CADERNOS DE MATEMÁTICA **16**, 81–92 May (2015)
ARTIGO NÚMERO SMA#5

Trudinger-Moser type inequality and applications to a class of quasilinear elliptic equations with exponential critical growth

José Francisco de Oliveira*

Department of Mathematics, Federal University of Piauí, 64049-550 Teresina, PI, Brazil
E-mail: jfoliveira@ufpi.edu.br

João Marcos do Ó

Department of Mathematics, Federal University of Paraíba, 58051-900 João Pessoa, PB, Brazil
E-mail: jmbo@pq.cnpq.br

We discuss extensions of the classic Trudinger-Moser inequality for a class of weighted Sobolev spaces. Moreover, we prove existence results for elliptic problems involving radial operators associated with this new Trudinger-Moser inequality. May, 2015 ICMC-USP

2000 Mathematics Subject Classification: 35J20, 46E35, 35B33.

Key Words: Trudinger-Moser inequality, critical growth, p-Laplacian equations, Hessian equations, variational methods

1. INTRODUCTION

Let $q \geq 1$, $\theta \geq 0$ and $R > 0$ be real numbers. Denote $L^q_\theta = L^q_\theta(0,R)$ the Banach space of Lebesgue measurable functions $u : (0,R) \to \mathbb{R}$ satisfying

$$\|u\|_{L^q_\theta} = \begin{cases} \left(\int_0^R |u(r)|^q \, \mathrm{d}\lambda_\theta\right)^{1/q} < \infty & \text{if } 1 \leq q < \infty, \\ \operatorname{ess} \sup\limits_{0<r<R} |u(r)| < \infty & \text{if } q = \infty \end{cases}$$

where λ_θ is the weighted Lebesgue measure defined by $\int_0^R f \, \mathrm{d}\lambda_\theta = \omega_\theta \int_A r^\theta f(r) \, \mathrm{d}r$ with

$$\omega_\theta = 2\frac{\pi^{\frac{\theta+1}{2}}}{\Gamma(\frac{\theta+1}{2})} \quad \text{and} \quad \Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \, \mathrm{d}t.$$

* Partially supported by CAPES and CNPq, Brazil.

Let us consider the weighted Sobolev space $X^{1,p}_R(\alpha,\theta)$ or simply X_R, the collection of all locally absolutely continuous functions $u:(0,R)\to\mathbb{R}$ for which $\lim_{r\to R^-}u(r)=0,\ u\in L^p_\theta$ and $u'\in L^p_\alpha$. In according to results in [5] and [15, pg.69] the X_R is a Banach space endowed with the norm $\|u\|_{X^{1,p}_R}=(\|u\|^p_{L^p_\theta}+\|u'\|^p_{L^p_\alpha})^{1/p}$. Moreover, we can distinguish two cases for X_R spaces:

(i) **Sobolev case** $\alpha-p+1>0$

(ii) **Trudinger-Moser case** $\alpha-p+1=0$.

For the *Sobolev case*, we have the following continuous embedding

$$X^{1,p}_R(\alpha,\theta)\hookrightarrow L^q_\nu \quad \text{if} \quad q\in(1,p^*] \quad \text{and} \quad \min\{\theta,\nu\}\geq\alpha-p, \tag{1.1}$$

where

$$p^*:=p^*(\alpha,p,\nu)=\frac{(\nu+1)p}{\alpha-p+1} \quad \text{with} \quad \alpha-p+1>0$$

is the Sobolev critical exponent for this class of spaces. Also, the embedding (1.1) are compact if $q<p^*$ (cf. [15, pg. 69 and 91]). On the other hand, in the *Trudinger-Moser case* we have $p^*\to\infty$ (formally) and it holds the compact embedding

$$X^{1,p}_R(\alpha,\theta)\hookrightarrow L^q_\nu \quad \text{for all} \quad q\in(1,\infty) \quad \text{and} \quad \nu\geq 0. \tag{1.2}$$

This suggests $X_R\subset L^\infty_\nu$, but it is not true as one can see taking $u(r)=\ln(\ln(\mathrm{e}R/r))$.

Question. What is the maximal growth available for a function g for which $g(u)\in L^1_\kappa$ whenever $u\in X_R$?

In order to answer this question the authors obtained a Trudinger-Moser type inequality for X_R spaces which ensures that exponential growth is available. More precisely,

Theorem 1.1 ([8, 9]). *Let $\alpha\geq 1,\ \theta,\kappa\geq 0$ and $p=\alpha+1$ be real numbers. Then, for any $\mu>0$ and $u\in X^{1,p}_R(\alpha,\theta),\ 0<R<\infty$, we have $\exp(\mu|u|^{p'})\in L^1_\kappa$. Moreover, there exists $c>0$ depending only on α, p and κ such that*

$$\sup_{\|u'\|_{L^p_\alpha}\leq 1}\int_0^R \mathrm{e}^{\mu|u|^{p'}}\,\mathrm{d}\lambda_\kappa \quad \begin{cases}\leq c & \text{if } \mu\leq\mu_{\alpha,\kappa}\\ =\infty & \text{if } \mu>\mu_{\alpha,\kappa},\end{cases} \tag{1.3}$$

where $\mu_{\alpha,\kappa}=(\kappa+1)\omega_\alpha^{1/\alpha}$ and $p'=p/(p-1)$.

