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Rigidity of least area hyperbolic surfaces in three-manifolds

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In this work we prove rigidity results for least area surfaces of genus greater than one in Riemannian three-manifolds with scalar curvature bounded from below by a negative constant. May, 2015 ICMC-USP

1. INTRODUCTION

Let (M, g) be a Riemannian three-manifold with lower bounded scalar curvature. It is an interesting fact in differential geometry that the existence of a least area surface can influence the geometry of M.

For example, in [SY3], Schoen and Yau proved that if M is a compact orientable three-manifold with scalar curvature $R \geqslant 0$ and $\Sigma \subset M$ is an incompressible two-torus (i.e., the fundamental group of Σ injects into that of M), then M is flat.

In [FCS], Fischer-Colbrie and Schoen conjectured that in the above result due to Schoen and Yau it is sufficient that M contains a least area two-torus (not necessarily incompressible). This conjecture was proved by Galloway and Cai in [CG].

THEOREM 1.1 (Cai and Galloway). *Let (M^3, g) be a Riemannian three-manifold with scalar curvature $R \geqslant 0$. If $\Sigma \subset M$ is an area-minimizing embedded two-torus, then M is flat in a neighborhood of Σ.*

It follows that the induced metric on Σ is flat and that locally M splits along Σ. The proof of Theorem 1.1 uses and argument based on a local deformation around Σ to obtain a metric with positive scalar curvature.

Recently, Bray, Brendle and Neves studied in [BBN] the case where (M, g) has scalar curvature $R_g \geqslant 2$ and Σ is a locally area-minimizing embedded two-sphere. In their case, the model is the standard Riemannian cylinder $(\mathbb{R} \times S^2, dt^2 + g)$, where g is the round metric on S^2 with constant Gaussian curvature equal to 1. They proved the following result.

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Theorem 1.2 (Bray, Brendle and Neves). *Let (M^3, g) be a Riemannian three-manifold with scalar curvature $R_g \geqslant 2$. If $\Sigma \subset M$ is an embedded two-sphere which is locally area-minimizing, then Σ has area less than or equal to 4π. Moreover, if equality holds, then Σ with the induced metric has constant Gauss curvature equal to 1 and locally M splits along Σ.*

The proof in [BBN] is based on a construction of a one-parameter family $\Sigma_t \subset M$ of constant mean curvature two-spheres along $\Sigma = \Sigma_0$.

A natural question is to know what happens when the model is a cylinder $(\mathbb{R}\times\Sigma, dt^2+g_\Sigma)$, where Σ is an orientable compact surface of genus greater than one and g_Σ is a metric on Σ with constant Gauss curvature equal to -1.

In the present work, we prove that the analogous result is true in this case.

Theorem 1.3 (N.). *Let (M^3, g) be a Rimeniann three-manifold with scalar curvature $R_g \geqslant -2$. If $\Sigma \subset M$ is an embedded orientable compact surface of genus $g(\Sigma) \geqslant 2$ which is locally area-minimizing, then*

$$|\Sigma|_g \geqslant 4\pi(g(\Sigma) - 1), \tag{1.1}$$

where $|\Sigma|_g$ is the area of Σ with respect to the induced metric. Moreover, if equality holds, then Σ has a neighboorhood in M which is isometric to $((-\epsilon, \epsilon) \times \Sigma, dt^2 + g_\Sigma)$, where $\epsilon > 0$ and g_Σ is the induced metric on Σ which has constant Gauss curvature equal to -1. More precisely, the isometry is given by $f(t, x) = \exp_x(t\nu(x))$, $(t, x) \in (-\epsilon, \epsilon) \times \Sigma$, where ν is the unit normal vector field along to Σ.

In the following we will give an idea of the proof of Theorem 1.3. The inequality (1.1) follows from the second variation of area using the Gauss equation, the lower bound of the scalar curvature and the Gauss-Bonnet theorem. In the equality case, we construct, using the implicit function theorem, a one-parameter family of constant mean curvature surfaces, denoted by Σ_t, with $\Sigma_0 = \Sigma$ and all having the same genus. Arguing by contradiction and using the solution of the Yamabe problem for compact manifolds with boundary and Hopf's maximum principle, we are able to conclude that each Σ_t has the same area. Finally, we obtain from this that Σ has a neighborhood isometric to $((-\epsilon, \epsilon) \times \Sigma, dt^2 + g_\Sigma)$ in a neighborhood of Σ.

