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Some recent results about the geometry of generalized flag manifolds

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To the memory of Professor Carlos Gutierrez

In this paper we survey some recent results about variational results on generalized flag manifolds: stability of harmonic maps, geodesics and Plücker formulae. These results are part of the author's PhD Thesis and received *"Menção Honrosa"* in the Carlos Gutierrez Prize 2012. May, 2015 ICMC-USP

1. INTRODUCTION

In this paper we study aspects of differential geometry in a class of homogeneous spaces called generalized flag manifolds (or Kähler C-spaces). This class of homogeneous spaces is defined taking the quotient G/P of a complex simple non-compact Lie group G by the normalizer of a parabolic sub-algebra $\mathfrak{p}$ of the Lie algebra $\mathfrak{g} = Lie(G)$. Equivalently, a generalized flag manifold is defined as U/K, where U is the maximal compact sub-group of G and $K = P \cap U$ is a centralizer of a torus. It is well known that generalized flag manifolds have a rich Riemannian and Hermitian geometry (see for instance [2]).

This paper deals with variational issues on classical subjects in Riemannian geometry: harmonic maps/minimal surfaces and geodesics.

We describe some recent results obtained in [10] and [15] concerning a special class of harmonic/minimal surfaces on several type of flag manifolds. We study the stability phenomena for a family of harmonic and holomorphic maps on flag manifolds. We also derive the so called Plücker formulae for full flag manifolds.

Plücker formulae for curves in projective spaces give a relationship between an intrinsic invariant (the genus of the curve) and a set of extrinsic invariants like the associated degrees

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and the ramification indices (see Griffiths-Harris [8]). In this paper we consider, instead of projective spaces, generalized flag manifolds of a complex semi-simple Lie group.

We also describe some results about *homogeneous equigeodesics*, namely homogeneous curves γ which are geodesic with respect to *each* G-invariant metric on the flag manifold. We discuss an algebraic approach in order to study equigeodesics on $SU(n)$-flags and flags manifolds with two isotropy summands. These results are proved in [5] and [9].

2. GENERALIZED FLAG MANIFOLDS

The purpose of this section is to fix notation and to state general results concerning to flag manifolds.

2.1. Flag manifolds

Let $\mathfrak{g}$ be a complex simple Lie algebra and take a Lie group G with Lie algebra $\mathfrak{g}$. Given a Cartan sub-algebra $\mathfrak{h}$ of $\mathfrak{g}$, denote by Π the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$, so that

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha, \tag{2.1}$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}; \forall H \in \mathfrak{h}, [H, X] = \alpha(H)X\}$ denotes the corresponding complex one-dimensional root space.

We denote by $\langle \cdot, \cdot \rangle$ the Cartan-Killing form of $\mathfrak{g}$ and fix once and for all a Weyl basis of $\mathfrak{g}$ which amounts to take $X_\alpha \in \mathfrak{g}_\alpha$ such that $\langle X_\alpha, X_{-\alpha} \rangle = 1$, and $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$ with $m_{\alpha,\beta} \in \mathbb{R}$, $m_{-\alpha,-\beta} = -m_{\alpha,\beta}$ and $m_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root (see Helgason [11], chapter IX).

Recall that $\langle \cdot, \cdot \rangle$ is non-degenerate on $\mathfrak{h}$. Given $\alpha \in \mathfrak{h}^*$ we let H_α be given by $\alpha(\cdot) = \langle H_\alpha, \cdot \rangle$, and denote by $\mathfrak{h}_\mathbb{R}$ the real subspace spanned by H_α, $\alpha \in \Pi$. Accordingly $\mathfrak{h}_\mathbb{R}^*$ stands for the real subspace of the dual $\mathfrak{g}^*$ spanned by the roots.

Let Π^+ be a choice of positive roots and Σ the corresponding set of simple roots. If Θ is a subset of Σ we put $\langle \Theta \rangle$ for the set of roots spanned by Θ, and $\langle \Theta \rangle^\pm := \langle \Theta \rangle \cap \Pi^\pm$. We have

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \langle \Theta \rangle^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \langle \Theta \rangle^+} \mathfrak{g}_{-\alpha} \oplus \sum_{\beta \in \Pi^+ \setminus \langle \Theta \rangle} \mathfrak{g}_\beta \oplus \sum_{\beta \in \Pi^+ \setminus \langle \Theta \rangle} \mathfrak{g}_{-\beta}. \tag{2.2}$$

Let

$$\mathfrak{p}_\Theta = \mathfrak{h} \oplus \sum_{\alpha \in \langle \Theta \rangle^-} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha \tag{2.3}$$

be the standard parabolic subalgebra determined by Θ. Put

$$\mathfrak{q}_\Theta = \sum_{\beta \in \Pi^+ \setminus \langle \Theta \rangle} \mathfrak{g}_{-\beta} \tag{2.4}$$

so that $\mathfrak{g} = \mathfrak{q}_\Theta \oplus \mathfrak{p}_\Theta$.

