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Horospherical and hyperbolic dual surfaces of spacelike curves in de Sitter space

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We define two surfaces, the horospherical surface and the hyperbolic dual surface of a spacelike curve in the de Sitter space S_1^3 in the Lorentzian-Minkowski 4-space. We study these surfaces by using technics of the singularity theory and furthermore, we give a relation between these surfaces from the view point of Legendrian dualities. May, 2016 ICMC-USP

1. INTRODUCTION

We investigate here spacelike curves in the de Sitter space S_1^3 in $\mathbb{R}_1^4$ and two special related surfaces from the view point of dual relations. For a curve $\gamma : I \to S_1^3$ with non-zero curvature, we define the horospherical surface in LC^* and the hyperbolic dual surface of γ in $H^3(-1)$. For the study of these surfaces we also use the technics of the singularity theory. In Sections 3 and 6, we define two families of the height functions on γ, a horospherical height function and a hyperbolic height function. Differentiating these functions we find an invariant related the each surface and we investigate the geometric meaning of these invariants. In Section 4, we prove propositions that give conditions for the curve γ be on a parabolic de Sitter quadric and we give also conditions for γ be part of a T-horoparabola or a S-horoparabola. Furthermore by using the theory of unfoldings (see [2]) we give a classification of the singularities of such surfaces. In Section 5 we give information about the geometry of the hyperbolic dual surface. In Section 7 we show that γ can be part of an elliptic de Sitter quadric by using an invariant of the curve and we prove a theorem that relates the contact of γ and an elliptic de Sitter quadric with

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the classification of singularities of hyperbolic dual surface of curve γ. Finally, in Section 8, we give a relation between the horospherical surface and the hyperbolic dual surface of the curve from the view point of Legendrian dualities which were introduced in [6]. Curves in the hyperbolic space $H^3(-1)$ in $\mathbb{R}^4_1$ and the de Sitter dual surface in S^3_1 and the horospherical surface in the lightcone LC^*, were investigated in the papers [3], [4], [8]. The duality relation between the curve and these surfaces were studied in [4].

2. PRELIMINARIES

The *Minkowski space* $\mathbb{R}^4_1$ is the vector space $\mathbb{R}^4$ endowsed with the pseudo-scalar product $\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$, for any $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$ in $\mathbb{R}^4_1$ (see, for example, [10]). We say that a non-zero vector $x \in \mathbb{R}^4_1$ is *spacelike* if $\langle x, x\rangle > 0$, *lightlike* if $\langle x, x\rangle = 0$ and *timelike* if $\langle x, x\rangle < 0$. We say that $\gamma : I \to \mathbb{R}^3_1$, $I \subset \mathbb{R}$ an open interval, is *spacelike* (resp. *timelike*) if $\gamma'(t)$ is a *spacelike* (resp. *timelike*) vector for any $t \in I$. The norm of a vector $x \in \mathbb{R}^3_1$ is defined by $\| x \| = \sqrt{|\langle x, x\rangle|}$. We now define the *hyperbolic space* by

$$H^3(-1) = \{x \in \mathbb{R}^4_1 \mid \langle x, x\rangle = -1\},$$

de Sitter space by

$$S^3_1 = \{x \in \mathbb{R}^4_1 \mid \langle x, x\rangle = 1\},$$

and *Lightcone* by

$$LC^* = \{x \in \mathbb{R}^4_1 \setminus \{0\} \mid \langle x, x\rangle = 0\}.$$

For any $x = (x_0, x_1, x_2, x_3)$, $y = (y_0, y_1, y_2, y_3)$, $z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4_1$, the pseudo product of x, y and z is defined as follows:

$$x \wedge y \wedge z = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{vmatrix},$$

where $\{e_0, e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^4$.

For a non-zero vector $v \in \mathbb{R}^4_1$ and a real number c, we define a *hyperplane* with *pseudo-normal* v by

$$HP(v, c) = \{x \in \mathbb{R}^4_1 \mid \langle x, v\rangle = c\}.$$

We call $HP(v, c)$ a *spacelike*, a *timelike* or *lightlike* hyperplane if v is timelike, spacelike or lightlike, respectively.

We have three kinds of surfaces in S^3_1 which are given by intersections of S^3_1 and hyperplanes in $\mathbb{R}^4_1$. A surface $S^3_1 \cap HP(v, c)$ is called an *elliptic de Sitter quadric*, a *hyperbolic de Sitter quadric* or a *parabolic de Sitter quadric* if $HP(v, c)$ is spacelike, timelike or lightlike respectively. We denote the parabolic de Sitter quadric by $QDP(v, c)$ and the elliptic de Sitter quadric by $QDE(v, c)$.

Let $\gamma : I \to S^3_1$ be a smooth and regular spacelike curve in S^3_1. Since γ is spacelike, we can parametrise it by arc length s, then we take the unit tangent vector $t(s) = \gamma'(s)$. Suppose that $\langle t'(s), t'(s)\rangle \neq 1$, then $\| t'(s) + \gamma(s) \| \neq 0$, and we have other unit vector $n(s) = \dfrac{t'(s) + \gamma(s)}{\| t'(s) + \gamma(s) \|}$. We also define an unit vector by $e(s) = \gamma(s) \wedge t(s) \wedge n(s)$, then we have an orthonormal basis $\{\gamma(s), t(s), n(s), e(s)\}$ of $\mathbb{R}^4_1$ along γ. The Frenet-Serret type formulae is given by

$$\begin{cases} \gamma'(s) = t(s) \\ t'(s) = -\gamma(s) + k_g(s)\, n(s) \\ n'(s) = -\delta(\gamma(s))\, k_g(s)\, t(s) + \tau_g(s)\, e(s) \\ e'(s) = \tau_g(s) n(s) \end{cases},$$

where $\delta(\gamma(s)) = sign(n(s))$ (we call δ for shorter), $k_g(s) = \| t'(s) + \gamma(s) \|$ and $\tau_g(s) = \dfrac{\delta(\gamma(s))}{k_g^2(s)} \det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))$, where det is the determinant of the 4×4 matrix. Here k_g is called a *geodesic curvature* and τ_g *geodesic torsion* of γ (see [7]).

Since $\langle t'(s) + \gamma(s), t'(s) + \gamma(s)\rangle = \langle t'(s), t'(s)\rangle - 1$, the condition $\langle t'(s), t'(s)\rangle \neq 1$ is equivalent to the condition $k_g(s) \neq 0$.

We define the maps

$$HS^{\pm}_{\gamma} : I \times J \to LC^* \quad \text{and} \quad HD^{\pm}_{\gamma} : I \times J \to H^3(-1)$$

by $HS^{\pm}_{\gamma}(s, \mu) = \gamma(s) + \mu n(s) + \lambda e(s)$ and $HD^{\pm}_{\gamma}(s, \mu) = \mu n(s) + \lambda e(s)$, respectively, with $\lambda^2 - \mu^2 = \delta(\gamma(s))$, where $\delta(\gamma(s)) = sign(n(s))$ is 1 if $n(s)$ is spacelike or -1 if $n(s)$ is timelike. In other words, $HS^{\pm}_{\gamma}(s, \mu) = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))} e(s)$ and $HD^{\pm}_{\gamma}(s, \mu) = \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))} e(s)$ with $\mu^2 + \delta(\gamma(s)) \geq 0$, i.e., $\mu \in \mathbb{R} = J$ for $n(s)$ spacelike or $\mu \in (-\infty, -1] \cup [1, \infty) = J$ for $n(s)$ timelike. We call $HS^{\pm}_{\gamma}$ the *horospherical surface of* γ and $HD^{\pm}_{\gamma}$ the *hyperbolic dual surface of* γ. We can think at λ and μ as $\cosh(t)$ and $\sinh(t)$ depending of $\delta(\gamma(s))$.

