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On C^r-closing for flows on 2-manifolds.

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For some full measure subset $\mathcal{B}$ of the set of *iet*'s (i.e. interval exchange transformations) the following is satisfied: Let X be a C^r, $1 \le r \le \infty$, vector field, with finitely many singularities, on a compact orientable surface M. Given a nontrivial recurrent point $p \in M$ of X, the holonomy map around p is semi-conjugate to an *iet* $E : [0,1) \to [0,1)$. If $E \in \mathcal{B}$ then there exists a C^r vector field Y, arbitrarily close to X, in the C^r-topology, such that Y has a closed trajectory passing through p. March, 2001 ICMC-USP

1. INTRODUCTION

The open problem "C^r-closing lemma" is stated as follows:

"Let M be a smooth compact manifold, $r \ge 2$ be an integer, $f \in \mathrm{Diff}^r(M)$ (resp. $X \in \mathfrak{X}^r(M)$) and p be a nonwandering point of f (resp. of X). There exists $g \in \mathrm{Diff}^r(M)$ (resp. $Y \in \mathfrak{X}^r(M)$) arbitrarily close to f (resp. to X) in the C^r-topology so that p is a periodic point of g (resp. of Y)".

To show the scope of the main result of this paper, let us consider the following *Localized C^r-Closing* problem:

"For all neighborhood V of p and, for all neighborhood $\mathcal{V}$ of X in $\mathfrak{X}^r(M)$, there exists $Y \in \mathcal{V}$ which is a perturbation of X supported in V such that Y has a closed trajectory passing through V."

It was proved by C. Pugh that once one obtains -by arbitrarily small C^r pertubations- closed orbits which are arbitrarily near to a non-wandering point point p, then one can also get -by arbitrarily small C^r pertubations- closed trajectories which pass through p.

Therefore, Localized C^r-closing implies C^r-closing. Nevertheless, Localized C^r-closing, when $r \ge 2$, is not always possible even for flows on the torus T^2. (see [Gu3]). On the other hand, the arguments of this work can trivially be modified to show that for every vector field -to which our C^r-closing (by global perturbations) applies- the Localized C^r-Closing is also valid. In this work we have chosen a global perturbation because when the starting vector field X is analytic, then the approximating vector field Y (in the C^∞-topology) is also analytic.

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C. Pugh proved that the Localized C^1-Closing is true (which implies the C^1-closing Lemma). There are few previous results when $r \geq 2$: Gutierrez [Gu1] showed similar results to this paper when the manifold is the torus T^2. Besides the negative answer to which we have referred above [Gu3], there are others: C. Carroll's [Car] proved that, even for flows with finitely many singularities, C^2-closing by a twist-perturbation (supported in a cylinder) is not always possible. Concerning hamiltonian flows, M. Herman [Her] has remarkable counter-examples to the C^r-closing lemma. Within the context of geodesic laminations, S. Aranson and E. Zhuzhoma anounced in 1988 [A-Z] the C^r-closing lemma for a class of flows on surfaces; however, their proofs have not been published yet. For basic definition the reader may consult [K-H].

From now on, we shall only consider vector fields *on compact surfaces.* It can be seen that (in this case) C^r closing is a problem only at non-trivial recurrent points [Pei]. A complete presentation of our result is the combination of Lemmas 2.1 and 2.2 and Theorem 2.3 . However we may summarize them as follows:

THEOREM 1.1. *For some full measure subset $\mathcal{B}$ of the set of iet's (i.e. interval exchange transformations) the following is satisfied:*

Let X be a C^r, $1 \leq r \leq \infty$, vector field, with finitely many singularities, on a compact orientable surface M. Given a nontrivial recurrent point $p \in M$ of X, the holonomy map around p is semi-conjugate to an iet $E : [0,1) \to [0,1)$. If $E \in \mathcal{B}$ then there exists a C^r vector field Y, arbitrarily close to X, in the C^r-topology, such that Y has a closed trajectory passing through p.