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The generalized flag manifold $\mathbb{F}_\Theta$ associated to $\mathfrak{p}_\Theta$ is defined as the homogeneous space

$$\mathbb{F}_\Theta = G/P_\Theta, \tag{2.5}$$

where P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G.

We take as compact real form of $\mathfrak{g}$ the real subalgebra

$$\mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{i\mathfrak{h}_{\mathbb{R}}, A_\alpha, iS_\alpha : \alpha \in \Pi\}$$

where $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = X_\alpha + X_{-\alpha}$. Denote by $U = \exp \mathfrak{u}$ the corresponding compact real form of G and write $K_\Theta = P_\Theta \cap U$. It is well known that U acts transitively on each $\mathbb{F}_\Theta$, that identifies with U/K_Θ.

Let $\mathfrak{k}_\Theta$ be the Lie algebra of K_Θ and write $\mathfrak{k}_\Theta^{\mathbb{C}}$ for its complexification. We have $\mathfrak{k}_\Theta = \mathfrak{u} \cap \mathfrak{p}_\Theta$ and

$$\mathfrak{k}_\Theta^{\mathbb{C}} = \mathfrak{h} \oplus \sum_{\alpha \in \langle\Theta\rangle} \mathfrak{g}_\alpha.$$

Denote by $o = eK_\Theta$ the origin of $\mathbb{F}_\Theta$. The tangent space $T_o\mathbb{F}_\Theta$ can be identified with the orthogonal complement of $\mathfrak{k}_\Theta$ in $\mathfrak{u}$, namely

$$T_o\mathbb{F}_\Theta = \mathfrak{m}_\Theta = \mathrm{span}_{\mathbb{R}}\{A_\alpha, iS_\alpha : \alpha \notin \langle\Theta\rangle\} = \sum_{\alpha \in \Pi \setminus \langle\Theta\rangle} \mathfrak{u}_\alpha,$$

where $\mathfrak{u}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{A_\alpha, iS_\alpha\}$. By complexifying $\mathfrak{m}_\Theta$ we obtain the complex tangent space of $T_o^{\mathbb{C}}\mathbb{F}_\Theta$, which can be identified with

$$\mathfrak{m}_\Theta^{\mathbb{C}} = \mathfrak{q}_\Theta = \sum_{\beta \in \Pi \setminus \langle\Theta\rangle} \mathfrak{g}_\beta.$$

The adjoint representations of $\mathfrak{k}_\Theta$ and K_Θ leave $\mathfrak{m}_\Theta$ invariant, so that we get a well defined representation of both $\mathfrak{k}_\Theta$ and K_Θ in $\mathfrak{m}_\Theta$. Analogously the complex tangent space $\mathfrak{q}_\Theta$ is invariant under the adjoint representation of $\mathfrak{k}_\Theta^{\mathbb{C}}$. This representation is semi-simple and hence decomposes into irreducible components (or $\mathrm{ad}(\mathfrak{k}_\Theta)$ sub-modules), each one is a sum of root spaces. In the sequel we write an irreducible component as $\mathfrak{m}_\sigma$, where σ is the set of roots α such that $\mathfrak{g}_\alpha \subset \mathfrak{m}_\sigma$, so that

$$\mathfrak{m}_\sigma = \sum_{\alpha \in \sigma} \mathfrak{g}_\alpha.$$

Also we write $\Pi(\Theta)$ for the collection of the sets σ giving rise to an irreducible component. With this notation we have

$$\mathfrak{q}_\Theta = \bigoplus_{\sigma \in \Pi(\Theta)} \mathfrak{m}_\sigma.$$

It is a standard fact that the roots appearing in an irreducible component $\sigma \in \Pi(\Theta)$ are either all positive or all negative. Hence it makes sense to write $\Pi(\Theta)^+$ and $\Pi(\Theta)^-$

for the set of those irreducible components containing positive roots and negative roots respectively. We denote by

$$\Sigma(\Theta) = \{\sigma \in \Pi(\Theta); \text{the height of } \sigma \text{ in } \Pi(\Theta) \text{ is } 1\}.$$

Each $\sigma \in \Pi(\Theta)$ defines a complex plane field on $\mathbb{F}_\Theta$ by

$$E_\sigma (k \cdot o) = k_* (\mathfrak{m}_\sigma),$$

which is well defined since $\mathrm{Ad}(k)(\mathfrak{m}_\sigma) = \mathfrak{m}_\sigma$, $\sigma \in \Pi(\Theta)$. Clearly, for any $x \in \mathbb{F}_\Theta$, we have

$$T_x^{\mathbb{C}} \mathbb{F}_\Theta = \sum_{\sigma \in \Pi(\Theta)} E_\sigma (x).$$

2.2. Almost complex structures

A U-invariant almost complex structure J on $\mathbb{F}_\Theta$ is completely determined by its value $J : \mathfrak{m}_\Theta \to \mathfrak{m}_\Theta$ in its tangent space at the origin. The map J satisfies $J^2 = -1$ and commutes with the adjoint action of K_Θ on $\mathfrak{m}_\Theta$. We also denote by J its complexification to $\mathfrak{q}_\Theta$.