We use families of height functions on curves in S^3_1, and technics of the singularity theory for the study of the horospherical surface and the hyperbolic dual surface. To use these technics, we show that the horospherical surface of γ is the discriminant set of the family of horospherical height functions (Corollary 3.3.2) and that the hyperbolic dual surface of γ is the discriminant set of the family of hyperbolic height functions (Corollary 6.6.2).

We observe that the discriminant sets of the family of horospherical height functions and of the family of hyperbolic height functions on timelike curves in S^3_1 are empty. Because of this reason, we are just considering spacelike curves in S^3_1.

The singularities of these surfaces can be A_k-type, that we define below.

Definition 2.2.1. *Let $F : \mathbb{R}^4_1 \to \mathbb{R}$ (respectively, $F\,|_{S^3_1} : S^3_1 \to \mathbb{R}$) be a submersion and $\gamma : I \to S^3_1$ be a regular curve. We say that γ and $F^{-1}(0)$ (respectively $F^{-1}(0) \cap S^3_1$) have contact of order k at s_0 if the function $g(s) = F \circ \gamma(s)$ satisfies $g(s_0) = g'(s_0) = \cdots = g^{(k)}(s_0) = 0$ and $g^{(k+1)}(s_0) \neq 0$, i.e., g has A_k-type singularity at s_0.*

Let $G : \mathbb{R} \times \mathbb{R}^r, (s_0, x_0) \to \mathbb{R}$ be a function germ. We call G an r-parameter unfolding of f if $f(s) = G_{x_0}(s)$ and f has an A_k-type singularity ($k \geq 1$) at s_0. We denote the $(k-1)$-jet with constant of the partial derivative $\frac{\partial G}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial G}{\partial x_i}(s, x_0))(s_0) = \sum_{j=0}^{k-1} \alpha_{ji}(s - s_0)^j$, for $i = 1, \dots, r$. Then G is called a *versal unfolding* if and only if the $k \times r$ matrix of coefficients (α_{ji}) has rank k ($k \leq r$) (see [2]).

The *discriminant set* of G is the set

$$\mathcal{D}_G = \{x \in \mathbb{R}^r \ \mid \ G = \frac{\partial G}{\partial s} = 0 \ at \ (s, x) \ for \ some \ s\}.$$

Theorem 2.2.2. *[2] Let $G : \mathbb{R} \times \mathbb{R}^r, (s_0, x_0) \to \mathbb{R}$ be an r-parameter unfolding of f which has A_k-type singularity at s_0. Suppose that F is a versal unfolding, then $\mathcal{D}_G$ is locally diffeomorphic to*

(1)$C \times \mathbb{R}^{r-2}$ if $k = 2$,
(2)$SW \times \mathbb{R}^{r-3}$ if $k = 3$,

where $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$ is the ordinary cusp and $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallowtail.

3. HOROSPHERICAL HEIGHT FUNCTION

In this section we introduce a family of functions on a curve that is useful for the study of the horospherical surface. For a spacelike curve $\gamma : I \to S_1^3$, we define a function $H : I \times LC^* \to \mathbb{R}$ by $H(s, v) = \langle \gamma(s), v \rangle - 1$. We call H a family of *horospherical height functions on* γ. We denote $h_v(s) = H(s, v)$ for any fixed $v \in LC^*$. By Definition 2.2.1, the family of horospherical height functions measures the contact of γ with a lightlike hyperplane in $\mathbb{R}_1^4$. Generically, this contact can be of order k, $1 \leq k \leq 3$.

From the next result we can obtain equivalent conditions for each A_k-type singularity, $1 \leq k \leq 3$. For example, h_v has A_2-type singularity at s if and only if $v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))} e(s)$, $\mu = \dfrac{1}{k_g(s)\delta(\gamma(s))}$ and $\sigma_{h_v}(s) \neq 0$.

Proposition 3.3.1. *Let $\gamma : I \to S_1^3$ be a parametrised by arc length curve, with $k_g(s) \neq 0$. Then*

(1)$h_v(s) = 0$ if and only if there exist real numbers μ, λ, η with $\eta^2 + \delta(\gamma(s))\mu^2 - \delta(\gamma(s))\lambda^2 = -1$ such that $v = \gamma(s) + \eta t(s) + \mu n(s) + \lambda e(s)$.

(2)$h_v(s) = h_v'(s) = 0$ if and only if there exist real numbers μ, λ such that $v = \gamma(s) + \mu n(s) + \lambda e(s)$ with $\lambda^2 - \mu^2 = \delta(\gamma(s))$.

(3)$h_v(s) = h_v'(s) = h_v''(s) = 0$ if and only if $v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))} e(s)$ with $\mu = \dfrac{1}{k_g(s)\delta(\gamma(s))}$.

(4)$h_v(s) = h_v'(s) = h_v''(s) = h_v^{(3)}(s) = 0$ *if and only if* $v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$, $\mu = \dfrac{1}{k_g(s)\delta(\gamma(s))}$ *and* $\sigma_{h_v}(s) = 0$, *where* $\sigma_{h_v}(s) = (k_g' \pm k_g\tau_g(-\delta)\sqrt{1 + k_g^2\delta})(s)$.

(5)(i) If $n(s)$ *is timelike with* $k_g(s) = 1$ *then* $h_v(s) = h_v'(s) = h_v''(s) = h_v^{'''}(s) = h_v^{(4)}(s) = 0$ *if and only if* $v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$, $\mu = \dfrac{1}{k_g(s)\delta(\gamma(s))}$, $\sigma_{h_v}(s) = 0$ *and* $k_g''(s) + \tau_g^2(s) = 0$.

(ii) Otherwise, if $n(s)$ *is timelike with* $k_g(s) \neq 1$ *or if* $n(s)$ *is spacelike,* $h_v(s) = h_v'(s) = h_v''(s) = h_v^{(3)}(s) = h_v^{(4)}(s) = 0$ *if and only if* $v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$, $\mu = \dfrac{1}{k_g(s)\delta(\gamma(s))}$ *and* $\sigma_{h_v}(s) = \sigma_{h_v}'(s) = 0$.

Proof. Since $h_v(s) = \langle\gamma(s), v\rangle - 1$, by using the Frenet-Serret type formulae we have

(a) $h_v'(s) = \langle t(s), v\rangle$,
(b) $h_v''(s) = \langle -\gamma(s) + k_g(s)n(s), v\rangle$,
(c) $h_v^{(3)}(s) = \langle(-1 - k_g^2(s)\delta(\gamma(s)))t(s) + k_g'(s)n(s) + k_g(s)\tau_g(s)e(s), v\rangle$,
(d) $h^{(4)}(s) = \langle(1 + k_g^2(s)\delta(\gamma(s)))\gamma(s) - 3\delta(\gamma(s))k_g'(s)k_g(s)t(s) + (-k_g(s) + k_g''(s) + k_g(s)\tau_g^2(s) - k_g^3(s)\delta(\gamma(s)))n(s) + (2k_g'(s)\tau_g(s) + k_g(s)\tau_g'(s))e(s), v\rangle$.

