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Regularity of invariant foliations and its relation to the dynamics

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We briefly survey some of the recent results concerning the metric behavior of the invariant foliations for a class of partially hyperbolic diffeomorphisms on a three-dimensional manifold and how it relates to the dynamics itself. The behavior of the invariant foliations ($\mathcal{F}^s, \mathcal{F}^c, \mathcal{F}^u$) treated here falls into two opposite classes: low regularity (atomic disintegration) and high regularity (smoothness). Low regularity can be obtained from the information on the Lyapunov exponents. And high regularity implies rigidity results, such as smooth conjugacy with a toy model (linear Anosov diffeomorphisms).

We propose a conjecture to characterize atomic behavior of center leaves for conservative partially hyperbolic homotopic to a linear Anosov (DA) on $\mathbb{T}^3$. On the other hand if one of the invariant foliations (stable, center or unstable) of a conservative DA diffeomorphism $\mathbb{T}^3$ is C^1 and transversely absolutely continuous with bounded Jacobians the Lyapunov exponent on this direction is defined everywhere and constant, which gives many consequences. We also propose many questions that we think might lead to better understanding the relation between the invariant foliations of a partially hyperbolic with its dynamics. May, 2015 ICMC-USP

1. INTRODUCTION

One of the goals in Ergodic Theory is somehow the understading of the orbits (i.e. $\{f^n(x)\}_n$) of a given diffeomorphism f from the statistical point of view. One way to understand these orbits, or the dynamics, is through some relevant statistical properties. One of such properties is called *ergodicity*. Given a diffeomorphism $f : M \to M$ we say that f preserves the probability μ if $\mu(f(A)) = \mu(A)$ for every measurable set A, denoted by $f_*\mu = \mu$. And we say that f is ergodic (for μ) if satisfies the following: given any invariant set A (i.e. $f(A) = A$), then $\mu(A) = 0$ or $\mu(A) = 1$. The Birkhoff's Ergodic Theorem (see [KH]) implies that for almost every point $x \in M$: $\sum_{n=0}^{\infty} \chi_B(f^n(x)) = \mu(B)$, where $B \subset M$ is a measurable set and χ_B is the characteristic function of B.

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All diffeomorphisms are assumed to be at least C^2. A classical result, due to Anosov [Ano], states that every volume preserving - what is now called an - Anosov diffeomorphism f on a compact manifold is ergodic. For the definition of Anosov diffeomorphism see Definition 1.1. The main technical ingredient of the proof given by Anosov is the absolute continuity of the stable and unstable foliation. It is worth pointing out that *foliation* in this paper has the meaning of a C^0 foliation, but with smooth leaves.

A foliation $\mathcal{F}$ is called *absolutety continuous* if given two transversals Σ_1 and Σ_2 to the foliation the holonomy map is absolutely continuous.

What Anosov did was to obtain a dynamical information (in his case ergodicity) from the regularity of the invariant foliations (the stable and unstable foliations are absolutely continuous). One may ask how much more information can one obtain by imposing stronger conditions on the stable and unstable foliations. For instance what would one get if we suppose smooth regularity for the stable and unstable foliations? For volume preserving Anosov diffeomorphisms on the $\mathbb{T}^2$ and for symplectic Anosov diffeomorphism on $\mathbb{T}^4$ Flaminio and Katok [FK] have shown that smooth regularity on the stable and unstable foliations implies smooth regularity of the conjugacy.

Instead of defining an Anosov diffeomorphism we will define a more general class of diffeomorphisms, which are known as partially hyperbolic diffeomorphisms. Let us first quickly motivate their existence. All known Anosov diffeomorphisms are robustly transitive. A diffeomorphism f is said to be transitive if there exist a dense orbit, and robustly transitive means that any diffeomorphism g C^1-close to f is transitve. Mañé proved that the only robustly transitive volume preserving diffeomorphism on the two-torus are Anosov diffeomorphisms (for more references see [BDV]). Mañé and Shub gave examples of non-anosov robustly transitive diffeomorphism on $\mathbb{T}^3$ and $\mathbb{T}^4$ respectively. It turns out that these two last examples are partially hyperbolic. In fact partially hyperbolic diffeomorphisms are in some sense natural in the robustly transitive world (for a discussion see [BDV]).

There are different, but similar, definitions of partially hyperbolic diffeomorphisms, we adopt the following:

Definition 1.1. A diffeomophism $f : M \to M$ on a compact manifold is called **partially hyperbolic** if the tangent bundle of M admits a Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that for any three $x, y, z \in M$

$$\|D_x f v^s\| < \|D_y f v^c\| < \|D_z f v^u\|$$

where v^s, v^c and v^u are unit vectors belonging respectively to E^s_x, E^c_y and E^u_z. An **Anosov diffeomorphism** is one for which it is possible to split the tangent space solely in an stable and unstable directions.

The subbundles E^s and E^u are the stable and unstable direction. These two subbundles integrate into invariant foliations, known as the stable $\mathcal{F}^s$ and unstable $\mathcal{F}^u$ foliation (see [BDV] and references therein). We point out that on Definition 1.1 the splitting of TM is not assumed any regularity, it follows a posteriori that these subbundles are Hölder continuous. In particular it is not possible to use Frobenius' theorem to integrate the invariant subbundles E^s and E^u. The existence of a center foliation (that is an invariant

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foliation tangent to the E^c direction) does not always exist. But for many cases it does exist, in particular all known transitive partially hyperbolic diffeomophism on a three-dimensional manifold M^3 admits a center foliation. One of such diffeomophisms, for which there exists $\mathcal{F}^c$, are the *Derived from Anosov* (DA) on $\mathbb{T}^3$.

DEFINITION 1.2. We say that $f : \mathbb{T}^n \to \mathbb{T}^n$ is a Derived from Anosov diffeomophism (or DA diffeomorphism) if f is partially hyperbolic and homotopic to a linear Anosov $A : \mathbb{T}^n \to \mathbb{T}^n$, that is $A \in SL(n, \mathbb{Z})$ and A is a hyperbolic matrix.

The automorphism $A : \mathbb{T}^n \to \mathbb{T}^n$ above, is the *linearization* of f and it is semi-conjugate to f: i.e. there exists a continuous and surjective map h such that $h \circ f = A \circ h$. When such an h is a homeomorphism, then h is called a conjugacy. A smooth conjugacy means that the map h is a smooth map.

Remark: We refer the reader to [Var1] for a deeper discussion on the importance of restricting attention to the class of DA diffeomorphisms and trying to study the dynamics through its foliations regularity.

If a foliation $\mathcal{F}$ is absolutely continuous, then given a set $A \subset M$ of full measure for volume almost every leaf $\mathcal{F}$ intersects the set A in a set of full leaf-volume (see [Var1] for a discussion). We know that the stable and unstable foliation always have this kind of Fubini property. It turns out that for the center foliation such behavior is not the expected one (e.g. Theorem 2.1).

A foliation $\mathcal{F}$ has *atomic disintegration* is there exist a set A of full volume such that each leaf of $\mathcal{F}$ intersects A in a countable set. The first example of atomic disintegration for a partially hyperbolic diffeomorphism with non-compact center leaves was given by Ponce, Tahzibi, Varão [PTV] (see Theorem 2.1). This is done by understanding the central Lyapunov exponent of the dynamics. Let us recall the definition of Lyapunov exponents of f along the invariant bundles E^s, E^c and E^u.

$$\lambda^\tau(x) := \lim_{n\to\infty} \frac{1}{n} \log ||Df^n(x) \cdot v||$$

where $v \in E^\tau$ and $\tau \in \{s, c, u\}$. If f is ergodic, the stable, center and unstable Lyapunov exponents are constant almost everywhere.

Before stating the results let us formalize the important notion of disintegration of a measure (we refer the reader to [Var1] for a more detailed discussion).

