CADERNOS DE MATEMÁTICA **16**, 39–69 May (2015)
ARTIGO NÚMERO SMA#3

Some topological and geometrical aspects of circle dynamics

Pablo Guarino

Instituto de Matemática e Estatística, Universidade Federal Fluminense, rua Mário Santos Braga S/N, 24020-140, Niterói, Rio de Janeiro, Brasil.
E-mail: pablo_guarino@id.uff.br

In these expository notes we survey some classical and recent results about the topology and geometry of circle dynamics without periodic orbits, emphasizing some connections with complex dynamics. May, 2015 ICMC-USP

Notation: Throughout these notes S^1 will denote the multiplicative group of complex numbers of modulus one, that is, the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. The Haar measure for this group, which is the normalized Lebesgue measure in the unit circle, will be denoted by λ. The length of an interval I will be denoted also by $|I|$. We will identify S^1 with $\mathbb{R}/\mathbb{Z}$ under the universal covering map $\pi : \mathbb{R} \to S^1$ given by $\pi(t) = \exp(2\pi i t)$. We denote by $\mathrm{Hom}_+(S^1)$ the group (under composition) of orientation preserving circle homeomorphisms, and by $\mathrm{Diff}^r_+(S^1)$ its subgroup of C^r diffeomorphisms for any $r \geq 1$. Finally, $\rho : \mathrm{Hom}_+(S^1) \to [0,1)$ will denote the rotation number.

1. INTRODUCTION

Let f be an orientation preserving circle homeomorphism, and assume that f has irrational rotation number $\rho(f) = \theta$. This is equivalent with the fact that f has no periodic orbits, and it implies that the non-wandering set of f is minimal, being a Cantor set or the whole circle [37, Chapter I].

As Poincaré showed, f is *semi-conjugate* to the rigid rotation of angle θ (denoted by R_θ): there exists a continuous surjective map $h : S^1 \to S^1$ such that $h \circ f = R_\theta \circ h$. Indeed, take a non-wandering point x in S^1 and consider its orbit under f, $\mathcal{O}_f(x) = \{f^n(x)\}_{n\in\mathbb{Z}}$. The map $h_x\big(f^n(x)\big) = \exp(2\pi i n\theta)$ sends the point x to the point 1, and conjugates f with R_θ along the orbit of x.

A crucial point is that f and R_θ are *combinatorially equivalent*, in the sense that for each $n \in \mathbb{N}$ the first n elements of the orbit of x under f are ordered in the same way as the first n elements of the orbit of the point 1 under the rotation R_θ (otherwise one can prove that f must have a periodic orbit, see [37, Chapter I]). The combinatorial equivalence between f and R_θ implies that the map h_x extends continuously to the closure of $\mathcal{O}_f(x)$. This

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

extension is surjective because any orbit of R_θ is dense in S^1, and then we can extend h_x as a constant function in any connected component of the complement of $\overline{\mathcal{O}_f(x)}$. This gives us a semi-conjugacy h_x between f and R_θ that sends the point x to the point 1 (given any other point $z \in S^1$ we have $h_z = R_\beta \circ h_x$ with $\exp(2\pi i\beta) = 1/h_x(z)$).

$$\begin{array}{ccc} S^1 & \xrightarrow{\;f\;} & S^1 \\ {\scriptstyle h_x}\big\downarrow & & \big\downarrow{\scriptstyle h_x} \\ S^1 & \xrightarrow[R_\theta]{} & S^1 \end{array}$$

Note that for every $y \in S^1$ the set $h_x^{-1}\big(\{y\}\big)$ is either a closed interval or a single point. If f is minimal in the whole circle we have $\overline{\mathcal{O}_f(x)} = S^1$, and in particular, h_x is a homeomorphism and then f is topologically conjugate to the rotation R_θ.

In any case f is *uniquely ergodic*: there exists a unique Borel probability measure μ in S^1 which is invariant under f, that is, $\mu(A) = \mu\big(f^{-1}(A)\big)$ for every Borel set $A \subset S^1$. The measure μ is just the pull-back of the Lebesgue measure under any semi-conjugacy: $\mu(A) = \lambda\big(h_x(A)\big)$ for any Borel set $A \subset S^1$ (since the semi-conjugacy h_x is unique up to post-composition with rotations, the measure μ is well-defined). In other words, the μ-measure of the arc between any point and its image under f, oriented in the counter-clockwise sense, equals the rotation number of f. Indeed:

$$\mu\big([x, f(x)]\big) = \lambda\big([1, h_x(f(x))]\big) = \lambda\big([1, R_\theta(1)]\big) = \lambda\big([1, \exp(2\pi i\theta)]\big) = \theta = \rho(f)\,.$$

Conversely, note that given any point $x \in S^1$ we can obtain the semi-conjugacy h_x from the measure μ defining:

$$h_x(y) = \exp\big(2\pi i\,\mu([x,y])\big) = \exp\left(2\pi i \int_x^y d\mu\right).$$

Indeed, since μ is f-invariant and f has no periodic orbits, μ has no points of positive measure and this implies that h_x is continuous and surjective (of course we have $h_x \circ f = R_\theta \circ h_x$). If f is minimal, any open interval has positive μ-measure (since the support of μ is an f-invariant compact set, it must be the whole circle). In particular h_x is an homeomorphism, and then f is topologically conjugate to the rotation R_θ.

Summarizing, an orientation preserving circle homeomorphism f with irrational rotation number θ is always semi-conjugate to the rigid rotation R_θ by a continuous surjective map h. If h is not a conjugacy, then there exists a point $y \in S^1$ such that $J = h^{-1}\big(\{y\}\big)$ is a non-degenerate closed interval. We call J a *wandering interval* since $f^n(J) \cap f^m(J) = \emptyset$ if $n \neq m \in \mathbb{Z}$, and since J is not contained in the basin of a periodic attractor.

In these expository notes we will deal with two kinds of smooth circle homeomorphisms with irrational rotation number: those without critical points (that is, diffeomorphisms) and those with a finite number of critical points of odd type, the so-called *critical circle maps*. In Section 2 we introduce two important tools for circle dynamics: the return times

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

given by the continued fraction expansion of the rotation number (Section 2.1) and the distortion of cross-ratio under iteration (Section 2.2). In Section 3 we prove the well-known Denjoy's theorem for smooth diffeomorphisms (Theorem 3.1), which asserts that C^2 diffeomorphisms have no wandering intervals. At the end of Section 3 (Section 3.2) we discuss some geometrical aspects of diffeomorphisms without periodic orbits. In Section 4 we focus on critical circle maps, which are circle dynamics that belong to the boundary of the space of diffeomorphisms. In Section 5 we carefully construct examples of real-analytic critical circle maps, emphasizing some connections with complex one-dimensional dynamics. In Section 6 we prove an extension of Denjoy's result to critical circle maps, due to J.-C. Yoccoz in the eighties. In Section 7 we discuss the geometrical classification of critical circle maps, providing statements of recent results. Finally, in Appendix 1, we briefly review the classical theory of continued fractions.

We refer the reader to the book of de Melo and van Strien [37] for general background in real one-dimensional dynamics.

2. PRELIMINARIES

2.1. Return times

Let $f \in \mathrm{Hom}_+(S^1)$. As we have pointed out already, our standing assumption is that the rotation number $\rho(f) = \theta \in [0,1)$ is irrational. Therefore it has an infinite continued fraction expansion, say

$$\theta = [a_0, a_1, ..., a_n, a_{n+1}, ...] = \lim_{n\to\infty} \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{1}{a_n}}}}} \,.$$

A classical reference for continued-fraction expansion is the monograph [26], see also Appendix 1 of these notes. We define recursively a sequence of *return times* of θ by:

$$q_0 = 1, \qquad q_1 = a_0 \quad \text{and} \quad q_{n+1} = a_n q_n + q_{n-1} \quad \text{for} \quad n \geq 1.$$

Recall that the numbers q_n are also obtained as the denominators of the truncated expansion of order n of θ:

$$\frac{p_n}{q_n} = [a_0, a_1, a_2, ..., a_{n-1}] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{1}{a_{n-1}}}}}} \,.$$

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

We recall also the following well-known estimates (see for instance Theorems 9 and 13 in [26, Ch. I], or Theorem 5 in [30, Ch. I]):

$$\frac{1}{q_n(q_n+q_{n+1})} < \left|\theta - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2} \qquad \text{for all } n \in \mathbb{N}.$$

The rational numbers p_n/q_n are called the *convergents* of the irrational θ. Each p_n/q_n is the best possible approximation to θ by rational numbers with denominator at most q_n [26, Chapter II, Theorem 15]:

$$\text{If } 0 < q < q_n \text{ then } \big|\theta - p_n/q_n\big| < \big|\theta - p/q\big| \text{ for any } p \in \mathbb{N}.$$

Now fix any point $x \in S^1$. The arithmetical properties of the continued fraction expansion described above imply that the iterates $\{R_\theta^{q_n}(x)\}_{n\in\mathbb{N}}$ are the closest returns of the orbit of x under the rigid rotation R_θ:

$$d\big(x, R_\theta^{q_n}(x)\big) < d\big(x, R_\theta^{j}(x)\big) \quad \text{for any} \quad j \in \{1, ..., q_n - 1\}$$

where d denote the standard distance in S^1. The sequence of return times $\{q_n\}$ grows at least exponentially fast as $n \to \infty$ (since $q_{n+1} = a_n q_n + q_{n-1} \geq 2q_{n-1}$), and the sequence of return distances $\{d(x, R_\theta^{q_n}(x))\}$ decreases to zero at least exponentially fast as $n \to \infty$. The sequence $\{R_\theta^{q_n}(x)\}_{n\in\mathbb{N}}$ approaches the point x alternating the order:

$$R_\theta^{q_1}(x) < R_\theta^{q_3}(x) < ... < R_\theta^{q_{2k+1}}(x) < ... < x < ... < R_\theta^{q_{2k}}(x) < ... < R_\theta^{q_2}(x) < R_\theta^{q_0}(x)\,.$$

By Poincaré's result this information remains true at the combinatorial level for any circle homeomorphism f with rotation number θ: for any $x \in S^1$ the interval $[x, f^{q_n}(x)]$ contains no other iterates $f^j(x)$ for $j \in \{1, ..., q_n - 1\}$, and if μ is the unique invariant Borel probability of f we can say that $\mu\big([x, f^{q_n}(x)]\big) < \mu\big([x, f^{j}(x)]\big)$ for any $j \in \{1, ..., q_n - 1\}$. A priori we cannot say anything about the usual distance in S^1.