Let us proceed to explain this result, provide the main idea of the proof and present related facts on the form of items:

(1) We explicitely describe a full measure subset $\mathcal{B}$ of *iet*'s (i.e. Interval Exchange Transformations). This set $\mathcal{B}$ does not depend on any vector field of $\mathfrak{X}^r(M)$

(2) C^r-closing at a non-trivial recurrent point p, of a vector field $X \in \mathfrak{X}^r(M)$, depends on the assymptotic itinerary of the trajectory passing through p. This itinerary is presented in terms of an *iet*, say $E = E(p) = E(p,[p,q))$ which is semiconjugate to the holonomy map $[p,q) \to [p,q)$ defined on a half-open interval $[p,q)$ transversal to X. It will be shown that the property that $E(p,[p,q))$ belongs to $\mathcal{B}$ does not depend on either the particular segment $[p,q)$ or the point of the trajectory passing through p. Furthermore, when $E \in \mathcal{B}$ we are able to find arbitrarily thin $\Sigma-$flow boxes. To visualize such a $\Sigma-$flow box, consider a standard flow box F with vertical transversal edges and horizontal trajectories; then glue together the following two points of F: its lower left corner with its upper right corner. The resulting set $\tilde{F}$ is the referred $\Sigma-$flow box the boundary of which $\partial\tilde{F}$ is the figure "8". The approximating vector field will have a closed trajectory, contained in $\tilde{F}$, transversal to the horizontal lines, passing through the vertex of $\partial\tilde{F}$. (the vertex is the point $\bar{b}$ of Fig. 1.b). The vertical edges of F make up just one interval $[\bar{a},\bar{b}] \cup [\bar{b},\bar{c}]$ of $\tilde{F}$ trasversal to X. The forward Poincaré map induced by X takes $[\bar{b},\bar{c}]$ onto $[\bar{a},\bar{b}]$ (see Fig. 1.b). These $\Sigma-$flow boxes are detected at the level of *iet*'s by the semiconjugation just described above. The domain of definition of such an *iet*, say E contains intervals such that if $[c,b]$ is one of

them, $E([c,b])$ is of the form $[a,b]$ and $[c,b] \cap [a,b] = \{b\}$. This interval $[c,b]$ (that will be called "virtual ortogonal edge") corresponds, by the semiconjugation, to $[\bar{c},\bar{b}]$ above.

(3) As we said above, nontrivial recurrence of a vector field of $\mathfrak{X}^r(M)$ is related to *iet*'s. The converse is also true: an *iet* E can be suspended to a smooth flow on a surface (see [Gu2]). In this sense, the set of vector fields having nontrivial recurrence to which our result applies is somehow big; nevertheles, we do not answer the question of how big is the referred set in more natural terms as the topology of $\mathfrak{X}^r(M)$ or within the framework of parametrized families of vector fieds.

2. STATEMENT OF THE RESULTS

Throughout this article, M will be a smooth, orientable, compact, two manifold and χ will be its Euler characteristic. We shall denote by $\mathfrak{X}^r(M)$ the space of vector field of class C^r, $1 \le r \le \infty$, with the C^r-topology. The trajectory of $X \in \mathfrak{X}^r(M)$ passing through $p \in M$ will be denoted by γ_p The domain of definition of a map S will be denoted by DOM(S). Smooth segments on M will be denoted and referred as (open, half-open, closed) intervals.

A bijective map $E : [0,1) \to [0,1)$ is said to be an *iet*, i.e. an *Interval Exchange Transformation* (with m intervals) if there exists a finite sequence $0 = a_1 < a_2 < \cdots < a_m < a_{m+1} = 1$ such that, for all $i \in \{1,2,\cdots,m\}$ and for all $x \in [a_i,a_{i+1})$, $E(x) = E(a_i) + x - a_i$, and moreover, E is discontinuous at exactly $a_2, a_3, \cdots, a_m$. This E will be identified with the pair $(\lambda,\pi) \in \Delta_m \times \mathfrak{S}_m$ made up of the positive probability vector $\lambda = \{|a_{i+1} - a_i|\}_{i=1}^m$ and the permutation π on the symbols $1,2,\cdots,m$, defined by $\pi(i) = \#\{j : E(a_j) \le E(a_i)\}$. The space of *iet*'s, with m intervals, defined in $[0,1)$, will be identified with the *measurable space* $\Delta_m \times \mathfrak{S}_m$ endowed with the product measure, where Δ_m is the simplex of positive probability vectors of $\mathbb{R}^m$, with Lebesgue measure, and $\mathfrak{S}_m$ is the finite set of permutations on m symbols with counting measure. Let $E : [a,b) \to [a,b)$ be an *iet*. We say that $[s,t] \subset [a,b)$ is a *virtual orthogonal edge for* E, if E restricted to $[s,t]$ is continuous and $s < E(s) < E^2(s) = t$. Given $k \in \mathbb{N}$, let $\mathcal{B}_k$ be the set of *iet*'s $E : [a,b) \to [a,b)$ such that for some sequence $b_n \to a$ of points of (a,b), and for every $n \in \mathbb{N}$, the *iet* $E_n : [a,b_n) \to [a,b_n)$, induced by E, has at least $\chi + k + 3$, pairwise disjoint, virtual orthogonal edges. Denote $\mathcal{B} = \cap_{k\ge1}\mathcal{B}_k$. It will be seen that, as a direct consequence of the work of W. A. Veech [Vee] and H. Masur [Mas],

Lemma 2.1. *For all* $m \ge 2$, $\Delta_m \times \mathfrak{S}_m \setminus \mathcal{B}$ *is a measure zero set.*

By transporting information along flow boxes, Item (a2) below follows from the definition of $\mathcal{B}_K$.