Remark 1. 1. The inequality (1.3) improves and complements the classical J. Moser inequality in [17] and the previous results in [19, 20, 21].

Remark 1. 2. In view of (1.3), we have the continuous embedding $X_R\hookrightarrow L_A(\kappa)$, where $L_A(\kappa)$ is the weighted Orlicz space defined by the N-function $A(t)=\mathrm{e}^{|t|^{p/(p-1)}}-1$.

On the spirit of the concentration-compacness principle due to P.-L. Lions [16] (see also [4]), the authors has studied the lack of compactness of the embedding $X_R\hookrightarrow L_A(\kappa)$. In [10], we have obtained recently an improvement of the Theorem 1.1, which yields a sharp upper bound for the value of μ in (1.3) along of some special sequences in X_R. Namely, THEOREM 1.2 ([8, 10]). *Let α, θ, κ and p under the hypotheses of Theorem 1.1. For any $(u_n) \subset X_R$ such that*

$$\|u_n'\|_{L^p_\alpha} \leq 1 \quad \text{and} \quad u_n \rightharpoonup u \quad \text{in} \quad X_R \quad \text{with} \quad u \not\equiv 0 \tag{1.4}$$

we have

$$\limsup_{n\to\infty} \int_0^R \mathrm{e}^{q\mu_{\alpha,\kappa}|u_n|^{p'}} \mathrm{d}\lambda_\kappa \quad \begin{cases} < \infty & \text{if } q < P \\ = \infty & \text{if } q \geq P, \end{cases} \tag{1.5}$$

where

$$P = P(\alpha, p, u) = \begin{cases} (1 - \|u'\|^p_{L^p_\alpha})^{-1/(p-1)} & \text{if } \|u'\|_{L^p_\alpha} < 1, \\ \infty & \text{if } \|u'\|_{L^p_\alpha} = 1. \end{cases} \tag{1.6}$$

In particular, $(\mathrm{e}^{\mu_{\alpha,\kappa}|u_n|^{p'}})$ converges to $\mathrm{e}^{\mu_{\alpha,\kappa}|u|^{p'}}$ in L^1_κ.

The above results are powerful tools to study the existence of extremals to functional inequalities and the existence of solutions to associated nonlinear elliptic equations. The main purpose of this paper is discuss how to apply those results to study a wide class of quasilinear elliptic equations.

For $\alpha, \kappa > -1$ and $p > 1$ real numbers we consider the class of operators $L = L(\alpha, p, \kappa)$ acting on ordinary functions as follows

$$Lu \stackrel{\text{def}}{=} -r^{-\kappa}\left(r^\alpha |u'|^{p-2}u'\right)'. \tag{1.7}$$

We point out that the class in (1.7) includes, when acting in symmetric function defined in ball $B_R \subset \mathbb{R}^N$, the following operators:

Operator	α	p	κ
Laplacian	$N-1$	2	$N-1$
q-Laplacian ($q > 1$)	$N-1$	q	$N-1$
k-Hessian	$N-k$	$k+1$	$N-1$

Remark 1. 3. This class of quasilinear operators has received considerable attention in recent years, the reader is referred to the works of de Figueiredo *et al.* [5, 6] where Brezis-Nirenberg type problems were studied. See also the papers of J. Jacobsen, K. Schmitt [13, 14].

We are interested in the following problem concerning the class of operators $L = L(\alpha, p, \kappa)$ given by(1.7):

$$\begin{cases} \left.\begin{aligned} &Lu = f(u)u^{p-2} \\ &u \geq 0 \end{aligned}\right\} \quad \text{in} \quad (0, R) \\ u(R) = u'(0) = 0, \end{cases} \tag{1.8}$$

where $f(t)$ has exponential critical growth and the Trudinger-Moser condition $\alpha - p + 1 = 0$ holds.

For $\alpha = \kappa = N - 1$, the problem (1.8) corresponds the N-Laplacian case on the ball $B_R \subset \mathbb{R}^N$. In this case, Adimurthi [1] was able to prove existence of solution for (1.8) even for smooth general bounded domain $\Omega \subset \mathbb{R}^N$. More recently, in the presence of the singular term, that is, $\kappa = N-1-a$ with $a \in [0, N)$ and $\alpha = N-1$, this problem was investigated in [2]. In the works [1, 2] was used the constraint method based in the Nehari works [18]. Due to radial nature of problem (1.8), for the special case of N-Laplacian, ODEs methods like as *shooting method* has been frequently the used for several authors, we distinguish [3, 7] for $N = 2$ and [11, 12] if $N \geq 2$. Here, motivated by initial works due to De Figueiredo *et al.* [5] and the Theorem 1.2, we are able to apply the variational approach in the same line of [1]. In order to state our result we make the following definition:

Definition 1.1. We say that $f : \mathbb{R} \to \mathbb{R}$ has critical growth if its can be written as $f(u) = h(u)\mathrm{e}^{b|u|^{p'}}$, $b > 0$ with h satisfying:

(H_1) $h \in C^1(\mathbb{R})$, $h(0) = 0$, $h > 0$ on $(0, \infty)$ and $h(-t) = (-1)^{p-1}h(t)$
(H_2) $f(t) < tf'(t)$, for all $t > 0$.
(H_3) $F(t) \leq M(1 + f(t)t^{p-2+\tau_0})$ for some $\tau_0 \in [0, 1)$ and $M > 0$, where $F(t) = \int_0^t f(s)s^{p-2}\mathrm{d}s$
(H_4) $\lim_{t\to\infty} h(t)\mathrm{e}^{-\epsilon t^{p'}} = 0$ and $\lim_{t\to\infty} h(t)\mathrm{e}^{\epsilon t^{p'}} = +\infty$.

Now, let $\Lambda_1 > 0$ be the weighted first eigenvalue for the class of operators (1.7), defined by

$$\Lambda_1 := \Lambda_1(\alpha, \kappa, p, R) = \inf_{u \in X_R \setminus \{0\}} \frac{\int_0^R r^\alpha |u'|^p \mathrm{d}r}{\int_0^R r^\kappa |u|^p \mathrm{d}r}.$$

The main result of this paper reads below

Theorem 1.3. *Suppose that f has critical growth and satisfies*

$$f'(0) < \Lambda_1 \quad \textit{and} \quad \limsup_{t\to\infty} h(t)\, t^{p-1} = +\infty. \tag{1.9}$$

Then, the class of problem (1.8) *has a nontrivial solution.*

Remark 1. 4. We would like to emphasize that our result improves and complements [1, 2, 3, 7, 11, 12] by consider non-integer parameters α, κ and includes the k-Hessian operator.

2. COMPACTNESS ANALYSIS

Let us consider $J : X_R \to \mathbb{R}$ the functional associated with the problem (1.8) defined by

$$J(u) = \frac{1}{p}\int_0^R |u'|^p r^\alpha \mathrm{d}r - \int_0^R F(u) r^\kappa \,\mathrm{d}r \tag{2.1}$$

where, F is the primitive of $f(s)s^{p-1}$ as in (H_3). We observe that a critical point of J is a solution to (1.8). Next, we give a useful characterization of the critical points of J. More precisely, set $\mathcal{N}$ the Nehari manifold given by

$$\mathcal{N} = \left\{ u \in X_R \setminus \{0\} \ ; \ \int_0^R |u'|^p r^\alpha \, \mathrm{d}r = \int_0^R f(u) u^{p-1} r^\kappa \, \mathrm{d}r \right\}. \tag{2.2}$$

Let us take S such that

$$S^p = p \inf_{u \in \mathcal{N}} J(u). \tag{2.3}$$

We note that the solutions of (1.8) is in $\mathcal{N}$. In the reverse direction we have the following result which the proof is analogous to [1, Lemma 3.5] and we omit here.

LEMMA 2.1. *Let $v \in \mathcal{N}$ be a extremals of* (2.3)*, then v is a critical point of J.*

The next result refers to the shape of $\mathcal{N}$.

LEMMA 2.2. *Assume f satisfying (H_1)-(H_4) and $f'(0) < \Lambda_1$. Then, for any $u \in X_R \setminus \{0\}$, there exists unique $\tau > 0$ such that $\tau u \in \mathcal{N}$.*

Proof. Since $f'(0) < \Lambda_1$, we have

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^R f(\tau u) u^{p-1} r^\kappa \, \mathrm{d}r = \int_0^R f'(0) u^p r^\kappa \, \mathrm{d}r < \int_0^R |u'|^p r^\alpha \mathrm{d}r.$$

Also, from (H_1) and (H_2), $\tau \mapsto \frac{f(\tau u) u^{p-1}}{\tau}$ is an increasing function for $\tau > 0$ and

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^R f(\tau u) u^{p-1} r^\kappa \, \mathrm{d}r = +\infty.$$

This proves the lemma. ∎

Now, we consider the functional $I : X_R \to \mathbb{R}$ given by

$$I(u) = \frac{1}{p} \int_0^R f(u) u^{p-1} r^\kappa \, \mathrm{d}r - \int_0^R F(u) r^\kappa \, \mathrm{d}r. \tag{2.4}$$

We note that $J(u) = I(u)$ for all $u \in \mathcal{N}$. Moreover, as we shall see later, we can reduce the compactness-analysis of J to the analysis of I. In the next Lemmas we will assume that $f(t) = h(t)\mathrm{e}^{b|t|^{p'}}$ has critical growth, see Definition 1.1.

