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Geometric approach to nonvariational singular elliptic equations

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In this paper we present briefly a part of the first author's PhD thesis conducted in the Department of Mathematics at Universidade Federal do Ceará, Brazil, in [3]. In this work we developed a systematic geometric approach to study fully nonlinear elliptic equations with singular absorption terms as well as their related free boundary problems. The magnitude of the singularity is measured by a negative parameter $(\gamma - 1)$, for $0 < \gamma < 1$, which reflects on lack of smoothness for an existing solution along the singular interface between its positive and zero phases. We establish existence as well sharp regularity properties of solutions. We further prove that minimal solutions are non-degenerate and obtain fine geometric-measure properties of the free boundary $\mathfrak{F} = \partial\{u > 0\}$. In particular we show sharp Hausdorff estimates which imply local finiteness of the perimeter of the region $\{u > 0\}$ and $\mathcal{H}^{n-1}$ a.e. weak differentiability property of $\mathfrak{F}$. May, 2015 ICMC-USP

1. INTRODUCTION

The aim of this present work is to study fine qualitative properties of nonvariational singular elliptic equations of the form

$$\begin{cases} F(D^2u) \sim u^{-\theta} \cdot \chi_{\{u>0\}} & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\theta = 1 - \gamma$, for $0 < \gamma < 1$, f is a positive, C^2 boundary datum and the governing operator F is assumed to be uniform elliptic, i.e.,

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there exist two constants $0 < \lambda \leq \Lambda$ such that

$$F(\mathcal{M} + \mathcal{N}) \leq F(\mathcal{M}) + \Lambda\|\mathcal{N}^+\| - \lambda\|\mathcal{N}^-\|, \quad \forall \mathcal{M}, \mathcal{N} \in \mathrm{Sym}(N). \tag{1.2}$$

The study of singular equations as (1.1) is motivated by applications in a number of problems in engineering sciences. For example, model fluids passing through a porous body Ω. More precisely, u could represent the density of a gas, or else the density of certain chemical specie, in reaction with a porous catalyst pellet, Ω.

From the singular equation

$$F(D^2u) = \gamma u^{\gamma - 1} \cdot \chi_{\{u>0\}}, \tag{1.3}$$

one notices that the Hessian of an existing solution blows-up along the free boundary $\mathfrak{F} = \partial\{u > 0\} \cap \Omega$; therefore, solutions cannot be of class C^2. In the fully nonlinear setting, the problem of optimal regularity for solutions to Equation (1.3) is a rather delicate issue and it will be discussed in Section 2.

The variational theory, $F(M) = \mathrm{Tr}(M)$, for the free boundary problem (1.1) is fairly well understood, nowadays. It appears as the Euler-Lagrange equation in the minimization of non-differentiable functionals:

$$\int \frac{1}{2}|\nabla u(X)|^2 + u(X)^\gamma dX \longrightarrow \min. \tag{1.4}$$

See, for instance [18], [19] and [2].

The case $\gamma = 1$ in (1.4) represents the obstacle problem, [7]; the case $\gamma = 0$ relates to the cavitation problem, [1]. Fully nonlinear version of the obstacle problem has been considered in [17]. Nonvariational cavitation problem has been recently studied in [20]. The delicate intermediary case, $0 < \gamma < 1$, addressed in this present work brings major novelty adversities as the equation satisfied within the positive set $\{u > 0\}$ is nonhomogeneous and blows-up along the *a priori* unknown quenching interface $\mathfrak{F} = \partial\{u > 0\} \cap \Omega$ - the so called free boundary of the problem. The lack of variational or energy approaches too implies significant difficulties in the problem and new, nonvariational solutions have to be established. In fact, since the free boundary problem considered in this paper has nonvariational character, one cannot use the powerful measure-distributional language to setup weak version of the problem. Instead we shall employ a perturbation scheme and will obtain uniform estimates with respect to the approximating parameter ε. A solution to the fully nonlinear free boundary problem (1.1) will therefore be obtained as the limit of appropriate approximating configurations.