The invariance of J entails that $J(\mathfrak{m}_\sigma) = \mathfrak{m}_\sigma$ for all $\sigma \in \Pi(\Theta)$. The eigenvalues of J are $\pm\sqrt{-1}$ and the eigenvectors in $\mathfrak{q}_\Theta$ are X_α, $\alpha \in \Pi_\Theta$. Hence, in each irreducible component $\mathfrak{m}_\sigma$, we have $J = \sqrt{-1}\epsilon_\sigma \mathrm{id}$ with $\epsilon_\sigma = \pm 1$ satisfying $\epsilon_{-\sigma} = -\epsilon_\sigma$. A U-invariant structure on $\mathbb{F}_\Theta$ is completely determined by the numbers $\varepsilon_\sigma = \pm 1$, $\sigma \in \Pi(\Theta)$.

As usual the eigenvectors associated to $\sqrt{-1}$ are said to be of type $(1,0)$ while the $-\sqrt{-1}$ eigenvectors are of type $(0,1)$. Thus the $(1,0)$ vectors at the origin are multiples of X_α, $\epsilon_\alpha = 1$, and the $(0,1)$ vectors are also multiples of X_α, $\epsilon_\alpha = -1$. Also,

$$T_x \mathbb{F}_\Theta^{(1,0)} = \sum_{\sigma \in \Pi(\Theta)^+} E_{\sigma \varepsilon_\sigma} (x) \qquad T_x \mathbb{F}_\Theta^{(0,1)} = \sum_{\sigma \in \Pi(\Theta)^+} E_{\sigma \varepsilon_{-\sigma}} (x) \tag{2.6}$$

Since $\mathbb{F}_\Theta$ is a homogeneous space of a complex Lie group it has a natural structure of a complex manifold. The associated integrable almost complex structure J_C is given by $\epsilon_\sigma = 1$ if $\sigma < 0$. The conjugate structure $-J_C$ is also integrable.

2.3. Invariant metrics

A U-invariant Riemannian metric ds^2_Λ on $\mathbb{F}_\Theta$ is completely determined by its value at the origin, that is, by an inner product $\langle \cdot, \cdot \rangle$ in $\mathfrak{m}_\Theta$ which is invariant under the adjoint action of K_Θ. Any such inner product has the form

$$\langle X, Y \rangle_\Lambda := -\langle \Lambda X, Y \rangle \tag{2.7}$$

with $\Lambda : \mathfrak{m}_\Theta \to \mathfrak{m}_\Theta$ positive-definite with respect to the Cartan-Killing form. The inner product $\langle \cdot, \cdot \rangle_\Lambda$ admits a natural extension to a symmetric bilinear form on $\mathfrak{q}_\Theta = \mathfrak{m}_\Theta^{\mathbb{C}}$. We do not change notation for objects in $\mathfrak{m}_\Theta$ and $\mathfrak{q}_\Theta$ either for the bilinear form $\langle \cdot, \cdot \rangle_\Lambda$ or

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for the corresponding complexified map Λ_Θ. The K_Θ-invariance of $\langle\cdot,\cdot\rangle_\Lambda$ is equivalent to the elements of the standard basis A_α, $\sqrt{-1}S_\alpha$, $\alpha\in\Pi_\Theta$, being eigenvectors of Λ, for the same eigenvalue. Thus, in each irreducible component of $\mathfrak{q}_\Theta$ we have $\Lambda=\lambda_\sigma\mathrm{id}$ with $\lambda_{-\sigma}=\lambda_\sigma>0$.

We denote either by $\langle X,Y\rangle_\Lambda$ or by ds^2_Λ the invariant metric given by Λ. In what follows we abuse notation and say that Λ itself is an invariant metric.

If τ is the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}$. Then $\langle\langle X,Y\rangle\rangle_\Lambda=\langle X,\tau Y\rangle_\Lambda$ is a Hermitian form on $\mathfrak{g}$ which restricts to a U-invariant Hermitian form on each $\mathbb{F}_\Theta$.

2.4. Kähler form and Borel type metrics

It is easy to see that any U-invariant metric ds^2_Λ is almost Hermitian with respect to an invariant almost complex structure J, that is, $ds^2_\Lambda(JX,JY)=ds^2_\Lambda(X,Y)$ (cf. [13], section 8 and [16]). Let $\Omega=\Omega_{J,\Lambda}$ stand for the corresponding Kähler form

$$\Omega(X_\alpha,X_\beta)=-\sqrt{-1}\lambda_\alpha\epsilon_\beta\langle X_\alpha,X_\beta\rangle. \tag{2.8}$$

Since $\langle X_\alpha,X_\beta\rangle=0$ unless $\beta=-\alpha$, it follows that Ω is not zero only on the pairs $(X_\alpha,X_{-\alpha})$. In this case $\Omega(X_\alpha,X_{-\alpha})=-\sqrt{-1}\lambda_\alpha\epsilon_\alpha$. Taking into account the expression for $d\Omega$ (cf. [16]) we make the following distinction between the triple of roots.