2.2. Cross-ratio distortion

Let $a < b < c < d$ be four distinct points in the real line. Let S_1 be the Möbius transformation determined by $S_1(a) = 0$, $S_1(c) = 1$ and $S_1(d) = \infty$. Note that S_1 has real coefficients since it preserves the real line. Define $\mathrm{Cr}_1(a, b, c, d) \in (0, 1)$ as $\mathrm{Cr}_1(a, b, c, d) = S_1(b)$, that is:

$$\mathrm{Cr}_1(a, b, c, d) = \left(\frac{d-c}{c-a}\right)\left(\frac{b-a}{d-b}\right).$$

If we denote by $T = (a, d)$ and by $M = (b, c)$ we have that:

$$\mathrm{Cr}_1(a, b, c, d) = \left(\frac{|L|}{|L| + |M|}\right)\left(\frac{|R|}{|R| + |M|}\right),$$

where L and R are the components of $T \setminus M$.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

The choice of the Möbius transformation S_1 was quite arbitrary. We can consider, for instance, the Möbius transformation S_2 determined by $S_2(a) = -1$, $S_2(b) = 0$ and $S_2(d) = \infty$, and define $\mathrm{Cr}_2(a,b,c,d) \in (0,+\infty)$ as $\mathrm{Cr}_2(a,b,c,d) = S_2(c)$, that is:

$$\mathrm{Cr}_2(a,b,c,d) = \left(\frac{d-a}{b-a}\right)\left(\frac{c-b}{d-c}\right).$$

As before, if we denote by $T = (a,d)$ and by $M = (b,c)$ we have that:

$$\mathrm{Cr}_2(a,b,c,d) = \frac{|M||T|}{|L||R|}.$$

Several different definitions of cross-ratio can be found in the literature, depending on the purposes of the authors. The first definition given here is the one used in [48], while the second definition is the one chosen in [34] and [10]. Of course both definitions are related by a Möbius transformation. Indeed, consider the orientation reversing real-analytic diffeomorphism $S : (0,1) \to (0,+\infty)$ given by $S(x) = \frac{1-x}{x}$, whose inverse is given by $S^{-1}(x) = \frac{1}{1+x}$. Then we have $S\big(\mathrm{Cr}_1(a,b,c,d)\big) = \mathrm{Cr}_2(a,b,c,d)$ for all $a < b < c < d$ in $\mathbb{R}$. Note that both $\mathrm{Cr}_1(a,b,c,d)$ and $\mathrm{Cr}_2(a,b,c,d)$ are invariant under Möbius transformations, that is, if S is any Möbius transformation and $a < b < c < d$ are four distinct real numbers, we have $\mathrm{Cr}_i\left(S(a), S(b), S(c), S(d)\right) = \mathrm{Cr}_i(a,b,c,d)$ for $i = 1, 2$. In these notes we will work with the second definition given above. More precisely:

Definition 2.1. Given intervals $M \subsetneq T \subset S^1$ we define the *cross-ratio* of M in T as:

$$\mathrm{Cr}[M,T] = \frac{|M||T|}{|L||R|},$$

where L and R are the components of $T \setminus M$. Suppose now that f is a homeomorphism in T, we define the *distortion of cross-ratio* of f in M and T as:

$$\mathrm{Cr}(f,M,T) = \frac{\mathrm{Cr}[f(M), f(T)]}{\mathrm{Cr}[M,T]}.$$

3. TOPOLOGICAL RIGIDITY OF CIRCLE DIFFEOMORPHISMS

In a classical article of 1932 [4], Denjoy proved the following topological rigidity result: any C^2 circle diffeomorphism with irrational rotation number is topologically conjugate to the corresponding rigid rotation.

At the time, Denjoy was interested in the dynamics of smooth flows on compact surfaces, where one-dimensional dynamics appear when considering first-return maps of local transverse sections (see also the work of Schwartz [46] of 1963).

Actually, the original result of Denjoy (see Theorem 3.1 below) is for C^1 diffeomorphisms such that $\log Df$ has *bounded variation*: there exists a constant $V = V(f) > 0$ such that given any ordered finite partition $\{x_0, x_1, ..., x_n\}$ of the circle we have that:

$$\sum_{i=0}^{n-1} \big| \log Df(x_{i+1}) - \log Df(x_i) \big| \leq V(f) = \operatorname{var}(\log Df)\,.$$

In this case we say that f is C^{1+bv}, or that $f \in \operatorname{Diff}_+^{1+bv}(S^1)$. If $f \in \operatorname{Diff}_+^2(S^1)$, then $f \in \operatorname{Diff}_+^{1+bv}(S^1)$ just by taking:

$$V(f) \geq \frac{\max_{x \in S^1} \big|D^2 f(x)\big|}{\min_{x \in S^1} \big|Df(x)\big|}\,, \text{ or even better } V(f) \geq \int_{S^1} \left| \frac{D^2 f(x)}{Df(x)} \right| dx\,.$$

The main result of this section is the following:

Theorem 3.1 (Denjoy, 1932). *Let f be a C^1 orientation preserving circle diffeomorphism such that $\log Df$ has bounded variation. If f has irrational rotation number θ, it is topologically conjugate to the rigid rotation of angle θ.*

Our proof of Theorem 3.1 wont be the easiest one (see for instance [37, Section I.2]), but it is more suitable to be adapted to the case of critical circle maps, see Section 6. The proof will be obtained by combining the following two results:

Proposition 3.2. *Let f be a C^1 orientation preserving circle diffeomorphism with irrational rotation number and such that $\log Df$ has bounded variation with some constant $V = V(f) > 0$. There exists a positive constant $\delta = \delta(V) > 0$ such that given any interval J and any sequence $\{T_n\}_{n\in\mathbb{N}}$ of intervals containing J such that for any $n \in \mathbb{N}$ the first $q_{n+1} - 1$ iterates of T_n have intersection multiplicity 2, we have that:*

$$\operatorname{Cr}(f^k, J, T_n) \geq \delta \quad \text{for any} \quad k \in \{0, ..., q_{n+1}\}\,.$$

The sequence $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of return times given by the rotation number of f (see Section 2.1). A family of intervals has *intersection multiplicity* $k \in \mathbb{N}$ if the maximum number of intervals from the family that has non-empty intersection is k.

Proposition 3.3. *Let $f \in \operatorname{Diff}_+^{1+bv}(S^1)$ with irrational rotation number. Suppose that f is not conjugate to the corresponding rigid rotation, and let J be a maximal wandering interval. There exists a decreasing sequence $\{T_n\}_{n\in\mathbb{N}}$ of open intervals such that:*

- $\overline{J} = \bigcap_{n\in\mathbb{N}} T_n$.
- *The family* $\big\{T_n, f(T_n), ..., f^{q_{n+1}-1}(T_n)\big\}$ *has intersection multiplicity* 2 *for all* $n \in \mathbb{N}$.
- $\lim_{n\to\infty} \operatorname{Cr}(f^{q_{n+1}}, J, T_n) = 0$.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Proof (Theorem 3.1). Combine Proposition 3.2 and Proposition 3.3. ∎

The remainder of this section is devoted to prove Proposition 3.2 and Proposition 3.3.

3.1. Proof of Proposition 3.2 and Proposition 3.3

Proof (Proposition 3.2). Fix $n \in \mathbb{N}$ and $k \in \{0, ..., q_{n+1}\}$, and note that:

$$\mathrm{Cr}\left(f^k, J, T_n\right) = \prod_{i=0}^{k-1} \mathrm{Cr}\left(f, f^i(J), f^i(T_n)\right).$$

In particular:

$$\left|\log\left(\mathrm{Cr}(f^k, J, T_n)\right)\right| \leq \sum_{i=0}^{k-1} \left|\log\left(\mathrm{Cr}(f, f^i(J), f^i(T_n))\right)\right|.$$

By the Mean Value Theorem there exist for any $i \in \{0, ..., k-1\}$ four points $x_i \in f^i(J), y_{i,n} \in f^i(T_n), z_{i,n}, w_{i,n} \in f^i(T_n) \setminus f^i(J)$ such that:

$$\mathrm{Cr}\left(f, f^i(J), f^i(T_n)\right) = \frac{|Df(x_i)||Df(y_{i,n})|}{|Df(z_{i,n})||Df(w_{i,n})|}.$$

Therefore:

$$\begin{aligned}\left|\log\left(\mathrm{Cr}(f^k, J, T_n)\right)\right| &\leq \sum_{i=0}^{k-1} \left|\log Df(x_i) + \log Df(y_{i,n}) - \log Df(z_{i,n}) - \log Df(w_{i,n})\right| \\ &\leq \sum_{i=0}^{k-1} \left|\log Df(x_i) - \log Df(z_{i,n})\right| + \left|\log Df(w_{i,n}) - \log Df(y_{i,n})\right|\end{aligned}$$

Now consider the finite partition $\mathcal{P}_k = \left\{x_i, y_{i,n}, z_{i,n}, w_{i,n}\right\}_{i=0}^{k-1}$. Since the family

$$\left\{T_n, f(T_n), ..., f^{q_{n+1}-1}(T_n)\right\}$$

has intersection multiplicity 2, the last term is less or equal than the double of the total variation of $\log Df$ in $\mathcal{P}_k$, and so we are done by taking $\delta = \exp(-2V)$. ∎

We focus now on the proof of Proposition 3.3. We will prove first (see Corollary 3.6 below) that if $f \in \mathrm{Diff}_+^{1+bv}(S^1)$, the sequence $\left\{\left|\log Df^{q_n}\right|\right\}_{n \in \mathbb{N}}$ is bounded. With this purpose we state and prove two well-known technical results:

LEMMA 3.4 (Denjoy-Koksma inequality). *Let $f \in \mathrm{Hom}_+(S^1)$ with $\rho(f) = \theta \in \mathbb{R} \setminus \mathbb{Q}$, and let μ be its unique invariant Borel probability measure. Let $\{q_n\}_{n \in \mathbb{N}}$ be the sequence of return times given by θ. For any $\psi : S^1 \to \mathbb{R}$ (non necessarily continuous) with finite total* *variation* $\operatorname{var}(\psi)$ *we have:*

$$\left|\sum_{j=0}^{q_n-1}\psi\big(f^j(x)\big)-q_n\int_{S^1}\psi\,d\mu\right|\leq\operatorname{var}(\psi)\quad\textit{for all } x\in S^1 \textit{ and all } n\in\mathbb{N}.$$

Proof (Lemma 3.4). Fix $x \in S^1$ and $n \in \mathbb{N}$. By combinatorics there exist q_n pairwise disjoint open intervals $\{I_0, I_1, ..., I_{q_n-1}\}$ in the unit circle such that $R_\theta^j(1) = e^{2\pi i j\theta} \in \overline{I_j}$ and $\lambda(I_j) = 1/q_n$ for all $j \in \{0, 1, ..., q_n - 1\}$ (just take the intervals determined by the q_n-roots of unity, and label them in order to have $e^{2\pi i j\theta} \in \overline{I_j}$ for all $j \in \{0, 1, ..., q_n - 1\}$). Let $h = h_x$ be the semi-conjugacy between f and R_θ that maps the point x to the point 1, and for each $j \in \{0, 1, ..., q_n - 1\}$ let $J_j = h^{-1}(I_j)$. Note that $f^j(x) \in J_j$ and $\mu(J_j) = 1/q_n$ for all $j \in \{0, 1, ..., q_n - 1\}$. Moreover $\left\{\overline{J_j}\right\}_{j=0}^{q_n-1}$ is a partition of the unit circle (modulo boundary points, whose μ-measure is zero since μ is f-invariant and f has no periodic orbits). Therefore:

$$\begin{aligned}\left|\sum_{j=0}^{q_n-1}\psi\big(f^j(x)\big)-q_n\int_{S^1}\psi\,d\mu\right| &= \left|\sum_{j=0}^{q_n-1}\left(\psi\big(f^j(x)\big)-q_n\int_{J_j}\psi\,d\mu\right)\right|\\ &\leq \sum_{j=0}^{q_n-1}\left|\psi\big(f^j(x)\big)-q_n\int_{J_j}\psi\,d\mu\right|\\ &= q_n\sum_{j=0}^{q_n-1}\left|\int_{J_j}\big(\psi\big(f^j(x)\big)-\psi\big)d\mu\right|\\ &\leq q_n\sum_{j=0}^{q_n-1}\int_{J_j}\left|\psi\big(f^j(x)\big)-\psi\right|d\mu\\ &\leq \sum_{j=0}^{q_n-1}\sup_{y\in J_j}\left|\psi\big(f^j(x)\big)-\psi(y)\right|\leq\operatorname{var}(\psi)\,.\end{aligned}$$

∎

The *Lyapunov exponent* of any C^1 diffeomorphism with irrational rotation number is zero. More precisely:

Lemma 3.5. *Let* $f \in \operatorname{Diff}_+^1(S^1)$ *with irrational rotation number, and let* μ *be its unique invariant Borel probability measure. Then:*

$$\int_{S^1}\log Df\,d\mu=0\,.$$

Proof (Lemma 3.5). If $f \in \mathrm{Diff}_+^1(S^1)$, the function $\psi : S^1 \to \mathbb{R}$ defined by $\psi = \log Df$ is a continuous function and therefore, by the unique ergodicity of f, the sequence of functions

$$\frac{1}{n}\sum_{j=0}^{n-1} \psi \circ f^j$$

converges uniformly to a constant [33, Chapter I, Section 9], and this constant must be $\int_{S^1} \log Df\, d\mu$. By the chain rule:

$$\sum_{j=0}^{n-1} \psi \circ f^j = \log(Df^n)\,. \tag{3.1}$$

Therefore the sequence of continuous functions $\log(Df^n)/n$ converges to the constant $\int_{S^1} \log Df\, d\mu$ uniformly in S^1. Since f^n is a diffeomorphism for all $n \in \mathbb{N}$, this constant must be zero. ▌

If $f \in \mathrm{Diff}_+^{1+bv}(S^1)$ we can put together Lemma 3.4 and Lemma 3.5 to obtain that the sequence of iterates $\{f^{q_n}\}_{n\in\mathbb{N}}$ is uniformly Lipschitz (and in particular equicontinuous) on the whole circle:

Corollary 3.6. *If $f \in \mathrm{Diff}_+^{1+bv}(S^1)$ has irrational rotation number, then:*

$$e^{-V(f)} \leq \big(Df^{q_n}\big)(x) \leq e^{V(f)} \quad \textit{for all } x \in S^1 \textit{ and all } n \in \mathbb{N}.$$

Proof (Corollary 3.6). Apply Lemma 3.4 with $\psi = \log Df$, and use (3.1) and Lemma 3.5. ▌

With Corollary 3.6 at hand, we are ready to prove Proposition 3.3.

Proof (Proposition 3.3). Let $J = (a, b)$ be a maximal wandering interval of f, and let $n \in \mathbb{N}$. The complement of the union of $\overline{J}$ and $f^{q_n}(\overline{J})$ are two open intervals. By combinatorics, the interval $f^{q_{n+1}+q_n}(J)$ is contained in one of them, and the interval $f^{q_{n+1}}(J)$ is contained in the other one. Since J is maximal as a wandering interval both a and b are recurrent for the future, and therefore the distance between $\overline{J}$ and $f^{q_n}(\overline{J})$ goes to zero as n goes to infinity. Let us suppose that for the fixed integer n we have that $f^{q_{n+1}+q_n}(J)$ is contained in the small component (this depends if n is even or odd, and the other case can be treated in the same way). Let $L_n = \big(f^{-q_n}(a), a\big)$, $R_n = \big(b, f^{q_n}(a)\big)$ and:

$$T_n = L_n \cup \overline{J} \cup R_n = \big(f^{-q_n}(a), f^{q_n}(a)\big).$$

By definition:

$$\mathrm{Cr}(f^{q_{n+1}}, J, T_n) = |L_n||R_n||f^{q_{n+1}}(T_n)| \left(\frac{|f^{q_{n+1}}(J)|}{|f^{q_{n+1}}(L_n)|}\right)\left(\frac{1}{|f^{q_{n+1}}(R_n)||J||T_n|}\right).$$

The key combinatorial point is that $J \subset f^{q_{n+1}}(R_n)$, and so we have $|f^{q_{n+1}}(R_n)| \geq |J|$. Since we also have $|T_n| \geq |J|$ and $|f^{q_{n+1}}(T_n)| \leq 1$ we obtain:

$$\operatorname{Cr}(f^{q_{n+1}}, J, T_n) \leq |R_n||f^{q_{n+1}}(J)| \left(\frac{|L_n|}{|f^{q_{n+1}}(L_n)|}\right)\left(\frac{1}{|J|^3}\right). \tag{3.2}$$

Estimate (3.2) holds for any homeomorphism f with irrational rotation number. Now we use the smoothness condition. By Corollary 3.6 the sequence:

$$\frac{|L_n|}{|f^{q_{n+1}}(L_n)|}$$

is bounded from above. Since $\sum_{n\in\mathbb{Z}} |f^n(J)| \leq 1$ we have that $|f^{q_{n+1}}(J)| \to 0$ as n goes to infinity, and since J is maximal (as a wandering interval) we have that $|R_n| \to 0$ as n goes to infinity. This proves that:

$$\lim_{n\to\infty} \operatorname{Cr}(f^{q_{n+1}}, J, T_n) = 0\,.$$

Finally, since $T_n = (f^{-q_n}(a), f^{q_n}(a))$, we know by combinatorics that the family:

$$\left\{T_n, f(T_n), ..., f^{q_{n+1}-1}(T_n)\right\}$$

has intersection multiplicity 2 for all $n \in \mathbb{N}$. ∎

Remark 3. 1. In [4] Denjoy also proved that some assumptions on the first derivative are needed: given any irrational number θ there exists a C^1 circle diffeomorphism with rotation number θ which has wandering intervals [37, Section I.2]. We remark that there are counterexamples even if the derivative is Hölder continuous [18]. See also [22], [23] and [20].

3.2. The geometrical classification

We finish Section 3 with some remarks concerning geometrical rigidity: let f be a circle homeomorphism topologically conjugate to R_θ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let μ be the unique Borel probability in S^1 invariant under f. One motivation to understand the measure μ is given by Birkhoff's Ergodic Theorem [33, Chapter II, Theorem 1.1]: given any point $x \in S^1$ and any interval $A \subset S^1$ we have that:

$$\lim_{n\to+\infty}\left[\left(\frac{1}{n}\right)\#\left\{j : 0 \leq j < n \quad \text{and} \quad f^j(x) \in A\right\}\right] = \mu(A).$$

As explained in the introduction, $\mu(A) = \lambda\big(h(A)\big)$ for any Borel set $A \subset S^1$, where h is a circle homeomorphism that conjugates f with R_θ. In particular μ has no atoms (i.e. no points of positive measure) and it gives positive measure to any open set.

$$\begin{array}{ccc} (S^1,\mu) & \xrightarrow{\ f\ } & (S^1,\mu) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ (S^1,\lambda) & \xrightarrow[R_\theta]{} & (S^1,\lambda) \end{array}$$

Note that if f is C^1 we have the following dichotomy: either μ is *absolutely continuous* with respect to Lebesgue, or μ is *singular* (otherwise we have a decomposition $\mu = \nu_1 + \nu_2$, where ν_1 is absolutely continuous with respect to Lebesgue, ν_2 is singular, and both are non-zero. Since f is C^1 it preserves sets of Lebesgue measure zero, and therefore both ν_1 and ν_2 are f-invariant, which contradicts the unique ergodicity of f).

If μ is absolutely continuous there exists a Lebesgue integrable function $\frac{d\mu}{d\lambda}$ such that $\mu(A) = \int_A \frac{d\mu}{d\lambda}\, d\lambda$, but in general this is not true: in 1961 Arnol'd gave examples ([1], see also [37, Section I.5]) of real-analytic circle diffeomorphisms that are minimal, but where the invariant probability μ is not absolutely continuous with respect to Lebesgue measure. In those examples the rotation number is Liouville (see Definition A.3 in Appendix 1) and any conjugacy with the corresponding rotation maps a set of zero Lebesgue measure in a set of positive Lebesgue measure.

In the same work Arnol'd showed that any real-analytic diffeomorphism with Diophantine rotation number (again, see Definition A.3) which is a small perturbation of a rigid rotation is conjugate to the corresponding rotation by a real-analytic diffeomorphism. He also conjectured that no restriction on being close to a rotation is needed. This was proved by Herman in 1979 [18] for a large class of Diophantine numbers, and extended by Yoccoz in 1984 [53] for all Diophantine numbers: any $C^{2+\varepsilon}$ diffeomorphism with Diophantine rotation number is conjugate to the corresponding rotation by a smooth diffeomorphism (see [37, Section I.3] for precise statements, see also [25] and the references therein). Moreover, C^∞ diffeomorphisms with the same Diophantine rotation number are C^∞-conjugate, and real-analytic diffeomorphisms with the same Diophantine rotation number are conjugate by a real-analytic diffeomorphism (again, see [37, Section I.3]).

In the remainder of these notes we will discuss certain circle dynamics that belong to the boundary of the space of circle diffeomorphisms, the so-called *critical circle maps*. In Section 6 we will prove an extension of Denjoy's result to critical circle maps due to J.-C. Yoccoz, and in Section 7 we will discuss their geometrical classification.

4. CRITICAL CIRCLE MAPS

By a *critical circle map* we mean an orientation preserving C^3 circle homeomorphism f, with finitely many non-flat critical points $c_1, c_2, \ldots, c_N$ $(N \geq 1)$ of odd type. A critical point c is called *non-flat* if in a neighbourhood of c the map f can be written as $f(t) = \big(\phi(t)\big)^d + f(c)$, where ϕ is a C^3 local diffeomorphism with $\phi(c) = 0$, and $d \in \mathbb{N}$ with $d \geq 2$. The *criticality* (or *order*, or *type*, or *exponent*) of such a critical point c is d.

Critical circle maps have been studied by several authors in the last three decades. From a strictly mathematical viewpoint, these studies started with basic topological aspects [17], [54], then evolved – in the special case of maps with a *single* critical point – to geometric bounds [19], [48], and further to geometric rigidity and renormalization aspects, see [2], [7], [8], [10], [11], [14], [15], [16], [24], [27], [48], [49], [50], [51] and [52]. Such brief account bypasses important numerical studies by several physicists, as well as computer-assisted and conceptual work by Feigenbaum, Kadanoff, Lanford, Rand, Epstein and others; see [8] and references therein.

In Section 5 of these notes we discuss some classical examples of real-analytic critical circle maps. Just as we did before with diffeomorphisms, we will focus on the case when the rotation number of f is irrational, in which case f is uniquely ergodic. In Section 6 we will prove a theorem due to J.-C. Yoccoz which asserts that f is minimal and therefore topologically conjugate to the corresponding rigid rotation (see Corollary 6.2). This result is an extension of Denjoy's theorem, stated and proved in Section 3. Yoccoz's theorem implies that the support of the unique invariant Borel probability measure of a critical circle map with irrational rotation number is the whole circle. Moreover, the analogue statement to Lemma 3.5 also holds for critical circle maps with irrational rotation number. More precisely:

LEMMA 4.1. *Let $f : S^1 \to S^1$ be a C^3 critical circle map with irrational rotation number, and let μ be its unique invariant Borel probability measure. Then $\log Df$ belongs to $L^1(\mu)$ and:*

$$\int_{S^1} \log Df \, d\mu = 0 \, .$$

The proof of Lemma 4.1 is much more difficult than the one of Lemma 3.5, since in this case $\log Df$ is not a continuous function (it is defined only in $S^1 \setminus \{c_1, c_2, \dots, c_N\}$, and it is unbounded). We refer the reader to [9, Theorem A].

5. EXAMPLES

In this section we provide some examples of real-analytic critical circle maps, emphasizing some connections with complex one-dimensional dynamics. We refer to the book of Milnor [38] for background in the area.