Lemma 2.2. *([Gu2, Structure Theorem, Section 3]) Let $X \in \mathfrak{X}^1(M)$. There are finitely many nontrivial recurrent trajectories $\gamma_{p_1}, \gamma_{p_2}, \cdots, \gamma_{p_\ell}$ of X such that if γ_p is any nontrivial recurrent trajectory of X, then $\overline{\gamma_p} = \overline{\gamma_{p_i}}$, for some $i = 1,2,\cdots,\ell$.*

Suppose that X has exactly $K \in \mathbb{N}$ singularities (K=0 is allowed). Let $p \in M$ be a nontrivial recurrent point of X. Take a half-open interval $[p,q) \subset M$ transversal to X, such

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that p is a cluster point of $\gamma_p \cap (p,q)$, Denote by $P_X : [p,q) \to [p,q)$ the forward Poincaré map induced by X. If $[p,q)$ is small enough, it can be associated to $(p,[p,q))$, an iet $E = E_{(p,[p,q))} : [0,1) \to [0,1)$ and a continuous monotone surjective map $h : [p,q) \to [0,1)$ such that $h(p) = 0$, h restricted to any given orbit of P_X is injective and, for all $x \in DOM(P_X)$, $E \circ h(x) = h \circ P_X(x)$; moreover,

(a1)there exists a subset $S \subset [0,1)$ of at most $\chi + K + 2$ elements such that if A is a connected component of $[0,1) \setminus S$, then $h^{-1}(A)$ is contained in $DOM(T)$;

(a2)Let $\overline{p} \in \overline{\gamma_p}$ be a nontrivial recurrent point of X and $(\overline{p},[\overline{p},\overline{q}))$ be a pair satisfying the same conditions as those of $[p,[p,q))$ above. Then the property that the iet $E_{(\overline{p},[\overline{p},\overline{q}))}$ belongs to $\mathcal{B}_K$ does not depend on $(\overline{p},[\overline{p},\overline{q}))$.

Under conditions of theorem above and if $E \in \mathcal{B}_K$, any nontrivial recurrent point of $\overline{\gamma_p}$ is said to be of $\mathcal{B}_K-$*type. Our result is the combination of Lemmas 2.1 and 2.2 and Theorem 2.3.*

Theorem 2.1. *Let $X \in \mathfrak{X}^r(M)$, $1 \le r \le \infty$, have $K \ge 0$ singularities. Let $p \in M$ be a $\mathcal{B}_K-$type nontrivial recurrent point of X. Then there exists $Y \in \mathfrak{X}^r(M)$, arbitrarily close to X, having a closed trajectory passing through p.*

Related to this theorem (see [Gu2]), we have that: For any $E \in \mathcal{B}$, it can be constructed $Y \in \mathfrak{X}^\infty(S)$, for some surface S, having a nontrivial recurrent point p_0 such that item (a1) is satisfied for some $h : [p_0, q_0) \to [0,1)$, and $P_Y : [p_0,q_0) \to [p_0,q_0)$. Here, P_Y can be obtained to be injective or not.

3. PROOF OF THE RESULTS

Suppose that M is endowed with an orientation and with a smooth riemannian metric $< \, , \, >$. Given a $X \in \mathfrak{X}^r(M)$, $1 \le r \le \infty$, we define $X^\perp \in \mathfrak{X}^r(M)$ by the following conditions: (a) $<X,X> = <X^\perp, X^\perp>$; and (b) when $p \in M$ is regular point of X, the ordered pair $(X(p), X^\perp(p))$ is an orthogonal positive basis of $T_p(M)$ (according to the given orientation of M). let Σ be an arc of trajectory of $X^\perp$. A $\Sigma-$*flow-box (for X)* is a compact subset $F \subset M$ whose interior is a flow box of X and whose boundary ∂F is a graph, homeomorphic to the figure "8", which is the union of arcs of trajectory $[\overline{c},\overline{a}]_X$ and $[\overline{a},\overline{c}]$ (connecting $\overline{a}$ and $\overline{c}$) of X and $X^\perp$, respectively. We shall refer to $[\overline{a},\overline{c}]$ (resp. $[\overline{c},\overline{a}]_X$) as the *orthogonal* (resp. *tangent) edge* of either F or ∂F. See Figs. 1.a and 1.b.