Definition 2.1. Let $J=(\epsilon_\alpha)$ be an invariant almost complex structure. A triple of roots α, β, γ with $\alpha+\beta+\gamma=0$ is said to be:

1. a $\{0,3\}$-triple for J if $\epsilon_\alpha=\epsilon_\beta=\epsilon_\gamma$, and
2. a $\{1,2\}$-triple otherwise.

An almost Hermitian manifold is said to be $(1,2)$-symplectic (or quasi-Kähler) if $d\Omega(X,Y,Z)=0$ when one of the vectors X, Y, Z is of type $(1,0)$ and the other two are of type $(0,1)$. If J is integrable and $d\Omega\equiv0$ we say that $(\mathbb{F}_\Theta,J,ds^2_\Lambda)$ is a Kähler manifold.

The next result was obtained, for $\Theta=\emptyset$, in [16]; for arbitrary Θ it was proved in [17].

Proposition 2.1. *$(\mathbb{F}_\Theta,J=(\epsilon_\alpha),ds^2_\Lambda=(\lambda_\alpha))$ is $(1,2)$-symplectic if and only if $\epsilon_\alpha\lambda_\alpha+\epsilon_\beta\lambda_\beta+\epsilon_\gamma\lambda_\gamma=0$, for every triple of roots α, β, γ with $\alpha+\beta+\gamma=0$.*

According to Borel (cf. [6]) we can describe precisely the Kähler structures on $\mathbb{F}_\Theta$. We can see that $(\mathbb{F}_\Theta,J,ds^2_\Lambda)$ is Kähler if and only if J is integrable, and if $\alpha\in\Pi(\Theta)$ is written as

$$\alpha=\sum_{i=1}^{k}n_i\alpha_i \tag{2.9}$$

where $\alpha_i\in\Sigma(\Theta)$, then $\lambda_\alpha=\sum_{i=1}^{k}n_i\lambda_{\alpha_i}$ with $n_i\geq0$ if α is positive.

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3. STABILITY OF HARMONIC SURFACES AND PLÜCKER FORMULAE ON FLAG MANIFOLDS

We will state some results that are deeply connect with the Hermitian Geometry on $\mathbb{F}_\Theta$ If $M = M^2$ is a Riemann surface and $\phi : M \to \mathbb{F}_\Theta$ is a differentiable map, we let $d^{\mathbb{C}}\phi$ be the complexification of the differential of ϕ. We endow $\mathbb{F}_\Theta$ with a complex structure J and, as usual decompose $d^{\mathbb{C}}\phi$ into $\dfrac{\partial\phi}{\partial z}(p) : T(M)^{(1,0)} \to T(\mathbb{F}_\Theta)^{(1,0)}$ and $\dfrac{\partial\phi}{\partial \bar z}(p) : T(M)^{(1,0)} \to T(\mathbb{F}_\Theta)^{(0,1)}$, which are identified with vectors in the complex tangent space. We use the decomposition of $T^{\mathbb{C}}\mathbb{F}_\Theta$ into irreducible components. By (2.6) we have

$$\frac{\partial\phi}{\partial z}(p) = \sum_{\sigma\in\Pi(\Theta)^+} \phi_{\epsilon_\sigma\sigma}(p) \qquad \frac{\partial\phi}{\partial \bar z}(p) = \sum_{\sigma\in\Pi(\Theta)^+} \phi_{\epsilon_{-\sigma}\sigma}(p) \tag{3.1}$$

where for each $\sigma \in \Pi(\Theta)$ the function $\phi_\sigma : M \to E_\sigma$ takes values in $E_\sigma(\phi(p))$, $p \in M$.

Given an almost complex structure on $\mathbb{F}_\Theta$, a map $\phi : M^2 \to \mathbb{F}_\Theta$ is is J-holomorphic if for all $p \in M$ it holds

$$\frac{\partial\phi}{\partial \bar z}(p) = \sum_{\sigma\in\Pi(\Theta)^+} \phi_{\epsilon_{-\sigma}\sigma}(p) = 0. \tag{3.2}$$

According to [14] for the flag manifolds of SU(n) and in [3] for the general case we have

Proposition 3.1. *A map $\phi : M^2 \to (\mathbb{F}_\Theta, J)$ is J-holomorphic in $p \in M$ if and only if for $\sigma \in \Pi(\Theta)$, $\phi_\sigma(p) \neq 0$, implies $\phi_{-\sigma}(p) = 0$.*