Let us turn our attention to the singularly perturbed strategy we shall use in order to grapple with the lack of variational approaches available. In this paper we suggest the following singular perturbation scheme to appropriately approach the free boundary problem (1.1):

$$\begin{cases} F(D^2u) = \beta_\varepsilon(u), & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{E_ε}$$

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The singular perturbation term β_ε is build up as follows: initially select your favorite function $\rho \in C_0^\infty[0,1]$ and set

$$\alpha := 1 + \frac{\gamma}{2-\gamma}. \tag{1.5}$$

Throughout the whole paper, α will always be the fixed value stated in (1.5). In the sequel, define

$$B_\varepsilon(t) = \int_0^{\frac{t-\varepsilon^\alpha \sigma_0}{\varepsilon^\alpha}} \rho(s)ds, \tag{1.6}$$

where $0 < \sigma_0 < \frac{1}{2}$ is an arbitrary technical choice. Notice that B_ε is a smooth approximation of $\chi_{(0,\infty)}$. Finally, we set

$$\beta_\varepsilon(t) = \gamma t^{\gamma-1} B_\varepsilon(t). \tag{1.7}$$

Such a construction is carefully carried out as to preserve the natural scaling of the desired equation (1.3).

To conclude this introduction, we comment briefly on the existence of a special viscosity solution to equation (E_ε). As we will show in Section 3, the family of minimal solutions turns out to satisfy the desired appropriate geometric features. Such properties will allow us to establish Hausdorff estimates of the free boundary in Section 4. Notice that because of the lack of monotonicity of equation (E_ε) with respect to the variable u, classical Perron's method cannot be directly employed. Following the arguments stated in [20], an adaptation of Perron's method, it is possible to assure the existence of a minimal viscosity solution to (E_ε). For full details see [3, Section 3]. In comparison with the variational theory, this choice is a replacement for the selection of minimizers of the Euler-Lagrange functional. Therefore, unless otherwise stated, whenever we mention viscosity solution to (E_ε), we mean the minimal solution provided by such arguments.

2. SHARP REGULARITY ESTIMATES

The first main problem to be addressed to this approach concerns the optimal regularity estimates for solutions to Equation (E_ε), uniformly on parameter ε. Optimal estimates for heterogeneous equations, $\mathrm{L}u = f(X,u)$ is in general a quite delicate issue. For the singular setting studied in this present work, optimal estimates are even more involved as they can be understood as invariant (tangential) equations for their own scaling. We show that solutions are locally of class $C^{1,\frac{\gamma}{2-\gamma}}$ at the free boundary. This result was only known in the variational setting, for minimizers of Euler-Lagrange functional, see [18], [19], [2] and [14].

Thus, we start off this Section by rather informal, heuristic considerations as to guide us through the genuine results to be established later on. Suppose 0 is a free boundary point and, say, $-\mathrm{e}_n$ is the unit outward normal pointing towards the quenching phase $\{u = 0\}$. If u is $C^{1,\beta}$ at 0, then, in a small neighborhood, say, $B_\rho \cap \{u > 0\}$, $\rho \ll 1$, u behaves like $\sim X_n^{1+\beta}$. Therefore, the singular potential of the equation in (1.1) is like $\sim X_n^{(1+\beta)\cdot(1-\gamma)}$. In view of the regularity theory for heterogeneous fully nonlinear equations $F(D^2u) = f(X)$, established in [8] and [22], we obtain the following implication

$$X_n^{(1+\beta)\cdot(1-\gamma)} \in L^\theta_{\text{weak}} \quad \text{implies} \quad u \in C^{1,1-\frac{1}{\theta}}.$$

The reasoning above gives the following system of algebraic equations

$$\begin{cases} \theta(1+\beta)(\gamma-1) &= -1 \\ \beta &= 1 - \frac{1}{\theta}. \end{cases}$$

Solving for β, revels, $\beta = \frac{\gamma}{2-\gamma}$, which agrees with the optimal regularity estimate established for the variational theory.