Let $X \in \mathfrak{X}^r(M)$, $1 \le r \le \infty$, and let $p \in M$ be a nontrivial recurrent point of X. We say that X is *T-closable* at p (i.e. twist-closable at p) if there exists a half-open interval $\Sigma = [p,q)$ tangent to $X^\perp$, such that, for any neighborhood V of p, there exists a $\Sigma-$flow-box for X having its orthogonal edge contained in $\Sigma \cap V$.

Proposition 3.1. *Let $X \in \mathfrak{X}^r(M)$, $1 \le r \le \infty$, and let $p \in M$ be a nontrivial recurrent point of X. Suppose that X is T-closable at p. Then there are sequences $t_n \to 0$, of real*

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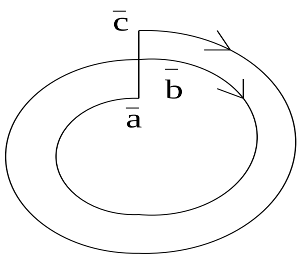

Fig. 1.a

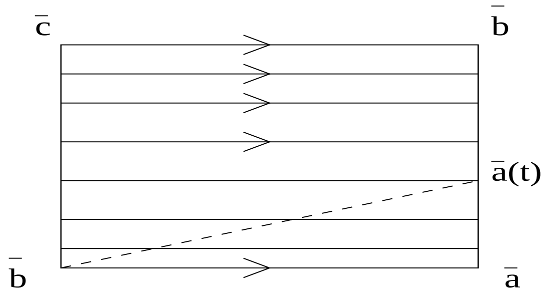

Fig. 1.b

numbers, and $p_n \to p$*, of points of* M*, such that* $X + t_n X^\perp$ *has a closed trajectory through* p_n

Proof: As X is T-closable at p, there exists a half-open interval $\Sigma = [p, q)$ tangent to $X^\perp$, such that, given neighborhoods $\mathcal{V}$ of X and V of p, we may choose a Σ−flow-box $F \subset M$ (for X) and $\sigma > 0$ such that if $[\overline{c}, \overline{a}]_X$ and $[\overline{a}, \overline{c}])$ are the tangent and orthogonal edges, respectively, of ∂F, and $\overline{b}$ is the vertex of ∂F, then:

(b1) $[\overline{a}, \overline{c}] \subset V$ and the flow of X enters into F through the closed subinterval $[\overline{b}, \overline{c}]$ of Σ; moreover, for all $t \in [-\sigma, \sigma]$, $X(t) := X + t\,X^\perp \in \mathcal{V}$;

(b2) both $X(\sigma)$ and $X(-\sigma)$ have an arc of trajectory contained in F, which is a global cross section for $X|_F$.

We shall continue considering only the case in which the flow of $X^\perp$ goes from $\overline{a}$ to $\overline{c}$. Let Γ be the set of real numbers $s \in [0, \sigma]$ such that when $t \in [0, s]$ there is an arc of trajectory $[\overline{b}, \overline{a}(t)]_{X(t)}$ of $X(t)$, joining $\overline{b}$ with $\overline{a}(t) \in [\overline{a}, \overline{b}]$, contained in F, with $\overline{a}(0) = \overline{a}$, and such that $\overline{a}(t)$ depends continuously on t. When $t \in \Gamma$, these conditions determine $\overline{a}(t)$ and also that $[\overline{b}, \overline{a}(t)]_{X(t)}$ is transversal to X. Therefore, by (b2), $\Gamma = [0, \sigma_1]$ is a closed interval, $\overline{a}(\sigma_1) = b$ and $[\overline{b}, \overline{a}(\sigma_1)]_{X(\sigma_1)}$ is a closed trajectory of $X(\sigma_1)$. See Fig. 1.b ▌

Under the assumptions and conclusions of this proposition, there exists a sequence $F_n : M \to M$ of C^r−diffeomorphisms, taking p_n to p. We may assume that F_n converges to the identity diffeomorphism in the C^{r+1}−topology. Therefore, the sequence of vector fields $(F_n)_*(X + t_n X^\perp) \to X$ in the C^r−topology and each $(F_n)_*(X + t_n X^\perp)$ has a closed trajectory passing through p. This proves the following