Let $(M, \mu, \mathcal{B})$ be a probability space, where M is a compact metric space, μ a probability and $\mathcal{B}$ the Borelian σ-algebra. Given a partition $\mathcal{P}$ of M by measurable sets, we associate the measurable set

$$(\mathcal{P}, \widetilde{\mu}, \widetilde{\mathcal{B}})$$

by the following way. Let $\pi : M \to \mathcal{P}$ be the canonical projection associate to a point of M the partition element that contains it. Then we define $\widetilde{\mu} := \pi_*\mu$ and $\widetilde{\mathcal{B}} := \pi_*\mathcal{B}$.

Given a partition $\mathcal{P}$. A family $\{\mu_P\}_{p\in\mathcal{P}}$ is a *system of conditional measures* for μ (with respect to $\mathcal{P}$) if

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i) given $\phi \in C^0(M)$, then $P \mapsto \int \phi \mu_P$ is measurable;
ii) $\mu_P(P) = 1$ $\widehat{\mu}$-a.e.;
iii) if $\phi \in C^0(M)$, then $\displaystyle\int_M \phi d\mu = \int_{\mathcal{P}} (\int_P \phi d\mu_P) d\widetilde{\mu}$.

Observe that the conditions *i)* and *iii)* also hold for bounded ϕ by the Dominated Convergence theorem. When it is clear which partition we are referring to, we say that the family $\{\mu_P\}$ *disintegrates* the measure μ.

Proposition 1.1. *If $\{\mu_P\}$ and $\{\nu_P\}$ are conditional measures that disintegrate μ, then $\mu_P = \nu_P$ $\widetilde{\mu}$-a.e.*

Notice that if $T : M \to M$ preserves a probability μ and the partition $\mathcal{P}$, then $T_*\mu_P = \mu_P$ $\widetilde{\mu}$-a.e.. In fact, it follows from the fact that $\{T_*\mu_P\}_{P\in\mathcal{P}}$ is also a disintegration of μ.

We say that a partition $\mathcal{P}$ is measurable if there exists a Borelian family $\{A_i\}_{i\in\mathbb{N}}$ such that

$$\mathcal{P} = \{A_1, A_1^c\} \vee \{A_2, A_2^c\} \vee \ldots \bmod 0.$$

Theorem 1.1 (Rohlin's disintegration [Roh]). *Let $\mathcal{P}$ be a measurable partition of a compact metric space M and μ a Borelian probability. Then there exists a disintegration by conditional measures for μ.*

By the discussion above if we disintegrate volume on a foliated box for the stable or unstable foliations the conditional measures are absolutely continuous to the lebesgue measure of the leaf. And atomic disintegration means that the conditional measures are sums of Dirac measures. Notice that to disintegrate a measure is a type of "Fubini" property.

2. RESULTS, PROBLEMS AND CONJECTURES

2.1. Low regularity of invariant foliations

We begin with a result on atomic disintegration.

Theorem 2.1. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a volume preserving, DA diffeomorphism. Suppose its linearization A has the splitting $T_A M = E^{su} \oplus E^{wu} \oplus E^s$ (su and wu represents strong unstable and weak unstable.) If f has $\lambda^c(x) < 0$ for Lebesgue almost every point $x \in \mathbb{T}^3$, then volume has atomic disintegration on $\mathcal{F}^c_f$, in fact the disintegration is mono atomic.*

By mono atomic we mean: there exists a set A of full volume which intersect each center leaf at one point.

The above result as well as the study disintegration done by [Var1] lead us to the following conjecture:

Conjecture: Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a volume preserving DA diffeomophisms, then the center foliation has atomic disintegration if and only if $\lambda^c_A \lambda^c_f < 0$, where λ_f is the center

Lyapunov exponent defined almost everywhere and λ^c_A is the center Lyapunov exponent for A.

This conjecture was first announced by the author in his talk at Gutierrez Prize symposium in 2013 at ICMC-USP.

As we have seen the invariant foliations can be absolutely continuous and they can have atomic disintegration. The first (non-trivial) example of a foliation in a partially hyperbolic diffeomorphism which is not absolutely continous nor have atomic disintegration was given by Varão [Var1, Var2]:

Theorem 2.2. *There exist volume preserving DA diffeomorphisms on $\mathbb{T}^3$ for which the center foliation is non-absolutely continuous and the disintegration of volume on center leaves are not atomic.*

We point out that these example is in accordance with the conjecture above. In fact it is obtained in the Anosov diffeomorsphism context. That is, f is taken to be an Anosov diffeomorphism with the subbundle having a partially hyperbolic splitting. A first step to get the conjecture above is to answer:

Question 2.2.1. *Let $A : \mathbb{T}^3 \to \mathbb{T}^3$ be a linear Anosov seen as a partially hyperbolic. Consider f a small perturbation of A and volume preserving. Can the center foliation of f have atomic disintegration?*

If the conjecture is right, the answer should negative.

2.2. High regularity of invariant foliations

It is conjectured for an Anosov diffeomorphism that if the stable and unstable foliations are C^r, then f is C^r conjugate to a linear Anosov (see [FK]). Recall the results in these case are for Anosov diffeomorphism on $\mathbb{T}^2$ and $\mathbb{T}^4$. This conjecture has been tackled in the context of DA diffeomorphism in $\mathbb{T}^3$ in [Var3].The stronger regularity in the case of DA diffeomorphisms will be C^1 foliations and the following uniform condition:

We say that a foliation $\mathcal{F}$ is *transversely absolutely continuous with bounded Jacobians* if: given an angle $\theta \in (0, \pi/2]$, there exists $K \geq 1$ such that, for any two transversals τ_1, τ_2 to $\mathcal{F}$ of angle at least θ with the leaves of $\mathcal{F}$, and any measurable set $A \subset \tau_1$ then

$$K^{-1} m_{\tau_1}(A) \leq m_{\tau_2}(h^{\mathcal{F}}_{1,2}(A)) \leq K m_{\tau_1}(A),$$

where m_{τ_i} is the Lebesgue measure on the transversal τ_i and $h^{\mathcal{F}}_{1,2}$ is the $\mathcal{F}$-holonomy from τ_1 to τ_2.

Theorem 2.3. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a volume preserving DA diffeomorphism, such that $\mathcal{F}^s_f$ and $\mathcal{F}^u_f$ are C^1 and transversely absolutely continuous foliation with bounded Jacobians, then f is smoothly conjugate to its linearization. In particular f is an Anosov diffeomorphism.*

The main ingredient of the proof of the above theorem is

Theorem 2.4. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a volume preserving DA diffeomorphism, such that there exists $* \in \{s, c, u\}$ for which $\mathcal{F}_f^*$ is C^1 and transversely absolutely continuous foliation with bounded Jacobians, then the Lyapunov exponent in the $*$-direction is constant and defined everywhere.*

We point out that we shall see A, the linearization of f, which is an Anosov diffeomorphism, as a partially hyperbolic diffeomorphism. We do so by considering each of the eigenvectors as the E^u, E^c, E^s directions for A. From the proof of Theorem 2.4 we can say exactly the value of the Lyapunov exponent

Corollary 2.1. $\lambda_f^* = \lambda_A^*$.

Remark 2. 1. Theorem 2.4 is false if we do not assume transversely absolutely continuous foliation with bounded Jacobians. In [Var1] is given an example of a DA diffeomorphism with C^1 center leaves, but which is not C^1 conjugate to its linearization, compare to Corollary 2.4 below.

Corollary 2.2. *It is not possible to construct a volume preserving Mañe's example (see [BDV] therein) preserving the center foliation.*

Proof. The Mañé's example, $f : \mathbb{T}^3 \to \mathbb{T}^3$ is a deformation of a linear Anosov automorphism, $A : \mathbb{T}^3 \to \mathbb{T}^3$, with three real distinct eigenvalues which is not an Anosov diffeomorphism, it is robustly transitive, and, by our hypothesis, it preserves the center foliation of the linear Anosov automorphism A. By absurd we assume it is volume preserving. Since $\mathcal{F}_f^c = \mathcal{F}_A^c$, then it is C^1 and transversely absolutely continuous. Hence it should have, by Theorem 2.4, the center Lyapunov exponent defined everywhere and constant, but there are two periodic points with distinct Lyapunov exponents. Absurd. ∎

Although the original construction of Mañé's example does not impose that it should have the center foliation coinciding with the center foliation of the linear Anosov automorphism A (once again see this linear Anosov automorphism as a partially hyperbolic diffeomorphism), but it is usually taken preserving the center leaf. In fact, on [BDV] they consider Mañé's example as one preserving the center foliation of the linear Anosov automorphism.

Corollary 2.3. *If f, as in Theorem 2.4, and $\mathcal{F}_f^* = \mathcal{F}_f^c$, then the center Lyapunov exponent of f is non-zero.*

Proof. It is straightforward from Theorem 2.4 and Corollary 2.1 ∎

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COROLLARY 2.4. *Let f be as in Theorem 2.4. If the center foliation is C^1 and transversely absolutely continuous with bounded Jacobians, then f is C^1 conjugated to a linear Anosov. In particular f is an Anosov diffeomorphism.*

Proof. We refer the reader to [Var3]. ∎

The rigidity result of Corollary 2.4 does not extend directly into higher dimension:

Example: Let $A : \mathbb{T}^4 \to \mathbb{T}^4$, $A \in SL(4, \mathbb{Z})$ be a hyperbolic matrix with eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4$, where $\lambda_1 < 1$ and $\lambda_4 > 1$ are the stable and unstable directions, respectively, and the sum of the eigenspaces of the eigenvalues λ_2 and λ_3 form the center direction. Let us consider a perturbation $A \circ h$ of A, where $h : \mathbb{T}^4 \to \mathbb{T}^4$ is a volume preserving diffeomorphism defined as follows, h is the identity map out side of some small ball of a fixed point $p \in \mathbb{T}^4$, h is volume preserving and preserves the E^c_A direction, then $\int \lambda^c_{A\circ h} < \int \lambda^c_A$ (see [Var1, Var3] for references in this part), hence A and $A \circ h$ are not C^1 conjugate, but $\mathcal{F}^c_{A\circ h}$ is a C^1 foliation transversely absolutely continuous with bounded Jacobians, because $\mathcal{F}^c_{A\circ h} = \mathcal{F}^c_A$.

Although the rigidity result of Corollary 2.4 is not valid as stated for higher dimension, Theorem 2.4 shed some light on what type of rigidity result to look for.

Question: Suppose $f : \mathbb{T}^4 \to \mathbb{T}^4$ is a volume preserving DA diffeomorphism with a two dimensional center foliation, if the center foliation as well as an invariant subfoliation of the center foliation are C^1 and transversely absolutely continuous with bounded Jacobian, does one get rigidity (smooth conjugacy with a linear Anosov)?

3. THE TECHNIQUES

We have selected some crucial points of the techniques to prove the results mentioned above.

3.1. Atomic disintegration

Theorem 2.1 has two parts, the first one is to prove atomic disintegration, the second one is to prove mono atomicity. The idea to prove atomicity comes from earlier works on the area (see [Var1] for a discussion), the novelty introduced by [Var1] was to obtain mono atomicity in a hyperbolic situation and [PTV] extends for the case of DA diffeomorphisms.

Suppose f is Anosov, consider $\{R_1, \ldots, R_k\}$ a Markov partition of $\mathbb{T}^3$. Let us suppose that $\mathcal{F}^c$ is expanding, the analogous argument works for the contracting case. Note that since center leaf goes to center leaf, a Markov property implies that $f(\mathcal{F}^c_{R(x)}(x)) \supset \mathcal{F}^c_{R(f(x))}(f(x))$.

The following is a key lemma:

LEMMA 3.1. *All the atoms have the same weight when considering the disintegration of volume on the center leaves of R_i.*

Proof. On each Markov rectangle we may apply Rokhlin's disintegration theorem on center leaves. Therefore, when writing m_x we mean the conditional measure for the disintegration on Markov rectangle that contains x. Consider the set $A_\delta = \{x \in A \mid m_x(x) \leq \delta\}$. Since $f(\mathcal{F}^c_{R(x)}(x)) \supset \mathcal{F}^c_{R(f(x))}(f(x))$, we have that $f_* m_x(I) \leq m_{f(x)}(I)$ where I is inside the connected component of $\mathcal{F}^c_{f(x)} \cap R(f(x))$ that contains $f^n(x)$. If $f(x) \in A_\delta$, then

$$m_x(x) = f_* m_x(f(x)) \leq m_x(f(x)) \leq \delta.$$

Hence, $f^{-1}(A_\delta) \subset A_\delta$.

By ergodicity, since our Anosov is volume preserving on $\mathbb{T}^3$, A_δ has full measure or zero measure. Let δ_0 be the discontinuity point of the function $\delta \in [0,1] \mapsto Vol(A_\delta)$. This implies that almost every atom has weight δ_0. ▮

If the center foliaton of f does not have one atom per center leaf it is possible by iterations to increase the number of atoms in each rectangle, which would imply an absurd, as it should be constant by the lemma above. This proves Theorem 2.1.

3.2. Conditional measures with dynamical meaning

A key step in the proof of Theorem 2.4 is the constructions of a measure m_x (not a probability) supported on the leaf $\mathcal{F}^*(x)$ having the following dynamical meaning $f_* m_x = \lambda m_{f(x)}$ and having uniformly bounded densities ρ_x. We refer the reader to [Var1, Var3, Var2] for the constructions of such measures. And let us finish the proof of Theorem 2.4.

Note that

$$\frac{df^n_* m_x}{dm_{f^n(x)}}(f^n(x)) = \lambda^{-n}.$$

Let us calculate the Radon-Nikodym derivative by another way. Let $I^n_\delta \subset \mathcal{F}^c_{f^n(x)}$ be a segment of length δ around $f^n(x)$. Then

$$\frac{df^n_* m_x}{dm_{f^n(x)}}(f^n(x)) = \lim_{\delta \to 0} \frac{f^n_* m_x(I^n_\delta)}{m_{f^n(x)}(I^n_\delta)}.$$

And

$$\begin{aligned}
\frac{df^n_* m_x}{dm_{f^n(x)}}(f^n(x)) &= \lim_{\delta \to 0} \frac{m_x(f^{-n}(I^n_\delta))}{m_{f^n(x)}(I^n_\delta)} = \lim_{\delta \to 0} \frac{\int_{f^{-n}(I^n_\delta)} \rho_x d\lambda_x}{\int_{I^n_\delta} \rho_{f^n(x)} d\lambda_{f(x)}} \\
&\approx \frac{\rho_x(x)}{\rho_{f^n(x)}(f^n(x))} \lim_{\delta \to 0} \frac{\int_{f^{-n}(I^n_\delta)} d\lambda_x}{\int_{I^n_\delta} d\lambda_{f(x)}} \approx \lim_{\delta \to 0} \frac{\rho_x(x)}{\rho_{f^n(x)}} \frac{\int_{I^n_\delta} ||Df^{-n}|| d\lambda_x}{\int_{I^n_\delta} d\lambda_{f(x)}} \\
&\approx \frac{\rho_x(x)}{\rho_{f^n(x)}(f^n(x))} ||Df^{-n}(x)||.
\end{aligned}$$

We then have

$$\lim_{\delta \to 0} \frac{df^n_* m_x}{dm_{f^n(x)}}(I^n_\delta) = \frac{\rho_x(x)}{\rho_{f^n(x)}(f^n(x))}||Df^{-n}(x)||.$$

From the other equalities we have

$$\frac{\rho_x(x)}{\rho_{f^n(x)}(f^n(x))}||Df^{-n}(x)|| = \lambda^{-n}.$$

By applying "$lim_{n\to\infty} 1/n \; log$" to the above equality we get

$$\lambda^c(x) = log\lambda,$$

since the densities of m_x are uniformly limited. The proof of Theorem 2.4 is now completed.

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