This Section is devoted to establish local $C^{1,\frac{\gamma}{2-\gamma}}$ regularity estimates for solutions u_ε to Equation (E_ε), uniform in ε at free boundary points. In fact we shall obtain a universal growth control on u_ε near the free boundary. The desired regularity along the free interface will then follow.

Hereafter, let us fix a point $X_0 \in \Omega$ and for simplicity take $X_0 = 0$. Our analysis will be based on the auxiliary function v_ε, defined by

$$v_\varepsilon(X) := u_\varepsilon^{\frac{2-\gamma}{2}}(X). \tag{2.1}$$

For the sake of notation convenience, let us omit the subscript ε in v_ε and in u_ε, writing simply v and u to denote these functions. By computing the gradient and hessian of v, we can notice that v satisfies the following equation

$$F\left(D^2v(X) + \frac{\gamma}{2-\gamma}v^{-1}(X)\cdot\nabla v(X)\otimes\nabla v(X)\right) = f(X)v^{-1}(X), \tag{2.2}$$

for a certain bounded function $f(X)$. It is standard to justify formally the Equation (2.2) using the language of viscosity solutions.

Our first result towards optimal regularity establishes equicontinuity for functions satisfying (2.2), which implies the same conclusion to the family of functions u_ε. The proof is an adaptation of the Ishii-Lions method [16], see also [4], [5] and [15]. For more details see [3, Proposition 1].

Proposition 2.1 (C^0-compactness). *Solutions to* (2.2) *are universally locally uniform continuous, that is, there exists a universal modulus of continuity, ϱ, such that $|v(X_1) - v(X_2)| \le C_{\Omega'}\varrho(|X_1 - X_2|)$, for $X_1, X_2 \in \Omega' \Subset \Omega$.*

Now, we are able to state the following regularity estimate for viscosity solutions of (E_ε).

Theorem 2.1 (Uniform optimal regularity). *Given a subset $\Omega' \Subset \Omega$, there exists a constant C depending on, $\|f\|_\infty$, γ, Ω', dimension, ellipticity, but independent of ε, such that, any family of viscosity solutions $\{u_\varepsilon\}$ of equation (E_ε) satisfies,*

$$\sup_{B_r(X)} u_\varepsilon \le C\left(r^{\frac{2}{2-\gamma}} + u_\varepsilon(X)\right), \quad \forall X \in \Omega'. \tag{2.3}$$

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Proof. Suppose, for the sake of contradiction, the thesis of Theorem 2.1 fails to hold. Combining discrete iterative techniques and a continuous methods, see [10], Lemma 3.3 and also [21] for similar reasoning, for each $k > 1$, it is possible to find $0 < r_k = \mathrm{o}(1)$, $X_k \in \Omega'$, $\varepsilon_k > 0$ such that the following two inequalities hold

$$s_k := \sup_{B_{r_k}(X_k)} u_{\varepsilon_k} > k\left(r_k^{\alpha} + u_{\varepsilon_k}(X_k)\right) \tag{2.4}$$

$$\sup_{B_{r_k}(X_k)} \left[u_{\varepsilon_k} - u_{\varepsilon_k}(X_k)\right] \geq 2^{-\alpha k} \sup_{B_{2^k r_k}(X_k)} \left[u_{\varepsilon_k} - u_{\varepsilon_k}(X_k)\right]. \tag{2.5}$$