Theorem 3.1. *Let* $X \in \mathfrak{X}^r(M)$*,* $1 \le r \le \infty$*. Let* $p \in M$ *be a nontrivial recurrent point of* X*. Suppose that that* X *is T-closable at* p*. Then there exists* $Y \in \mathfrak{X}^r(M)$ *arbitrarily close to* X *having a closed trajectory through* p*.*

Proof of Lemma 2.1: We shall prove that: For all $m \ge 2$, $\Delta_m \times \mathfrak{S}_m \setminus \mathcal{B}$ is a measure zero set. It was proved by W. A. Veech [Vee] and H. Masur [Mas] that the Rauzy operator

$\mathcal{R} : \mathcal{M} \to \mathcal{M}$, defined in a full measure subset $\mathcal{M}$ of $\Delta_m \times \mathfrak{S}_m$, is ergodic and has the following property:

(c) Given $E \in \mathcal{M}$, there exists a sequence $\{[0, a_n)\}$ of subintervals of $[0,1)$ such that $a_n \to 0$ and, if $\tilde{E_n} : [0, a_n) \to [0, a_n)$ denotes the *iet* induced by E, then, up to re-scaling, $\mathcal{R}^n(E)$ coincides with $\tilde{E_n}$; more precisely, $\mathcal{R}^n(E)(z) = (1/a_n)\tilde{E}_n(a_n z)$, for all $z \in [0,1)$.

Given $k \geq 1$, let A_k be the set of $E \in \Delta_m \times \mathfrak{S}_m$ such that for some $a \in (16^{-k} - 32^{-k}, 16^{-k} + 32^{-k})$, $E(x) = a + x$, for all $x \in [0, 1/2]$. We observe that A_k is open and so it has positive measure. Let $\tilde{\mathcal{B}_k}$ be the set of $E \in \mathcal{M}$ such that the positive $\mathcal{R}-$orbit of E visits A_k infinitely many often. As A_k has positive measure and $\mathcal{R}$ is ergodic, the complement of $\tilde{\mathcal{B}_k}$ has measure zero. Therefore, the complement of $\tilde{\mathcal{B}} = \cap_{k\geq 2}\tilde{\mathcal{B}_k}$ has measure zero. Observe that if and *iet* $E \in A_k$, then E has more than k, pairwise disjoint, virtual orthogonal edges. Therefore, as $\mathcal{R}$ satisfy (c) right above and since the positive $\mathcal{R}-$orbit of any given $E \in \tilde{\mathcal{B}}$ visits every A_k infinitely many often, we obtain that $\tilde{\mathcal{B}} \subset \mathcal{B}$. this proves the lemma. ▮

Proof of Theorem 2.3: This theorem is stated as follows: Let $p \in M$ be a $\mathcal{B}_K-$type nontrivial recurrent point of $X \in \mathfrak{X}^r(M)$, $1 \leq r \leq \infty$. Suppose that X has $K \geq 0$ singularities. Then there exists a $Y \in \mathfrak{X}^r(M)$ arbitrarily close to X, having a closed trajectory passing through p.

By theorem 3.2, it is enough to prove that X is T-closable at p. Let $\Sigma = [p,q)$, $T : [p,q) \to [p,q)$, $E : [0,1) \to [0,1)$, $h : [p,q) \to [0,1)$ be as in Lemma 2.2. As $E \in \mathcal{B}_K$, given a neighborhood V of p, there exist $b \in (0,1)$ and an *iet* $E_V : [0,b) \to [0,b)$, such that:

(e) E_V has at least $\chi + K + 3$ pairwise disjoint virtual orthogonal edges contained in $[0,b)$; moreover, the interval $\Sigma_V = h^{-1}([0,b))$ is contained in V.

Let $T_V : \Sigma_V \to \Sigma_V$ be the map induced by T. As X has K singularities, (e) and Lemma 2.2 imply that E_V has a virtual orthogonal edge $[a, E_V(a)] \subset [0,b)$ such that, for some $\overline{a} \in \mathrm{DOM}((T_{\Sigma_V}))$, $[\overline{a}, T_V(\overline{a})] = h^{-1}([a, E(a)]) \subset \mathrm{DOM}(T|_{\Sigma_V})$. Therefore, there exists a $\Sigma-$flow-box bounded by $[\overline{a}, T_V^2(\overline{a})] \cup [\overline{a}, T_V^2(\overline{a})]_X$. As V is arbitrary, this proves that X is T-closable at p. ▮

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