LEMMA 2.3. *Let (u_n) and (v_n) two sequence in X_R converging weakly and for almost every $r \in (0, R)$ to u and v respectively.*

(1)*If* $\sup_n \int_0^R f(u_n) u_n^{p-1} r^\kappa \, \mathrm{d}r < \infty$ *then, for any $\tau \in [0, 1)$*

$$\lim_{n \to \infty} \int_0^R f(|u_n|) |u_n|^{p-2+\tau} r^\kappa \, \mathrm{d}r = \int_0^R f(|u|) |u|^{p-2+\tau} r^\kappa \, \mathrm{d}r \tag{2.5}$$

and

$$\lim_{n \to \infty} \int_0^R F(u_n) r^\kappa \, \mathrm{d}r = \int_0^R F(u) r^\kappa \, \mathrm{d}r. \tag{2.6}$$

(2)*If* $\limsup_{n\to\infty}\|u_n'\|^p_{L^p_\alpha} < \left(\frac{\mu_{\alpha,\kappa}}{b}\right)^{p-1}$ *and* $\ell \geq 0$ *is an integer number, then*

$$\lim_{n\to\infty}\int_0^R \frac{f(u_n)v_n^\ell\, r^\kappa}{u_n}\,\mathrm{d}r = \int_0^R \frac{f(u)v^\ell r^\kappa}{u}\,\mathrm{d}r. \tag{2.7}$$

(3)$I(u) \geq 0$ *for all* $u \in X_R$ *and* $I(u) > 0$ *if* $u \not\equiv 0$*. Further, there exists a constant* $M > 0$ *such that, for all* $u \in X_R$

$$\int_0^R f(u)u^{p-1}r^\kappa \mathrm{d}r \leq M(1 + I(u)). \tag{2.8}$$

Proof.

(1). For $T > 0$ we can write

$$\int_0^R f(|u_n|)|u_n|^{p-2+\tau}r^\kappa\,\mathrm{d}r = \int_{|u_n|\leq T} f(|u_n|)|u_n|^{p-2+\tau}r^\kappa\,\mathrm{d}r + \int_{|u_n|>T} f(|u_n|)|u_n|^{p-2+\tau}r^\kappa\,\mathrm{d}r. \tag{2.9}$$

Also,

$$\sup_n \int_{|u_n|>T} f(|u_n|)|u_n|^{p-2+\tau}r^\kappa\,\mathrm{d}r \leq T^{\tau-1}\sup_n \int_{|u_n|>T} f(|u_n|)|u_n|^{p-1}r^\kappa\,\mathrm{d}r = O\left(T^{\tau-1}\right).$$

Thus, with the help of dominated convergence theorem, (2.5) it follows choosing T large enough and passing to the limit in (2.9). Now, in view of the (H_3), (2.6) it follows by (2.5) and the dominated convergence theorem again.

(2). From (H_4), for $\epsilon > 0$

$$|f(t)| \leq C(\epsilon)\mathrm{e}^{(b+\epsilon)|t|^{p'}}, \tag{2.10}$$

for some $C(\epsilon) > 0$. By hypothesis, we can take $C > 0$ with

$$\|u_n'\|_{L^p_\alpha} \leq C^{p'} < \frac{\mu_{\alpha,\kappa}}{b}.$$

Thus, taking $q > 1$ and $\epsilon > 0$ such that $q(1 + \frac{\epsilon}{b})C^{p'} < \frac{\mu_{\alpha,\kappa}}{b}$, we obtain

$$|f(u_n)|^q \leq C(\epsilon)\mathrm{e}^{q(b+\epsilon)C^{p'}|\frac{u_n}{C}|^{p'}}, \quad \text{with} \quad q(b+\epsilon)C^{p'} < \mu_{\alpha,\kappa}.$$

Hence, using the Trudinger-Moser type inequality Theorem 1.1, we get

$$\sup_n \int_0^R |f(u_n)|^q r^\kappa \mathrm{d}r < \infty.$$

In view of (1.2) we have v_n^ℓ is bounded in $L^{q_1}_\nu$ for all $q_1 \in [1,\infty)$. Then, for any $T > 0$ the Hölder inequality implies

$$\left|\int_{|u_n|>T} \frac{f(u_n)v_n^\ell\, r^\kappa}{u_n}\,\mathrm{d}r\right| \leq \frac{1}{T}\int_0^R |f(u_n)||v_n^\ell|\, r^\kappa\,\mathrm{d}r \leq \frac{1}{T}\|f(u_n)\|_{L^q_\kappa}\|v_n^\ell\|_{L^{q'}_\kappa}.$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Thus, we can conclude that

$$\int_0^R \frac{f(u_n)v_n^\ell\, r^\kappa}{u_n}\,\mathrm{d}r = \int_{|u_n|\le T} \frac{f(u_n)v_n^\ell\, r^\kappa}{u_n}\,\mathrm{d}r + O\left(\frac{1}{T}\right).$$

We recall that $\frac{f(t)}{t}$ is bounded on $[-T,T]$, $v_n^\ell \to v^\ell$ in L^q_ν for $q\in[1,\infty)$. Hence, by the dominated convergence theorem, letting $n\to\infty$ and $T\to\infty$ the above equation provides (2.7).