The normalized function $\varphi_k \colon B_1 \to \mathbb{R}$ given by

$$\varphi_k(Y) := \frac{u_{\varepsilon_k}(X_k + r_k Y)}{s_k}, \tag{2.6}$$

satisfies

$$0 \leq \varphi_k(Y) \leq C|Y|^{\alpha}\ , \tag{2.7}$$

$$\varphi_k(0) = \mathrm{o}(1), \tag{2.8}$$

$$\sup_{B_1} \varphi_k = 1. \tag{2.9}$$

In addition, the following equation is satisfied in the viscosity sense

$$F_k(D^2\varphi_k(Y)) = \left(\frac{r_k^{\alpha}}{s_k}\right)^{\frac{2}{\alpha}} \beta_{\varepsilon_k}(\varphi_k), \tag{2.10}$$

where

$$F_k(\mathcal{M}) := \frac{r_k^2}{s_k} \cdot F\left(\frac{s_k}{r_k^2}\mathcal{M}\right), \tag{2.11}$$

which is a (λ, Λ)-elliptic operator. Thus, from Proposition 2.1, up to a subsequence, φ_k converges locally uniformly to an entire function $\varphi_0 \colon \mathbb{R}^N \to \mathbb{R}$. From hypothesis of contradiction (2.4),

$$\left(\frac{r_k^{\alpha}}{s_k}\right)^{\frac{2}{\alpha}} = \mathrm{o}(1). \tag{2.12}$$

Passing to another subsequence, if necessary, F_k converges locally uniformly to a limiting *recession* operator $\tilde{F}$, which is (λ, Λ)-elliptic and homogeneous of degree one for nonnegative scalars. Passing the limit as $k \to \infty$ in (2.10) yields

$$\varphi_0^{1-\gamma} \cdot \tilde{F}(D^2\varphi_0(Y)) = 0. \tag{2.13}$$

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Notice, in view of (2.8) and (2.9), we have

$$\varphi_0(0) = 0, \quad \sup_{B_1} \varphi_0 = 1. \tag{2.14}$$

We now revisit the proof of Proposition 2.1, see [3, Proposition 1]. Defining $\psi := \varphi_0^{1/\alpha}$, we find

$$\tilde{F}(D^2\psi + c_\gamma \psi^{-1} \nabla\psi \otimes \nabla\psi) = 0. \tag{2.15}$$

By running the same reasonings of the proof of Proposition 2.1, for ψ, with $\omega = t^\theta$, $0 < \theta < 1$, $\delta = 0$, $f = 0$ and with no localization term, gives $C^{0,\theta}$ estimates for ψ, for any $\theta < 1$. In fact, for $L \gg 1$, depending only on ellipticity, estimate (39) in [3, Proposition 1], becomes

$$0 < cL^2 \cdot \left(\psi^{-1}(\bar{X}) - \psi^{-1}(\bar{Y})\right) < 0, \tag{2.16}$$

since the contradiction assumption in the reasoning of the proof of Proposition 2.1 implies $\psi(\bar{X}) > \psi(\bar{Y})$. We now choose θ_0 so that

$$\frac{1}{\alpha} < \theta_0 < 1. \tag{2.17}$$

A final contradiction is then obtained when we confront (2.14) with the C^{0,θ_0} regularity for ψ. Indeed, select a point Z in $\{\varphi_0 > 0\}$ and $Z_0 \in \{\varphi_0 = 0\}$, satisfying $\operatorname{dist}(Z, \{\varphi_0 = 0\}) = |Z - Z_0| < \frac{1}{2}$. It follows from Hopf maximum principle that

$$0 < \liminf_{h\to 0} \frac{\varphi(he + Z_0)}{h}, \tag{2.18}$$

where e is the inward normal vector to the ball $B_{|Z-Z_0|}(Z)$ at Z_0. On the other hand, we have

$$\begin{aligned} \frac{\varphi_0(he + Z_0)}{|h|} &= \sqrt[\theta_0]{\frac{\varphi_0^{\frac{1}{\alpha}}(he + Z_0)}{|h|^{\theta_0}}} \cdot \varphi_0^{1 - \frac{1}{\alpha\theta_0}}(he + Z_0) \\ &\leq C \cdot \varphi_0^{\delta_0}(he + Z_0) \\ &\to 0, \end{aligned} \tag{2.19}$$

