_X^s(A) = \bigcup_{y \in Y} \widetilde{W}_y^s(A)$.*

Proof. Let $x \in \widetilde{W}^s_X(A)$ then $\lim\limits_{t \to +\infty} \rho_X(\widetilde{\pi}(x,t), A) = 0$. We claim that

$$\lim_{t \to +\infty} \rho_X(\widetilde{\pi}(x,t), A_{\widetilde{\sigma}(h(x),t)}) = 0.$$

In fact, suppose to the contrary that there are $\epsilon_0 > 0$ and a sequence $\{t_n\}_{n \geq 1} \subset \mathbb{R}_+$ such that $t_n \overset{n \to +\infty}{\longrightarrow} +\infty$ and $\rho_X(\widetilde{\pi}(x,t_n), A_{\widetilde{\sigma}(h(x),t_n)}) \geq \epsilon_0$ for each $n = 1, 2, \ldots$. Since A is compact, we may suppose that $\widetilde{\pi}(x,t_n) \overset{n \to +\infty}{\longrightarrow} a \in A$ and $\widetilde{\sigma}(h(x),t_n) \overset{n \to +\infty}{\longrightarrow} h(a)$. By condition (C) of A we have

$$\rho_X(\widetilde{\pi}(x,t_n), A_{\widetilde{\sigma}(h(x),t_n)}) \overset{n \to +\infty}{\longrightarrow} \rho_X(a, A_{h(a)}) = 0,$$

which is a contradiction. Hence, $\widetilde{W}^s_X(A) \subset \bigcup\limits_{y \in Y} \widetilde{W}^s_y(A)$. The other set inclusion follows from the definition. $\qquad\square$

Assume that A is asymptotically $\widetilde{\pi}-$stable with respect to the system $(X, \pi; M_X, I_X)$. In Theorem 5.4, we present sufficient conditions for A to be asymptotically orbitally stable with respect to the system (5.1).

Theorem 5.4. *Let $A \subset X$ be a nonempty compact set satisfying the condition (C). Assume that A is asymptotically $\widetilde{\pi}-$stable with respect to the system $(X, \pi; M_X, I_X)$ and $h(A) = Y$, then A is asymptotically orbitally stable with respect to the system (5.1).*

Proof. First, let us prove that A is orbitally stable with respect to the system (5.1). Let $\epsilon > 0$ be given. By Lemma 5.1, there is $\gamma = \gamma(\epsilon) > 0$ such that

$$B_X(A, \gamma) \subset \bigcup_{y \in Y} B_y(A, \epsilon). \tag{5.4}$$

Since A is orbitally $\widetilde{\pi}-$stable, there is $\delta = \delta(\gamma) > 0$ such that $\rho_X(w, A) < \delta$ implies

$$\rho_X(\widetilde{\pi}(w,t), A) < \gamma \quad \text{for all} \quad t \geq 0. \tag{5.5}$$

Now, let $x \in X$ and $y = h(x)$ such that $\rho_X(x, A_y) < \delta$. Then

$$\rho_X(x, A) \leq \rho_X(x, A_y) < \delta,$$

and consequently by (5.5), we have $\rho_X(\widetilde{\pi}(x,t), A) < \gamma$ for all $t \geq 0$. Using (5.4) we obtain

$$\rho_X(\widetilde{\pi}(x,t), A_{\widetilde{\sigma}(y,t)}) < \epsilon \quad \text{for all} \quad t \geq 0.$$

Hence, A is orbitally stable with respect to the system (5.1).