Using (a) to (d), by simple calculations, we can show *(1)* to *(5)*. ▮

Corollary 3.3.2. *The horospherical surface of* γ *is the discriminant set,* $\mathcal{D}_H$*, of the family of horospherical height functions* H*.*

Proof. The proof follows from the definition of discriminant set given in the Section 2 and Proposition 3.3.1, *(2)*. ▮

From the above proposition, we define the invariant

$$\sigma_{h_v}(s) = (k_g' \pm k_g\tau_g(-\delta)\sqrt{1 + k_g^2\delta})(s)$$

of the curve. In the next result we show that the horospherical height function on a curve in S_1^3, is a versal unfolding of an A_k-type singularity ($k = 2, 3$). We want to study the geometric meaning of this invariant.

Proposition 3.3.3. *With the same assumptions in Proposition 3.3.1, let* $H : I \times LC^* \to \mathbb{R}$ *be a family of horospherical height functions on* γ*. If* h_v *has an* A_2*-type singularity at* s_0*, then* H *is a versal unfolding of* h_v*. If* h_v *has an* A_3*-type singularity at* s_0 *and* $n(s_0)$ *is timelike with* $k_g(s_0) \neq 1$ *(generic condition) or if* $n(s_0)$ *is spacelike, then* H *is a versal unfolding of* h_v*.*

Proof. The family is given by

$$H(s,v) = -v_1 x_1(s) + v_2 x_2(s) + v_3 x_3(s) + v_4 x_4(s) - 1,$$

where $v = (v_1, v_2, v_3, v_4)$, $\gamma(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$ is the curve parametrised by the arc length, $v_1 = \sqrt{v_2^2 + v_3^2 + v_4^2}$ and $x_1(s) = \sqrt{x_2^2(s) + x_3^2(s) + x_4^2(s) - 1}$.

Writing $H(s,v) = H(s, v_2, v_3, v_4)$, we have

$$\frac{\partial H}{\partial v_i} = x_i(s) - \frac{v_i}{v_1} x_1(s),$$

for $i = 2, 3, 4$. Therefore, the 2-jet of $\dfrac{\partial H}{\partial v_i}$ at s_0 is:

$$x_i(s_0) - \frac{v_i}{v_1} x_1(s_0) + \left(x_i'(s_0) - \frac{v_i}{v_1} x_1'(s_0) \right)(s - s_0) + \frac{1}{2}\left(x_i''(s_0) - \frac{v_i}{v_1} x_1''(s_0) \right)(s - s_0)^2.$$

We assume first that h_v has an A_3-type singularity at $s = s_0$, and we show that the determinant of the 3×3 matrix

$$A = \begin{pmatrix} x_2(s_0) - \frac{v_2}{v_1} x_1(s_0) & x_3(s_0) - \frac{v_3}{v_1} x_1(s_0) & x_4(s_0) - \frac{v_4}{v_1} x_1(s_0) \\ x_2'(s_0) - \frac{v_2}{v_1} x_1'(s_0) & x_3'(s_0) - \frac{v_3}{v_1} x_1'(s_0) & x_4'(s_0) - \frac{v_4}{v_1} x_1'(s_0) \\ x_2''(s_0) - \frac{v_2}{v_1} x_1''(s_0) & x_3''(s_0) - \frac{v_3}{v_1} x_1''(s_0) & x_4''(s_0) - \frac{v_4}{v_1} x_1''(s_0) \end{pmatrix}$$

is nonzero. Denote

$$a = \begin{pmatrix} x_1(s_0) \\ x_1'(s_0) \\ x_1''(s_0) \end{pmatrix}, b_i = \begin{pmatrix} x_i(s_0) \\ x_i'(s_0) \\ x_i''(s_0) \end{pmatrix},$$

for $i = 2, 3, 4$. Then

$$\det A = \frac{v_1}{v_1}\det(b_2\ b_3\ b_4) - \frac{v_2}{v_1}\det(a\ b_3\ b_4) - \frac{v_3}{v_1}\det(b_2\ a\ b_4) - \frac{v_4}{v_1}\det(b_2\ b_3\ a).$$

On the other hand,

$$(\gamma \wedge \gamma' \wedge \gamma'')(s_0) = (-\det(b_2\ b_3\ b_4), -\det(a\ b_3\ b_4), -\det(b_2\ a\ b_4), -\det(b_2\ b_3\ a)).$$

Therefore

$$\begin{aligned} \det A &= \left\langle \left(\frac{v_1}{v_1}, \frac{v_2}{v_1}, \frac{v_3}{v_1}, \frac{v_4}{v_1} \right), (\gamma \wedge \gamma' \wedge \gamma'')(s_0) \right\rangle \\ &= \frac{1}{v_1}\langle \gamma(s_0) + \mu n(s_0) \pm \sqrt{\mu^2 + \delta}\, e(s_0), k_g(s_0) e(s_0) \rangle \\ &= \pm \frac{1}{v_1}(-\delta)\sqrt{k_g^2(s_0)\delta + 1}. \end{aligned}$$

In the case that $n(s_0)$ is a spacelike vector, we have $\det A = \mp\frac{1}{v_1}\sqrt{k_g^2(s_0)+1} \neq 0$ and therefore, H is a versal unfolding of h_v at $s = s_0$. If $n(s_0)$ is a timelike vector, then we have $\det A = \pm\frac{1}{v_1}\sqrt{1-k_g^2(s_0)}$ and therefore $\det A \neq 0$ under the condition that $k_g(s_0) \neq 1$ and H is a versal unfolding of h_v at $s = s_0$.

In the case $k = 2$, we require the 2×3 matrix

$$B = \begin{pmatrix} x_2(s_0) - \frac{v_2}{v_1}x_1(s_0) & x_3(s_0) - \frac{v_3}{v_1}x_1(s_0) & x_4(s_0) - \frac{v_4}{v_1}x_1(s_0) \\ x_2'(s_0) - \frac{v_2}{v_1}x_1'(s_0) & x_3'(s_0) - \frac{v_3}{v_1}x_1'(s_0) & x_4'(s_0) - \frac{v_4}{v_1}x_1'(s_0) \end{pmatrix}$$

to be nonsingular. Since B is the first and second line of A, we have that if $n(s_0)$ is a spacelike vector, then the matrix B is nonsingular because $\det A \neq 0$. If $n(s_0)$ is a timelike vector, the matrix B is nonsingular if $k_g(s_0) \neq 1$. For the case $k_g(s_0) = 1$, we calculate the determinant of the Gram-Schimidt matrix of B which is equal to $\frac{2(x_1(s_0)-v_1)}{v_1}$. Then it is enough to show that $x_1(s_0) \neq v_1$. As $k_g(s_0) = 1$, we have

$$v(s_0) = \gamma(s_0) - n(s_0) = \gamma(s_0) - (t'(s_0) + \gamma(s_0))$$

by Proposition 3.3.1 *(2)* and the Frenet-Serret type formulae. Therefore $v_1 = -x_1''(s_0)$. Since $t'(s_0) = n(s_0) - \gamma(s_0)$, we have that $x_1''(s_0) = n_1(s_0) - x_1(s_0)$, i.e, $v_1 = x_1(s_0) - n_1(s_0)$, where $n(s_0) = (n_1(s_0), n_2(s_0), n_3(s_0), n_4(s_0))$. Without lost of generality we can suppose $n_1(s_0) \neq 0$. ▌

Using Theorem 2.2.2 and Proposition 3.3.3 we can know the geometric shape of the horospherical surface. The main result in this section is given as follows.