(3). From (H_2) we have, for all $t>0$

$$\frac{d}{dt}\left[f(t)t^{p-1} - pF(t)\right] = t^{p-1}[f'(t) - f(t)t] > 0.$$

Therefore, from (H_1) and (H_2), $f(t)t^{p-1} - pF(t)$ is an even positive function and increasing for $t>0$. This implies that $I(u)\ge 0$ and $I(u)=0$ if and only if $u=0$. Moreover, from (H_3) we have

$$pI(u) = \int_0^R \left[f(u)u^{p-1} - pF(u)\right] r^\kappa\,\mathrm{d}r \ge \int_0^R \left[f(u)u^{p-1} - pM(1+|f(u)|u^{p-2+\tau})\right] r^\kappa\,\mathrm{d}r.$$

Hence, for some constants C_1 and $C_2>0$

$$pI(u) \ge C_1 + \frac{1}{2}\int_{|u|\ge C_2} f(u)u^{p-1}r^\kappa\,\mathrm{d}r$$

which ensures

$$\int_0^R f(u)u^{p-1}r^\kappa \mathrm{d}r \le M(1+I(u)).$$

for some constant $M>0$. ▮

Next, we analyze the non-compact case. We would like to emphasize that the Theorem 1.2 plays a crucial role in this discussion.

Lemma 2.4 (Compactness Lemma). *Let $u_n \rightharpoonup u \not\equiv 0$ in X_R and $u_n(r)\to u(r)$ a.e in $(0,R)$. Assume that*

(i) There exists $c\in(0,\frac{1}{p\omega_\alpha}(\frac{\mu_{\alpha,\kappa}}{b})^{p-1}]$ such that $J(u_n)\to c$ as $n\to\infty$

(ii) $\int_0^R |u'|^p r^\alpha\,\mathrm{d}r \ge \int_0^R f(u)u^{p-1}r^\kappa\,\mathrm{d}r$

(iii) $\sup_n \int_0^R f(u_n)u_n^{p-1}r^\kappa\,\mathrm{d}r < \infty$.

Then

$$\lim_{n\to\infty}\int_0^R f(u_n)u_n^{p-1}r^\kappa\,\mathrm{d}r = \int_0^R f(u)u^{p-1}r^\kappa\,\mathrm{d}r.$$

Proof. As in the previous Lemma is sufficient to show that $\int_{|u_n|\ge T}^R f(u_n)u_n^{p-1}r^\kappa\,\mathrm{d}r = o(1)$ as $T\to\infty$ uniformly in n. By (3) in Lemma 2.3 and assumption (ii) we have $J(u)\ge I(u)>0$. Also from (2.6), using the weak lower semi-continuity of the norm $0<J(u)\le \liminf_{n\to\infty} J(u_n) = c$. Hence we can choose $\epsilon>0$ satisfying

$$(c - J(u))(1+\epsilon)^{p-1} < \frac{1}{p\omega_\alpha}\left(\frac{\mu_{\alpha,\kappa}}{b}\right)^{p-1}. \tag{2.11}$$

Set $\eta = \int_0^R F(u) r^\kappa \,\mathrm{d}r$. Combining our assumption (iii) and (2.6), we obtain

$$\lim_{n\to\infty} \|u_n'\|^p_{L^p_\alpha} = p\omega_\alpha \lim_{n\to\infty} \left\{ J(u_n) + \int_0^R F(u_n) r^\kappa \,\mathrm{d}r \right\} = p\omega_\alpha (c+\eta). \tag{2.12}$$

From (2.11) and (2.12), for n sufficiently large,

$$(1+\epsilon)^{p-1} \left(\frac{b}{\mu_{\alpha,\kappa}}\right)^{p-1} \|u_n'\|^p_{L^p_\alpha} < \frac{(c+\eta)}{c - J(u)} = \left(1 - \frac{\|u'\|^p_{L^p_\alpha}}{p\omega_\alpha (c+\eta)}\right)^{-1}.$$

We note that $\frac{u_n}{\|u_n'\|_{L^p_\alpha}}$ converges weakly to $\frac{u}{[p\omega_\alpha(c+\eta)]^{1/p}}$ in X_R. Thus, we can apply Theorem 1.2 and conclude

$$\sup_n \int_0^R \mathrm{e}^{(1+\epsilon)b|u_n|^{p'}} r^\kappa \,\mathrm{d}r < \infty. \tag{2.13}$$

Let $M_1 = \sup_{t\in\mathbb{R}} |h(t) t^{p-1}| \, \mathrm{e}^{-\epsilon \frac{b}{2}|t|^{p'}}$. From (2.13) we have

$$\int_{|u_n|\geq T}^R f(u_n) u_n^{p-1} r^\kappa \,\mathrm{d}r \leq M_1 \int_{|u_n|\geq T}^R \mathrm{e}^{-\epsilon \frac{b}{2}|u_n|^{p'}} \mathrm{e}^{(1+\epsilon)b|u_n|^{p'}} r^\kappa \,\mathrm{d}r = O\left(\mathrm{e}^{-\epsilon \frac{b}{2} T^{p'}}\right).$$

which completes the lemma. ∎

LEMMA 2.5. *Suppose* $\limsup_{t\to\infty} h(t) t^{p-1} = \infty$. *Then, for any* $\Lambda \geq 0$,

$$\sup_{\|u'\|_{L^p_\alpha}\leq 1} \int_0^R f(\Lambda u) u^{p-1} r^\kappa \,\mathrm{d}r < \infty \tag{2.14}$$

if and only if $\Lambda < \left(\frac{\mu_{\alpha,\kappa}}{b}\right)^{1/p'}$.