By hypothesis, we may obtain $\gamma > 0$ such that $B_X(A, \gamma) \subset \widetilde{W}^s_X(A)$. Let $x \in X_y$ with $\rho_X(x, A_y) < \gamma$. Then $\rho_X(x, A) \leq \rho_X(x, A_y) < \gamma$ and hence $x \in \widetilde{W}^s_X(A)$. By Lemma 5.2 we have $x \in \widetilde{W}^s_y(A)$, that is, $\rho(\widetilde{\pi}(x,t), A_{\widetilde{\sigma}(y,t)}) \overset{t \to +\infty}{\longrightarrow} 0$. Therefore, A is asymptotically orbitally stable with respect to system (5.1). $\qquad\square$

5.2 Attractors

In all this section, we consider a non-autonomous impulsive semidynamical system

$$\langle (X, \pi; M_X, I_X), (Y, \sigma; M_Y, I_Y), h \rangle, \tag{5.6}$$

where (X, ρ_X) is a complete metric space, (Y, ρ_Y) is a compact metric space and (X, h, Y) is a locally trivial Banach fiber bundle (see [16]). Let $|\cdot| : X \to \mathbb{R}_+$ be a mapping defined by $|x| = \rho_X(x, \theta_{h(x)})$ for all $x \in X$ (θ_y is a null element of the space X_y, $y \in Y$). Let Θ be its trivial section, i.e., $\Theta = \{\theta_y : y \in Y\}$.

According to our assumptions, the set

$$A(R) := \{x \in X : |x| \leq R\} = \bigcup_{y \in Y} \{x \in X_y : \rho_X(x, \theta_y) \leq R\}$$

is bounded for all $R > 0$. In fact, since Y is compact and the Banach fiber bundle (X, h, Y) is locally trivial, then its null section $\Theta = \{\theta_y : y \in Y\}$ is compact. Hence, $A(R)$ is bounded. This remark is described in [9].

Definition 5.10. A non-autonomous impulsive system $\langle (X, \pi; M_X, I_X), (Y, \sigma; M_Y, I_Y), h \rangle$ admits a global attractor if $\langle (X, \pi; M_X, I_X), (Y, \sigma; M_Y, I_Y), h \rangle$ is compactly $k-$dissipative and

$$\lim_{t \to +\infty} \sup_{\substack{x \in X \\ |x| \leq R}} \rho_X(\widetilde{\pi}(x, t), J) = 0$$

for any $R > 0$, where J is the center of Levinson of system $(X, \pi; M_X, I_X)$.

The first result gives necessary and sufficient conditions for a non-autonomous impulsive system to admit a global attractor.

Theorem 5.5. *Consider the system (5.6) with $(X, \pi; M_X, I_X)$ completely continuous. Then the following conditions are equivalent:*

a) *there exists $r > 0$ such that for any $x \in X$ there will be $\tau = \tau(x) > 0$ and $\delta = \delta(x) > 0$ for which $|\widetilde{\pi}(y, t)| < r$ for all $y \in B_X(x, \delta)$ and for all $t \geq \tau$;*

b) *system (5.6) admits a global attractor.*

Proof. Let us show that condition a) implies condition b). First, we prove that system $(X, \pi; M_X, I_X)$ is compact $k-$dissipative. In fact, by the comments done in the beginning of this section, the set

$$A(r) := \{x \in X : |x| \leq r\} = \bigcup_{y \in Y} \{x \in X_y : \rho_X(x, \theta_y) \leq r\}$$

is bounded. Consequently, by hypothesis, there is $\ell = \ell(A(r)) > 0$ such that $K := \overline{\widetilde{\pi}(A(r), \ell)}$ is compact and $K \cap M_X = \emptyset$.

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Now, given $x \in X$, it follows by condition $a)$ that there are $\tau = \tau(x) > 0$ and $\delta = \delta(x) > 0$ such that

$$|\widetilde{\pi}(y,t)| \leq r$$

for all $y \in B_X(x,\delta)$ and for all $t \geq \tau$. It means that $\widetilde{\pi}(B_X(x,\delta),[\tau,+\infty)) \subset A(r)$. Thus, $\widetilde{\pi}(B_X(x,\delta),[\tau+\ell,+\infty)) \subset K$. Hence, $(X,\pi;M_X,I_X)$ is locally $k-$dissipative. Consequently, $(X,\pi;M_X,I_X)$ is compact $k-$dissipative, see Lemma 3.7 in [7]. Let J be its center of Levinson.