THEOREM 3.3.4. *With the same assumptions in Proposition 3.3.1, let $HS^{\pm}_{\gamma}$ be the horospherical surface of γ. Then we have the following:*

(1)The singular points of $HS^{\pm}_{\gamma}$ are given by

$$h^{\pm}_{\mu}S_{\gamma}(s) = \gamma(s) + \frac{1}{k_g(s)\delta(\gamma(s))}n(s) \pm \sqrt{\frac{1}{k_g^2(s)} + \delta(\gamma(s))}e(s).$$

(2)$HS^{\pm}_{\gamma}$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, μ_0) if and only if $\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))}$ and $\sigma_{h_v}(s_0) \neq 0$.

(3)$HS^{\pm}_{\gamma}$ is locally diffeomorphic to the swallowtail SW at (s_0, μ_0) if and only if $\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))}$, $\sigma_{h_v}(s_0) = 0$ and $\sigma'_{h_v}(s_0) \neq 0$, for $n(s_0)$ timelike with $k_g(s_0) \neq 1$ or for $n(s_0)$ spacelike.

Proof. Consider the horospherical surface given by $HS^{\pm}_{\gamma}(s,\mu) = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$. Then, we have

$$\frac{\partial HS^{\pm}_{\gamma}}{\partial s}(s,\mu) = (1 - \mu\delta(\gamma(s))k_g(s))t(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s) \quad \text{and}$$
$$\frac{\partial HS^{\pm}_{\gamma}}{\partial \mu}(s,\mu) = n(s) \pm \frac{\mu}{\sqrt{\mu^2 + \delta(\gamma(s))}}e(s).$$

The vectors $\left\{\frac{\partial HS^{\pm}_{\gamma}}{\partial s}(s_0,\mu_0), \frac{\partial HS^{\pm}_{\gamma}}{\partial \mu}(s_0,\mu_0)\right\}$ are linearly dependent if and only if $\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))}$. Then the singular points of $HS^{\pm}_{\gamma}$ are given by $h^{\pm}_{\mu_0}S_{\gamma}(s_0) = HS^{\pm}_{\gamma}(s_0,\mu_0)$ and assertion *(1)* holds. By the Corollary 3.3.2, the discriminant set $\mathcal{D}_H$ of the family of horospherical height functions H of γ is the horospherical surface of γ. It also follows from assertions *(3)* and *(4)* of Proposition 3.3.1 that h_v has the A_2-type singularity (respectively, the A_3-type singularity) at $s = s_0$ if and only if $\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))}$ and $\sigma_{h_v}(s_0) \neq 0$ (respectively, $\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))}$, $\sigma_{h_v}(s_0) = 0$ and $\sigma'_{h_v}(s_0) \neq 0$). By Theorem 2.2.2 and Proposition 3.3.3, we have assertions *(2)* and *(3)*. We observe that in *(3)* if $n(s_0)$ is timelike is necessary $k_g(s_0) \neq 1$ in order to obtain Proposition 3.3.3. ▮

4. INVARIANTS AND SPECIAL GEOMETRY OF THE HOROSPHE-RICAL SURFACE

We would like to study the geometric meaning of the invariant $\sigma_{h_v}(s)$ defined above. Let v be a lightlike vector, w be a spacelike vector and z be a timelike vector. Remember that the surface $QDP(v,1)$ is the parabolic de Sitter quadric given by $S^3_1 \cap HP(v,1)$ where $HP(v,1)$ is a lightlike hyperplane. We call the de Sitter space curve given by the intersections $QDP(v,1) \cap HP(w,0)$ and $QDP(v,1) \cap P(z,0)$, of T-horoparabolas and S-horoparabolas, respectively.

Given a unit speed spacelike curve γ in S^3_1, the unit normal vector n can be a timelike or a spacelike vector. We prove the following propositions that give conditions for the curve γ be on a parabolic de Sitter quadric. Besides, we give also conditions for γ be part of a T-horoparabola or a S-horoparabola. These facts are related to the invariants $\sigma_{h_v}(s)$ and $\tau_g(s)$. It was necessary to divide in two cases: $n(s)$ is timelike (Proposition 4.4.1) and $n(s)$ is spacelike (Proposition 4.4.2).

We observe that for a curve in hyperbolic 3-space (see [8]), there is only one case bacause $n(s)$ is always spacelike. The technique of the proof of the next result is similar to that for a curve in hyperbolic space in [8].

PROPOSITION 4.4.1. *Let* $\gamma : I \to S^3_1$ *be a spacelike curve parametrised by arc length such that* $n(s)$ *are timelike vectors,* $k_g(s) \leq 1$ *and* $k_g(s) \neq 0$*. Consider the singular points* $h^{\pm}_{\mu} S_{\gamma}(s)$ *of the horospherical surface.*

(1)Suppose that $k_g(s) \equiv 1$*. Then the following conditions are equivalent:*

(a)$h^{\pm}_{\mu} S_{\gamma}(s)$ *is a constant vector.*

(b)$\tau_g(s) \equiv 0$.

(c)γ *is a part of a T-horoparabola.*

(2)Suppose that the set $\{s \in I \mid k_g(s) = 1\}$ *consists of isolated points. The following conditions are equivalent:*

(a)$h^{\pm}_{\mu} S_{\gamma}(s)$ *is a constant vector* $v_0 \in LC^*$.

(b)$\sigma_{h_v}(s) \equiv 0$.

(c)γ *is located on a parabolic de Sitter quadric* $QDP(v_0, 1)$.

Proof. Consider the singular points $h^{\pm}_{\mu} S_{\gamma}(s)$ of the surface that we call here of $v(s) = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 - 1}e(s)$ with $\mu = -\dfrac{1}{k_g(s)}$. Suppose that $k_g(s) \equiv 1$. Then $v(s) = \gamma(s) - n(s)$, and $v'(s) = -\tau_g(s)e(s)$. Therefore $v(s)$ is constant if and only if $\tau_g(s) \equiv 0$ and the conditions *(a)* and *(b)* of *(1)* are equivalent. If $v(s)$ is constant, then $\tau_g(s) \equiv 0$ and as $e'(s) = \tau_g(s)n(s)$, this means that $e(s)$ is constant. Thus the hyperplane $P(e(s), 0)$ generated by $\gamma(s)$, $t(s)$ and $n(s)$ is constant. In this case the parabolic de Sitter quadric $QDP(v(s), 1)$ is also constant. Thus the image of γ is a part of a horoparabola given by $QDP(v(s), 1) \cap P(e(s), 0)$. Therefore *(a)* implies *(c)*. If γ is a part of a T-horoparabola, then it is a de Sitter plane curve. Therefore $\tau_g(s) \equiv 0$ and as $v'(s) = -\tau_g(s)e(s)$ then *(c)* implies *(b)*. This completes the proof of the assertion *(1)*.