as $h \to 0$. This concludes the proof of Theorem 2.1. ∎

3. NONDEGENERACY OF MINIMAL SOLUTIONS

The next principal result delivered in [3] states that minimal solutions, i.e., solutions obtained from Perron's type method do grow precisely as $\operatorname{dist}(X, \{u_\varepsilon \sim \varepsilon^\alpha\})^{1+\frac{\gamma}{2-\gamma}}$, which corresponds to the maximum growth rate allowed. Such a result implies a quite restrictive geometry for the free quenching interface $\{u_\varepsilon \sim \varepsilon^\alpha\}$. Also, we shall establish a stronger

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nondegeneracy property of minimal solutions, which also has fundamental importance in our blow-up analysis.

To simplify the statement of the results, we introduce some definitions and notations. Hereafter we shall use systematically we following notations:

$$\begin{aligned} \{u_\varepsilon > \kappa\} &:= \{x \in \Omega \mid u_\varepsilon(x) > \kappa\}, \\ \{\tau > u_\varepsilon > \lambda\} &:= \{x \in \Omega \mid \tau > u_\varepsilon(x) > \lambda\}, \\ d_\varepsilon(X) &:= \operatorname{dist}(X, \partial\{u_\varepsilon > \varepsilon^\alpha\}), \end{aligned}$$

The nondegeneracy feature of minimal solutions is based on the construction of appropriate viscosity supersolution whose value within an inter disk is much smaller than its value on the boundary of an outer disk. Under scaling properties, we obtain, for each parameter $\varepsilon > 0$ fixed, a radial function θ_ε satisfying

- ✓ $\theta_\varepsilon = 2\sigma_0\varepsilon^\alpha$ in $B_{c_1\varepsilon\eta}$;
- ✓ $\theta_\varepsilon \geq c_2\eta^\alpha$ in $\Omega \setminus B_{\varepsilon\eta}$;
- ✓ $\theta_\varepsilon \in C^{1,1}(\Omega)$ and it is a supersolution to (E_ε).

The technical details are stated in [3, Proposition 2].

Therefore, given θ_ε as above, we establish strong nondegeneracy of minimal solutions to the singularly perturbed problem (E_ε).

Theorem 3.1 (Strong nondegeneracy). *Let $X_0 \in \{u_\varepsilon > \varepsilon^\alpha\}$. There exist two universal positive constants $c_0 > 0$ and $r_0 > 0$ such that if $r < r_0$, there holds*

$$\sup_{B_r(X_0)} u_\varepsilon \geq c_0 r^\alpha,$$

for α as in (1.5).

Proof. Given $r < r_0$, we construct θ_ε for $\eta = r/\varepsilon$. By minimality of u_ε,

$$u_\varepsilon(Z) > \theta_\varepsilon(Z),$$

for some point $Z \in \partial B_r(X_0)$. Indeed, suppose for the sake of contradiction that $u_\varepsilon \leq \theta_\varepsilon$ along ∂B_r. Define

$$w_\varepsilon = \begin{cases} \min\{\theta_\varepsilon, u_\varepsilon\} & \text{in } \overline{B_r}; \\ u_\varepsilon & \text{in } \Omega \setminus \overline{B_r}. \end{cases}$$

Thus, w_ε is supersolution to (E_ε); however in $B_{c_1 r}$, we have,

$$u_\varepsilon > \varepsilon^\alpha > 2\sigma_0\varepsilon^\alpha \equiv \theta_\varepsilon = w_\varepsilon,$$

which contradicts the minimality of u_ε. In conclusion,

$$c_2 r^\alpha \leq \theta_\varepsilon(Z) < u_\varepsilon(Z) \leq \sup_{B_r} u_\varepsilon,$$

and the Theorem is proven. ▮

An immediate Corollary of Theorem 3.1 is the upper and lower control of u_ε by r^α in $B_r \subset \{u_\varepsilon > \varepsilon^\alpha\}$.

