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Minimal and constant mean curvature surfaces in homogeneous 3-manifolds

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In this work we present some results on minimal and constant mean curvature surfaces in homogeneous 3-manifolds that were part of the author's PhD thesis. First, we classify the compact embedded surfaces with constant mean curvature in the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by a subgroup of isometries generated by a horizontal translation along horocycles of $\mathbb{H}^2$ and a vertical translation. Moreove, we construct new examples of periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and we prove a multi-valued Rado theorem for small perturbations of the helicoid.

In some metric semidirect products, we construct new examples of complete minimal surfaces similar to the doubly and singly periodic Scherk minimal surfaces in $\mathbb{R}^3$. In particular, we obtain these surfaces in the Heisenberg space with its canonical metric, and in Sol_3 with a one-parameter family of non-isometric metrics.

After that, we prove a half-space theorem for an ideal Scherk graph $\Sigma \subset M \times \mathbb{R}$ over a polygonal domain $D \subset M$, where M is a Hadamard surface with bounded curvature. More precisely, we show that a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from Σ is a translate of Σ.

Finally, based in a joint paper with L. Hauswirth, we prove that if a properly immersed minimal surface in the quotient space $\mathbb{H}^2 \times \mathbb{R}/G$ has finite total curvature then its total curvature is a multiple of 2π and, moreover, we understand the geometry of the ends. Here G is a subgroup of isometries generated by a vertical translation and a horizontal isometry in $\mathbb{H}^2$ without fixed points.

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1. INTRODUCTION

One of the most natural and established topics in the differential geometry of surfaces is the global theory of minimal and constant mean curvature surfaces in the space forms $\mathbb{R}^3, \mathbb{S}^3$ and $\mathbb{H}^3$. This is a classic field that remains very active nowadays and uses a wide variety of techniques from different subjects, for example, variational calculus, complex analysis, topology, elliptic PDE theory and others.

The extension of this classic global theory for the case of immersed surfaces in homogeneous Riemannian three-dimensional manifolds has attracted the attention of many researchers in the last decade. These homogeneous manifolds are the most simple and

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symmetric Riemannian manifolds that we can consider besides the space forms, together forming the eight 3-dimensional Thurston geometries.

This theory is extremely rich, with lots of beautiful examples. Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, for instance, have been used by Collin and Rosenberg [2] to give counterexamples to a well-known conjecture of Schoen and Yau about harmonic diffeomorphisms between the complex plane and the disk.

In this work we will present some of our contributions to the theory of minimal and constant mean curvature surfaces. We will prove some results associated to uniqueness questions, classification problems, construction of new examples of minimal surfaces, half-space theorems and related themes.

2. THE ALEXANDROV PROBLEM IN A QUOTIENT SPACE OF $\mathbb{H}^2 \times \mathbb{R}$

Alexandrov, in 1962, proved that the only compact embedded constant mean curvature hypersurfaces in $\mathbb{R}^n, \mathbb{H}^n$ and $\mathbb{S}^n_+$ are the round spheres. Since then, many people have proved an Alexandrov type theorem in other spaces.

For instance, W.T. Hsiang and W.Y. Hsiang [13] showed that a compact embedded constant mean curvature surface in $\mathbb{H}^2 \times \mathbb{R}$ or in $\mathbb{S}^2_+ \times \mathbb{R}$ is a rotational sphere. They used the Alexandrov reflection method with vertical planes in order to prove that for any horizontal direction, there is a vertical plane of symmetry of the surface orthogonal to that direction.

To apply the Alexandrov reflection method we need to start with a vertical plane orthogonal to a given direction that does not intersect the surface, and in $\mathbb{S}^2 \times \mathbb{R}$ this fact is guaranteed by the hypothesis that the surface is contained in the product of a hemisphere with the real line. We remark that in $\mathbb{S}^2 \times \mathbb{R}$, we know that there are embedded rotational constant mean curvature tori, but the Alexandrov problem is not completely solved in $\mathbb{S}^2 \times \mathbb{R}$. In other simply connected homogeneous spaces with a 4-dimensional isometry group ($\mathrm{Nil}_3, \widetilde{\mathrm{PSL}}_2(\mathbb{R})$, some Berger spheres), we do not know if the solutions to the Alexandrov problem are spheres.