Let $\phi : (M, g) \to (\mathbb{F}_\Theta, ds^2_\Lambda)$ be a differentiable map. Using our notations its energy is given by

$$\begin{aligned} E(\phi) &= \frac{1}{2}\int_M \left(\left|\frac{\partial\phi}{\partial z}(p)\right|^2_\Lambda + \left|\frac{\partial\phi}{\partial \bar z}(p)\right|^2_\Lambda \right)\nu_g \\ &= \frac{1}{2}\sum_{\sigma\in\Pi(\Theta)^+}\int_M \langle\phi_\sigma(p),\phi_\sigma(p)\rangle_\Lambda + \langle\phi_{-\sigma}(p),\phi_{-\sigma}(p)\rangle_\Lambda\,\nu_g, \end{aligned} \tag{3.3}$$

Taking into account that $\lambda_{-\sigma} = \lambda_\sigma$, the above expression simplifies to

$$E(\phi) = \sum_{\sigma\in\Pi(\Theta)}\int_M \langle\phi_\sigma(p),\phi_\sigma(p)\rangle_\Lambda\,\nu_g \tag{3.4}$$

We are now in condition to derive the Euler-Lagrange equations for our variational problem. As we know a map $\phi : (M^2, g) \to (\mathbb{F}_\Theta, ds^2_\Lambda)$ is harmonic if and only if it is a critical point of the energy functional (cf. [7]).

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Proposition 3.2. *The map $\phi : (M, g) \to (\mathbb{F}_\Theta, ds^2_\Lambda)$ is harmonic if and only if*

$$\mathrm{Re}\left(\sum_{\sigma\in\Pi(\Theta)} \lambda_\sigma(\nabla_{\bar{z}}\phi_\sigma)(p))\right) = 0. \tag{3.5}$$

We will now study the stability phenomenon on flags and showing its relation with the Hermitian Geometry on $\mathbb{F}_\Theta$.

Given $\phi : M^2 \to \mathbb{F}_\Theta$ we take perturbations of the type

$$\phi^t(p) = e^{tq(p)} \cdot \phi(p) \qquad -\epsilon < t < \epsilon,$$

where $q : M \to \mathfrak{u}$ is a smooth map. If $\phi : (M^2, g) \to (\mathbb{F}_\Theta, ds^2_\Lambda)$ is a harmonic map we denote by $I^\phi_\Lambda(q)$ its second variation, that is,

$$I^\phi_\Lambda(q) = \frac{d^2}{dt^2}\bigg|_{t=0} E(\phi^t),$$

Proposition 3.3. *We have*

$$\begin{aligned} I^\phi_\Lambda(q) = \frac{1}{2}\left(\int_M \langle \frac{\partial q}{\partial z}(p), \frac{\partial q}{\partial z}(p)\rangle_\Lambda \nu_g\right) + \mathrm{Re}\left(\int_M \langle [q(p), \frac{\partial q}{\partial z}(p)], \frac{\partial \phi}{\partial z}(p)\rangle_\Lambda \nu_g\right) \\ + \frac{1}{2}\left(\int_M \langle \frac{\partial q}{\partial \bar{z}}(p), \frac{\partial q}{\partial \bar{z}}(p)\rangle_\Lambda \nu_g\right) + \mathrm{Re}\left(\int_M \langle [q(p), \frac{\partial q}{\partial \bar{z}}(p)], \frac{\partial \phi}{\partial \bar{z}}(p)\rangle_\Lambda \nu_g\right). \end{aligned}$$

Starting with an invariant metric on $\mathbb{F}_\Theta$ defined by $\Lambda = (\lambda_\sigma)_{\sigma\in\Pi(\Theta)}$ let $\mathcal{P}$ be a subset of $\Pi(\Theta)$. We say that the metric $\Lambda^\# = (\lambda^\#_\sigma)_{\sigma\in\Pi(\Theta)}$ is a $\mathcal{P}$-perturbation of Λ in case

1. $\lambda^\#_\sigma = \lambda_\sigma$ if $\sigma \in \mathcal{P}$ and
2. $\lambda^\#_\sigma = \lambda_\sigma + \xi_\sigma > 0$, $\xi_\sigma \in \mathbb{R}$ if $\sigma \in \Pi(\Theta) \setminus \mathcal{P}$.

Definition 3.1. The map $\psi : M^2 \to \mathbb{F}_\Theta$ is said to be subordinate to $\mathcal{P}$ if $\psi_\sigma = 0$ when $\sigma \in \Pi(\Theta) \setminus \mathcal{P}$. Here, as before, ψ_σ denotes the E_σ-component of the derivative of ψ.

A crucial result is the following perturbation lemma.

Lemma 3.1. *Let $\phi : (M^2, g) \to (\mathbb{F}_\Theta, ds^2_\Lambda)$ a map subordinate to $\mathcal{P}$ and $\Lambda^\# = (\lambda^\#_\sigma)$, $\lambda^\#_\sigma = \xi_\sigma + \lambda_\sigma > 0$, a $\mathcal{P}$-perturbation of $\Lambda = (\lambda_\sigma)$. Suppose that ϕ is harmonic with respect to both Λ and $\Lambda^\#$. Then*

$$I^\phi_{\Lambda^\#}(q) = I^\phi_\Lambda(q) + \sum_{\sigma\in\Pi(\Theta)\setminus\mathcal{P}} \xi_\sigma \int_M \langle\langle q_\sigma(p), q_\sigma(p)\rangle\rangle_\Lambda \nu_g \tag{3.6}$$

where $\frac{\partial q}{\partial \bar{z}}(p) = \sum_{\alpha \in \Pi(\Theta)^+} q_{\epsilon_{-\sigma}\sigma}(p)$ *and* $\frac{\partial q}{\partial z}(p) = \sum_{\alpha \in \Pi(\Theta)^+} q_{\epsilon_\sigma \sigma}(p)$.