Proof. Arguing as in (2.10), we have

$$|f(t)| \leq C(\epsilon) \mathrm{e}^{(b+\epsilon)|t|^{p'}},$$

for some $C(\epsilon) > 0$. Hence, if $\Lambda^{p'} < \frac{\mu_{\alpha,\kappa}}{b}$ we can choose $\epsilon > 0$ and $q > 1$ such that

$$q(b+\epsilon)\Lambda^{p'} < \mu_{\alpha,\kappa}.$$

Thus, (2.14) it follows from the Theorem 1.1 and Hölder inequality. Now, assume $\Lambda^{p'} \geq \frac{\mu_{\alpha,\kappa}}{b}$. For $0 < l < R$, we set

$$u_l(r) = \frac{1}{\omega_\alpha^{\frac{1}{p}}} \begin{cases} \left(\ln \dfrac{R}{l}\right)^{\frac{1}{p'}} & \text{if} \quad 0 \leq r \leq l, \\ \dfrac{\ln \frac{R}{r}}{\left(\ln \frac{R}{l}\right)^{\frac{1}{p}}} & \text{if} \quad l \leq r \leq R. \end{cases}$$

Using $\alpha - p + 1 = 0$, it is easy to verify that $\|u_l'\|_{L^p_\alpha} = 1$. Moreover, since u_l is constant on $(0,l)$ we obtain

$$\int_0^R f(\Lambda u_l) u_l^{p-1} r^\kappa \,\mathrm{d}r \geq \int_0^l h(\Lambda u_l) \mathrm{e}^{b|\Lambda u_l|^{p'}} u_l^{p-1} r^\kappa \,\mathrm{d}r = h(t) t^{p-1} \frac{R^{\frac{b\Lambda^{p'}}{\mu_{\alpha,\kappa}}(\kappa+1)}}{\Lambda^{p-1}(\kappa+1)} l^{(\kappa+1)\left[1 - \frac{b\Lambda^{p'}}{\mu_{\alpha,\kappa}}\right]},$$

where $t = \frac{\Lambda}{\omega_\alpha^{1/p}}(\ln \frac{R}{l})^{1/p'}$. Since $t \to \infty$ if $l \to 0$ and $\limsup_{t\to\infty} h(t)t^{p-1} = \infty$, we get

$$\limsup_{l\to 0} \int_0^R f(\Lambda u_l) u_l^{p-1} r^\kappa \, \mathrm{d}r = +\infty.$$

∎

3. EXISTENCE OF SOLUTIONS

This section is devoted to prove the existence result Theorem 1.3.

Lemma 3.1. *Suppose* $f(u) = h(u)\mathrm{e}^{b|u|^{\frac{p}{p-1}}}$ *with critical growth and satisfying* (1.9). *Then,* $0 < S < \frac{1}{\omega_\alpha^{1/p}} \left(\frac{\mu_{\alpha,\kappa}}{b}\right)^{\frac{p-1}{p}}$, *where* S *is given in* (2.3).

Proof. Using Lemma 2.3, item (3) we have $S \geq 0$. Firstly, we will prove that $S > 0$. Indeed, suppose $S = 0$. Then, there exists (u_n) be a sequence in $\mathcal{N}$ such that

$$J(u_n) = I(u_n) \to 0 \quad \text{if} \quad n \to \infty. \tag{3.1}$$

Hence, by (2.8) we obtain

$$\sup_n \int_0^R f(u_n) u_n^{p-1} r^\kappa \, \mathrm{d}r < \infty$$

and thus

$$\sup_n \|u_n'\|_{L^p_\alpha} < \infty.$$

Therefore, up to a subsequence we can assume $u_n \rightharpoonup u$ in X_R and $u_n(r) \to u(r)$ a.e in $(0, R)$. Now, the Fatou's Lemma and Lemma 2.3 item (1) imply

$$0 \leq I(u) \leq \liminf_{n\to\infty} I(u_n) = 0.$$

This implies $u \equiv 0$. Combining Lemma 2.3 and (3.1) it follows

$$\lim_{n\to\infty} \|u_n'\|^p_{L^p_\alpha} = \omega_\alpha p \lim_{n\to\infty} \left\{ J(u_n) + \int_0^R F(u_n) r^\kappa \, \mathrm{d}r \right\} = 0.$$