5.1. The Arnol'd family

Classical examples of critical circle maps are obtained from the two-parameter family $\widetilde{f}_{a,b} : \mathbb{C} \to \mathbb{C}$ of entire maps in the complex plane:

$$\widetilde{f}_{a,b}(z) = z + a - \left(\frac{b}{2\pi}\right) \operatorname{sen}(2\pi z) \quad \text{for } a \in [0,1) \text{ and } b \geq 0.$$

Since each $\widetilde{f}_{a,b}$ commutes with unitary horizontal translation, it is the lift of a holomorphic map of the punctured plane $f_{a,b} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ under the holomorphic

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

universal cover $z \mapsto e^{2\pi i z}$. Since $\widetilde{f}_{a,b}$ preserves the real axis, $f_{a,b}$ preserves the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. This classical family was introduced by Arnol'd in [1], and it is called the *Arnol'd family*.

For $b = 0$ the family $f_{a,b} : S^1 \to S^1$ is the family of rigid rotations $z \mapsto e^{2\pi i a} z$, and for $b \in (0, 1)$ the family is still contained in the space of real-analytic circle diffeomorphisms.

For $b = 1$ each $\widetilde{f}_{a,b}$ still restricts to an increasing real-analytic homeomorphism of the real line, that projects to an orientation preserving real-analytic circle homeomorphism $f_{a,1}$. Each $f_{a,1}$ presents one critical point of cubic type at the point 1, the projection of the integers.

Denote by $\rho(a)$ the rotation number of the circle homeomorphism $f_{a,1}$. It is well-known that $a \mapsto \rho(a)$ is continuous, non-decreasing and it maps $[0, 1)$ onto itself. The interval $\rho^{-1}(\theta) \subset [0, 1)$ degenerates to a point whenever $\theta \in [0, 1) \setminus \mathbb{Q}$ [18]. Moreover, the set $\{a \in [0, 1) : \rho(a) \in \mathbb{R} \setminus \mathbb{Q}\}$ has zero Lebesgue measure [48].

For $0 \leq p < q$ coprime integers we know that $\rho^{-1}\left(\{\frac{p}{q}\}\right)$ is always a non-degenerate closed interval. In the interior of this interval we find critical circle maps with two periodic orbits (of period q), one attracting and one repelling, which collapse to a single parabolic orbit in the boundary of the interval [6].

For $b > 1$ the maps $f_{a,b} : S^1 \to S^1$ are not invertible any more (they present two quadratic critical points). These examples show how critical circle maps arise as bifurcations from circle diffeomorphisms to endomorphisms, and in particular, from zero to positive topological entropy (compare with infinitely renormalizable unimodal maps [37, Chapter VI]). This is one of the main reasons why critical circle maps attracted the attention of physicists and mathematicians interested in understanding the *boundary of chaos* ([5], [12], [21], [28], [29], [31] [32], [39], [43], [44], [45], [47]).

Remark 5. 1. In the same way one can define a one-parameter family of critical circle maps with $N \geq 1$ cubic critical points:

$$\widetilde{f}_a : \mathbb{R} \to \mathbb{R} \quad \text{given by} \quad \widetilde{f}_a(t) = t + a + \left(\frac{1}{2N\pi}\right) \operatorname{sen}(2N\pi t) \quad \text{for all } a \in [0, 1).$$

As before, each $\widetilde{f}_a$ is an increasing real-analytic homeomorphism of the real line verifying $\widetilde{f}_a(t + 1) = \widetilde{f}_a(t) + 1$ for every $t \in \mathbb{R}$ and every $a \in [0, 1]$, and then it projects under the exponential map to a real-analytic homeomorphism of the circle that preserves orientation, presenting N critical points of cubic type: $\{\exp(2\pi i c_j)\}_{j=1}^{j=N}$, where $c_j = \frac{2j-1}{2N}$ for every $j \in \{1, ..., N\}$.

5.2. Blaschke products

Consider the Blaschke product $f_0 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the Riemann sphere given by:

$$f_0(z) = z^2 \left(\frac{z - 3}{1 - 3z}\right). \tag{5.1}$$

Lemma 5.1. *The rational function f_0 defined in (5.1) preserves the unit circle. Its restriction is a critical circle map with a unique critical point of cubic type, and with rotation number equal to zero.*

Proof (Lemma 5.1). Note that f_0 commutes with the *conformal* involution around the unit circle $z \mapsto 1/z$, and also with the *geometric* involution around the unit circle $\Phi(z) = 1/\bar{z}$ (note that Φ is the identity in the unit circle). This implies that f_0 preserves the unit circle.

The fixed points of f_0 are -1, 0, 1 and the point at infinity. The point -1 is repelling, whereas 0, 1 and ∞ are superattracting. Moreover, 0 and ∞ are quadratic critical points, while 1 is a cubic critical point.

Now we claim that the restriction of f_0 to S^1 is a homeomorphism. Indeed, the topological degree of f_0 in the unit circle is equal to the *index* (or *winding number*) of the closed curve $f_0(S^1)$ around the origin, and this index can be computed with the help of the holomorphic 1-form $\omega(z) = \frac{1}{2\pi i}\frac{dz}{z}$ as follows:

$$\operatorname{index}\big(f_0(S^1), 0\big) = \int_{f_0(S^1)} \omega = \int_{S^1} f_0^*\omega = \frac{1}{2\pi i}\int_{S^1} \frac{Df_0(z)}{f_0(z)}\, dz\,,$$

where $f_0^*\omega$ denotes the pull-back of ω under f_0. By the Argument Principle, the last term is equal to 1, since f_0 has two zeros (both at the origin) and one pole (at $1/3$) in the unit disk. Therefore f_0 has degree one on the unit circle, as claimed. Thus $f_0|_{S^1}$ is a critical circle map with a unique critical point at 1 (which, as we said, is of cubic type). Finally, note that the rotation number of $f_0|_{S^1}$ is zero because it has a fixed point at 1. ∎

Now we consider the one-parameter family $f_\gamma : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of Blaschke products in the Riemann sphere given by:

$$f_\gamma(z) = e^{2\pi i\gamma} z^2 \left(\frac{z-3}{1-3z}\right) = e^{2\pi i\gamma} f_0(z) \quad \text{for } \gamma \in [0,1). \tag{5.2}$$

Every map in this family leaves invariant the unit circle (Blaschke products *are* the rational maps leaving invariant the unit circle), and its restriction to S^1 is a real-analytic critical circle map with a unique critical point at 1, which is of cubic type, and with critical value $e^{2\pi i\gamma}$.

Furthermore, by monotonicity of the rotation number (see for instance [18] and [37]) we know that for each irrational number θ in $[0,1)$ there exists a unique γ in $[0,1)$ such that the rotation number of $f_\gamma|_{S^1}$ is θ. By a theorem of Yoccoz [54] that we will discuss in the next section (see Corollary 6.3) the restriction of f_γ to the unit circle is minimal, and therefore, it is topologically conjugate to the corresponding rigid rotation.

5.2.1. Dynamics in $\widehat{\mathbb{C}}$

Let us (try to) describe the dynamics of f_γ on the Riemann sphere, when $f_\gamma|_{S^1}$ has irrational rotation number. We start with points on the Fatou set F_γ, and we denote by

$\mathcal{B}_0$ and $\mathcal{B}_\infty$ the corresponding immediate basin of attraction of 0 and ∞ (both are simply-connected, one being the image of the other under the geometric involution Φ). There is an open set (compactly contained in the unit disk) around $1/3$ which is mapped by f_γ onto $\mathcal{B}_\infty$, whereas there is an open set (S^1-symmetric with respect to the previous one, and compactly contained in the complement of the unit disk) around 3 which is mapped by f_γ onto $\mathcal{B}_0$ (see Figure 1). We claim that $\mathcal{B}_0$ and $\mathcal{B}_\infty$ are the only periodic components of the Fatou set F_γ. More precisely:

LEMMA 5.2. *Let γ in $[0,1)$ such that the rotation number of $f_\gamma|_{S^1}$ is irrational. Then:*

$$F_\gamma = \bigcup_{n\in\mathbb{N}} \left(f_\gamma^{-n}(\mathcal{B}_0) \cup f_\gamma^{-n}(\mathcal{B}_\infty)\right).$$

In other words, any point in the Fatou set of f_γ converges either to the origin or to infinity under forward iteration.

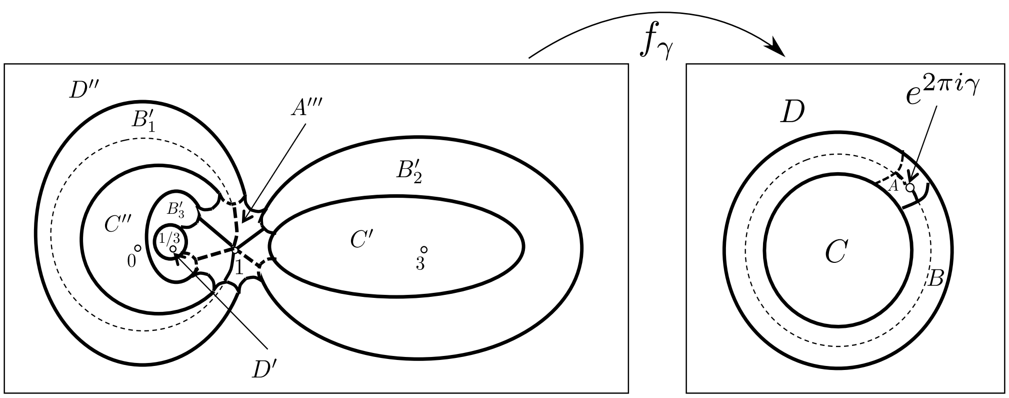

Figure 1: Topological behaviour of the Blaschke product f_γ (5.2) around the unit circle, for γ approximately equal to $1/8$. At the left of Figure 1 we see the preimage under f_γ of the annulus around the unit circle drawn at the right (in both planes, the unit circle is dashed). The complement of the annulus $A \cup B$ in the complex plane has two connected components, C and D. The preimage of C is the union $C' \cup C''$, where the notation C' means that $f_\gamma : C' \to C$ has topological degree 1 (equivalently $f_\gamma : C'' \to C$ has topological degree 2). In the same way, the preimage of D is the union $D' \cup D''$, the preimage of B is $B_1' \cup B_2' \cup B_3'$ and the preimage of A is A'''.

Proof (Lemma 5.2). By Sullivan Nonwandering Theorem [38, Theorem 16.4] it is enough to prove that $\mathcal{B}_0$ and $\mathcal{B}_\infty$ are the only periodic components of F_γ. Note first that, other than 0 and ∞, there are no attracting, superattracting nor parabolic periodic points for f_γ, since their immediate basin of attraction would contain a critical point (besides 0 and ∞, the only critical point of f_γ is at 1, whose positive orbit is dense in the unit circle.). If f_γ admits a Siegel disk, its boundary would be the unit circle (because it must be contained in the closure of the post-critical set [38, Theorem 11.17]) and in that case the Siegel disk would be the unit disk $\mathbb{D}$ or $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, but both domains contain superattractors and this gives us a contradiction. Finally, f_γ have no Herman ring since its boundary would belong to the closure of the post-critical set [38, Lemma 15.7], but there is only one closed curve in it, which is precisely the unit circle. By the classification of periodic components of the Fatou set [38, Theorem 16.1] we are done. ∎

As it is usually the case, the dynamics on the Julia set J_γ are much more difficult to describe. Note that for the original map f_0 we have $S^1 \cap J_0 \neq \emptyset$ (it contains -1) and $S^1 \cap F_0 \neq \emptyset$ (it contains 1), where J_0 and F_0 denote respectively the Julia and Fatou set of f_0 in $\widehat{\mathbb{C}}$. We focus, again, on the case that the rotation number of $f_\gamma|_{S^1}$ is irrational. For each $n \in \mathbb{N}$, the finite union:

$$L_n(\gamma) = \bigcup_{j=1}^{j=n} f_\gamma^{-j}(S^1)$$

is a forward invariant compact and connected set, which is the union of 3^n Jordan curves, one being the unit circle itself and the rest being real-analytic except at one point (which is a preimage under f_γ of the critical point placed at 1). The increasing union:

$$L_\gamma = \bigcup_{n \in \mathbb{N}} L_n(\gamma) = \bigcup_{n=1}^{\infty} f_\gamma^{-n}(S^1)$$

is also a connected set, which is *totally invariant* under f (that is, invariant both for forward and backward iterates).