In Sol_3, Rosenberg proved that an embedded compact constant mean curvature surface is a sphere [5].

Recently, Mazet, Rodríguez and Rosenberg [16] considered the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by a discrete group of isometries of $\mathbb{H}^2 \times \mathbb{R}$ generated by a horizontal translation along a geodesic of $\mathbb{H}^2$ and a vertical translation. They classified the compact embedded constant mean curvature surfaces in the quotient space. Moreover, they constructed examples of periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, where by periodic we mean a surface which is invariant by a non-trivial discrete group of isometries of $\mathbb{H}^2 \times \mathbb{R}$.

We also consider periodic surfaces in $\mathbb{H}^2 \times \mathbb{R}$. The discrete groups of isometries of $\mathbb{H}^2 \times \mathbb{R}$ we consider are generated by a horizontal translation ψ along horocycles $c(s)$ of $\mathbb{H}^2$ and/or a vertical translation $T(h)$ for some $h > 0$. In the case the group is the $\mathbb{Z}^2$ subgroup generated by ψ and $T(h)$, the quotient space $\mathcal{M} = \mathbb{H}^2 \times \mathbb{R}/[\psi, T(h)]$ is diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$, where $\mathbb{T}^2$ is the 2-torus. Moreover, $\mathcal{M}$ is foliated by the family of tori $\mathbb{T}(s) = c(s) \times \mathbb{R}/[\psi, T(h)]$ which are intrinsically flat and have constant mean curvature 1/2. In this quotient space $\mathcal{M}$, we prove the following Alexandrov type theorem.

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Theorem 2.1. *Let $\Sigma \subset \mathcal{M}$ be a compact immersed surface with constant mean curvature H. Then $H \geq \frac{1}{2}$. Moreover,*

1.If $H = \frac{1}{2}$, then Σ is a torus $\mathbb{T}(s)$, for some s;
2.If $H > \frac{1}{2}$ and Σ is embedded, then Σ is either the quotient of a rotational sphere, or the quotient of a vertical unduloid (in particular, a vertical cylinder over a circle).

The proof of the first part is a consequence of the Maximum Principle using the family of constant mean curvature tori $\mathbb{T}(s)$; while the second part uses the Alexandrov reflection technique. The main point in the proof is that since in the space $\mathcal{M}$ we do not have many reflectional symmetries, instead of applying the Alexandrov reflection in this space, we consider a lift of our surface to the cover space of $\mathcal{M}$ which is just $\mathbb{H}^2 \times \mathbb{R}$ quotient by the vertical translation, and we apply the Alexandrov procedure in that space. See [19], Section 2.3, for more details.

3. NEW EXAMPLES OF PERIODIC MINIMAL SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

Periodic minimal surfaces in $\mathbb{R}^3$ have received great attention since Riemann, Schwarz, Scherk (and many others) studied them. They also appear in the natural sciences. In [18], Meeks and Rosenberg proved that a periodic properly embedded minimal surface of finite topology (in $\mathbb{R}^3/G, G$ a discrete group of isometries acting properly discontinuously on $\mathbb{R}^3, G \neq (1)$) has finite total curvature and the ends are asymptotic to standard ends (planar, catenoidal, or helicoidal). In a joint paper with Hauswirth [7], we consider the same study for periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. The first step is to understand what are the possible models for the ends in the quotient. This is one reason to construct examples.