We note that recently M. Micallef and V. Moraru proved the above rigidity result by using an alternative argument (see [MM]).

In recent years, several results were obtained concerning the problem of recognizing the geometry of a compact manifold with boundary, provided the geometry of the boundary is known and some curvature conditions are satisfied. In [Miao], Miao observed that the positive mass theorem (see [SY1, Wtt]) implies the following rigidity result for the unit ball $B^n \subset \mathbb{R}^n$.

Theorem 1.4 (Miao). *Let g be a smooth Riemannian metric on B^n with nonnegative scalar curvature such that $\partial B^n = S^{n-1}$ with the induced metric has mean curvature greater*

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than or equal to $n-1$ *and is isometric to* S^n *with the standard metric. Then* g *is isometric to the standard metric of* B^n.

The theorem above was generalized by Shi and Tam in [ST]. In [Min], Min-Oo proved a scalar rigidity result for the hyperbolic space. Moreover, analogues of the positive mass theorem for asymptotically hyperbolic manifolds were obtained in [CH] and [Wa]. We note that these results imply the analogue of the Miao's theorem for geodesic balls in the hyperbolic space.

Inspired by the above results, Min-Oo conjectured in [Min2] the following scalar curvature rigidity for the hemisphere

Conjecture 1.1. Let g be a smooth metric on the hemisphere S^n_+ with scalar curvature $R_g \geqslant n(n-1)$ such that the induced metric on ∂S^n_+ agrees with the standard metric on ∂S^n_+ and is totally geodesic. Then g is isometric to the standard metric on S^n_+.

This conjecture is true for $n = 2$ (see [Top, HW]) and recently counterexamples were constructed by Brendle, Marques e Neves [BMN] for $n \geqslant 3$. We refer the reader to [Eich, HW, HWu] for partial results concernig Min-Oo's conjecture.

Our next theorem is a rigidity result for cylinders $([a, b] \times \Sigma, dt^2 + g_\Sigma)$, where (Σ, g_Σ) is a compact hyperbolic surface. This is similar to Miao's result and Min-Oo's conjecture, the difference being that in our setting we are dealing with cylinders instead of geodesic balls.

Theorem 1.5 (N.). *Let* Σ *be a compact orientable surface with Gauss curvature* $K_\Sigma \equiv -1$. *Let* (Ω^3, g) *be a compact orientable irreducible connected Riemannian three-manifold with boundary satisfying the following properties:*

- $R_g \geqslant -2$.
- $H_g \geqslant 0$.
- $\partial\Omega$ *tem uma componente incompressível e isométrica a* (Σ, g_Σ).

Moreover, suppose that Ω *does not contain any one-sided compact embedded surface. Then* (Ω, g) *is isometric to* $([a, b] \times \Sigma, dt^2 + g_\Sigma)$.

2. AREA ESTIMATE AND INFINITESIMAL RIGIDITY

Let (M, g) be a Riemannian three-manifold with scalar curvature $R_g \geqslant -2$ and let $\Sigma \subset M$ be a compact stable minimal surface of genus $g(\Sigma) \geqslant 2$.

Let ν be the unit vector field along to Σ. Since Σ is stable we have, by the second variation of area, that

$$\int_\Sigma (\mathrm{Ric}(\nu, \nu) + |A|^2)\varphi^2 \, d\sigma \leqslant \int_\Sigma |\nabla\varphi|^2 \, d\sigma. \tag{2.1}$$

for all $\varphi \in C\infty(\Sigma)$.

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Choosing $\varphi = 1$, we obtain

$$\int_\Sigma (\mathrm{Ric}(\nu,\nu) + |A|^2)\, d\sigma \leqslant 0. \tag{2.2}$$

Now, the Gauss equation implies

$$\mathrm{Ric}(\nu,\nu) = \frac{R_g}{2} - K_\Sigma - \frac{|A|^2}{2}. \tag{2.3}$$

Substituting (2.3) em (2.2), we get

$$\frac{1}{2}\int_\Sigma R_g + |A|^2\, d\sigma \leqslant \int_\Sigma K_\Sigma = 4\pi(1 - g(\Sigma)), \tag{2.4}$$

where the equality follows from the Gauss-Bonnet theorem.