Lemma 5.3. *Let γ in $[0,1)$ such that the rotation number of $f_\gamma|_{S^1}$ is irrational. Then L_γ is strictly contained in the Julia set J_γ of f_γ, and it is dense in J_γ (in particular L_γ is non compact).*

In other words, Lemma 5.3 gives us $S^1 \subsetneq L_\gamma \subsetneq \overline{L_\gamma} = J_\gamma$.

Proof (Lemma 5.3). Let us prove first that the unit circle is contained in the Julia set: since S^1 is f_γ-invariant and $f_\gamma|_{S^1}$ is minimal, either we have $S^1 \subset J_\gamma$ or $S^1 \cap J_\gamma = \emptyset$. Suppose, by contradiction, that $S^1 \cap J_\gamma = \emptyset$, and let U be the Fatou component of f_γ containing S^1. By the invariance of the unit circle, U is mapped into itself by f_γ, and therefore it must be a Siegel disk or a Herman ring (precisely because it has an invariant simple closed curve on its interior, see [38, Theorem 16.1]). But in that case $f_\gamma : U \to U$ would be a biholomorphism, which is impossible since it has a critical point in the unit circle. Therefore $S^1 \subset J_\gamma$. By the invariance of the Julia set we have $L_\gamma \subset J_\gamma$. The difference $J_\gamma \setminus L_\gamma$ is non-empty because L_γ contains no periodic orbits, which are dense in J_γ [38, Theorem 14.1]. Finally, L_γ is dense in J_γ since the preorbit of any point in the unit circle is dense in J_γ [38, Corollary 4.13]. ▮

Note that the original map f_0 defined in (5.1) is uniformly hyperbolic (Axiom A) in the whole Riemann sphere since all its critical points are fixed points. Therefore, for γ sufficiently close to 0, we also have that f_γ is uniformly hyperbolic (and that $\rho\big(f_\gamma|_{S^1}\big) = 0$). When $\rho\big(f_\gamma|_{S^1}\big)$ is irrational this is no longer true, since by Lemma 5.3 the Julia set contains the critical point at 1. Moreover, for any γ in $[0,1)$ such that the rotation number of $f_\gamma|_{S^1}$ is irrational, Lemma 4.1 gives us a non-atomic Borel probability measure which is f_γ-invariant and ergodic, whose support is contained in the Julia set J_γ (recall that $S^1 \subset J_\gamma$ by Lemma 5.3) and with Lyapunov exponent equal to zero. By the main result of [42], and since there is only one critical point in the Julia set J_γ, we deduce that f_γ does not satisfy any of the standard definitions of non-uniform hyperbolicity for rational maps in the Riemann sphere (Collet-Eckmann condition, topological Collet-Eckmann, uniform hyperbolicity of periodic orbits in the Julia set, etc.). See [9, Section 6] for more details.

Finally, let us point out that with the family (5.2) at hand the developments on rigidity of critical circle maps were very useful in the study of local connectivity and Lebesgue measure of Julia sets associated to generic quadratic polynomials with Siegel disks ([40], [35], [49], [41]).

Remark 5. 2. We can embed the family (5.2) into the two-parameter family $g_{a,\gamma} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of Blaschke products given by:

$$g_{a,\gamma}(z) = e^{2\pi i \gamma} z^2 \left(\frac{z-a}{1-az}\right) \quad \text{for } a \geq 3 \text{ and } \gamma \in [0,1).$$

Note that $g_{3,\gamma} = f_\gamma$ for any $\gamma \in [0,1)$, and in particular $g_{3,0} = f_0$ as defined in (5.1). Again every map in this family commutes with $\Phi(z) = 1/\bar{z}$ and therefore leaves invariant the unit circle. Moreover, its restriction to S^1 is a real-analytic circle homeomorphism (as before, the fact that each $g_{a,\gamma}$ has topological degree one, when restricted to the unit circle, follows from the Argument Principle since $g_{a,\gamma}$ has two zeros and one pole in the unit disk).

For $a > 3$ each $g_{a,\gamma}$ has four critical points in the Riemann sphere which are all different and non-degenerate (quadratic). In particular, the restriction of $g_{a,\gamma}$ to the unit circle is a real-analytic diffeomorphism for any $a > 3$. Indeed, besides 0 and ∞ the others two critical points are:

$$w_a = \frac{a^2+3}{4a} + \frac{\sqrt{(a+3)(a+1)(a-1)(a-3)}}{4a} > 1 \quad \text{and} \tag{5.3}$$

$$1/w_a = \frac{a^2+3}{4a} - \frac{\sqrt{(a+3)(a+1)(a-1)(a-3)}}{4a} \in (0,1)\,. \tag{5.4}$$

Fix any Diophantine number $\theta \in (0,1)$. For any $a \geq 3$ we can choose γ_a such that the rotation number of the restriction $g_{a,\gamma_a}|_{S^1}$ is equal to θ (again by monotonicity of the rotation number).

By Arnol'd-Herman-Yoccoz theory already mentioned at Section 3.2 (see [37, Section I.3] and the references therein) there exists a topological annulus around the unit circle where g_{a,γ_a} is conformally conjugate to the corresponding irrational rotation on a round annulus. Therefore, by Montel theorem, the unit circle is contained in an invariant component of the Fatou set of g_{a,γ_a}, that must be a rotation domain. Note that the unit circle cannot be contained in a Siegel disk, since both critical points 0 and ∞ are fixed points for g_{a,γ_a} and, therefore, superattracting. This implies that the unit circle is contained in a Herman ring of g_{a,γ_a}. Both critical points $w_a > 1$ and $1/w_a \in (0,1)$ belong to the boundary of this Herman ring, each one on a different component, and their future orbits are dense on each corresponding component (see [38, Figure 32, page 164], which shows the Julia set of g_{a,γ_a} for $\theta = (\sqrt{5}-1)/2$ and $a = 4$).

We are interested, however, in the limit case: when $a \to 3$ both critical points $w_a > 1$ and $1/w_a \in (0,1)$ collapse to the critical point of $g_{3,\gamma_3} \equiv f_{\gamma_3}$ placed at 1 (as we can see from (5.3) and (5.4)), which is of cubic type. From the point of view of circle dynamics, when $a \to 3$, the family g_{a,γ_a} converges to the boundary of the space of real-analytic circle diffeomorphisms. Moreover, as we saw in Lemma 5.2 and Lemma 5.3, the Herman ring described above disappear when $a = 3$, and the unit circle belongs now to the Julia set.

6. TOPOLOGICAL RIGIDITY OF CRITICAL CIRCLE MAPS

In 1984, J.-C. Yoccoz [54] extended Denjoy's result to critical circle maps:

THEOREM 6.1 (Yoccoz, 1984). *Let f be a C^1 orientation preserving circle homeomorphism with irrational rotation number θ and with N critical points $c_1, c_2, ..., c_N$. Suppose that:*

- $\log Df$ *has bounded variation in any compact interval that contains no critical points.*
- *Given any $j \in \{1,2,...,N\}$ there exist constants $\varepsilon_j, A_j, B_j > 0$ and $s_j \in \mathbb{N}$ such that:*

  1. *$(Df)^{-1/2}$ is a convex function in $(c_j - \varepsilon_j, c_j)$ and in $(c_j, c_j + \varepsilon_j)$.*
  2. *Given $|t| < \varepsilon_j$ we have that $A_j|t|^{s_j} \leq (Df)(c_j + t) \leq B_j|t|^{s_j}$.*

Then f has no wandering intervals, that is, it is topologically conjugate to the rigid rotation of angle θ.

Note that Theorem 6.1 implies Denjoy's result discussed in Section 3. We have proved Theorem 3.1 independently as a courtesy to the reader. Two other consequences of Theorem 6.1 are the following:

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Corollary 6.2. *Any C^3 orientation preserving circle homeomorphism with irrational rotation number and with only non-flat critical points is topologically conjugate to the corresponding rigid rotation.*

Proof (Corollary 6.2). Since f is C^3, the map $g = (Df)^{-1/2}$ is C^2 away from the critical points of f. Moreover $D^2 g = (-1/2)\, g\, Sf$, where Sf denote the Schwarzian derivative of f. In particular g is a strictly convex function if and only if $Sf < 0$, and a straightforward computation shows that this always happens in a punctured neighbourhood of a non-flat critical point. Property (2.) of Theorem 6.1 also holds around any non-flat critical point, since f is a power-law in suitable local charts (indeed, if ϕ_j is C^1 and $f(c_j + t) = \big(\phi_j(c_j + t)\big)^{d_j} + f(c_j)$ for t close to zero, then $\lim_{t \to 0} \left\{ \dfrac{Df(c_j + t)}{t^{d_j - 1}} \right\} = d_j \big(D\phi_j(c_j)\big)^{d_j}$, which is different from zero since ϕ_j is a local diffeomorphism around c_j). ∎

We remark that the condition of non-flatness on the critical points cannot be removed: in [17] Hall was able to construct C^∞ homeomorphisms of the circle with no periodic points and no dense orbits.

Corollary 6.3. *Any real-analytic orientation preserving circle homeomorphism with irrational rotation number is topologically conjugate to the corresponding rigid rotation.*

Corollary 6.3 is just a particular case of Corollary 6.2, since flat critical points do not exist for non-constant real-analytic maps.

6.1. Degenerated cross-ratio

Given an interval $J \subset S^1$ with boundary points $a \neq b$ we define:

$$M(f, J) = \frac{|f(J)|}{|J|} \big(Df(a) Df(b)\big)^{-1/2},$$

with the convention that $M(f, J) = +\infty$ if a or b are critical points of f.

Let $\varepsilon > 0$ and $J_\varepsilon \supsetneq J$ be such that both components of $J_\varepsilon \setminus J$ (call them L_ε and R_ε) have length ε. Then:

$$\mathrm{Cr}(f, J, J_\varepsilon) = \frac{|f(J)||f(J_\varepsilon)|}{|J|\big(|J| + 2\varepsilon\big) Df(x_\varepsilon) Df(y_\varepsilon)},$$

where Cr is the cross-ratio defined in Section 2.2, and $x_\varepsilon \in L_\varepsilon$ and $y_\varepsilon \in R_\varepsilon$ are given by the Mean Value Theorem. Since f is C^1 we have that:

$$\lim_{\varepsilon \to 0} \mathrm{Cr}(f, J, J_\varepsilon) = \big(M(f, J)\big)^2.$$

For this reason we call M the *degenerated cross-ratio*.

6.2. Yoccoz's proof

The main point in the proof of Theorem 6.1 is the following estimate on the distortion of the degenerated cross-ratio of large iterates (compare with Proposition 3.2).