Definition 3.2. Let $\phi : M^2 \to \mathbb{F}_\Theta$. We say that a map ϕ is equi-harmonic if is harmonic for each invariant metric ds^2_Λ on $\mathbb{F}_\Theta$.

The study of equi-harmonic maps started with [14] for flag manifolds of $SU(n)$ type. In [3] was studied these maps for any flag manifolds using the Lie theoretical description of flags and the theory of f-structures. In particular was proved an interesting necessary condition for a map $\phi : M^2 \to \mathbb{F}_\Theta$ to be an equi-harmonic map.

Remark 3. 1. The hypothesis in the perturbation lemma that ϕ is harmonic with respect to both metrics Λ and $\Lambda^\#$ is essential in the computations, in order that the first derivatives of the energy function annihilates. This assumption is fulfilled by the equi-harmonic maps. ▮

Now we apply the perturbation lemma to holomorphic maps on $\mathbb{F}_\Theta$. Several invariant Hermitian structures are considered and examples of stable as well as of unstable (equi-) harmonic maps are given.

The maps we take as examples are the holomorphic-horizontal ones, in the following sense.

Definition 3.3. A map $\phi : M^2 \to (\mathbb{F}_\theta, J)$ is called generalized holomorphic-horizontal if it is J-holomorphic and satisfies $\phi_\sigma = 0$ if $\sigma \in \Pi(\Theta) \setminus \Sigma(\Theta)$.

One of their main features is the following result proved in [10], base on techniques developed by Black in [3].

Theorem 3.1. *If $\phi : M \to \mathbb{F}_\Theta$ is an generalized holomorphic-horizontal map then ϕ is equi-harmonic.*

By the very definition a generalized holomorphic-horizontal map is subordinate to any subset $\mathcal{P} \subset \Sigma(\Theta)$, in particular to $\mathcal{P} = \Sigma(\Theta)$. Hence for these maps and for $\mathcal{P} \subset \Sigma(\Theta)$ the perturbation lemma applies.

We start by considering an *iacs* $J = (\epsilon_\alpha)$ and denote by $C(J)$ the subset of roots α such that there exists a $\{0, 3\}$-triple $\{\alpha, \beta, \gamma\}$.

We consider $ds^2_{\Lambda^0 = (\lambda^0_\alpha)}$ given by $\lambda^0_\alpha = k > 0$ for each $\alpha \in \Sigma \cup C(J)$ and $0 < \lambda^0_\alpha \le k$ otherwise.We can prove :

Proposition 3.4. *Let $\psi :: M^2 \to \mathbb{F}$ be an arbitrary holomorphic-horizontal frame. then $\psi : (M^2, g) \to (\mathbb{F}, ds^2_{\Lambda^0})$ is unstable.*

THEOREM 3.2. *According to [1], $\mathbb{F}(n) = SU(n)/T$ for $n = 3$ admits the normal and the Kähler-Einstein metrics and for $n \geq 4$ admits at least $n+1$ Einstein non- Kähler metrics. One is the usual normal metric and the remaining n are given explicitly as follows:*
$\lambda_{si} = \lambda_{sj} = n - 1$, $i \neq s$, $j \neq s$
$\lambda_{kl} = n + 1$, $k, l \neq s$ $\quad (1 \leq s \leq n)$

We can now prove the following result:

THEOREM 3.3. *Consider on $(\mathbb{F}(n), ds^2_{\Lambda=(\lambda_{ij})})$ equipped with any of the $n + 1$ Einstein non- Kähler metrics described above . Let $\psi : (M^2, g) \to (\mathbb{F}(n), ds^2_{\Lambda=(\lambda_{ij})})$ an arbitrary holomorphic-horizontal frame.Then ψ is unstable.*

Finally we compute the Plücker formulae for any full flag manifold $\mathbb{F}$. Let G be a complex simple Lie group and P be a Borel (parabolic minimal) subgroup of G. Then $\mathbb{F} = G/P = U/T$, where U is a compact real form of G and $T = U \cap P$ is a maximal torus of U.

We take a complex structure J on $\mathbb{F}$. Let $\mathcal{P} = \Sigma$ be the set of simple roots induced by the invariant complex structure J and consider a holomorphic map f subordinated to $\mathcal{P}$.

DEFINITION 3.4. Let $f : M \to (\mathbb{F}, J)$ be an horizontal holomrphic map and rank $U = n$. We say f is *non-degenerate* if $f(M)$ does not lie in any $U'/(T \cap U')$, where U' is a closed subgroup of U with rank $U' < n$.

Plücker formulae for curves on the projective space give a precise relationship between an intrinsic invariant (the genus) and a set of extrinsic invariants (associated degrees and ramification indices), see [8]. Here we state these formulae for holomorphic-horizontal curves on full flag manifolds.

THEOREM 3.4. *Let $(\mathbb{F}, J, ds^2_\Lambda)$ be a full flag manifold equiped with invariant metric and complex structure. Let $f : M \to (\mathbb{F}, J, ds^2_\Lambda)$ be a non-degenerated horizontal-holomorphic map. Then the Plücker formulae for f are given by*

$$2g - 2 - \sharp_i = -\sum_{j=1}^{k} \alpha_i(H_{\alpha_j}) d_j, \quad 1 \leq i \leq k, \tag{3.7}$$

where g is the genus of the Riemann surface M, α_i denote the simple roots, H_{α_j} represents the dual of the root α_j (via Cartan-Killing form) and k is the number of simple roots of $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}}$, d_j and $\sharp_j$ $(1 \leq j \leq k)$ are the volume and the number of zeros of a pseudometric on M induced by the horizontal distribution $\mathcal{P}$, respectively.

We now compute the Plücker formulae in some examples. The Cartan matrix of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is used in order to compute $\alpha_i(H_{\alpha_j})$. Example 3.1 was already studied in [18]. In the following examples, we consider J to be a complex structure.

EXAMPLE 3.1. [Flags of D_l-type] Consider the full flag manifold $\mathrm{SO}(12)/T$. The Cartan matrix of $\mathfrak{so}(12,\mathbb{C}) = D_6$ is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}.$$

Then the equations (3.7) are given by

$$\begin{array}{rcl} 2g-2-\sharp_1 & = & -2d_1+d_2 \\ 2g-2-\sharp_2 & = & d_1-2d_2+d_3 \\ 2g-2-\sharp_3 & = & d_2-2d_3+d_4 \\ 2g-2-\sharp_4 & = & d_3-2d_4+d_5+d_6 \\ 2g-2-\sharp_5 & = & d_4-2d_5 \\ 2g-2-\sharp_6 & = & d_4-2d_6. \end{array}$$

EXAMPLE 3.2. [Flags of A_l-type, [8]] Consider the full flag manifold $SU(4)/T$. The Cartan matrix of $\mathfrak{sl}(4,\mathbb{C})$ is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

and therefore the equations (3.7) are given by

$$\begin{array}{rcl} 2g-2-\sharp_1 & = & -2d_1+d_2 \\ 2g-2-\sharp_2 & = & d_1-2d_2+d_3 \\ 2g-2-\sharp_3 & = & d_2-2d_3. \end{array}$$

EXAMPLE 3.3. [Case G_2] Consider the full flag manifold G_2/T. The Cartan matrix of $\mathfrak{g}_2$ is

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},$$

hence, the equations (3.7) are given by

$$\begin{array}{rcl} 2g-2-\sharp_1 & = & -2d_1+d_2 \\ 2g-2-\sharp_2 & = & 3d_1-2d_2. \end{array}$$

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4. EQUIGEODESICS

Let $\mathbb{F}_\Theta$ be a generalized flag manifold. A curve of the form $\gamma(t) = (\exp tX) \cdot o$ is said an *equigeodesic* on $\mathbb{F}_\Theta$ if it is a geodesic with respect to each invariant metric on $\mathbb{F}_\Theta$. The vector X is called *equigeodesic vector*. The study of equigeodesics in generalized flag manifolds started in [5] with the description of equigeodesics on $SU(n)$-flags. All results in this section are proved in [5] and [9].

We have the following algebraic characterization of equigeodesic vectors.

Proposition 4.1 ([5]). *Let $\mathbb{F}_\Theta$ be a generalized flag manifold and $X \in \mathfrak{m}_\Theta$ be a non-zero vector. Then X is an equigeodesic vector if, and only if,*

$$[X, \Lambda X]_{\mathfrak{m}_\Theta} = 0, \tag{4.1}$$

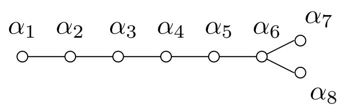
for each invariant metric Λ.

We remark that to solve equation (4.1) is equivalent to solve a non-linear algebraic system of equations whose variables are the coefficients of the vector X. Using the real Weyl basis $\{A_\alpha, S_\alpha; \alpha \in \Pi^+ \setminus \langle\Theta\rangle\}$ of the tangent space $\mathfrak{m}_\Theta$ we remark that the underlying semisimple Lie algebra structure play an important role. For example, analysing the Lie bracket of the form $[A_\alpha, S_\beta]$, $[A_\alpha, A_\beta]$ and $[S_\alpha, S_\beta]$ is clear that if the structural of structure $m_{\alpha,\beta}$, $m_{-\alpha,\beta}$, $m_{\alpha,-\beta}$ vanish (e.g. if $\alpha \pm \beta$ is not a root) then these bracket also vanish and the system can be simplified.

Theorem 4.1. *Let $\mathbb{F} = U/T$ be a full flag manifold . Let $Z = Z_{\alpha_1} + \ldots + Z_{\alpha_r}$ such that $Z_{\alpha_i} \in \mathfrak{u}_{\alpha_i}$ for all i. If $\alpha_p \pm \alpha_q$ are not roots for every $p, q \in \{1, \ldots, r\}$ then Z is an equigeodesic vector.*

Example 4.1. Consider the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ over $\mathbb{C}$, R being an associated root system, and Σ a simple root system. Two simple roots are said to be *orthogonal* if they are not joined in the Dynkin diagram. If α_1 and α_2 are two orthogonal simple roots then $\alpha_1 \pm \alpha_2$ are not roots, see [12].

For example, on the full flag manifold $SO(16)/T$ we have $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(16, \mathbb{C})$ (a Lie algebra of type D_l) and the associated Dynkin diagram is given by



Hence, any element in the set $\mathfrak{u}_{\alpha_1} \oplus \mathfrak{u}_{\alpha_3}$ is an equigeodesic vector since $\alpha_1 \pm \alpha_3$ are not roots. In the same way, any element in the set $\mathfrak{u}_{\alpha_2} \oplus \mathfrak{u}_{\alpha_4} \oplus \mathfrak{u}_{\alpha_7}$ is equigeodesic vector.

We remark that as consequence of the invariance of the metric Λ, we have $\Lambda|_{\mathfrak{m}_i} = \lambda_i \mathrm{Id}_{\mathfrak{m}_i}$, for each irreducible component of the isotropy representation. Therefore if $X \in \mathfrak{m}_i$ the equation (4.1) is satisfied trivially.

Definition 4.1. An equigeodesic vector $X \in \mathfrak{m}_\Theta$ is *nontrivial* if $X \in \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_k$ with $k > 1$; otherwise if $X \in \mathfrak{m}_i$ for some i, X is said a *trivial* equigeodesic vector.

Proposition 4.2. *In the full flag manifold $\mathbb{F} = U/T$ if X is a trivial equigeodesic vector then corresponding geodesic, $\gamma(t) = \exp(tX) \cdot o$, is closed.*

Now we focus our attention to generalized flag manifolds with two isotropy summands. In this case the tangent space at origin splits into $\mathfrak{m}_\Theta = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ and a vector $X \in \mathfrak{m}$ is written as $X = X_{\mathfrak{m}_1} + X_{\mathfrak{m}_2}$ with $X_{\mathfrak{m}_i} \in \mathfrak{m}_i$, $i = 1, 2$.

The generalized flag manifolds with two isotropy summands are listed in Table 4. For a complete classification of all generalized flag manifolds using the *painted Dynkin diagrams* see [4].

Proposition 4.3. *Let $\mathbb{F}_\Theta$ be a generalized flag manifold with two isotropy summands. A vector $X = X_{\mathfrak{m}_1} + X_{\mathfrak{m}_2} \in \mathfrak{m}_\Theta$ is equigeodesic if, and only if*

$$[X_{\mathfrak{m}_1}, X_{\mathfrak{m}_2}] = 0. \tag{4.2}$$

The next result provide a complete classification of flag manifolds with two isotropy summands that admits only trivial equigeodesics.

Theorem 4.2. *The generalized flag manifolds with two isotropy summands*

$G_2/U(2)$,	$F_4/Sp(3) \times U(1)$,	$E_6/SU(6) \times U(1)$,
$E_7/SO(12) \times U(1)$,	$E_8/E_7 \times U(1)$,	$Sp(l)/U(1) \times Sp(l-1)$,
$SO(2l+1)/U(2) \times SO(2l-3)$,	$SO(2l)/U(2) \times SO(2l-4)$	

admits only trivial equigeodesic vectors.

Generalized flag manifold with two isotropy summands.

$\mathbb{F}_\Theta = U/K_\Theta$	Dimension
$SO(2l+1)/U(p) \times SO(2(l-p)+1)$	$4p(l-p)+p^2+p$
$Sp(l)/U(p) \times Sp(l-p)$	$4p(l-p)+p^2+p$
$SO(2l)/U(p) \times SO(2(l-p))$	$4p(l-p)+p^2-p$
$E_6/SU(5) \times SU(2) \times U(1)$	50
$E_6/SU(6) \times U(1)$	42
$E_7/SO(10) \times SU(2) \times U(1)$	84
$E_7/SO(12) \times U(1)$	66
$E_7/SU(7) \times U(1)$	84
$E_8/E_7 \times U(1)$	114
$E_8/SO(14) \times U(1)$	156
$F_4/SO(7) \times U(1)$	30
$F_4/Sp(3) \times U(1)$	30
$G_2/U(2)$ ($U(2)$ is represented by the short root of G_2)	10

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