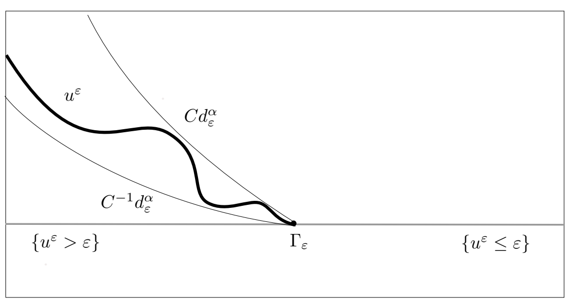
Corollary 3.1. *Given a subdomain $\Omega' \Subset \Omega$, there exists a universal constant $C = C(\Omega')$ such that for $X_0 \in \Omega' \cap \{u_\varepsilon > \varepsilon^\alpha\}$ and $r \le r_0$,*

$$C^{-1} r^\alpha \le \sup_{B_r(X_0)} u_\varepsilon \le C(r^\alpha + u_\varepsilon(X_0)).$$

Another important consequence shows that u_ε does growth at the sharp rate away from the free boundary, that is $\sim d_\varepsilon^\alpha$. See [3, Corollary 2].

Corollary 3.2. *Given a subdomain $\Omega' \Subset \Omega$, there exists a universal constant $C = C(\Omega')$ such that for $X \in \Omega' \cap \{u_\varepsilon > \varepsilon^\alpha\}$ and $\varepsilon \le d_\varepsilon(X)$,*

$$C^{-1} d_\varepsilon(X)^\alpha \le u_\varepsilon(X) \le C d_\varepsilon(X)^\alpha.$$



Total control of the minimal solution u_ε.

4. LIMITING FREE BOUNDARY PROBLEM

In this Section, we address the fully nonlinear free boundary problem obtained by letting $\varepsilon \to 0$. The ultimate goal is to find a solution to the free boundary problem (1.1) that enjoys all the desired analytic and geometric properties.

Our analysis starts off by the compactness of minimal solutions to Equation (E_ε). In fact, Proposition (2.1) implies that $\{u_\varepsilon\}_{\varepsilon>0}$ is a compact sequence and up to a subsequence,

$$\lim_{\varepsilon \to 0} u_\varepsilon =: u_0. \tag{4.1}$$

Here, we address the study of geometric properties to the limiting function u_0.

Next Theorem recovers the fully nonlinear equation satisfied by u_0 within its positive set as well as its precise growth behavior near the free boundary, $\mathfrak{F}(u_0)$.

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THEOREM 4.1. *The limiting function u_0 defined in (4.1) is a viscosity solution to*

$$F(D^2u) = \gamma u^{\gamma-1} \quad \text{in } \{u > 0\}. \tag{4.2}$$

Moreover, for a fixed $\Omega' \Subset \Omega$, there exists a constant $C = C(\Omega')$ that depends on Ω' and universal constants such that for any $X \in \Omega' \cap \{u_0 > 0\}$, there holds

$$Cd_0(X)^\alpha \leq u_0(X) \leq C^{-1}d_0(X)^\alpha,$$

as well as,

$$C^{-1}\rho^\alpha \leq \sup_{B_\rho(X)} u_0 \leq C(\rho^\alpha + u_0(X)) \tag{4.3}$$

for $d_0(X), \rho$ positive numbers sufficiently small.