Therefore, since $R_g \geqslant -2$, we have

$$|\Sigma|_g \geqslant 4\pi(g(\Sigma) - 1). \tag{2.5}$$

Now, suppose that $|\Sigma|_g = 4\pi(g(\Sigma) - 1)$. In this case, inequalities (2.2) and (2.4) become equalities.

The equality in (2.2) implies that the constant function $\varphi = 1$ is in the kernel of the Jacobi operator $L = \Delta_\Sigma + \mathrm{Ric}(\nu,\nu) + |A|^2$ of Σ. Therefore, $\mathrm{Ric}(\nu,\nu) + |A|^2 = 0$ on Σ.

In turn, from equality (2.4) we have that Σ is totally geodesic, that is, $A = 0$, and $R_g(x) = -2$ for all $x \in \Sigma$. Note that $A = 0$ implies $\mathrm{Ric}(\nu,\nu) = 0$ on Σ. Finally, by equation (2.3) we get that $K_\Sigma = -1$.

From the above we obtain the following proposition.

PROPOSITION 2.1. *Let (M, g) be a Riemannian three-manifold and let $\Sigma \subset M$ be a compact minimal surface of genus $g(\Sigma) \geqslant 2$. If Σ is stable, then $|\Sigma|_g \geqslant 4\pi(g(\Sigma) - 1)$. Moreover, equality holds if and only if Σ is totally geodesic, $\mathrm{Ric}(\nu,\nu) = 0$ and $R_g = -2$ on Σ, and Σ has constant Gauss curvature equal to -1 with the induced metric.*

3. LOCAL RIGIDITY

In this section, we will conclude the proof of Theorem 1.3. The following proposition is fundamental to obtain the local rigidity.

PROPOSITION 3.1. *If $|\Sigma|_g = 4\pi(g(\Sigma) - 1)$, then there exist $\epsilon > 0$ and a smooth family $\Sigma_t \subset M$, $t \in (-\epsilon, \epsilon)$, of compact embedded surfaces satisfying:*

•$\Sigma_t = \{\exp_x(w(t,x)\nu(x)) : x \in M\}$, *where $w : (-\varphi, \varphi) \times \Sigma \to \mathbb{R}$ is a smooth function such that*

$$w(0,x) = 0, \quad \frac{\partial w}{\partial t}(0,x) = 1 \quad \text{and} \quad \int_\Sigma (w(t,\cdot) - t)\, d\sigma = 0.$$

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•Σ_t *has constant mean curvature for all* $t \in (-\varepsilon, \varepsilon)$.

Proof. See [Nun], Proposition 2. ∎

Let $\nu(t)$ denote the unit normal vector field along Σ_t such that $\nu(0) = \nu$. In our convention, the mean curvature $H(t)$ of Σ_t satisfies $\vec{H}(t) = -H(t)\nu(t)$, where $\vec{H}(t)$ is the mean curvature vector of Σ_t. In this case, we have

$$\frac{d}{dt}|\Sigma|_g = H(t) \int_{\Sigma_t} \left\langle \nu(t), \frac{\partial f}{\partial t}(t, \cdot) \right\rangle d\sigma_t, \tag{3.1}$$

where $f(t,x) = \exp_x(w(t,x)\nu(x))$, $x \in \Sigma$. Notice that $\frac{\partial f}{\partial t}(0,x) = \nu(x)$, so we can suppose, decreasing $\epsilon > 0$ if necessary, that $\int_{\Sigma_t} \langle \nu(t), \frac{\partial f}{\partial t}(t,\cdot)\rangle \, d\sigma_t$ is positive for all $t \in (-\epsilon, \epsilon)$. Moreover, we can assume that $|\Sigma|_g \leqslant |\Sigma_t|_g$ for all $t \in (-\epsilon, \epsilon)$, because Σ is locally area-minimizing.

Next we will prove that $|\Sigma_t|_g \leqslant |\Sigma|_g$ for each t sufficiently small. We will restrict ourselves to the case $t \geqslant 0$ since the proof in the case $t \leqslant 0$ is analogous. By equation (3.1) it is enough to prove the following proposition.

Proposition 3.2. *There exists* $0 < \epsilon_1 < \epsilon$ *such that* $H(t) \leqslant 0$ *for all* $t \in [0, \epsilon_1)$.