We construct some new examples of periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant by a subgroup of isometries, which is either isomorphic to $\mathbb{Z}^2$, or generated by a vertical translation, or generated by a screw motion. In fact, we only consider subgroups generated by a parabolic translation ψ along a horocycle and/or a vertical translation $T(h)$, for some $h > 0$. We cons

We construct this examples by considering the solution to the Plateau problem for a geodesic polygonal contour Γ, and letting some sides of Γ tend to infinity in length, so that the associated Plateau solutions all pass through a fixed compact region (this will be assured by the existence of minimal annuli playing the role of barriers). Then a subsequence of the Plateau solutions will converge to a minimal surface bounded by a geodesic polygon with edges of infinite length. We complete this surface by symmetry across the edges. The whole construction requires precise geometric control and uses curvature estimates for stable minimal surfaces.

We construct examples of three different types of periodic minimal surfaces. More precisely, we prove the following results.

Proposition 3.1. *There exists a doubly periodic minimal surface (invariant by horizontal translations along a horocycle and by a vertical translation) such that, in the quotient space, this surface is topologically a sphere minus four points, with two ends asymptotic* *to vertical planes and two asymptotic to horizontal planes, all of them with finite total curvature.*

Proposition 3.2. *There exists a singly periodic minimal surface (invariant by a vertical translation) such that, in the quotient space, this surface has two ends, one end is asymptotic to a vertical plane and has finite total curvature, while the other one is topologically an annular end and has infinite total curvature.*

Proposition 3.3. *There exists a minimal surface invariant by a screw motion such that, in the quotient space, this minimal surface has four ends. Two vertical ends and two helicoidal ends, all of them with finite total curvature.*

For more details about the construction, see [19], Section 2.4.

4. A MULTI-VALUED RADO THEOREM IN $\mathbb{H}^2 \times \mathbb{R}$

The well known Rado's theorem in the Euclidean space says that minimal surfaces over a convex domain with graphical boundaries must be disks which are themselves graphical. In $\mathbb{H}^2\times\mathbb{R}$ we prove a multi-valued Rado theorem for small perturbations of the helicoid, more precisely, we prove that for certain small perturbations of the boundary of a (compact) helicoid there exists only one compact minimal disk with that boundary. By a compact helicoid we mean the intersection of a helicoid with certain compact regions in $\mathbb{H}^2\times\mathbb{R}$. The idea here originated in the work of Hardt and Rosenberg [6]. We apply this multi-valued Rado theorem to construct an embedded minimal surface in $\mathbb{H}^2\times\mathbb{R}$ whose boundary is a small perturbation of the boundary of a complete helicoid.

Consider Y the Killing field whose flow $\phi_\theta, \theta\in[0,2\pi)$, is given by rotations around the z-axis. For some $0<c<1$, let $D=\{(x,y)\in\mathbb{H}^2; x^2+y^2\leq c\}$. Take a helix h_0 of constant pitch contained in a solid cylinder $D\times[0,d]$, so that the vertical projection of h_0 over $\mathbb{H}^2\times\{0\}$ is ∂D, and the endpoints of h_0 are in the same vertical line. Let us denote by Γ_0 the Jordan curve which is the union of h_0, the two horizontal geodesic arcs joining the endpoints of h_0 to the z-axis, and the part of the z-axis. Call $\mathcal{H}$ the compact part of the helicoid that has Γ_0 as its boundary. We know that $\mathcal{H}$ is a minimal surface transversal to the Killing field Y at the interior points. Take $\theta<\pi/4$, and consider $\mathcal{H}_1=\phi_{-\theta}(\mathcal{H})$ and $\mathcal{H}_2=\phi_\theta(\mathcal{H})$. Hence $\mathcal{H}_1,\mathcal{H}_2$ are two compact helicoids with boundary $\partial\mathcal{H}_1=\phi_{-\theta}(\Gamma_0)$, $\partial\mathcal{H}_2=\phi_\theta(\Gamma_0)$.

Consider h a small smooth perturbation of the helix h_0 with fixed endpoints such that h is transversal to Y and h is contained in the region between $\phi_{-\theta}(h_0)$ and $\phi_\theta(h_0)$ in $\partial D\times[0,d]$. Call Γ the Jordan curve which is the union of h, the two horizontal geodesic arcs and a part of the z-axis, hence $\Gamma=(\Gamma_0\setminus h_0)\cup h$.