Suppose now that $k_g(s) \neq 1$. Since $\mu(s) = -\dfrac{1}{k_g(s)}$, we have

$$v(s) = \gamma(s) - \frac{1}{k_g(s)} n(s) \pm \frac{\sqrt{1 - k^2_g(s)}}{k_g(s)} e(s).$$

Then

$$v'(s) = \left(\frac{k'_g \pm k_g \tau_g \sqrt{1 - k^2_g}}{k^2_g} \right)(s) n(s) - \left(\frac{\sqrt{1 - k^2_g} k_g \tau_g \pm k'_g}{k^2_g \sqrt{1 - k^2_g}} \right)(s) e(s).$$

Therefore, $v'(s) \equiv 0$ if and only if $\sigma_{h_v}(s) \equiv 0$ and the conditions *(a)* and *(b)* of *(2)* are equivalent at any point $s \in I$.

We now consider the family of horospherical height functions $H(s,v)$ on γ. If γ is located on a parabolic de Sitter quadric $QDP(v_0,1)$, then this means that $H(s,v_0)\equiv 0$. By Proposition 3.3.1 *(4)*, we have $(k_g' \pm k_g\tau_g\sqrt{1-k_g^2})(s)\equiv 0$, therefore *(c)* implies (b). If v is a constant vector v_0, then $\langle\gamma(s),v_0\rangle = 1$ for all $s\in I$ and thus $\gamma(s)\in QDP(v_0,1)$ for all $s\in I$. Therefore γ is located on a parabolic de Sitter quadric. ▮

Proposition 4.4.2. *Let $\gamma : I \to \mathbb{R}^3_1$ be a spacelike curve parametrised by arc length such that $n(s)$ are spacelike vectors and $k_g(s)\neq 0$. Consider the singular points $h^{\pm}_{\mu}S_{\gamma}(s)$ of the horospherical surface. The following conditions are equivalent:*

(a)$h^{\pm}_{\mu}S_{\gamma}(s)$ is a constant vector $v_0\in LC^*$.

(b)$\sigma_{h_v}(s)\equiv 0$.

(c)γ is located on a parabolic de Sitter quadric $QDP(v_0,1)$ for some v_0.

Furthermore, when $\gamma\subset QDP(v_0,1)$ and $\tau_g(s)\equiv 0$ then γ is part of a S-horoparabola.

Proof. The proof is analogous to the proof of the Proposition 4.4.1 *(2)*. ▮

5. HYPERBOLIC DUAL SURFACE

Let $\gamma : I \to S^3_1$ be a spacelike curve parametrised by the arc length. Remember that we are supposing $\langle t'(s),t'(s)\rangle \neq 1$ (generic condition), that is equivalent to $k_g(s)\neq 0$, to define $n(s)=\dfrac{t'(s)+\gamma(s)}{\parallel t'(s)+\gamma(s)\parallel}$. Then $n(s)$ is a spacelike normal vector field or a timelike normal vector field of γ. The next proposition gives information about the geometry of the hyperbolic dual surface.

Proposition 5.5.1. *Let $\gamma : I \to S^3_1$ be a spacelike curve parametrised by arc length such that $k_g(s)\neq 0$ for all $s\in I$.*

(1)If $n(s)$ is a spacelike normal vector field, the hyperbolic dual surface $HD^{\pm}_{\gamma}$ of γ is singular at (s_0,μ_0) if and only if $\mu_0=0$. In other words, the singular points of the hyperbolic dual surface are given by $h^{\pm}_{\mu_0}D_{\gamma}(s)=HD^{\pm}_{\gamma}(s,0)$ with $s\in I$ and $\mu_0=0$.

(2)If $n(s)$ is a timelike normal vector field, the hyperbolic dual surface $HD^{\pm}_{\gamma}$ of γ does not have singular points.

Proof. Consider the hyperbolic dual surface of γ given by

$$HD^{\pm}_{\gamma}(s,\mu)=\mu n(s)\pm\sqrt{\mu^2+\delta(\gamma(s))}e(s).$$

Then, we have

$$\frac{\partial HD_\gamma^\pm}{\partial s}(s,\mu) = -\delta(\gamma(s))\mu k_g(s)t(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s) \quad \text{and}$$
$$\frac{\partial HD_\gamma^\pm}{\partial \mu}(s,\mu) = n(s) \pm \frac{\mu}{\sqrt{\mu^2+\delta(\gamma(s))}}e(s).$$

If $n(s)$ is a spacelike normal vector field, the proof of *(1)* is similar to the proof of Theorem 3.3.4 *(1)*. However, if $n(s)$ is a timelike normal vector field, we have that for $\mu_0 = 0$ the hyperbolic dual surface is not defined and therefore assertion *(2)* holds. ▌

From the above proposition, we have that hyperbolic dual surface of a spacelike curve with $n(s)$ a timelike normal vector field does not have singular points. Thus, in the next section, we use techniques of the singularities theory for study the hyperbolic dual surface of a spacelike curve with a spacelike normal vector field $n(s)$.

6. HYPERBOLIC HEIGHT FUNCTION

In this section we introduce a family of functions on a curve which is useful to study the singularities of the hyperbolic dual surface of a spacelike unit speed curve γ. From Proposition 5.5.1 the surface is $HD_\gamma^\pm(s,\mu) = \mu n(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}e(s)$ with a spacelike normal vector field $n(s)$.

Let $\gamma : I \to S_1^3$ be a spacelike curve. We define a family of functions $H : I \times H^3(-1) \to \mathbb{R}$ by $H(s,v) = \langle\gamma(s), v\rangle$. We call H a family of *hyperbolic height functions on* γ and denote $h_v(s) = H(s,v)$ for any fixed $v \in H^3(-1)$. By Definition 2.2.1, the hyperbolic height function measures the contact of γ with a spacelike hyperplane. Generically, the order of this contact can be k, $1 \leq k \leq 3$.

From the next result we can obtain equivalent conditions for each A_k-type singularity, $1 \leq k \leq 3$.

Proposition 6.6.1. *Let $\gamma : I \to S_1^3$ be a parametrised by the arc length spacelike curve with $n(s)$ spacelike vectors and $k_g(s) \neq 0$ for all $s \in I$. Then we have the following:*

(1)$h_v(s) = 0$ if and only if there exist real numbers μ, λ, η with $\eta^2+\mu^2-\lambda^2 = -1$ such that $v = \eta t(s) + \mu n(s) + \lambda e(s)$.

(2)$h_v(s) = h_v'(s) = 0$ if and only if there exist real numbers μ, λ such that $v = \mu n(s) + \lambda e(s)$ with $\lambda^2 - \mu^2 = 1$.

(3)$h_v(s) = h_v'(s) = h_v''(s) = 0$ if and only if $v = \pm e(s)$.

(4)$h_v(s) = h_v'(s) = h_v''(s) = h_v^{(3)}(s) = 0$ if and only if $v = \pm e(s)$ and $\tau_g(s) = 0$.

(5)$h_v(s) = h_v'(s) = h_v''(s) = h_v^{(3)}(s) = h_v^{(4)}(s) = 0$ if and only if $v = \pm e(s)$ and $\tau_g(s) = \tau_g'(s) = 0$.

Proof. Since $h_v(s) = \langle\gamma(s), v\rangle$, we have

(a) $h'_v(s) = \langle t(s), v\rangle$,
(b) $h''_v(s) = \langle -\gamma(s) + k_g(s)n(s), v\rangle$,
(c) $h^{(3)}_v(s) = \langle(-1 - k^2_g(s))t(s) + k'_g(s)n(s) + k_g(s)\tau_g(s)e(s), v\rangle$,
(d) $h^{(4)}(s) = \langle(1+k^2_g(s))\gamma(s) - 3k'_g(s)k_g(s)t(s) + (-k_g(s) + k''_g(s) + k_g(s)\tau^2_g(s) - k^3_g(s))n(s) + (2k'_g(s)\tau_g(s) + k_g(s)\tau'_g(s))e(s), v\rangle$.

Now, by simple calculations we can show *(1)* to *(5)*. ▮

Corollary 6.6.2. *The hyperbolic dual surface of γ is the discriminant set, $\mathcal{D}_H$, of the family of hyperbolic height functions H.*

Proof. The proof follows from the definition of discriminant set given in the Section 2 and Proposition 6.6.1 *(2)*. ▮

On the next result we show that the hyperbolic height function on a curve is a versal unfolding of an A_k-type singularity ($k = 2, 3$).

Proposition 6.6.3. *Let $\gamma : I \to S^3_1$ be a parametrised by arc length spacelike curve with $n(s)$ spacelike vectors, $k_g \neq 0$ and $H : I \times H^3(-1) \to \mathbb{R}$ the family of the hyperbolic height functions on $\gamma(s)$. If h_v has an A_k-type singularity ($k = 2, 3$) at s_0, then H is a versal unfolding of h_v.*

Proof. The technique of the proof is similar to that of Proposition 3.3.3. Here the case $k = 2$ follows directly without extra conditions. ▮

The above proposition is fundamental for the proof of the next result, that gives the geometric shape of the hyperbolic dual surface with singularities.

Theorem 6.6.4. *Let $\gamma : I \to S^3_1$ be a spacelike curve parametrised by arc length with a spacelike vector field $n(s)$ and $k_g(s) \neq 0$ for all $s \in I$. Consider the hyperbolic dual surface $HD^{\pm}_{\gamma}$ of γ.*

(1)The singular points of $HD^{\pm}_{\gamma}$ are given by $h^{\pm}_{\mu}D_{\gamma}(s) = \pm e(s)$.
(2)$HD^{\pm}_{\gamma}$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, μ_0) if and only if $\mu_0 = 0$ and $\tau_g(s_0) \neq 0$.
(3)$HD^{\pm}_{\gamma}$ is locally diffeomorphic to the swallowtail SW at (s_0, μ_0) if and only if $\mu_0 = 0$, $\tau_g(s_0) = 0$ and $\tau'_g(s_0) \neq 0$.

Proof. By Corollary 6.6.2, the discriminant set $\mathcal{D}_H$ of the family of the hyperbolic height functions H of γ is the hyperbolic dual surface of γ. It Follows from Proposition 6.6.1, *(3)* and *(4)*, that h_v has A_2-type singularity (respectively, A_3-type singularity) at s_0 if and only if $\mu_0 = 0$ and $\tau_g(s_0) \neq 0$ (respectively, $\mu_0 = 0$, $\tau_g(s_0) = 0$ and $\tau'_g(s_0) \neq 0$). The Theorem 2.2.2 and Proposition 6.6.3 complete the proof. ▮

7. INVARIANT AND SPECIAL GEOMETRY OF THE HYPERBOLIC DUAL SURFACE

In this section we investigate the geometric properties of hyperbolic dual surface $HD^{\pm}_{\gamma}$ near its singularities, by using the invariant τ_g of γ. The de Sitter focal surface of hyperbolic space curves is studied in [3].

Remember that $QDE(v,0) = S^3_1 \cap HP(v,0)$ is the elliptic de Sitter quadric, where $HP(v,0)$ is a spacelike hyperplane, that is, v is a timelike vector.

Proposition 7.7.1. *Let $\gamma : I \to S^3_1$ be a parametrised by arc length spacelike curve with $n(s)$ a spacelike vector field and $k_g(s) \neq 0$ for all $s \in I$. Consider the singular points $h^{\pm}_{\mu}D_{\gamma}(s)$ of the hyperbolic dual surface. The following conditions are equivalent:*

(a)$h^{\pm}_{\mu}D_{\gamma}(s)$ is a constant vector $v_0 \in H^3(-1)$,
(b)$\tau_g(s) \equiv 0$,
(c)γ is part of the elliptic de Sitter quadric $QDE(v_0,0)$.

Proof. If the hyperbolic dual surface is singular at (s,μ) then $\mu = 0$. Therefore $h^{\pm}_{\mu}D_{\gamma}(s) = HD^{\pm}_{\gamma}(s,\mu) = \pm e(s)$ and $\dfrac{\partial HD^{\pm}_{\gamma}}{\partial s}(s,\mu) = \pm\tau_g(s)n(s) \equiv 0$ if and only if $\tau_g(s) \equiv 0$. This means that the condition *(a)* is equivalent to the condition *(b)*. Suppose that $\tau_g(s) \equiv 0$ then $h^{\pm}_{\mu}D_{\gamma}(s) = \pm e(s) = \pm v_0$ is constant. Since $\langle \gamma(s), \pm e(s)\rangle = 0$, then $\gamma(s) \in S^3_1 \cap HP(e(s),0)$, where $v_0 = e(s)$ that is a timelike vector. Therefore the condition *(b)* implies the condition *(c)*.

On the other hand, suppose that $Im\gamma \subset QDE(v,0) = S^3_1 \cap HP(v,0)$ where v is a timelike fix vector. Then we have $h_v(s) = \langle \gamma(s), v\rangle = 0$ for all $s \in I$. By the Proposition 6.6.1, *(4)*, $\tau_g(s) \equiv 0$. This complete the proof. ▌

By the above result, we characterize when the curve γ is contained in the elliptic de Sitter quadric. This means that $\tau_g(s) \equiv 0$. Otherwise, if $\tau_g(s) \not\equiv 0$ the theorem below shows that the degeneracy of singularities of $HD^{\pm}_{\gamma}$ characterize the contact of the curve with a elliptic de Sitter quadric.

Theorem 7.7.2. *Let $\gamma : I \to S^3_1$ be a spacelike curve parametrised by arc length with spacelike vectors field $n(s)$, $k_g \neq 0$ and $\tau_g \not\equiv 0$. For $v_0 = HD^{\pm}_{\gamma}(s_0,\mu_0)$, we have the following:*

(1)γ has at least 2-point contact with $QDE(v_0,0)$ at s_0 if and only if $\mu_0 = 0$ if and only if the hyperbolic dual surface of γ is singular at (s_0,μ_0).

(2)γ has 2-point contact with $QDE(v_0,0)$ at s_0 if and only if $\mu_0 = 0$ and $\tau_g(s_0) \neq 0$ if and only if the hyperbolic dual surface of γ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0,μ_0) .

(3)γ has 3-point contact with $QDE(v_0,0)$ at s_0 if and only if $\mu_0 = 0$, $\tau_g(s_0) = 0$ and $\tau'_g(s_0) \neq 0$ if and only if the hyperbolic dual surface of γ is locally diffeomorphic to the swallowtail SW at (s_0,μ_0).

Proof. For $v_0 = HD^{\pm}_{\gamma}(s_0, \mu_0)$, we define the map $\widetilde{h}_{v_0} : S^3_1 \to \mathbb{R}$ by $\widetilde{h}_{v_0}(x) = \langle x, v_0\rangle$. Thus, we have $(\widetilde{h}_{v_0})^{-1}(0) = QDE(v_0, 0)$. In this case $g(s) = \widetilde{h}_{v_0} \circ \gamma(s) = h_{v_0}(s)$ and then the proof of the first part (the first equivalences as *(1)*, *(2)* or *(3)*) of this proposition follows from Definition 2.2.1 and from Proposition 6.6.1. The proof of the second part (the second equivalences as *(1)*, *(2)* or *(3)*) follows from Proposition 6.6.1 and Theorem 6.6.4. ▌

8. DUAL RELATIONS ON HOROSPHERICAL AND HYPERBOLIC DUAL SURFACES

We require some properties of contact manifolds and Legendrian submanifolds for the duality results in this section and we now review these concepts (for more details see for example [1]). Let N be a $(2m+1)$-dimensional smooth manifold and K be a field of tangent hyperplanes on Δ. Locally such a field is defined as the field of zeros of a 1-form α. The tangent hyperplane field K is said to be *non-degenerate* if $\theta \wedge (d\theta)^m \neq 0$ at any point on Δ. The pair (Δ, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case K is called a *contact structure* and θ a *contact form.* A submanifold $i : L \subset \Delta$ of a contact manifold (Δ, K) is said to be *Legendrian* if $dimL = m$ and $di_x(T_xL) \subset K_{i(x)}$ at any $x \in L$, where i is an immersion . A smooth fibre bundle $\pi : E \to M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and the fibers of π are Legendrian submanifolds. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \to M$ is called a *Legendrian map.* The image of the Legendrian map $\pi \circ i$ is called a *wavefront* set of i and is denoted by $W(i)$.

The duality concepts we use here are those introduced in [6] and [5] (the Legendrian dualities between pseudo spheres in Lorentz-Minkowski space), where five Legendrian double fibrations are considered on the subsets Δ_i, $i = 1, \ldots, 5$ of the product of two of the pseudo spheres $H^n(-1)$, S^n_1 and LC^*. The geometric ideas behind the choice of the subsets Δ_i and the Legendrian double fibrations are as follows (see [11]). Here we use only $i = 1, 2, 3$.

Let M be a hypersurface embedded in $H^n(-1)$. We consider an embedding $x : U \to H^n(-1)$ from an open subset $U \subset \mathbb{R}^{n-1}$. We write $M = x(U)$. Since $\langle x, x\rangle \equiv -1$, we have $\langle x_{u_i}, x\rangle \equiv 0$, for $i = 1, \ldots, n-1$, where $u = (u_1, \ldots, u_{n-1}) \in U$. We can construct a spacelike unit normal vector $e(u)$ to M at $x(u)$ by

$$e(u) = \frac{x(u) \wedge x_{u_1}(u) \wedge \ldots \wedge x_{u_{n-1}}(u)}{\| x_{u_1}(u) \wedge \ldots \wedge x_{u_{n-1}}(u) \|},$$

where $\wedge$ denotes the wedge product of n vetors in $\mathbb{R}^{n+1}_1$. Then we have $\langle e, x_{u_i}\rangle = 0$, $\langle e, x\rangle = 0$ and $\langle e, e\rangle = 1$. It follows that the vector $x \pm e$ is a lightlike vector. Let

$$\mathbb{E} : U \to S^n_1 \text{ and } \mathbb{L}^{\pm} : U \to LC^*$$

be the maps defined by $\mathbb{E}(u) = e(u)$ and $\mathbb{L}^{\pm}(u) = x(u) \pm e(u)$. These are called, respectively, the *de Sitter Gauss map* and *lightcone Gauss map* of M.

Consider the pair of embeddings $\mathcal{L}_1 : U \to H^n(-1) \times S^n_1$ by $\mathcal{L}_1(u) = (x(u), E(u))$. We can show that $\mathcal{L}_1$ is a Legendrian embedding into the subset $\Delta_1 = \{(v, w) \in H^n(-1) \times S^n_1 \mid \langle v, w\rangle = 0\}$. (The contact structure on Δ_1 is given bellow). This means that $M = x(U)$ and $M^* = \mathbb{E}(U)$ are dual. We call this duality the Δ_1-duality.

Consider now the lightcone Gauss map $\mathbb{L}^\pm : U \to LC^*$ which satisfies $\langle x(u), \mathbb{L}^\pm(u)\rangle = -1$. The pair $(x, \mathbb{L}^\pm) : U \to H^n(-1) \times LC^*$ determines a Legendrian embedding into the set $\Delta_2 = \{(v, w) \in H^n(-1) \times LC^* \mid \langle v, w\rangle = -1\}$, so $M = x(U)$ and $M^* = \mathbb{L}^\pm(U)$ are dual. We call this duality the Δ_2-duality. Similarly, we have $\langle \mathbb{E}(u) \pm x(u), \mathbb{E}(u)\rangle = 1$ that lead to the concepts of Δ_3-duality.

In this section, we define one-forms $\langle dv, w\rangle = w_0dv_0 + \sum_{i=1}^n w_idv_i$, $\langle v, dw\rangle = v_0dw_0 + \sum_{i=1}^n v_idw_i$ on $\mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1$, and consider the following three Legendrian double fibrations.

$$\begin{aligned}
(1)\ \ &(a)\ H^n(-1) \times S^n_1 \supset \Delta_1 = \{(v, w) \mid \langle v, w\rangle = 0\},\\
&(b)\ \pi_{11} : \Delta_1 \to H^n(-1),\ \pi_{12} : \Delta_1 \to S^n_1,\\
&(c)\ \theta_{11} = \langle dv, w\rangle\,|_{\Delta_1},\ \theta_{12} = \langle v, dw\rangle\,|_{\Delta_1}.
\end{aligned}$$

$$\begin{aligned}
(2)\ \ &(a)\ H^n(-1) \times LC^* \supset \Delta_2 = \{(v, w) \mid \langle v, w\rangle = -1\},\\
&(b)\ \pi_{21} : \Delta_2 \to H^n(-1),\ \pi_{22} : \Delta_2 \to LC^*,\\
&(c)\ \theta_{21} = \langle dv, w\rangle\,|_{\Delta_2},\ \theta_{22} = \langle v, dw\rangle\,|_{\Delta_2}.
\end{aligned}$$

$$\begin{aligned}
(3)\ \ &(a)\ LC^* \times S^n_1 \supset \Delta_3 = \{(v, w) \mid \langle v, w\rangle = 1\},\\
&(b)\ \pi_{31} : \Delta_3 \to LC^*,\ \pi_{32} : \Delta_3 \to S^n_1,\\
&(c)\ \theta_{31} = \langle dv, w\rangle\,|_{\Delta_3},\ \theta_{32} = \langle v, dw\rangle\,|_{\Delta_3}.
\end{aligned}$$

Here, $\pi_{i1}(v, w) = v$, $\pi_{i2}(v, w) = w$ are the canonical projections. We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i, $(i = 1, 2, 3)$. It has been shown in [6] that each (Δ_i, K_i) $(i = 1, 2, 3)$ is a contact manifold and π_{i1} and π_{i2} $(i = 1, 2, 3)$ are Legendrian fibrations. Moreover the contact manifolds (Δ_1, K_1), (Δ_2, K_2) and (Δ_3, K_3) are contact diffeomorphic to each other.

For a given Legendrian embedding $\mathcal{L}_i : U \to \Delta_i$, $i = 1, 2, 3$, we say that $\pi_{i1}(\mathcal{L}_i(U))$ is the Δ_i-dual of $\pi_{i2}(\mathcal{L}_i(U))$ and vice-versa (see [4]). In the next result, for showing the duality we have to show that the immersion $\mathcal{L}_i : U \to \Delta_i$, $i = 1, 2, 3$ is a Legendrian immersion , i.e., $dim\, U = m$ and $(d\mathcal{L}_i)_x(T_x(U)) \subset K_{\mathcal{L}_i(x)}$ for all $x \in L$ (see also [6]). Equivalently, $\mathcal{L}_i$ is a Legendrian immersion if $dim\, U = m$ and ${\mathcal{L}_i}^*\theta_{i1} = 0$ (see for example [9]). Therefore, we can show that a submanifold is Legendrian using the second definition.

Since $\mathcal{L}_1$ is a Legendrian embedding, we have $\langle dx(u), \mathbb{E}(u)\rangle = 0$, so that $\mathbb{E}(u)$ belongs to the normal plane in $\mathbb{R}^{n+1}$. Thus, we have the following geometric properties for a Legendrian submanifold $\mathcal{L}_1(U) \subset \Delta_1$. If $\pi_{11}(\mathcal{L}_1(U))$ is smooth at a point $\pi_{11}(\mathcal{L}_1(u))$, then $\pi_{12}(\mathcal{L}_1(u))$ is the normal vector to the hypersurface $\pi_{11}(\mathcal{L}_1(U)) \subset H^n(-1)$ at $\pi_{11}(\mathcal{L}_1(u))$. Conversely, if $\pi_{12}(\mathcal{L}_1(U))$ is smooth at a point $\pi_{12}(\mathcal{L}_1(u))$, then $\pi_{11}(\mathcal{L}_1(u))$ is the normal vector to the hypersurface $\pi_{12}(\mathcal{L}_1(U)) \subset S^n_1$. For the Δ_i-duality, $i = 2, \ldots, 5$, we can think of the same way.

Then we have the following relations on horospherical and hyperbolic dual surfaces. We observe that here $n = 3$, $m = 2$ and $dim\,\Delta_i = 5$, $i = 1, 2, 3$. We observe also that for hyperbolic curves γ in [4], the authors prove duality results for hyperbolic focal surface and de Sitter focal surface of γ.

THEOREM 8.8.1. *Let $\gamma : I \to S^3_1$ be a spacelike curve parametrised by arc length with $k_g(s) \neq 0$ for all $s \in I$. Then*

(1)γ is Δ_1-dual of $HD^{\pm}_{\gamma}$.
(2)γ is Δ_3-dual of $HS^{\pm}_{\gamma}$.
(3)$HD^{\pm}_{\gamma}$ is Δ_2-dual of $HS^{\pm}_{\gamma}$.

Proof.
(1) Consider the mapping $\mathcal{L}_1 : I \times J \to \Delta_1$, $U = I \times J$ defined by $\mathcal{L}_1(s, \mu) = (HD^{\pm}_{\gamma}(s, \mu), \gamma(s))$, where $M = \pi_{11}(\mathcal{L}_1(I \times J)) = HD^{\pm}_{\gamma}(s, \mu) = \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$ and $M^* = \pi_{12}(\mathcal{L}_1(I \times J)) = \gamma(s)$. Then $\langle HD^{\pm}_{\gamma}(s, \mu), \gamma(s)\rangle = 0$ and the mapping is well-defined, i.e., $\mathcal{L}_1(s, \mu) \in \Delta_1$. We have

$$\frac{\partial \mathcal{L}_1}{\partial s}(s, \mu) = (-\delta(\gamma(s))\mu k_g(s)t(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s), t(s))$$
$$\frac{\partial \mathcal{L}_1}{\partial \mu}(s, \mu) = (n(s) \pm \frac{\mu}{\sqrt{\mu^2 + \delta(\gamma(s))}}e(s), 0),$$

then $\mathcal{L}_1$ is an immersion. Since $\mathcal{L}_1^*\theta_{12} = \langle HD^{\pm}_{\gamma}(s, \mu), t(s)\rangle ds = 0$, then by definition $\mathcal{L}_1(I \times J)$ is a Legendrian summanifold in Δ_1.
(2) We also consider the mapping $\mathcal{L}_3 : I \times J \to \Delta_3$ defined by $\mathcal{L}_3(s, \mu) = (HS^{\pm}_{\gamma}(s, \mu), \gamma(s))$, where $HS^{\pm}_{\gamma}(s, \mu) = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$.
Thus, $\langle HS^{\pm}_{\gamma}(s, \mu), \gamma(s)\rangle = 1$, i.e., $\mathcal{L}_3(s, \mu) \in \Delta_3$ and the proof follows analogous to *(1)*.
(3) Now consider the mapping $\mathcal{L}_2 : I \times J \to \Delta_2$ defined by $\mathcal{L}_2(s, \mu) = (HD^{\pm}_{\gamma}(s, \mu), HS^{\pm}_{\gamma}(s, \mu))$. Then we have

$$\langle HD^{\pm}_{\gamma}(s, \mu), HS^{\pm}_{\gamma}(s, \mu))\rangle = \mu^2\delta(\gamma(s)) + (\mu^2 + \delta(\gamma(s)))(-\delta(\gamma(s))) = -1.$$

Thus, $\mathcal{L}_2(s, \mu) \in \Delta_2$ and the mapping is well-defined. Since

$$\frac{\partial \mathcal{L}_2}{\partial s}(s, \mu) = (-\delta(\gamma(s))\mu k_g(s)t(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s), (1 - \delta(\gamma(s))\mu k_g(s))t(s)$$
$$\pm \sqrt{\mu^2 + \delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s))$$
$$\frac{\partial \mathcal{L}_2}{\partial \mu}(s, \mu) = (n(s) \pm \frac{\mu}{\sqrt{\mu^2 + \delta(\gamma(s))}}e(s), n(s) \pm \frac{\mu}{\sqrt{\mu^2 + \delta(\gamma(s))}}e(s)),$$

$\mathcal{L}_2$ is an immersion, because $-\delta(\gamma(s))\mu k_g(s) \neq 0$ or $1 - \delta(\gamma(s))\mu k_g(s) \neq 0$. Moreover

$$
\begin{aligned}
&\mathcal{L}_2^*\theta_{21} = \left\langle d(HD^{\pm}_{\gamma}(s,\mu)), HS^{\pm}_{\gamma}(s,\mu)\right\rangle \\
&= \left\langle \frac{\partial HD^{\pm}_{\gamma}}{\partial s}(s,\mu)ds + \frac{\partial HD^{\pm}_{\gamma}}{\mu}(s,\mu)d\mu, HS^{\pm}_{\gamma}(s,\mu)\right\rangle \\
&= \left\langle -\mu\delta(\gamma(s))k_g(s)t(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s), \gamma(s)\right\rangle ds \\
&+ \left\langle -\mu\delta(\gamma(s))k_g(s)t(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s), \mu n(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}e(s)\right\rangle ds \\
&+ \left\langle n(s) \pm \frac{\mu}{\sqrt{\mu^2+\delta(\gamma(s))}}e(s), \gamma(s) + \mu n(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}e(s)\right\rangle d\mu = 0.
\end{aligned}
$$

Therefore, $\mathcal{L}_2(I \times J)$ is a Legendrian summanifold in Δ_2. ▮

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