Proof.
Suppose, by contradiction, that there exists a sequence $\epsilon_k \to 0$, $\epsilon_k > 0$, such that $H(\epsilon_k) > 0$ for all k. Consider (V_k, g_k), where $V_k = [0, \epsilon_k] \times \Sigma$ e $g_k = (f|_{V_k})^* g$. Therefore, V_k is a compact three-manifold with boundary satisfying:

• $R_{g_k} \geqslant -2$.
• The mean curvature of ∂V_k with respect to the outward normal vector, denoted by H_{g_k}, is nonnegative. More precisely, $\partial V_k = \Sigma \cup \Sigma_{\varepsilon_k}$, where Σ is a minimal surface and Σ_{ε_k} has positive constant mean curvature.
• $|\Sigma|_{g_k} = 4\pi(g(\Sigma) - 1)$.

By using the solution to the Yamabe problem on manifolds with boundary which was first studied by Escobar [Esc], we can prove (see [Nun], Proposition 3) that if $k > 0$ is large enough, then exists a positive function $u \in C^\infty(V_k)$ such that the conformal metric $\overline{g} = u^4 g_k$ onV_k satisfies

$$R_{\overline{g}} = -2 \text{ on } V_k, \text{ e } H_{\overline{g}} = 0 \text{ on } \partial V_k.$$

In analytic terms, this mean that u solves

$$\begin{cases} \Delta_{g_k} u - \frac{1}{8} R_{g_k} u - \frac{1}{4} u^5 = 0 \text{ em } V_k, \\ \frac{\partial u}{\partial \eta} + \frac{1}{4} H_{g_k} u = 0 \text{ em } \partial V_k. \end{cases} \tag{3.2}$$

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It follows from the maximum principle together with the curvature conditions of g_k that $u < 1$. Therefore, we have that $|\Sigma|_{\overline{g}} < |\Sigma|_{g_k} = 4\pi(g(\Sigma) - 1)$.

Finally, we can apply the Existence Theorem 5.1 in [MSY] to obtain that there exists minimal surface $\overline{\Sigma}$ in $(\overline{V}, \overline{g})$ of least area in the isotopy class of Σ in V_k. In particular, $|\overline{\Sigma}|_{\overline{g}} \leqslant |\Sigma|_{\overline{g}} < 4\pi(g(\Sigma) - 1)$. But this is a contradiction with (1.1), since we have proven, by using the lower bound $R_{\overline{g}} \geqslant -2$ and the second variation of area, that we must have $|\overline{\Sigma}|_{\overline{g}} \geqslant 4\pi(g(\Sigma) - 1)$. This concludes the proof of the proposition. ∎

By the proposition above, we conclude that $|\Sigma_t|_g = |\Sigma|_g$ for all $t \in (-\epsilon, \epsilon)$, where $\epsilon > 0$ is sufficiently small. Therefore, by Proposition 2.1, we have that Σ_t is totally geodesic, $\mathrm{Ric}(\nu(t), \nu(t))$ and $R_g = -2$ on Σ_t, and $K_{\Sigma_t} \equiv -1$ for all $t \in [0, \epsilon_1)$. This implies that $w(t, \cdot) = t$ and $f(t, x) = \exp_x(t\nu(x))$ is an isometry between $((-\epsilon, \epsilon) \times \Sigma, dt^2 + g_\Sigma)$, where g_Σ denotes the induce metric on Σ, and a neighborhood of Σ in M.

4. PROOF OF THEOREM 1.5

In this section, we will give an idea of the proof of Theorem 1.5.

First, by using the Existence Theorem 5.1 in [MSY], we can find a minimal surface $\overline{\Sigma}$ of least area in the isotopy class of $\partial\Omega^{(1)}$, where $\partial\Omega^{(1)}$ is the component of $\partial\Omega$ which is isometric to (Σ, g_Σ). Since, from the area estimate 1.1, the area of $\partial\Omega^{(1)}$ is the minimum value for the area of a stable minimal surface in its isotopy class, we assumte that $\overline{\Sigma} = \partial\Omega^{(1)}$.

Now, since $\overline{\Sigma} = 4\pi(g(\Sigma) - 1)$, we have local rigidity in a neighborhood of $\overline{\Sigma}$ in Ω as a consequence of Theorem 1.3. Finally, by a standard continuation argument (see [Nun], Section 4) we obtain that (Ω, g) is isometric to $([0, a] \times \overline{\Sigma}, dt^2 + g_{\overline{\Sigma}})$.

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