Denote by R the convex region bounded by $\mathcal{H}_1$ and $\mathcal{H}_2$ in the solid cylinder $D\times[0,d]$. The Jordan curve Γ is contained in the simply connected region R which has mean convex boundary. Then we can consider the solution to the Plateau problem in this region R, and we get a compact minimal disk H contained in R with boundary $\partial H = \Gamma$.

THEOREM 4.1 (A multi-valued Rado Theorem). *Under the assumptions above, H is the unique compact minimal disk with boundary Γ.*

To prove this result, we first observe that H is transversal to the Killing field Y at the interior points and the family $(\phi_\theta(H))_{\theta\in[0,2\pi)}$ foliates $D \times [0, d] \setminus \{z\text{-axis}\}$. Then the proof follows by contradiction using arguments of foliations. For more details, see Section 2.5 in [19].

5. PERIODIC MINIMAL SURFACES IN SEMIDIRECT PRODUCTS

We construct examples of periodic minimal surfaces in some semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$, depending on the matrix A. By periodic surface we mean a properly embedded surface invariant by a nontrivial discrete group of isometries.

One of the most simple examples of semidirect product is $\mathbb{H}^2 \times \mathbb{R} = \mathbb{R}^2 \rtimes_A \mathbb{R}$, when we take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. In this space, Mazet, Rodríguez and Rosenberg [16] proved some results about periodic constant mean curvature surfaces and they constructed examples of such surfaces. One of their methods is to solve a Plateau problem for a certain contour. In [29], using a similar technique, Rosenberg constructed examples of complete minimal surfaces in $M^2 \times \mathbb{R}$, where M is either the two-sphere or a complete Riemannian surface with nonnegative curvature or the hyperbolic plane.

Meeks, Mira, Pérez and Ros [17] have proved results concerning the geometry of solutions to Plateau type problems in metric semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$, when there is some geometric constraint on the boundary values of the solution.

The first example that we construct is a complete periodic minimal surface similar to the doubly periodic Scherk minimal surface in $\mathbb{R}^3$. It is invariant by two translations that commute and it is a four punctured sphere in the quotient of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ by the group of isometries generated by the two translations.

THEOREM 5.1. *In any semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, there exists a periodic minimal surface similar to the doubly periodic Scherk minimal surface in $\mathbb{R}^3$.*

We also obtain a complete periodic minimal surface analogous to the singly periodic Scherk minimal surface in $\mathbb{R}^3$.

THEOREM 5.2. *In any semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, there exists a periodic minimal surface similar to the singly periodic Scherk minimal surface in $\mathbb{R}^3$.*

These surfaces are obtained by solving the Plateau problem for a geodesic polygonal contour Γ (it uses a result by Meeks, Mira, Pérez and Ros [17] about the geometry of solutions to the Plateau problem in semidirect products), and letting some sides of Γ tend to infinity in length, so that the associated Plateau solutions all pass through a fixed compact region (this will be assured by the existence of minimal annuli playing the role of barriers). Then a subsequence of the Plateau solutions will converge to a minimal surface bounded by a geodesic polygon with edges of infinite length. We complete this surface by symmetry across the edges. The whole construction requires precise geometric control and uses curvature estimates for stable minimal surfaces.

We obtain periodic minimal surfaces in the Heisenberg space, when $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and in Sol_3, when $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with their well known Riemannian metrics. When we consider the one-parameter family of matrices $A(c) = \begin{pmatrix} 0 & c \\ \frac{1}{c} & 0 \end{pmatrix}$, $c \geq 1$, we get a one-parameter family of metrics in Sol_3 which are not isometric.

6. A HALF-SPACE THEOREM FOR IDEAL SCHERK GRAPHS IN $M \times \mathbb{R}$

A well known result in the global theory for proper minimal surfaces in the Euclidean 3-space is the so called *half-space theorem* due to Hoffman and Meeks [12], which says that if a properly immersed minimal surface S in $\mathbb{R}^3$ lies on one side of some plane P, then S is a plane parallel to P. Moreover, they also proved the *strong half-space theorem*: two properly immersed minimal surfaces in $\mathbb{R}^3$ that do not intersect must be parallel planes.

The problem of giving conditions which force two minimal surfaces of a Riemannian manifold to intersect has received considerable attention, and many people have worked on this subject.

Notice that there is no half-space theorem in Euclidean spaces of dimensions greater than 4 since there exist rotational proper minimal hypersurfaces contained in a slab.

Similarly, there exists no half-space theorem for horizontal slices in $\mathbb{H}^2 \times \mathbb{R}$ since rotational minimal surfaces (catenoids) are contained in a slab [23, 24]. However, there are half-space theorems for constant mean curvature (CMC) 1/2 surfaces in $\mathbb{H}^2 \times \mathbb{R}$ [10, 25]. For instance, Hauswirth, Rosenberg, and Spruck [10] proved that if S is a properly immersed CMC 1/2 surface in $\mathbb{H}^2 \times \mathbb{R}$, contained on the mean convex side of a horocylinder C, then S is a horocylinder parallel to C; and if S is embedded and contains a horocylinder C on its mean convex side, then S is also a horocylinder parallel to C. Nelli and Sa Earp [25] showed that in $\mathbb{H}^2 \times \mathbb{R}$ the mean convex side of a simply connected rotational CMC 1/2 surface can not contain a complete CMC 1/2 surface besides the rotational simply connected ones.

Other examples of homogeneous manifolds where there are half-space theorems for minimal surfaces are Nil_3 and Sol_3 [1, 3, 4]. For instance, we know that if a properly immersed minimal surface S in Nil_3 lies on one side of some entire minimal graph Σ, then S is the image of Σ by a vertical translation.

Mazet [15] proved a general half-space theorem for constant mean curvature surfaces. Under certain hypothesis, he proved that in a Riemannian 3-manifold of bounded geometry,

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a constant mean curvature H surface on one side of a parabolic constant mean curvature H surface Σ is an equidistant surface to Σ.

We consider the half-space problem for an ideal Scherk graph Σ over a polygonal domain $D \subset M$, where M denotes a Hadamard surface whose curvature is bounded above by a negative constant, that is, M is a complete simply connected Riemannian surface with curvature $K_M \leq -a^2 < 0$ for some constant $a \in \mathbb{R}$. More precisely, we prove the following result.

Theorem 6.1. *Let M denote a Hadamard surface with curvature bounded above by a negative constant, and let $\Sigma = Graph(u)$ be an ideal Scherk graph over an admissible polygonal domain $D \subset M$. If S is a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from Σ, then S is a translate of Σ.*

We remark that Mazet's theorem does not apply in our case for Scherk surfaces. In fact, in the case of minimal surfaces, one of his hypotheses on the geometry of equidistant surfaces to the parabolic one is that the mean curvature points away from the original surface. However, since an end of a Scherk surface is asymptotic to some vertical plane $\gamma \times \mathbb{R}$, where γ is a geodesic, we know that an equidistant surface is asymptotic to $\gamma_s \times \mathbb{R}$, where γ_s is an equidistant curve to γ. Hence, in the case of a Scherk surface, the mean curvature vector of an equidistant surface points toward the Scherk surface.

The idea to prove this result is based on the proof of the classical half-space theorem in the Euclidean three-space due to Hoffman and Meeks [12]. In their proof, they use as barrier a family of minimal surfaces (obtained from the catenoid by homothety) that converges to the plane minus a point, where the plane is the minimal surface for which they want to prove the half-space theorem. Hence, in order to prove our result, using their ideas, we need a family of minimal surfaces that play the role of barriers and converge to our ideal Scherk graph, at least outside a compact set. To construct such a family, we follow an idea of Rosenberg, Schulze, and Spruck [30] by constructing a discrete family of minimal graphs in $D \times \mathbb{R}$. For more details, see [21].

7. ON DOUBLY PERIODIC MINIMAL SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$ WITH FINITE TOTAL CURVATURE IN THE QUOTIENT SPACE

Among all the minimal surfaces in $\mathbb{R}^3$, the ones of finite total curvature are the best known. In fact, if a minimal surface in $\mathbb{R}^3$ has finite total curvature then this minimal surface is either a plane or its total curvature is a non-zero multiple of 2π. Moreover, if the total curvature is -4π, then the minimal surface is either the Catenoid or the Enneper's surface [26].

In the homogeneous space $\mathbb{H}^2 \times \mathbb{R}$, B. Nelli and H. Rosenberg [23] obtained the first examples of minimal surfaces with finite total curvature which are graphs over certain polygonal domains. Later, the first author jointly with H. Rosenberg [9] developed the theory of complete embedded minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$. In that work they proved that the total curvature of such surfaces must be a multiple of 2π, and

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they gave simply connected examples whose total curvature is $-2\pi m$, for each nonnegative integer m.

In the last few years, many people have worked on this subject, constructed new examples and classified some minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ (see [8, 11, 14, 22, 27, 28, 31]).

We consider the space $\mathbb{H}^2 \times \mathbb{R}$ quotiented by a subgroup of isometries $G \subset \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$ generated by a horizontal isometry in $\mathbb{H}^2$ without fixed points, ψ, and a vertical translation, $T(h)$, for some $h > 0$. The isometry ψ can be either a parabolic translation along horocycles in $\mathbb{H}^2$ or a hyperbolic translation along a geodesic in $\mathbb{H}^2$. We prove that if a properly immersed minimal surface in $\mathbb{H}^2 \times \mathbb{R}/G$ has finite total curvature then its total curvature is a multiple of 2π, and moreover, we understand the geometry of the ends. More precisely, we prove the following result.

Theorem 7.1. *Let $X : \Sigma \hookrightarrow \mathcal{M} = \mathbb{H}^2 \times \mathbb{R}/[\psi, T(h)]$ be a properly (oriented) immersed minimal surface with finite total curvature. Then*

1. Σ is conformally equivalent to a compact Riemann surface $\overline{M}$ with genus g minus a finite number of points, that is, $\Sigma = \overline{M} \setminus \{p_1, ..., p_k\}$.

2. The total curvature satisfies

$$\int_\Sigma K d\sigma = 2\pi(2 - 2g - k).$$

3. The ends contained in $\mathcal{M}_-$ are necessarily asymptotic to a vertical plane $\gamma \times \mathbb{S}^1$ and the ends contained in $\mathcal{M}_+$ are asymptotic to either

- *a horizontal slice $\mathbb{H}^2/[\psi] \times \{c\}$, or*
- *a vertical plane $\gamma \times \mathbb{S}^1$, or*
- *the quotient of a Helicoidal plane.*

4. If we parametrize each end by a punctured disk then either Q extends to zero at the origin (in the case where the end is asymptotic to a horizontal slice) or Q extends meromorphically to the puncture with a double pole and residue zero. In this last case, the third coordinate satisfies $h(z) = b \arg(z) + O(|z|)$ with $b \in \mathbb{R}$.

In this result, $\mathcal{M}_-$ denotes an end of $\mathcal{M}$ whose injectivity radius is strictly positive, $\mathcal{M}_+$ denotes an end of $\mathcal{M}$ whose injectivity radius is zero, and *Helicoidal plane* refers to a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ which is parametrized by $X(x, y) = (x, y, ax + b)$ when it is considered the halfplane model for $\mathbb{H}^2$.

Let us mention that this result holds true for properly immersed minimal surfaces in $M \times \mathbb{S}^1$, where M is a hyperbolic surface ($K_M = -1$) with finite topology whose ends are either isometric to $\mathcal{M}_+$ or $\mathcal{M}_-$.

The proof of this result uses a great amount of complex analysis, combined with elliptic regularity theory and delicate geometric arguments. For more details, see [7].

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