Set $v_n = \frac{u_n}{\|u_n'\|_{L^p_\alpha}}$. We can assume that v_n converges weakly and for almost every $r \in (0, R)$ to $v \in X_R$. Since $u_n \in \mathcal{N}$, the Lemma 2.3 item (2) and the assumption $f'(0) < \Lambda_1$ imply

$$1 = \omega_\alpha \lim_{n\to\infty} \int_0^R \frac{f(u_n) v_n^p r^\kappa}{u_n} \, \mathrm{d}r = \omega_\alpha \int_0^R f'(0) v^p r^\kappa \, \mathrm{d}r < \Lambda_1 \omega_\alpha \int_0^R v^p r^\kappa \, \mathrm{d}r \leq \|v'\|^p_{L^p_\alpha} = 1.$$

This contradiction ensures that $S > 0$.

Let $u \in X_R$ such that $\|u'\|_{L^p_\alpha} = 1$. From Lemma 2.2, we can choose a $\tau > 0$ such that $\tau u \in \mathcal{N}$. Hence

$$\frac{S^p}{p} \leq J(\tau u) \leq \frac{\tau^p}{p} \int_0^R |u'|^p r^\alpha \mathrm{d}r = \frac{\tau^p}{p\omega_\alpha}$$

which implies $\omega_\alpha^{1/p} S \le \tau$. Using that $t \mapsto \frac{f(tu)u^{p-1}}{t}$ is an increasing function on $(0,\infty)$ and $\tau u \in \mathcal{N}$, we have

$$\int_0^R \frac{f(\omega_\alpha^{1/p} S u)u^{p-1} r^\kappa}{\omega_\alpha^{1/p} S}\,\mathrm{d}r \le \int_0^R \frac{f(\tau u)u^{p-1} r^\kappa}{\tau}\,\mathrm{d}r = \int_0^R |u'|^p r^\alpha \mathrm{d}r = 1/\omega_\alpha.$$

This implies

$$\sup_{\|u'\|_{L^p_\alpha} \le 1} \int_0^R f(\omega_\alpha^{1/p} S u)u^{p-1} r^\kappa\,\mathrm{d}r \le \omega_\alpha^{\frac{1}{p}-1} S.$$

From Lemma 2.5, we obtain $\omega_\alpha^{1/p} S < (\frac{\mu_{\alpha,\kappa}}{b})^{1/p'}$. ▮

Proof of Theorem 1.3. According to the Lemma 2.1, it sufficient to prove that there is $u_0 \in \mathcal{N}$ with $u_0 \not\equiv 0$ and

$$J(u_0) = \frac{S^p}{p}.$$

Also, since $J(u_0) = J(|u_0|)$ we will can take $u_0 \ge 0$. Let $u_n \in \mathcal{N}$ such that $J(u_n) \to \frac{S^p}{p}$ as $n \to \infty$. Thus, $J(u_n) = I(u_n)$ and (2.8) imply

$$\sup_n \int_0^R f(u_n)u_n^{p-1} r^\kappa\,\mathrm{d}r < \infty$$

and

$$\sup_n \|u_n'\|_{L^p_\alpha} < \infty.$$

Therefore, by passing to a subsequence, we can assume $u_n \rightharpoonup u_0$ weakly in X_R and $u_n(r) \to u_0(r)$ a.e in $(0,R)$. We only need to show that $u_0 \in \mathcal{N} \setminus \{0\}$ and $J(u_0) = \frac{S^p}{p}$. First, from (2.6) we have

$$\lim_{n\to\infty} \int_0^R F(u_n) r^\kappa\,\mathrm{d}r = \int_0^R F(u_0) r^\kappa\,\mathrm{d}r. \tag{3.2}$$

Next, we will prove that $u_0 \not\equiv 0$. Suppose $u_0 \equiv 0$. Then, (3.2) and Lemma 3.1

$$\lim_{n\to\infty} \|u_n'\|^p_{L^p_\alpha} = \omega_\alpha p \lim_{n\to\infty} \left\{ J(u_n) + \int_0^R F(u_n) r^\kappa\,\mathrm{d}r \right\} = \omega_\alpha S^p < \left(\frac{\mu_{\alpha,\kappa}}{b}\right)^{p-1}.$$

Hence, we can apply Lemma 2.3, item (2) and conclude

$$\lim_{n\to\infty} \int_0^R f(u_n)u_n^{p-1} r^\kappa\,\mathrm{d}r = 0.$$

This implies $\frac{S^p}{p} = \lim_n I(u_n) = 0$ which contradicts the Lemma 3.1. Thus, $u_0 \not\equiv 0$.

Now, from Lemma 2.2 we can take $\tau > 0$ with $\tau u_0 \in \mathcal{N}$. It remains to prove that $\tau = 1$ and $J(u_0) = \frac{S^p}{p}$. We next claim that

$$\int_0^R |u_0'|^p r^\alpha\,\mathrm{d}r \le \int_0^R f(u_0)u_0^{p-1} r^\kappa\,\mathrm{d}r. \tag{3.3}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Indeed, suppose that (3.3) is not satisfied. Then, u_n and u_0 satisfy all the assumptions of Lemma 2.4 and consequently

$$\int_0^R |u_0'|^p r^\alpha \,\mathrm{d}r \le \liminf_{n\to\infty} \int_0^R |u_n'|^p r^\alpha \,\mathrm{d}r = \liminf_{n\to} \int_0^R f(u_n) u_n^{p-1} r^\kappa \,\mathrm{d}r = \int_0^R f(u_0) u_0^{p-1} r^\kappa \,\mathrm{d}r$$

which is a contradiction. Therefore, since both $\frac{f(tu_0)u_0^{p-1}}{t}$ and $I(tu_0)$ are increasing functions on $(0,\infty)$, using (3.3) we obtain $0 < \tau \le 1$. Moreover,

$$\frac{S^p}{p} \le J(\tau u_0) = I(\tau u_0) \le I(u_0) \le \liminf_{n\to\infty} I(u_n) = \lim_{n\to\infty} J(u_n) = \frac{S^p}{p}.$$

Hence $I(\tau u_0) = I(u_0)$ and thus $\tau = 1$. Also, $u_0 \in \mathcal{N}$ and the above inequalities imply $J(u_0) = \frac{S^p}{p}$.

REFERENCES

1. Adimurthi, *Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4** 17 (1990), 393–413.
2. Adimurthi. K. Sandeep, *A singular Moser-Trudinger embedding and its applications*, NoDEA Nonlinear Differential Equations Appl. **13** (2007), 585–603.
3. F. V. Atkinson, L. A. Peletier, *Ground states and Dirichlet problems for $-\Delta u = f(u)$ in $\mathbb{R}^2$*, Arch. Ration. Mech. Anal. **96** (1986), 147–165.
4. R. Černý, A. Cianchi, S. Hencl, *Concentration-compactness principles for Moser-Trudinger inequalities: new results and proofs*, Ann. Mat. Pura Appl. **2** (2013), 225–243.
5. P. Clément, D. G, Figueiredo, E. Mitidieri, *Quasilinear elliptic equations with critical exponents*, Topol. Methods Nonlinear Anal. **7** (1996), 133–170.
6. D.G. de Figueiredo, J.V. Gonçalves, O.H. Miyagaki, *On a class of quasilinear elliptic problems involving critical Sobolev exponents*, Commun. Contemp. Math. **2** (2000), 47–59.
7. D. G. de Figueiredo, B. Ruf, *Existence and non-existence of radial solutions for elliptic equations with critical exponent in $\mathbb{R}^2$*, Comm. Pure Appl. Math. **48** (1995), 639-655.
8. J. F. de Oliveira, *Desigualdades do tipo Trudinger-Moser para uma classe de espaços de Sobolev*, Tese de Doutorado, Universidade Federal de Pernambuco, Recife, 2013
9. J. F. de Oliveira, J. M. do Ó, *Trudinger–Moser type inequalities for weighted Sobolev spaces involving fractional dimensions*, Proc. Amer. Math. Soc., **142** (8) (2014), 2813–2828
10. J. F. de Oliveira, J. M. do Ó, *Concentration-compactness principle and extremal functions for a sharp Trudinger-Moser inequality*, Calc. Var. Partial Differential Equations, **52** (2015) 125-163
11. M. García–Huidobro, R. Manásevich, J. Serrin, M. Tang, C.S. Yarur, *Ground states and free boundary value problems for the n-Laplacian in n dimensional space*, J. Funct. Anal. **172** (2000), 177–201.
12. F. Gazzola, J. Serrin, M. Tang, *Existence of ground states and free boundary value problems for quasilinear elliptic operators*, Adv. Differential Equations, **5** (2000), 1–30.
13. J. Jacobsen, K. Schmitt, *Radial solutions of quasilinear elliptic differential equations.* Handbook of differential equations, Amsterdam, (2004), 359-435.
14. J. Jacobsen, K. Schmitt, *The Liouville–Bratu–Gelfand Problem for Radial Operators*, J. Differential Equations **184**, (2002), 283-298
15. A. Kufner, B. Opic, *Hardy–type inequalities*, Pitman Res. Notes in Math., vol. 219, Longman Scientific and Technical, Harlow, 1990.
16. P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case, Part 1* Rev. Mat. Iberoamericana **1**, (1985), 145–201

17. J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077–1092.

18. Z. Nehari, *On a class of nonlinear second-order differential equations*, Trans. Amer. Math. Soc. **95** (1960) 101–123.

19. S. I. Pohozaev, *The Sobolev embedding in the case $pl = n$*, Proceedings of the Technical Scientific Conference on Advances of Scientific Research 1964–1965. Mathematics Section, Moscov. Energet. Inst. (1965), 158–170.

20. N. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.

21. V. I. Yudovich, *Some estimates connected with integral operators and with solutions of elliptic equations*. Dok. Akad. Nauk SSSR **138**, 804–808 (1961) [English translation in Soviet Math. Doklady 2, 746–749 (1961)].

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC