CADERNOS DE MATEMÁTICA **16**, 169–177 May (2015)
ARTIGO NÚMERO SMA#11

Algebraic topology applied to equilibrium problems

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This paper is a survey about my joint work with Professor Carlos Biasi on the relationship between the existence of local equilibria in non-cooperative games and the existence of fixed points of applications. Once this relationship is clear, we can apply methods of algebraic topology to ensure the existence of equilibria. May, 2015 ICMC-USP

1. INTRODUCTION

Given a set X and a real function $p : X \to \mathbb{R}$, one can ask if f has maxima (minima) points. From general topology, it is well know that the answer is affirmative if X is a compact topological space and p is a continuous function. A more complicated situation is when we have $n \geq 2$ functions, $p_1, \ldots, p_n : X \to \mathbb{R}$, and we try to find a point $x \in X$ maximizing simultaneously all functions p_i, $1 \leq i \leq n$. This type of question arises naturally in the study of game theory and mathematical economics, as we will see below.

An n-person non-cooperative game can be characterized by the following elements: there are a set of n players, $\{1, 2, \ldots, n\}$. Each player i has a set S_i of possible strategies, and it chooses some element $s_i \in S_i$. These choices are to be made simultaneously by all the players. The profit of the competition to the player i is a function $p_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ which depends on the choices $s_1, \ldots, s_n$ of all players, not just on the choice s_i of the player i. The function p_i is called the payoff function of player i. An assumption made - called axiom of rationality - is that the aim of player i is to make a choice of strategy $s_i \in S_i$ good enough to maximize his profit, with the understanding that each player $j \neq i$ is simultaneously trying to make a choice $s_j \in S_j$ in order to maximize his own profit. An example is the following: imagine an one-dimensional beach. In this place, there are two sellers of ice cream. A person in that beach who wants an ice cream will look to the nearest seller. The sellers need to choose in which place of the beach they will stay (set of strategies). Assuming that the distribution of people on the beach is a constant function,

* Partially supported by CAPES, Brazil.

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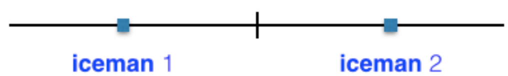
the goal of each one of the sellers is to stay in a place which is closer to the most people. The above figure represents a situation in which each vendor meets half the beach. Of course that the iceman 1 can decide to take a position on the right of his initial position, but this will cause a movement of the iceman 2 to the left, and so on. Question: when do these movements stop? With this question, we go to the brilliant concept explicited by John Nash about what would be an equilibrium in a non-cooperative game.



In his PhD thesis, John Forbes Nash Jr. established a concept of solution to non-cooperative games, nowadays called Nash equilibrium, which says: a point $(\tilde{s}_1, \dots, \tilde{s}_n) \in S_1 \times \cdots \times S_n$ is a Nash equilibrium to $p_1, \dots, p_n$ if

$$p_i(\tilde{s}_1, \dots, \tilde{s}_n) \geq p_i(\tilde{s}_1, \dots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \dots, \tilde{s}_n), \ \forall\ s_i \in S_i,\ 1 \leq i \leq n.$$

The interpretation is that a Nash equilibrium is a solution where there is no motivation to any player changes its strategy if the other do not.

The Nash equilibrium concept is in itself a finding. According to John Milnor [2]

"...the ideas in Nash's thesis are simple and rigorous, and provide a firm background, not only for economic theory but also for research in evolutionary biology, and more generally for the study of any situation in which human or nonhuman beings face competition or conflict."

Nash proved that:

THEOREM 1.1 (Nash's Theorem). *Let $S_1, \dots, S_n$ be compact convex subsets of an Euclidean space. Suppose that $p_1, \dots, p_n : S = S_1 \times \cdots \times S_n \to \mathbb{R}$ are maps such that, for each $i = 1, \dots, n$, $p_i(s_1, \dots, s_n)$ is linear(afim) as a function of s_i. Then there exists at least one equilibrium to $p_1, \dots, p_n$.*

The proof is an application of Brouwer's fixed point theorem. The idea is the following:
Let $S_i \subset \mathbb{R}^{d_i}$, where d_i is the dimension of S_i. Thus, $S \subset \mathbb{R}^d$, where $d = d_1 + \cdots + d_n$. From the hypothesis, the payoff functions are of the type

$$p_i(s) = v_i(s) \cdot s_i + u_i(s)$$

where $v_i : S \to \mathbb{R}^{d_i}$ and $u_i : S \to \mathbb{R}$ are maps which do not depend on the coordinate s_i, $i = 1, \dots, n$. Let $v : S \to \mathbb{R}^d$ be the vector field defined by $v(s) = (v_1(s), \dots, v_n(s))$. Let

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$r : \mathbb{R}^d \to S$ be the natural retraction that assigns each point $p \in \mathbb{R}^d$ to the point $r(p) \in S$ which realizes the distance of p to S. Finally, let $f : S \to S$ be defined by $f(s) = r(s+v(s))$. Then, one can shown that $\tilde{s} \in S$ is a Nash equilibrium to $p_1, \ldots, p_n$ if and only if $\tilde{s}$ is a fixed point of f. Note that the existence of a fixed point to f is assured by Brouwer's fixed point theorem. Then, Nash concludes the existence of equilibrium for such maps.

Based on the above proof, Professor Carlos Biasi put the question of existence of equilibrium points in the setting that the spaces of strategies are just compact ENR's, not necessarily convex. That means that each space S_i is a subset of some euclidean space $\mathbb{R}^{d_i}$ and there is an open neighborhood V_i of S_i in $\mathbb{R}^{d_i}$ and a retraction $r_i : V_i \to S_i$. The next sections are dedicated to explain what kind of results we achieve with respect to this question.

2. PROPERTY OF CONVENIENT RETRACTION

Definition 2.1. We say that a subset X of $\mathbb{R}^m$ has the **property of convenient retraction (abbrev., p.c.r.)** if there exists a retraction $r : V \to X$, where V is an open neighborhood of X in $\mathbb{R}^m$, satisfying: given $x_0 \in V$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle x_0 - r(x_0), x - r(x_0)\rangle \leq \varepsilon \|x - r(x_0)\|,$$

for all $x \in X$ with $\|x - r(x_0)\| < \delta$, where $\langle\ ,\ \rangle$ denotes the inner product in $\mathbb{R}^m$ and $\|\cdot\|$ denotes the the norm induced by it. In this case, we say that $r : V \to X$ is a convenient retraction.

Another class of space with the p.c.r. is:

Proposition 2.1. *Every submanifold M of $\mathbb{R}^n$, of class C^2, with or without boundary, has the p.c.r..*

Proof. Let $M \subset \mathbb{R}^n$ be a C^2 manifold. Then, M has a tubular neighborhood V in $\mathbb{R}^n$. The retraction $r : V \to M$ that, to each point $x \in V$ assigns the foot of the single normal segment that contains it, is a convenient retraction. In fact, we have that r is a differentiable map of class C^1. Thus, given $x_0 \in V$ and $x \in M$, we have

$$\begin{aligned} x = r(x) &= r(r(x_0)) + r'(r(x_0)) \cdot (x - r(x_0)) + \rho(x - r(x_0)) \\ &= r(x_0) + r'(r(x_0)) \cdot (x - r(x_0)) + \rho(x - r(x_0)) \end{aligned}$$

with

$$\lim_{h \to 0} \frac{\rho(h)}{\|h\|} = 0,$$

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where $r'(r(x_0)) : \mathbb{R}^n \to T_{r^2(x_0)}M$ is the derivative of r at $r(x_0)$, noting that $r^2(x_0) = r(x_0)$. Since $r'(r(x_0)) \cdot (x - r(x_0)) \in T_{r(x_0)}M$, it follows that

$$\langle x_0 - r(x_0), r'(r(x_0)) \cdot (x - r(x_0))\rangle = 0.$$

Thus,

$$\langle x_0 - r(x_0), x - r(x_0)\rangle = \langle x_0 - r(x_0), \rho(x - r(x_0))\rangle \leq \|x_0 - r(x_0)\|\|\rho(x - r(x_0))\|.$$

Since $\lim_{h\to 0} \rho(h)/\|h\| = 0$, given $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x - r(x_0)\| < \delta$ then

$$\|\rho(x - r(x_0))\| \leq \frac{\epsilon}{\|x_0 - r(x_0)\|}\|x - r(x_0)\|.$$

Hence, for $x \in M$ with $\|x - r(x_0)\| < \delta$, we have

$$\langle x_0 - r(x_0), x - r(x_0)\rangle \leq \epsilon\|x - r(x_0)\|.$$

Therefore, M has the p.c.r.. ∎

Let X be a closed subset of the euclidean space $\mathbb{R}^n$ and let V be an open neighborhood of X in $\mathbb{R}^n$. A map $r : V \to X$ is called a proximative retraction (or metric projection) if

$$\|r(y) - y\| = \text{dist}(y, X), \text{ for every } y \in V,$$

where

$$\text{dist}(y, X) = inf\{\|x - y\| \mid x \in X\}$$

is the distance of y to X.

Evidently, every proximative retraction is a retraction map but not conversely.

A compact subset $K \subset \mathbb{R}^n$ is called a proximative neighborhood retract (written $K \in$ PANR) if there exists an open neighborhood V of K in $\mathbb{R}^n$ and a proximative retraction $r : V \to K$.

The following result give us more examples of spaces with the p.c.r..

PROPOSITION 2.2. *Let K be a compact subset of $\mathbb{R}^n$. If $K \in$ PANR then K is an ENR with the p.c.r..*

Proof. Suppose $K \in$ PANR and let $r : V \to K$ be a proximative retraction. Then, r is a convenient retraction. Indeed, let $x_0 \in V$ and $x \in K$ be arbitraries. Since r is a proximative retraction, $\|x_0 - x\| \geq \|x_0 - r(x_0)\|$. Given $\varepsilon > 0$, if $\varepsilon \geq \|x_0 - r(x_0)\|$ then

$$\langle x_0 - r(x_0), x - r(x_0)\rangle \leq \varepsilon\|x - r(x_0)\|, \text{ for every } x \in K.$$

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Suppose $0 < \varepsilon < \|x_0 - r(x_0)\|$. Let $0 < \theta < \pi/2$ be such that

$$\cos\theta = \frac{\varepsilon}{\|x_0 - r(x_0)\|}.$$

Now, if we take $\delta = 2\varepsilon$, it is easy to see that, for every $x \in K$ such that $\|x - r(x_0)\| < \delta$, the angle α between $(x_0 - r(x_0))$ and $(x - r(x_0))$ is in $(\theta, \pi]$. Thus, for every $x \in K$ such that $\|x - r(x_0)\| < \delta$, we have

$$\begin{aligned}\langle x_0 - r(x_0), x - r(x_0)\rangle &\leq \frac{\varepsilon}{\|x_0 - r(x_0)\|}\|x_0 - r(x_0)\|\|x - r(x_0)\| \\ &= \varepsilon\|x - r(x_0)\|.\end{aligned}$$

Hence, r is a convenient retraction and, therefore, K is an ENR with the p.c.r.. ∎

A clear property of the ENR's with the p.c.r. is that the cartesian product of a finite number of spaces with the p.c.r. also has the p.c.r..

The above results show us that the class of compact ENR's with the p.c.r. is a relatively larger class than the class of compact convex subsets of euclidean spaces. It is in this class of space that we focus on our investigations. Remember that for spaces of strategies that are compact and convex, the equilibrium problem was obtained by applying Brouwer's fixed point theorem. Since now we are considering a more general class of spaces, the fixed point theorem that we will need must be more general than Brouwer's fixed point theorem. In fact, Lefschetz fixed point theorem will be useful to us.

3. LOCAL EQUILIBRIA

Suppose that the sets of strategies $S_1, \ldots, S_n$ are metric spaces. It seems natural to think of a local equilibrium for $p_1, \ldots, p_n : S_1 \times \cdots \times S_n \to \mathbb{R}$ to be a point $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n) \in S_1 \times \cdots \times S_n$ such that: there exists $\varepsilon > 0$ with

$$p_i(\tilde{s}_1, \ldots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \ldots, \tilde{s}_n) \leq p_i(\tilde{s}), \text{ for any } s_i \in B(\tilde{s}_i, \varepsilon),\ 1 \leq i \leq n,$$

where $B(\tilde{s}_i, \varepsilon)$ is the open ball in S_i with center in $\tilde{s}_i$ and radius ε. Thus, in a competition situation, the interpretation is that in a local Nash equilibrium neither player has incentive to chance its strategy to a close strategy if the other players kept fixed in its strategies. In this sense, we can say that a local Nash equilibrium is resistent to small unilateral changes.

A weaker kind of local equilibrium is the following.

Definition 3.1. Let $(S_1, d_1), \ldots, (S_n, d_n)$ be metric spaces and $p_1, \ldots, p_n : S_1 \times \cdots \times S_n \to \mathbb{R}$ real functions. We say that $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n) \in S$ is a **weak local equilibrium (abbrev., w.l.e.)** for $p_1, \ldots, p_n$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$p_i(\tilde{s}_1, \ldots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \ldots, \tilde{s}_n) \leq p_i(\tilde{s}) + \varepsilon d_i(s_i, \tilde{s}_i),$$

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for every $s_i \in B(\tilde{s}_i, \delta)$, $1 \leq i \leq n$.

Local equilibrium and weak local equilibrium have the same interpretation in the study of competitions. In both cases, there is no motivation to small unilateral changes of strategy. Moreover, every weak local equilibrium is a candidate to be a local equilibrium.

We prove that:

THEOREM 3.1 ([1]). *Let $p_1, \ldots, p_n : S_1 \times \ldots \times S_n \to \mathbb{R}$ be functions, where each $S_i \subset \mathbb{R}^{m_i}$ is a compact ENR with the p.c.r., $1 \leq i \leq n$. Also, suppose $p_i : S \to \mathbb{R}$ continuous as a function of n variables and $p_i(s_1, \ldots, s_n)$ continuously differentiable in a neighborhood of s_i when the other variables are kept fixed, $1 \leq i \leq n$. If $\chi(S_i) \neq 0$ for $1 \leq i \leq n$ then $p_1, \ldots, p_n$ have at least one w.l.e. ($\chi(S_i)$ denotes the Euler characteristic of S_i).*

EXAMPLE 3.1. Let $p_1, p_2 : [-1,1] \times [-1,1] \to \mathbb{R}$ be the functions given by

$$p_1(x,y) = -xy \quad \text{and} \quad p_2(x,y) = (2y+x)^2.$$

One can show that p_1, p_2 do not have Nash equilibria. However, by the above theorem, they have at least one w.l.e.. To find them, we take the natural retraction $r : \mathbb{R}^2 \to [-1,1] \times [-1,1]$, which is given by

$$r(x,y) = \begin{cases} (x,y), & \text{if } x, y \in [-1,1] \\ (x,1), & \text{if } x \in [-1,1] \text{ and } y \geq 1 \\ (x,-1), & \text{if } x \in [-1,1] \text{ and } y \leq -1 \\ (1,y), & \text{if } x \geq 1 \text{ and } y \in [-1,1] \\ (-1,y), & \text{if } x \leq -1 \text{ and } y \in [-1,1] \\ (1,1), & \text{if } x \geq 1 \text{ and } y \geq 1 \\ (1,-1), & \text{if } x \geq 1 \text{ and } y \leq -1 \\ (-1,-1), & \text{if } x \leq -1 \text{ and } y \leq -1 \\ (-1,1), & \text{if } x \leq -1 \text{ and } y \geq 1 \end{cases}$$

We consider the vector field

$$v(x,y) = \left(\frac{\partial p_1}{\partial x}(x,y), \frac{\partial p_2}{\partial y}(x,y) \right) = (-y, 8y+4x),$$

and we solve the equation

$$r\left((x,y) + v(x,y)\right) = (x,y),$$

whose solutions are $(0,0)$, $(-1, \frac{1}{2})$, $(1, -\frac{1}{2})$ and $(-1,1)$.

Then, $(0,0)$, $(-1, \frac{1}{2})$, $(1, -\frac{1}{2})$ and $(-1,1)$ are all w.l.e. for p_1, p_2. Moreover, $(-1,1)$ is a local equilibrium. Indeed, note that

$$p_1(x,1) = -x \leq 1 = p_1(-1,1), \text{ for all } x \in [-1,1]$$

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and

$$p_2(-1,y) = (2y-1)^2 \leq 1 = p_2(-1,1), \text{ for all } y \in [0,1].$$

The calculations in the present example are to illustrate the proof of Theorem 3.1.

4. SKETCH OF THE PROOF OF THEOREM 3.1

We are considering singular homology with coefficients in $\mathbb{Q}$ - the field of rational numbers.

If $X \subset \mathbb{R}^m$ is a compact ENR, then it is well known that $\mathrm{H}_i(X)$ is a finite dimensional vector space for every i and that $\mathrm{H}_j(X) = 0$ for sufficiently large j. Thus, it is well defined the number

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{Q}} \mathrm{H}_i(X),$$

called Euler characteristic of X. Moreover, if $f : X \to X$ is a continuous function, it is well defined the number

$$\Lambda(f) = \sum_{i=0}^{\infty} (-1)^i \mathrm{trace}(f_{*i}),$$

called Lefschetz number of f. Note that if f is homotopic to the identity map then $\Lambda(f) = \chi(X)$.

The Lefschetz fixed point theorem asserts that if $\Lambda(f) \neq 0$ then f has at least one fixed point.

Proof of Theorem 3.1. First, we construct the vector field $v : S \to \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$ defined by

$$s = (s_1, \ldots, s_n) \mapsto (v_1(s), \ldots, v_n(s)) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n},$$

where the component $v_i(s)$ in direction $\mathbb{R}^{m_i}$ is the gradient vector $\vec{\nabla}_{s_i} p_i(s)$ at the point s_i for the function $p_i(s_1, \ldots, s_n)$ considered as a function of s_i, when s_j is kept fixed for $j \neq i$.

We are assuming that the spaces $S_1, \ldots, S_n$ have the p.c.r.. So, also the cartesian product $S = S_1 \times \cdots \times S_n$ has it. Thus, let us consider $r : V \to S$ a convenient retraction.

Since S is a compact subset of $\mathbb{R}^m$, $m = m_1 + \cdots + m_n$, V is an open neighborhood of S in $\mathbb{R}^m$ and $v : S \to \mathbb{R}^m$ is a continuous vector field, one can show that there exists $t_1 > 0$ such that $x + tv(x) \in V$ for all $x \in X$ and all $t \in [0, t_1]$.

Let $f : S \to S$ be the map defined by

$$f(s) = r(s + t_1 v(s)).$$

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The above map is homotopic to the identity map via the homotopy $H : X \times [0, t_1] \to X$ given by

$$H(x,t) = r(x + tv(x))$$

for all $x \in X$ and all $t \in [0, t_1]$. Hence, $\Lambda(f) = \chi(S)$.

If we suppose $\chi(S_i) \neq 0$, $1 \leq i \leq n$, then, $\chi(S) = \chi(S_1) \cdot \cdots \cdot \chi(S_n) \neq 0$. It follows that $\Lambda(f) \neq 0$. Thus, by Lefschetz fixed point theorem, f has a fixed point.

Now, one can show that every fixed point of f is a w.l.e. of $p_1, \ldots, p_n$. ∎

Remark 4. 1. Let $p_1, \ldots, p_n : S_1 \times \cdots \times S_n \to \mathbb{R}$ be maps as in Theorem 3.1. If $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n)$ is a w.l.e. of $p_1, \ldots, p_n$ and each $\tilde{s}_i \in S_i \setminus \partial S_i$, one can show that $v(\tilde{s}) = 0$, where v is the vector field defined in the proof of Theorem 3.1. Conversely, if $v(\tilde{s}) = 0$, we see from the proof of Theorem 3.1 that $\tilde{s}$ must be a w.l.e. for $p_1, \ldots, p_n$. Thus, if we denote by

$$WLE(p_1, \ldots, p_n) = \{\tilde{s} \in S \mid \tilde{s} \text{ is a w.l.e. of } p_1, \ldots, p_n\}$$

then

$$WLE(p_1, \ldots, p_n) \cap (S \setminus \partial S) = \{\tilde{s} \in S \setminus \partial S \mid v(\tilde{s}) = 0\}.$$

Example 4.1. Consider the classical Cournot oligopoly model: there are $n \geq 2$ firms, $\{1, 2, \ldots, n\}$, producing a homogeneous commodity. Let s_i be the production of firm i and let f_i be its production cost function. Let q be the demand price function. Suppose that the market consumes the total amount of production, $\sum s_j$. Thus, the profit of firm i is expressed by

$$p_i(s_1, \ldots, s_m) = q\left(\sum_{j=1}^{n} s_j\right) \cdot s_i - f_i(s_i).$$

Let $I_1, I_2, \ldots, I_n$ be compact intervals in $\mathbb{R}$. If q and f_i, $1 \leq i \leq n$, are continuously differentiable, by Theorem 3.1, there exists at least one solution $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_m) \in I_1 \times \cdots \times I_n$ resistent to small unilateral changes in $I_1 \times \cdots \times I_n$. This solutions are exactly the solutions to the variational problem

$$\sum_{i=1}^{n} \left[q'\left(\sum_{j=1}^{n} \tilde{s}_j\right) \cdot \tilde{s}_i + q\left(\sum_{j=1}^{n} \tilde{s}_j\right) - f_i'(\tilde{s}_i) \right] \cdot (s_i - \tilde{s}_i) \leq 0, \qquad (1)$$

for every $s = (s_1, \ldots, s_n) \in S$, $S = I_1 \times \cdots \times I_n$.

It is well known that if the functions p_i are differentiable and concave in s_i then the solutions to (1) are exactly the Nash equilibria to $p_1, \ldots, p_n$. Without the assumption of

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concavity of p_i, we know at least that the solutions of (1) are resistant to small unilateral deviations.

ACKNOWLEDGMENT

I am grateful to Professor Carlos Biasi for his advising during my PhD studies and for his generosity in sharing with me some of his ideas.

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