Proof. Let us fix a point $X_0 \in \{u_0 > 0\}$ and let $u_0(X_0) := \sigma > 0$. By continuity $u_0 \geq \frac{1}{2}\sigma$ in $B_\rho(X_0)$ for same $\rho > 0$. Since $u_\varepsilon \to u_0$ uniformly over compact sets, for $\varepsilon \ll 1$ we have

$$u_\varepsilon \geq \frac{1}{8}\sigma > (1+\sigma_0)\varepsilon^\alpha.$$

That is, u_ε satisfies

$$F(D^2u_\varepsilon) = \gamma u_\varepsilon^{\gamma-1} \quad \text{in} \quad B_{\frac{1}{2}\rho}(X_0).$$

By the stability of viscosity solutions under uniform limits, we conclude u_0 is indeed a viscosity solution to Equation (4.2).

Let us now turn our attention to the growth rate controls. For that, fix $X_0 \in \Omega' \cap \{u_0 > 0\}$, with $d_0(X_0) \leq \frac{1}{4}\text{dist}(\Omega', \partial\Omega)$ and label $u_0(X_0) = s > 0$. For $\varepsilon \ll 1$ we have

$$u_\varepsilon(X_0) \geq \frac{s}{2} > \varepsilon^\alpha.$$

Thus, according to Corollary 3.2, we obtain

$$u_\varepsilon(X_0) \geq Cd_\varepsilon(X_0)^\alpha.$$

Let $Y_\varepsilon \in \partial\{u_\varepsilon > \varepsilon^\alpha\}$ be such that $d_\varepsilon(X_0) = |X_0 - Y_\varepsilon|$. By uniform convergence, it clearly follows that $Y_\varepsilon \to Y_0$ and $u_0(Y_0) = 0$. In conclusion,

$$u_0(X_0) \geq C|X_0 - Y_0|^\alpha \geq Cd_0(X_0)^\alpha.$$

The upper estimate is obtained similarly. The proof of the estimate (4.3) follows similarly. ▮

It also follows as consequence that the set $\{u_0 > 0\}$ has uniform positive density along the free boundary $\mathfrak{F}(u_0)$.

Theorem 4.2. *Given $\Omega' \Subset \Omega$ there exists a constant $0 < c \leq 1$, depending on Ω' and universal parameters, such that*

$$\frac{\mathfrak{L}\left(B_\delta(X) \cap \{u_0 > 0\}\right)}{\mathfrak{L}\left(B_\delta(X)\right)} \geq c,$$

for all $X \in \mathfrak{F}(u_0) \cap \Omega'$.

As mentioned, one of the major mathematical difficulties in dealing with singular equations as in (1.1) is the fact that right hand side blows-up near the quenching region. In particular, if one tries to interpret the singular term $\gamma u^{\gamma-1}$ as a right hand side $f(X)$ for the equation, classical Harnack inequality gives no information near the free boundary. we obtain a *clean* Harnack inequality valid near the free boundary $\mathfrak{F}(u_0)$. Such a result is quite surprising a first view, as the nonlinear source of the equation is of order $\sim u^{\gamma-1}$ and thus it blows up near the boundary of the quenching region.

Theorem 4.3 (Harnack Inequality for tangential balls). *Let $X_0 \in \{u_0 > 0\}$ and $d := d_0(X_0)$. Then, there exist a universal constant $C > 0$ such that*

$$\sup_{B_{\frac{d}{2}}(X_0)} u_0 \leq C \inf_{B_{\frac{d}{2}}(X_0)} u_0.$$

5. GEOMETRIC ESTIMATES OF THE FREE BOUNDARY

In this section, we turn our attention to uniform geometric-measure properties of the limiting free boundary $\mathfrak{F} := \partial\{u_0 > 0\}$. Through this section we shall work under the following extra structural condition on the operator F:

Definition 5.1. We say a uniformly elliptic operator $F\colon \mathrm{Sym}(N) \to \mathbb{R}$ is asymptotically concave if there exists a positive definite matrix $\mathcal{F} = (f_{ij})_{ij}$ and a nonnegative constant $C_F \geq 0$ such that

$$f_{ij}\mathcal{M}_{ij} - F(\mathcal{M}) \geq -C_F, \tag{AC}$$

for all matrix $\mathcal{M} \in \mathrm{Sym}(N)$,

Initially, let us point out that indeed hypothesis (AC) is an asymptotic condition as $\|M\| \gg 1$, as it suffices to hold in the limit for $\|M\| \to +\infty$. It represents a sort of concavity condition at infinity of F. For concave operators, $C_F = 0$. The structural condition (AC) arises from recent considerations on the *recession* operator

$$F^\star(\mathcal{M}) := \lim_{\mu \to 0} \mu F(\mu^{-1}\mathcal{M}).$$

The limiting operator $F^\star$ should be interpreted as the tangential equation for the natural elliptic scaling on F. For example, for a number of elliptic operators, it is possible to verify the existence of the limit

$$b_{ij} := \lim_{\|\mathcal{M}\| \to \infty} F_{ij}(\mathcal{M}).$$

In this case, $F^\star(\mathcal{M}) = \mathrm{tr}(b_{ij}\mathcal{M})$ and (AC) is automatically satisfied. A particularly interesting example is the class of Hessian operators of the form

$$F_\iota(M) = f_\iota(\lambda_1, \lambda_2, \cdots \lambda_N) := \sum_{j=1}^{N} (1 + \lambda_j^\iota)^{1/\iota},$$

where ι is an odd natural number. For this family of operators, we have $F_\iota^\star = \Delta$ and condition (AC) is satisfied.

In this final Section we explain further fine geometric-measure properties of the free boundary $\mathfrak{F}(u_0)$. Hereafter we shall work under the addition structural assumption (AC). Thanks the previous results, we show concerns the local finiteness of the $\mathcal{H}^{N-1}$-Hausdorff measure of the free boundary $\mathfrak{F}(u_0)$.

Theorem 5.1. *Given $\Omega' \Subset \Omega$ there exists a constant $C = C(\Omega') > 0$, depending on Ω' and universal constants, such that*

$$\mathfrak{L}\left(\mathcal{N}_\mu(\{u_0 > 0\}) \cap B_\rho(X_0)\right) \le C\mu\rho^{N-1},$$

whenever, $X_0 \in \Omega' \cap \partial\{u_0 > 0\}$, $d_0(X_0) < \frac{1}{10} dist(\Omega', \partial\Omega)$, $\mu \ll \rho$ and ρ is universally small. In particular,

$$\mathcal{H}^{N-1}\left(B_\rho(X_0) \cap \mathfrak{F}(u_0)\right) \le C\rho^{N-1}.$$

A consequence of Theorem 5.1 is that the limiting region $\{u_0 > 0\}$ has locally finite perimeter. The key final result we will show here states that the reduced free boundary, $\partial_{\text{red}}\{u_0 > 0\}$ has total measure. More importantly, we prove that around points Z of the reduced free boundary, there holds

$$\mathcal{H}^{N-1}\left(B_\rho(Z) \cap \mathfrak{F}(u_0)\right) \sim \rho^{N-1}.$$

In particular the free boundary has a theoretical measure outward unit vector for $\mathcal{H}^{N-1}$ almost all points in $\mathfrak{F}(u_0)$.

Theorem 5.2. *Given $\Omega' \Subset \Omega$, there exists a positive constant $C = C(\Omega')$, that depends only on Ω' and universal constants, such that for any ball $B_\rho(X_0)$, with ρ universally small, centered at a free boundary point $x_0 \in \partial\{u_0 > 0\}$, there holds*

$$C^{-1}\rho^{N-1} \le \mathcal{H}^{N-1}(\partial_{red}\{u_0 > 0\} \cap B_\rho(X_0)) \le C\rho^{N-1}.$$

In particular,

$$\mathcal{H}^{N-1}\left(\partial\{u_0 > 0\} \setminus \partial_{red}\{u_0 > 0\}\right) = 0.$$

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