LEMMA 6.4. *Let f be as in Theorem 6.1. There exists a positive constant $\delta > 0$ such that for any $n \geq 1$, any $p \in \{0, ..., q_{n+1}\}$, any $x \in S^1$ and any interval $J = (a, b)$ contained in the arc $\big(f^{-q_n}(x), f^{q_n}(x)\big)$ we have that $M(f^p, J) \geq \delta$.*

Proof (Lemma 6.4). Following Yoccoz we split the family $\{J, f(J), ..., f^{p-1}(J)\}$ in four disjoint families $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ as follows:

- $\mathcal{F}_1$ contains the intervals $f^i(J)$ that are disjoint of the intervals:

$$\left[c_j - \frac{\varepsilon_j}{2}, c_j + \frac{\varepsilon_j}{2}\right] \quad \text{for any } j \in \{1, ..., N\}.$$

- $\mathcal{F}_2$ contains the intervals $f^i(J)$ that contain some interval of the form:

$$\left[c_j - \varepsilon_j, c_j - \frac{\varepsilon_j}{2}\right]$$

or some interval of the form:

$$\left[c_j + \frac{\varepsilon_j}{2}, c_j + \varepsilon_j\right].$$

Note that $f^i(J)$ may contain the critical point c_j.

- $\mathcal{F}_3$ contains the intervals $f^i(J)$ that are contained on an interval $(c_j - \varepsilon_j, c_j + \varepsilon_j)$, that intersect $[c_j - \varepsilon_j/2, c_j + \varepsilon_j/2]$ but do not contain the critical point c_j.

- $\mathcal{F}_4$ contains the intervals $f^i(J)$ that are contained in some interval $(c_j - \varepsilon_j, c_j + \varepsilon_j)$ and that contain the critical point c_j.

Note that:

$$M(f^p, J) = \prod_{i=0}^{p-1} M\big(f, f^i(J)\big) = \prod_{l=1}^{l=4} \left(\prod_{f^i(J) \in \mathcal{F}_l} M\big(f, f^i(J)\big) \right).$$

Note also that $\#\mathcal{F}_2 \leq 4N$ and $\#\mathcal{F}_4 \leq 2N$ since the family $\{J, f(J), ..., f^{q_{n+1}-1}(J)\}$ has intersection multiplicity 2.

• The intervals in the family $\mathcal{F}_1$ are treated in the same way as in Proposition 3.2: by the Mean Value Theorem for each $f^i(J) \in \mathcal{F}_1$ there exists a point $x_i \in f^i(J)$ such that:

$$\log M\big(f, f^i(J)\big) = \log Df(x_i) - \frac{1}{2}\log Df(f^i(a)) - \frac{1}{2}\log Df(f^i(b))\,.$$

Let $V > 0$ be the total variation of $\log Df$ in the compact set:

$$K = S^1 \setminus \bigcap_{j=1}^{j=N}\left(c_j - \frac{\varepsilon_j}{2}, c_j + \frac{\varepsilon_j}{2}\right).$$

Then:

$$\prod_{f^i(J)\in\mathcal{F}_1} M\big(f, f^i(J)\big) \geq \exp(-V).$$

• Let $\varepsilon = \min\{\varepsilon_1, ..., \varepsilon_N\}$, $D = \max_{z\in S^1}\{Df(z)\}$ and $d = \min_{z\in K}\{Df(z)\} > 0$ with K as above. For any interval $f^i(J)$ in $\mathcal{F}_2$ we have $|f^i(J)| \geq \frac{\varepsilon}{2}$ and so $|f(f^i(J))| \geq \frac{d\varepsilon}{2}$. In particular $M\big(f, f^i(J)\big) \geq \left(\frac{d}{D}\right)\left(\frac{\varepsilon}{2}\right)$. Then:

$$\prod_{f^i(J)\in\mathcal{F}_2} M\big(f, f^i(J)\big) \geq \left(\frac{d}{D}\right)^{4N}\left(\frac{\varepsilon}{2}\right)^{4N}.$$

• Since $(Df)^{-1/2}$ is well defined and convex in any member of $\mathcal{F}_3$ we easily obtain that $M\big(f, f^i(J)\big) \geq 1$ for any $f^i(J) \in \mathcal{F}_3$. Indeed, let g be the affine map that coincide with $(Df)^{-1/2}$ at both points $f^i(a)$ and $f^i(b)$, and note that:

$$M\big(f, f^i(J)\big) = \left(\frac{g\big(f^i(a)\big)\,g\big(f^i(b)\big)}{|f^i(J)|}\right)\left(\int_{f^i(a)}^{f^i(b)} Df(t)dt\right).$$

Since $(Df)^{-1/2}$ is convex, $Df \geq 1/g^2$ between $f^i(a)$ and $f^i(b)$ and then:

$$M\big(f, f^i(J)\big) \geq \left(\frac{g\big(f^i(a)\big)\,g\big(f^i(b)\big)}{f^i(b) - f^i(a)}\right)\left(\int_{f^i(a)}^{f^i(b)} \frac{dt}{g^2(t)}\right) = 1$$

since g is affine. Then:

$$\prod_{f^i(J)\in\mathcal{F}_3} M\big(f, f^i(J)\big) \geq 1\,.$$

• Now we claim that if $f^i(J) \in \mathcal{F}_4$ we have that:

$$M\big(f, f^i(J)\big) \geq \frac{A_j}{2B_j(s_j+1)}\,,$$

where the constants A_j, B_j and s_j are associated with the critical point c_j that belongs to $f^i(J)$. Indeed, let us suppose that $|f^i(a) - c_j| \leq |f^i(b) - c_j|$ and let $I = [c_j, f^i(b)]$. Note that:

$$\begin{aligned} Df\big(f^i(a)\big)\, Df\big(f^i(b)\big) &\leq B_j^2 |f^i(a) - c_j|^{s_j} |f^i(b) - c_j|^{s_j} \\ &\leq B_j^2 |f^i(b) - c_j|^{2s_j} = \left(B_j |I|^{s_j}\right)^2 . \end{aligned}$$

Also:

$$\begin{aligned} |f(I)| &= \int_{c_j}^{f^i(b)} Df(t)dt \geq A_j \int_{c_j}^{f^i(b)} |t - c_j|^{s_j} dt \\ &= A_j \left(\frac{|f^i(b) - c_j|^{s_j+1}}{s_j + 1} \right) = \left(\frac{A_j}{s_j + 1} \right) |I|^{s_j+1}. \end{aligned}$$

Combining these two estimates we obtain:

$$\begin{aligned} M\big(f, f^i(J)\big) &= \frac{|f(f^i(J))|}{|f^i(J)|} \big(Df(f^i(a))\, Df(f^i(b))\big)^{-1/2} \\ &\geq \frac{|f(I)|}{2|I|} \big(Df(f^i(a))\, Df(f^i(b))\big)^{-1/2} \\ &\geq \frac{A_j}{2B_j(s_j + 1)}\, , \quad \text{as claimed.} \end{aligned}$$

In particular if we define:

$$\alpha = \min_{j \in \{1,\dots,N\}} \left\{ \frac{A_j}{2B_j(s_j + 1)} \right\} ,$$

we have that:

$$\prod_{f^i(J) \in \mathcal{F}_4} M\big(f, f^i(J)\big) \geq \alpha^{2N}.$$

We finish the proof by taking $\delta = \exp(-V) \left(\frac{d}{D}\right)^{4N} \left(\frac{\varepsilon}{2}\right)^{4N} \alpha^{2N}$. ∎

Note that the last estimates in the family $\mathcal{F}_4$ are false if we allow flat critical points. With Lemma 6.4 at hand we are ready to prove Theorem 6.1.

Proof (Theorem 6.1). Suppose that there exists a wandering interval $I \subset S^1$: $f^n(I) \cap f^m(I) = \emptyset$ for all $n \neq m \in \mathbb{Z}$.

Fix $n \geq 1$. By the Mean Value Theorem there exist $a \in f^{-q_n - q_{n+1}}(I)$ and $b \in f^{-q_{n+1}}(I)$ such that:

$$Df^{q_{n+1}}(a) = \frac{|f^{-q_n}(I)|}{|f^{-q_n - q_{n+1}}(I)|} \quad \text{and} \quad Df^{q_{n+1}}(b) = \frac{|I|}{|f^{-q_{n+1}}(I)|}\, .$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Let J be the compact interval with boundary points a and b that contains I, and let $x \in I$. By combinatorics, those four intervals are ordered as follows: $f^{-q_n}(I)$, $f^{-q_n-q_{n+1}}(I)$, I, $f^{-q_{n+1}}(I)$ and $f^{q_n}(I)$ (or the opposite depending if n is even or odd). In any case $J \subset \big(f^{-q_n}(x), f^{q_n}(x)\big)$, and so Lemma 6.4 gives us $M(f^{q_{n+1}}, J) \geq \delta$, that is:

$$\left(\frac{|f^{q_{n+1}}(J)|}{|J|}\right)^2 \frac{|f^{-q_n-q_{n+1}}(I)||f^{-q_{n+1}}(I)|}{|f^{-q_n}(I)||I|} \geq \delta^2 .$$

Equivalently:

$$\frac{|f^{-q_{n+1}}(I)|}{|f^{-q_n}(I)|} \geq \left(\frac{|J|}{|f^{q_{n+1}}(J)|}\right)^2 \left(\frac{|I|}{|f^{-q_n-q_{n+1}}(I)|}\right) \delta^2 .$$

Since I is a wandering interval there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have that: $|f^{-q_n-q_{n+1}}(I)| \leq |I|^3\delta^2$. Using that $|f^{q_{n+1}}(J)| \leq 1$ for all $n \in \mathbb{N}$ we obtain:

$$\frac{|f^{-q_{n+1}}(I)|}{|f^{-q_n}(I)|} \geq \left(\frac{|J|}{|I|}\right)^2 > 1$$

since I is stricltly contained in J. This contradicts the fact that the sequence $\big\{|f^{-q_n}(I)|\big\}_{n\in\mathbb{N}}$ goes to zero as n goes to infinity. ▮

Let us point out, finally, that in 1992 Martens, de Melo and van Strien [34] proved that any C^2 map of the circle or any compact interval with only non-flat critical points has no wandering interval. In other words: any open interval for which all positive iterates are mutually disjoint is contained in the basin of a periodic (maybe one-sided) attractor (see also [36] and [37, Chapter IV]).

7. GEOMETRICAL RIGIDITY OF CRITICAL CIRCLE MAPS

Numerical observations ([12], [39], [47]) suggested in the early eighties some strong rigidity for the geometry of the orbits of C^3 critical circle maps with a single critical point and with rotation number of bounded type (see Definition A.2 in Appendix 1). This was posed as a conjecture in several works by Lanford ([28], [29]), Rand ([43], [44] and [45], see also [39]) and Shenker ([47], see also [12]) among others. The most recent results on this area are the following:

Theorem 7.1. *Let f and g be two C^4 circle homeomorphisms with the same irrational rotation number and with a unique critical point of the same odd type. Let h be the unique topological conjugacy between f and g that maps the critical point of f to the critical point of g. Then:*

1. *h is a C^1 diffeomorphism.*
2. *For a full Lebesgue measure set of rotation numbers, h is a $C^{1+\alpha}$ diffeomorphism.*

Therefore, inside each topological class of C^4 critical circle maps, the order of the critical point is the unique invariant of the C^1-conjugacy classes (the topology of the system determines its geometry).

By comparing with what we discussed in Section 3.2 we see that, on one hand, the presence of the critical point gives us more rigidity than in the case of diffeomorphisms: smooth conjugacy is obtained for all irrational rotation numbers, with no Diophantine conditions. On the other hand, there exist examples ([2], [10]) showing that h may not be $C^{1+\alpha}$ in general, even for real-analytic dynamics (see Corollary A.4 in Appendix 1 for the definition of the rotation numbers referred at item (2.) of Theorem 7.1).