Since $A(R)$ is bounded for all $R > 0$ and the system $(X,\pi;M_X,I_X)$ is completely continuous, there is $\ell_R > 0$ such that $W := \overline{\widetilde{\pi}(A(R),\ell_R)}$ is compact and $W \cap M_X = \emptyset$. By the compact $k-$dissipativity of $(X,\pi;M_X,I_X)$ we have $\lim\limits_{t\to+\infty} \sup\limits_{x \in W} \rho_X(\widetilde{\pi}(x,t),J) = 0$. Then

$$\lim_{t\to+\infty} \sup_{x \in A(R)} \rho_X(\widetilde{\pi}(x,t),J) = 0$$

and the result follows.

Now, let us show that condition $b)$ implies condition $a)$. Let $\epsilon > 0$ be given. Since J is compact, there is $r > 0$ such that $B_X(J,\epsilon) \subset A(r)$.

On the other hand, given $x \in X$, there is $R > 0$ such that $|x| < R$. By condition $b)$ we have

$$\lim_{t\to+\infty} \sup_{y \in A(R)} \rho_X(\widetilde{\pi}(y,t),J) = 0.$$

Since $|x| < R$ then there is $\delta = \delta(x) > 0$ such that $B_X(x,\delta) \subset A(R)$. Then,

$$\lim_{t\to+\infty} \sup_{y \in B_X(x,\delta)} \rho_X(\widetilde{\pi}(y,t),J) = 0,$$

that is,

$$\lim_{t\to+\infty} \beta_X(\widetilde{\pi}(B_X(x,\delta),t),J) = 0.$$

For the $\epsilon > 0$ chosen above, there is $\tau > 0$ such that

$$\widetilde{\pi}(B_X(x,\delta),t) \subset B_X(J,\epsilon) \subset A(r) \quad \text{for all} \quad t \geq \tau$$

and we have condition $a)$. $\qquad\square$

Let $\Omega_X = \overline{\cup\{\widetilde{L}_X^+(x) : x \in X\}}$. If we assume that system $(X,\pi;M_X,I_X)$ satisfies the condition of Ladyzhenskaya and $\widetilde{D}_X^+(\Omega_X) \cap M_X = \emptyset$, then we can obtain the same equivalence in Theorem 5.5, see the next result.

Theorem 5.6. *Consider system* (5.6) *and assume that* $(X,\pi;M_X,I_X)$ *satisfies the condition of Ladyzhenskaya and* $\widetilde{D}_X^+(\Omega_X) \cap M_X = \emptyset$. *Then conditions* $a)$ *and* $b)$ *of Theorem 5.5 are equivalent.*

Proof. It is enough to show that condition a) implies b). By the proof of Theorem 5.5, the condition a) implies that system $(X, \pi; M_X, I_X)$ is locally dissipative. Thus by Theorem 3.12 from [8], $(X, \pi; M_X, I_X)$ is bounded $k-$dissipative. Now, by Theorem 3.7 from [8], the system $(X, \pi; M_X, I_X)$ is compact $k-$dissipative and its center of Levinson J attracts all the bounded subsets from X. Since $A(R)$ is bounded for all $R > 0$, we have

$$\lim_{t \to +\infty} \beta_X(\widetilde{\pi}(A(R), t), J) = 0.$$

The result is proved. $\qquad\square$

Next, we suppose that the autonomous system $(X, \pi; M_X, I_X)$ is $\widetilde{\pi}-$asymptotically compact and we get necessary and sufficient conditions to obtain a global attractor for the system (5.6).

Theorem 5.7. *Consider system* (5.6) *and let* $(X, \pi; M_X, I_X)$ *be* $\widetilde{\pi}-$*asymptotically compact and* $\widetilde{D}_X^+(\Omega_X) \cap M_X = \emptyset$. *Then the following conditions are equivalent:*

a) *there is a number* $R_0 > 0$ *such that for all* $R > 0$ *there is* $l = l(R) > 0$ *satisfying* $|\widetilde{\pi}(x, t)| \leq R_0$ *for all* $t \geq l(R)$ *and* $|x| \leq R$;

b) *system* (5.6) *admits a global attractor.*

Proof. It is enough to show that a) implies b). Let $B_0 \subset X$ be a bounded set. Then there is $R > 0$ such that

$$B_0 \subset A(R).$$

By condition a), there is $l = l(R) > 0$ such that $|\widetilde{\pi}(x, t)| \leq R_0$ for all $t \geq l$ and $|x| \leq R$. In particular, $W := \widetilde{\pi}(B_0, [l, +\infty))$ is bounded and positively $\widetilde{\pi}-$invariant. Since $(X, \pi; M_X, I_X)$ is $\widetilde{\pi}-$asymptotically compact, there is a compact $K = K(B_0)$ such that $K \cap M_X = \emptyset$ and

$$\lim_{t \to +\infty} \beta_X(\widetilde{\pi}(W, t), K) = 0,$$

consequently,

$$\lim_{t \to +\infty} \beta_X(\widetilde{\pi}(B_0, t), K) = 0.$$

Hence, $(X, \pi; M_X, I_X)$ satisfies the condition of Ladyzhenskaya and the result follows by Theorem 5.6. $\qquad\square$

If Y minimal and $I_Y(M_Y) \cap M_Y = \emptyset$, we establish sufficient conditions for the system (5.6) to obtain a global attractor, see Theorem 5.8. But before that we show an auxiliary result.

Lemma 5.3. *Let system* (5.6) *be such that* Y *is minimal and* $I_Y(M_Y) \cap M_Y = \emptyset$. *Assume that the following conditions hold:*

a) $(X, \pi; M_X, I_X)$ is completely continuous;

b) $\sup\limits_{t \geq 0} |\widetilde{\pi}(x, t)| < +\infty$ for all $x \in X$;

c) there are $y_0 \in Y \setminus M_Y$ and $R_0 > 0$ such that for any $x \in X_{y_0}$ there is $\tau = \tau(x) \geq 0$ for which $|\widetilde{\pi}(x, \tau)| < \dfrac{R_0}{3}$.

Then $(X, \pi; M_X, I_X)$ is weak $k-$attractor and its weak $k-$attractor does not intercept M_X.

Proof. Let $x \in X$ and $y = h(x)$. By condition b), there is a bounded set $B = B(x)$ such that $\widetilde{\pi}^+(x) \subset B$. Since $(X, \pi; M_X, I_X)$ is completely continuous, there is $\ell > 0$ such that $\overline{\widetilde{\pi}(B, \ell)}$ is relatively compact and $\overline{\widetilde{\pi}(B, \ell)} \cap M_X = \emptyset$.

On the other hand, since $\widetilde{\sigma}(y, \ell) \notin M_Y$ and Y is minimal we have

$$y_0 \in Y = \overline{\widetilde{\sigma}^+(\widetilde{\sigma}(y, \ell))},$$

where y_0 is given by condition c). Then there is a sequence $\{t_n\}_{n \geq 1} \subset \mathbb{R}_+$ such that

$$\widetilde{\sigma}(y, \ell + t_n) \overset{n \to +\infty}{\longrightarrow} y_0.$$

Now, since $\{\widetilde{\pi}(x, \ell + t_n)\}_{n \geq 1} \subset \overline{\widetilde{\pi}(B, \ell)}$ we may assume that

$$\widetilde{\pi}(x, \ell + t_n) \overset{n \to +\infty}{\longrightarrow} x^* \in \overline{\widetilde{\pi}(B, \ell)}.$$

Then
$$h(x^*) = \lim_{n \to +\infty} h(\widetilde{\pi}(x, \ell + t_n)) = \lim_{n \to +\infty} \widetilde{\sigma}(y, \ell + t_n) = y_0$$

and we conclude that $x^* \in X_{y_0}$. By condition c), there is $\tau = \tau(x^*) > 0$ such that $|\widetilde{\pi}(x^*, \tau)| = \rho_X(\widetilde{\pi}(x^*, \tau), \theta_{\widetilde{\pi}(x^*, \tau)}) < \dfrac{R_0}{3}$. Since $\overline{\widetilde{\pi}(B, \ell)} \cap M_X = \emptyset$, it follows by Lemma 3.2 from [6] that there is a sequence $\{\epsilon_n\}_{n \geq 1} \subset \mathbb{R}$ such that $\epsilon_n \overset{n \to +\infty}{\longrightarrow} 0$ and

$$\widetilde{\pi}(x, \ell + t_n + \tau + \epsilon_n) \overset{n \to +\infty}{\longrightarrow} \widetilde{\pi}(x^*, \tau). \tag{5.7}$$

Let $n_1 > 0$ be such that $\rho_X(\widetilde{\pi}(x, \ell + t_n + \tau + \epsilon_n), \widetilde{\pi}(x^*, \tau)) < \dfrac{R_0}{3}$ for all $n \geq n_1$.

Also, by (5.7) and since (X, h, Y) is local trivial, we have

$$\theta_{\widetilde{\pi}(x, \ell + t_n + \tau + \epsilon_n)} \overset{n \to +\infty}{\longrightarrow} \theta_{\widetilde{\pi}(x^*, \tau)}.$$

Let $n_2 > 0$ be such that $\rho_X(\theta_{\widetilde{\pi}(x, \ell + t_n + \tau + \epsilon_n)}, \theta_{\widetilde{\pi}(x^*, \tau)}) < \dfrac{R_0}{3}$ for all $n \geq n_2$.
Set $n_0 = \max\{n_1, n_2\}$ and $\tau(x) := \ell + t_{n_0} + \tau + \epsilon_{n_0}$. Then

$$|\widetilde{\pi}(x, \tau(x))| = \rho_X(\widetilde{\pi}(x, \tau(x)), \theta_{\widetilde{\pi}(x, \tau(x))}) \leq$$

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$$\leq \rho_X(\widetilde{\pi}(x, \tau(x)), \widetilde{\pi}(x^*, \tau)) + \rho_X(\widetilde{\pi}(x^*, \tau), \theta_{\widetilde{\pi}(x^*, \tau)}) + \rho_X(\theta_{\widetilde{\pi}(x^*, \tau)}, \theta_{\widetilde{\pi}(x, \tau(x))}) < R_0.$$

Thus, $A(R_0)$ is a bounded weak b−attractor of system $(X, \pi; M_X, I_X)$. By Theorem 3.10 from [8], it follows that $(X, \pi; M_X, I_X)$ is weak k−attractor and its weak k−attractor does not intercept M_X. $\qquad\square$

Theorem 5.8. *Let system* (5.6) *be such that Y is minimal and $I_Y(M_Y) \cap M_Y = \emptyset$. Suppose the three conditions of Lemma 5.3 hold. If the weak k−attractor of $(X, \pi; M_X, I_X)$ is uniform $\widetilde{\pi}$−attracting, then system* (5.6) *admits a global attractor.*

Proof. By Lemma 5.3, the system $(X, \pi; M_X, I_X)$ is weakly k−dissipative and its weak k−attractor does not intercept M_X. By Theorems 3.7 and 3.11 from [8], the system $(X, \pi; M_X, I_X)$ is compact k−dissipative with its center of Levinson J attracting all the bounded subsets from X. Since $A(R)$ is bounded for all $R > 0$ we have

$$\lim_{t \to +\infty} \sup_{x \in A(R)} \rho_X(\widetilde{\pi}(x, t), J) = 0.$$

Therefore, the result is proved. $\qquad\square$

5.3 Construction of a non-autonomous impulsive system

In this subsection, we consider an impulsive differential equation and we obtain a non-autonomous impulsive semidynamical system associated.

Consider the impulsive system

$$\begin{cases} u' = f(t, u), \\ I : M \to \mathbb{R}^n, \end{cases} \tag{5.8}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $M \subset \mathbb{R}^n$ is an impulsive set and the continuous map I is an impulse function. We also consider its H−class given by

$$\begin{cases} v' = g(t, v), \\ I : M \to \mathbb{R}^n, \end{cases} \tag{5.9}$$

where $g \in H(f) = \overline{\{f_\tau : \tau \in \mathbb{R}\}}$, $f_\tau(t, u) = f(t + \tau, u)$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^n$ and f is regular, that is, for every equation $v' = g(t, v)$ (without impulses) and for every system (5.9) the conditions of existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $\varphi(\cdot, v, g)$ the solution of equation $v' = g(t, v)$ passing through the point $v \in \mathbb{R}^n$ at the initial moment $t = 0$. It is well-know (see [9] for instance) that the continuous mapping

$$\varphi : \mathbb{R}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n,$$

satisfies the following conditions:

a) $\varphi(0, v, g) = v$ for all $v \in \mathbb{R}^n$ and for all $g \in H(f)$,

b) $\varphi(t, \varphi(\tau, v, g), g_\tau) = \varphi(t + \tau, v, g)$ for every $v \in \mathbb{R}^n$, $g \in H(f)$ and $t, \tau \in \mathbb{R}_+$.

Moreover, the mapping $\pi : \mathbb{R}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n \times H(f)$ given by

$$\pi(t, v, g) = (\varphi(t, v, g), g_t), \tag{5.10}$$

defines a continuous semidynamical system in $\mathbb{R}^n \times H(f)$.

Denote $\mathbb{R}^n \times H(f)$ by X and define the function $\phi_X : X \to (0, +\infty]$ by

$$\phi_X(v, g) = \begin{cases} s, & \text{if } \varphi(s, v, g) \in M \text{ and } \varphi(t, v, g) \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } \varphi(t, v, g) \notin M \text{ for all } t > 0. \end{cases}$$

Now, we define $M_X = M \times H(f)$ and $I_X : M_X \to X$ by

$$I_X(v, g) = (I(v), g).$$

It follows that $(X, \pi; M_X, I_X)$ is an impulsive semidynamical system on X, where the impulsive trajectory of a point $(v, g) \in X$ is an $X-$valued function $\widetilde{\pi}(\cdot, v, g)$ defined inductively as described in the following lines.

Let $(v, g) \in X$ be given. If $\phi_X(v, g) = +\infty$, then

$$\widetilde{\pi}(t, v, g) = \pi(t, v, g) \quad \text{for all} \quad t \in \mathbb{R}_+.$$

However, if $\phi_X(v, g) = s_0$, that is, $\varphi(s_0, v, g) = v_1 \in M$ and $\varphi(t, v, g) \notin M$ for $0 < t < s_0$, we define $\widetilde{\pi}(\cdot, v, g)$ on $[0, s_0]$ by

$$\widetilde{\pi}(t, v, g) = \begin{cases} \pi(t, v, g), & 0 \le t < s_0, \\ (v_1^+, g_{s_0}), & t = s_0, \end{cases}$$

where $v_1^+ = I(v_1)$.

Since $s_0 < +\infty$, the process now continues from (v_1^+, g_{s_0}) onwards. If $\phi_X(v_1^+, g_{s_0}) = +\infty$ then we define

$$\widetilde{\pi}(t, v, g) = \pi(t - s_0, v_1^+, g_{s_0})$$

for $s_0 \le t < +\infty$. If $\phi_X(v_1^+, g_{s_0}) = s_1$, that is, $\varphi(s_1, v_1^+, g_{s_0}) = v_2 \in M$ and $\varphi(t - s_0, v_1^+, g_{s_0}) \notin M$ for $s_0 < t < s_0 + s_1$, then we define $\widetilde{\pi}(\cdot, v, g)$ on $[s_0, s_0 + s_1]$ by

$$\widetilde{\pi}(t, v, g) = \begin{cases} \pi(t - s_0, v_1^+, g_{s_0}), & s_0 \le t < s_0 + s_1, \\ (v_2^+, g_{s_0 + s_1}), & t = s_0 + s_1, \end{cases}$$

where $v_2^+ = I(v_2)$, and so on.

Let $M_Y \subset H(f)$ be an impulsive set and $I_Y : M_Y \to H(f)$ be the identity $I_Y(g) = g$. Note in this case that the impulsive system $(H(f), \sigma; M_Y, I_Y)$ is a continuous semidynamical system. Then

$$\langle (X, \pi; M_X, I_X), (H(f), \sigma; M_Y, I_Y), h \rangle,$$

where h is the projection in the second coordinate, is a non-autonomous impulsive semidynamical system associated to the system (5.8).

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