In the C^3 class the following is known:

Theorem 7.2. *Any two C^3 critical circle maps with a single critical point, with the same irrational rotation number of bounded type (see Definition A.2 in Appendix 1 of these notes) and with the same odd criticality are conjugate to each other by a $C^{1+\alpha}$ circle diffeomorphism, for some universal $\alpha > 0$.*

We remark that the statement of Theorem 7.2 was actually the precise statement of the *rigidity conjecture* mentioned above. Theorem 7.1 was proved in [15], and Theorem 7.2 was proved in [16] (see also [14]). Both results build on earlier work of Herman, Świątek, de Faria, de Melo, Khmelev, Yampolsky, Khanin and Teplinsky among others (see [19], [48], [7], [8], [10], [11], [49], [50], [51], [52], [27] and [24]). Its proofs are far beyond the scope of these notes.

We finish this section with the following question: let f and g be two orientation preserving C^3 circle homeomorphisms with the same irrational rotation number, and with $N \geq 2$ non-flat critical points of odd type. Denote by $S_f = \{c_1, ..., c_N\}$ the ordered critical set of f, by $S_g = \{c'_1, ..., c'_N\}$ the ordered critical set of g, and suppose that the criticalities of c_i and c'_i are the same for all $i \in \{1, ..., N\}$ (the cubic case is the generic one). Finally, denote by μ_f and μ_g the corresponding unique invariant measures of f and g.

By Yoccoz's result (Theorem 6.1) we know that f and g are topologically conjugate to each other. By elementary reasons, the condition $\mu_f\big([c_i, c_{i+1}]\big) = \mu_g\big([c'_i, c'_{i+1}]\big)$ for all $i \in \{1, ..., N-1\}$ is necessary (and sufficient) in order to have a topological conjugacy between f and g that sends the critical points of f to the critical points of g. Under this assumption, it turns out that this conjugacy is in fact a *quasisymmetric* homeomorphism (this follows from a recent general result of Clark and van Strien [3]).

Question 7.3. *Is this conjugacy a smooth diffeomorphism?*

To the best of our knowledge, Question (7.3) remains completely open.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

APPENDIX: CONTINUED FRACTIONS

Due to its importance for this subject (and in particular for these notes), we briefly review in this appendix some classical facts about approximations of irrational numbers by continued fractions, mostly from a probabilistic viewpoint. For any positive number θ denote by $\lfloor\theta\rfloor$ the *integer part* of θ:

$$\lfloor\theta\rfloor \in \mathbb{N} \quad \text{and} \quad \lfloor\theta\rfloor \leq \theta < \lfloor\theta\rfloor + 1\,.$$

Define the *Gauss map* $G : [0,1] \to [0,1]$ by:

$$G(\theta) = \frac{1}{\theta} - \left\lfloor\frac{1}{\theta}\right\rfloor \quad \text{for} \quad \theta \neq 0\,, \quad \text{and} \quad G(0) = 0\,.$$

For $k \geq 1$ consider $I_k = \left(\frac{1}{k+1}, \frac{1}{k}\right)$. Then G is an expanding orientation reversing real-analytic diffeomorphism between each I_k and $(0,1)$, and the union $\bigcup_{k\geq 1} I_k$ is a Markov partition for G. After a well-known folklore theorem in one-dimensional dynamics (see [33, Chapter III, Theorem 1.2] or [37, Chapter V, Theorem 2.2]) the map G has a unique[1] invariant ergodic Borel probability ν (called the *Gauss measure*) which is equivalent to the Lebesgue measure Leb in $[0,1]$ (they share the same null sets). A straightforward computation shows that for any Borel set $A \subset [0,1]$ we have:

$$\nu(A) = \left(\frac{1}{\log 2}\right) \int_A \left(\frac{1}{1+\theta}\right) d\,\mathrm{Leb}\,.$$

Both $\mathbb{Q} \cap [0,1]$ and $[0,1] \setminus \mathbb{Q}$ are G-invariant. Under the action of G, all rational numbers in $[0,1]$ eventually land on the fix point at the origin, while the irrationals remain in the union $\bigcup_{k\geq 1} I_k$.

Definition A.1. The *continued fraction expansion* of an irrational number in $[0,1]$ is the sequence given by its itinerary under G according to the partition $\bigcup_{k\geq 1} I_k$.

More precisely, to any irrational number θ in $[0,1]$ we associate the sequence $\{a_n\}_{n\in\mathbb{N}}$ defined by $G^n(\theta) \in I_{a_n}$ for all $n \in \mathbb{N}$, that is:

$$a_n = \left\lfloor\frac{1}{G^n(\theta)}\right\rfloor \quad \text{for all} \quad n \in \mathbb{N}\,.$$

Since G is expanding on the Markov partition $\bigcup_{k\geq 1} I_k$, the map h from $[0,1] \setminus \mathbb{Q}$ to $\mathbb{N}^{\mathbb{N}}$ (endowed with the product topology) that associates any irrational number to its itinerary

[1] The map G has infinitely many invariant ergodic Borel probabilities since it has infinitely many periodic orbits (dense in $[0,1]$). Uniqueness comes from the equivalence with respect to Lebesgue measure.

is a well-defined homeomorphism. Therefore, the action of G on $[0,1]\setminus\mathbb{Q}$ is topologically conjugate to the left shift $\sigma:\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ that sends $\{a_n\}_{n\in\mathbb{N}}$ to $\{a_{n+1}\}_{n\in\mathbb{N}}$:

$$\begin{array}{ccc} [0,1]\setminus\mathbb{Q} & \xrightarrow{\;G\;} & [0,1]\setminus\mathbb{Q} \\ \Big\downarrow h & & \Big\downarrow h \\ \mathbb{N}^{\mathbb{N}} & \xrightarrow{\quad\sigma\quad} & \mathbb{N}^{\mathbb{N}} \end{array}$$

We will use the classical notation $\theta=\big[a_0,a_1,...,a_n,a_{n+1},...\big]$. The natural numbers a_n are called the *partial quotients* of θ.

Definition A.2. A real number θ is of *bounded type* if there exists a constant $M>0$ such that $a_n<M$ for all $n\in\mathbb{N}$.

Since periodic orbits of σ are dense in $\mathbb{N}^{\mathbb{N}}$ (endowed with the product topology), irrational numbers with periodic continued fraction expansion are dense in $[0,1]$, and therefore bounded type numbers are dense in $[0,1]$. On the other hand:

Lemma A.1. *The set of numbers of bounded type has zero Lebesgue measure in* $[0,1]$.

Proof (Lemma A.1). Consider the increasing sequence $\{K_m\}_{m\in\mathbb{N}}$ of Cantor sets in $[0,1]$ defined by:

$$K_m=\big\{\theta\in[0,1]\setminus\mathbb{Q}\;:\;\theta=[a_0,a_1,...]\quad\text{with}\quad a_n<m\quad\text{for all}\quad n\in\mathbb{N}\big\}.$$

It is enough to prove that $\operatorname{Leb}\big(K_m\big)=0$ for each $m\in\mathbb{N}$, and since Gauss measure is equivalent with Lebesgue, it is enough to prove that $\nu\big(K_m\big)=0$ for each $m\in\mathbb{N}$. But this follows from the ergodicity of ν under G, since each K_m is G-invariant and contained in $\big(\frac{1}{m},1\big)$. ∎

For the classical proof of Lemma A.1, with no dynamical arguments, we refer the reader to [26, Chapter III, Theorem 29]. Birkhoff's Ergodic Theorem [33, Chapter II, Theorem 1.1] gives us a much more precise statement:

Theorem A.2. *For Lebesgue almost every θ in $[0,1]$ we have that every integer $k\geq 1$ must appear infinitely many times in the continued fraction expansion of $\theta=\big[a_0,a_1,...\big]$. Moreover if we define:*

$$\tau_n(\theta,k)=\left(\frac{1}{n}\right)\#\big\{0\leq j<n:a_j=k\big\}\,,$$

we have that $\big\{\tau_n(\theta,k)\big\}_{n\in\mathbb{N}}$ converges to the positive value:

$$\left(\frac{1}{\log 2}\right)\log\left(\frac{(k+1)^2}{k(k+2)}\right),$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

that only depends on k*.*

Proof (Theorem A.2). By definition of the continued fraction expansion we have:

$$\tau_n(\theta, k) = \left(\frac{1}{n}\right) \#\{0 \leq j < n : G^j(\theta) \in I_k\}\,.$$

By Birkhoff's Ergodic Theorem:

$$\lim_{n \to +\infty} \tau_n(\theta, k) = \nu(I_k) = \left(\frac{1}{\log 2}\right) \log\left(\frac{(k+1)^2}{k(k+2)}\right), \quad \text{for } \nu \text{ almost every } \theta \text{ in } [0,1].$$

Again by equivalence this is true for Lebesgue almost every θ in $[0,1]$. ∎

Since the asymptotic frequency given by Theorem A.2 is strictly decreasing in k, one should expect that *typical* numbers, even having unbounded partial quotients, have slow growth:

Lemma A.3. *Let* $\{b_n\}_{n\in\mathbb{N}}$ *be any increasing sequence of positive real numbers such that* $\sum_{n\in\mathbb{N}} 1/b_n < \infty$*. For Lebesgue almost every* $\theta = [a_0, a_1, \ldots]$ *in* $[0,1]$ *we have* $a_n < b_n$ *for all* n *large enough.*

Proof (Lemma A.3). For each $n \in \mathbb{N}$ let:

$$U_n = \{\theta : a_n > b_n\} \quad \text{and} \quad V_n = \{\theta : a_0 > b_n\}$$

We want to prove that:

$$\mathrm{Leb}\left(\bigcap_{k\in\mathbb{N}} \bigcup_{n \geq k} U_n\right) = 0$$

Since $G^{-n}(V_n) = U_n$ and $V_n \subset \left(0, \frac{1}{b_n}\right)$ we get:

$$\nu(U_n) \leq \nu\left(0, \frac{1}{b_n}\right) = \left(\frac{1}{\log 2}\right) \log\left(1 + \frac{1}{b_n}\right) \leq \left(\frac{1}{\log 2}\right)\left(\frac{1}{b_n}\right)$$

for all n large enough. In particular:

$$\sum_{n\in\mathbb{N}} \nu(U_n) < \infty$$

and therefore the claim follows by Borel-Cantelli Lemma and the equivalence between Gauss and Lebesgue measures. ∎

Corollary A.4. *For Lebesgue almost every $\theta = [a_0, a_1, ..., a_n, ...]$ in $[0,1]$ we have:*

$$\lim_{n\to\infty} \frac{1}{n}\log a_n = 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log a_j < \infty.$$

We also have:

$$\frac{1}{n}\sum_{j=k+1}^{k+n} \log a_j \leq C(\theta)\left[1 - \log\left(\frac{n}{k}\right)\right] \quad \text{for all} \quad 0 < n \leq k,$$

where $C(\theta) > 0$ depends on θ.

As mentioned in Section 7, the set of numbers given by Corollary A.4 was introduced by de Faria and de Melo in [10, Section 4.4, page 361], and is precisely the set referred in the statement of Theorem 7.1, item (2.). The first and second condition in Corollary A.4 follow straightforward from Lemma A.3 by taking, say, $b_n = n^{1+\varepsilon}$ for any $\varepsilon > 0$. The fact that the third condition holds Lebesgue almost everywhere also follows from Lemma A.3, but with more involved arguments [10, Proposition C.2, page 390].

We finish this appendix with the definitions of Diophantine and Liouville numbers.

Definition A.3. An irrational number in $[0,1]$ is said to be *Diophantine* if there exist constants $C > 0$ and $\delta \geq 0$ such that:

$$\left|\theta - \frac{p}{q}\right| \geq \frac{C}{q^{2+\delta}}\,,$$

for any natural numbers p and $q \neq 0$. Irrational numbers which are not Diophantine are called *Liouville* numbers.

An irrational number is of bounded type if it satisfies Definition A.3 for $\delta = 0$, that is, θ in $[0,1]$ is of bounded type if there exists $C > 0$ such that:

$$\left|\theta - \frac{p}{q}\right| \geq \frac{C}{q^{2}}\,,$$

for any natural numbers p and $q \neq 0$ (for the equivalence between this definition and Definition A.2 see [26, Chapter II, Theorem 23]). As we saw in Lemma A.1 the set of numbers of bounded type has zero Lebesgue measure. However for any small $\delta > 0$ in Definition A.3 we capture Lebesgue almost every real number in $[0,1]$:

Lemma A.5. *Given any $\delta > 0$ the set:*

$$D_\delta = \left\{\theta \in [0,1] : \exists\, C > 0 \quad \text{such that} \quad \left|\theta - \frac{p}{q}\right| \geq \frac{C}{q^{2+\delta}} \quad \forall\, p, q \in \mathbb{N}\right\}$$

has full Lebesgue measure in $[0,1]$*. In particular the set of Diophantine numbers in* $[0,1]$ *has full Lebesgue measure.*

Proof (Lemma A.5). Fix some decreasing sequence $\{C_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that $C_n \to 0$ as $n \to +\infty$, and consider:

$$U_n = \left\{\theta \in [0,1] : \exists\, p, q \in \mathbb{N} \quad \text{such that} \quad \left|\theta - \frac{p}{q}\right| < \frac{C_n}{q^{2+\delta}}\right\}.$$

Note that $\{U_n\}_{n\in\mathbb{N}}$ is a decreasing sequence and:

$$\bigcap_{n\in\mathbb{N}} U_n = [0,1] \setminus D_\delta.$$

Therefore it is enough to prove that $\lim_{n\to+\infty} \operatorname{Leb}(U_n) = 0$. With this purpose fix $n \in \mathbb{N}$ and consider for any $q \in \mathbb{N} \setminus \{0\}$:

$$U_n(q) = \left\{\theta \in [0,1] : \exists\, p \in \{0,1,...,q-1,q\} \quad \text{such that} \quad \left|\theta - \frac{p}{q}\right| < \frac{C_n}{q^{2+\delta}}\right\}.$$

Since:

$$U_n = \bigcup_{q\in\mathbb{N}\setminus\{0\}} U_n(q) \quad \text{and} \quad \operatorname{Leb}\big(U_n(q)\big) = \frac{2C_n}{q^{1+\delta}},$$

we obtain:

$$\operatorname{Leb}(U_n) \leq 2C_n \left(\sum_{q\in\mathbb{N}\setminus\{0\}} \frac{1}{q^{1+\delta}}\right)$$

and this goes to zero as n goes to infinity by the choice of $\{C_n\}_{n\in\mathbb{N}}$ and the fact that $\delta > 0$. ∎

Incidentally we have proved that $[0,1] \setminus D_\delta$ is a residual set, in the Baire sense, since each U_n is open and dense in $[0,1]$. This proves, with the obvious adaptations, that the set of Liouville numbers is a residual set in $[0,1]$, and in particular is uncountable and dense in $[0,1]$.

REFERENCES

1. Arnol'd, V. I., Small denominators I. Mappings of the circle onto itself, *Izv. Akad. Nauk. Math. Series*, **25**, 21-86, (1961) [*Translations of the Amer. Math. Soc. (series 2)*, **46**, 213-284, (1965)].
2. Avila, A., On rigidity of critical circle maps, *Bull. Braz. Math. Soc.*, **44**, 611-619, (2013).
3. Clark, T., van Strien, S., Quasisymmetric rigidity in one-dimensional dynamics, *manuscript*.
4. Denjoy, A., Sur les courbes définies par les équations différentielles à la surface du tore, *J. Math. Pure et Appl.*, **11**, 333-375, (1932).

5. Dixon, T. W., Gherghetta, T., Kenny, B.G., Universality in the quasiperiodic route to chaos, *Chaos*, **6**, 32-42, (1996).
6. Epstein, A., Keen, L., Tresser, C., The set of maps $F_{a,b} : x \mapsto x + a + \frac{b}{2\pi}\sin(2\pi x)$ with any given rotation interval is contractible, *Commun. Math. Phys.*, **173**, 313-333, (1995).
7. de Faria, E., Proof of universality for critical circle mappings, *PhD Thesis*, CUNY, 1992.
8. de Faria, E., Asymptotic rigidity of scaling ratios for critical circle mappings, *Ergod. Th. & Dynam. Sys.*, **19**, 995-1035, (1999).
9. de Faria, E., Guarino, P., Real bounds and Lyapunov exponents, to appear in *Disc. and Cont. Dyn. Sys. A*, available at `arXiv:1506.01087`.
10. de Faria, E., de Melo, W., Rigidity of critical circle mappings I, *J. Eur. Math. Soc.*, **1**, 339-392, (1999).
11. de Faria, E., de Melo, W., Rigidity of critical circle mappings II, *J. Amer. Math. Soc.*, **13**, 343-370, (2000).
12. Feigenbaum, M., Kadanoff, L., Shenker, S., Quasi-periodicity in dissipative systems. A renormalization group analysis, *Physica*, **5D**, 370-386, (1982).
13. Graczyk, J., Świątek, G., Critical circle maps near bifurcation, *Commun. Math. Phys.*, **176**, 227-260, (1996).
14. Guarino, P., *Rigidity conjecture for C^3 critical circle maps*, Ph.D. thesis IMPA (2012), available at `www.preprint.impa.br`.
15. Guarino, P., Martens, M., de Melo, W., Rigidity of critical circle maps: unbounded combinatorics, in preparation.
16. Guarino, P., de Melo, W., Rigidity of smooth critical circle maps, available at `arXiv:1303.3470`.
17. Hall, G. R., A C^∞ Denjoy counterexample, *Ergod. Th. & Dynam. Sys.*, **1**, 261-272, (1981).
18. Herman, M., Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, *Publ. Math. IHES*, **49**, 5-234, (1979).
19. Herman, M., Conjugaison quasi-simétrique des homéomorphismes du cercle à des rotations (manuscript), (1988).
20. Hu, J., Sullivan, D., Topological conjugacy of circle diffeomorphisms, *Ergod. Th. & Dynam. Sys.*, **17**, 173-186, (1997).
21. Kadanoff, L., Shenker, S., Critical behavior of a KAM surface. I. Empirical results, *J. Statist. Phys.*, **27**, 631-656, (1982).
22. Katznelson, Y., Ornstein, D., The differentiability of the conjugation of certain diffeomorphisms of the circle, *Ergod. Th. & Dynam. Sys.*, **9**, 643-680, (1989).
23. Katznelson, Y., Ornstein, D., The absolute continuity of the conjugation of certain diffeomorphisms of the circle, *Ergod. Th. & Dynam. Sys.*, **9**, 681-690, (1989).
24. Khanin, K., Teplinsky, A., Robust rigidity for circle diffeomorphisms with singularities, *Invent. Math.*, **169**, 193-218, (2007).
25. Khanin, K., Teplinsky, A., Herman's theory revisited, *Invent. Math.*, **178**, 333-344, (2009).
26. Khinchin, A. Ya., *Continued fractions*, (reprint of the 1964 translation), Dover Publications, Inc., 1997.
27. Khmelev, D., Yampolsky, M., The rigidity problem for analytic critical circle maps, *Mosc. Math. J.*, **6**, 317-351, (2006).
28. Lanford, O. E., Renormalization group methods for critical circle mappings with general rotation number, *VIIIth Int. Congress on Mathematical Physics (Marseille, 1986)*, World Scientific, Singapore, 532-536, (1987).
29. Lanford, O. E., Renormalization group methods for circle mappings, *Nonlinear Evolution and Chaotic Phenomena (NATO Adv. Sci. Inst. Ser. B: Phys., 176)*, Plenum, New York, 25-36, (1988).
30. Lang, S., *Introduction to diophantine approximations*, (new expanded edition), Springer-Verlag, (1995).

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

31. MacKay, R. S., A renormalisation approach to invariant circles in area-preserving maps, *Physica*, **7D**, 283-300, (1983).

32. MacKay, R. S., *Renormalisation in Area-Preserving Maps*, World-Scientific, 1993.

33. Mañé, R., *Ergodic theory and differentiable dynamics*, Springer-Verlag, 1987.

34. Martens, M., de Melo, W., van Strien, S., Julia-Fatou-Sullivan theory for real one-dimensional dynamics, *Acta Math.*, **168**, 273-318, (1992).

35. McMullen, C.T., Self-similarity of Siegel disks and Hausdorff dimension of Julia sets, *Acta Math.*, **180**, 247-292, (1998).

36. de Melo, W., van Strien, S., A structure theorem in one-dimensional dynamics, *Annals of Math.*, **129**, 519-546, (1989).

37. de Melo, W., van Strien, S., *One-dimensional dynamics*, Springer-Verlag, 1993.

38. Milnor, J., *Dynamics in One Complex Variable*, Annals of Math. Studies, Princeton University Press, **160**, 2006.

39. Ostlund, S., Rand, D., Sethna, J., Siggia, E., Universal properties of the transition from quasi-periodicity to chaos in dissipative systems, *Physica*, **8D**, 303-342, (1983).

40. Petersen, C. L., Local connectivity of some Julia sets containing a circle with an irrational rotation, *Acta Math.*, **177**, 163-224, (1996).

41. Petersen, C. L., Zakeri, S., On the Julia set of a typical quadratic polynomial with a Siegel disk, *Annals of Math.*, **159**, 1-52, (2004).

42. Przytycki, F., Rivera-Letelier, J., Smirnov, S., Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps, *Invent. Math.*, **151**, 29-63, (2003).

43. Rand, D., Universality and renormalisation in dynamical systems, *New Directions in Dynamical Systems*, Eds. T. Bedford and J.W. Swift. Cambridge University Press, Cambridge, (1987).

44. Rand, D., Global phase space universality, smooth conjugacies and renormalisation: I. The $C^{1+\alpha}$ case, *Nonlinearity*, **1**, 181-202, (1988).

45. Rand, D., Existence, non-existence and universal breakdown of dissipative golden invariant tori: I. Golden critical circle maps, *Nonlinearity*, **5**, 639-662, (1992).

46. Schwartz, A. J., A generalization of a Poincaré-Bendixson theorem to closed two-dimensional manifolds, *Amer. J. Math.*, **85**, 453-458, (1963).

47. Shenker, S., Scaling behavior in a map of a circle onto itself: empirical results, *Physica*, **5D**, 405-411, (1982).

48. Świątek, G., Rational rotation numbers for maps of the circle, *Commun. Math. Phys.*, **119**, 109-128, (1988).

49. Yampolsky, M., Complex bounds for renormalization of critical circle maps, *Ergod. Th. & Dynam. Sys.*, **19**, 227-257, (1999).

50. Yampolsky, M., The attractor of renormalization and rigidity of towers of critical circle maps, *Commun. Math. Phys.*, **218**, 537-568, (2001).

51. Yampolsky, M., Hyperbolicity of renormalization of critical circle maps, *Publ. Math. IHES*, **96**, 1-41, (2002).

52. Yampolsky, M., Renormalization horseshoe for critical circle maps, *Commun. Math. Phys.*, **240**, 75-96, (2003).

53. Yoccoz, J.-C., Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne, *Ann. scient. Éc. Norm. Sup.*, **17**, 333-359, (1984).

54. Yoccoz, J.-C., Il n'y a pas de contre-exemple de Denjoy analytique, *C.R. Acad. Sc. Paris*, **298**, 141-144, (1984).

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC