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Brunella's Local Alternative

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Este é um breve resumo do trabalho realizado na minha Tese de Doutoramento e seus artigos subseqüentes. A tese foi realizada em modo de co-tutela entre a UFMG, sob orientação de Márcio Gomes Soares, e a Universidad de Valladolid (Espanha), sob orientação de Felipe Cano, e foi concluída em Fevereiro de 2013. A tese foi laureada com a Menção Honrosa do Premio Gutierrez 2014.
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1. FIRST RESULTS - NODAL COMPONENTS

The motivation of our work was the following question, proposed by Marco Brunella:

Let $\mathcal{F}$ be a singular holomorphic foliation of codimension one in the projective space $\mathbb{P}^3_{\mathbb{C}}$. If there is no projective algebraic surface invariant by $\mathcal{F}$, then each leaf of $\mathcal{F}$ is a union of algebraic curves.

This question is relevant because it is widely believed that the dynamics of the leaves of a given foliation is organized around its *separatrices*, which are the surfaces invariant for the foliation (these surfaces are *algebraic* in the global case, and *analytical* in the local case). Hence the importance of knowing whether a foliation has an invariant surface, and what happens to its leaves in the case it does not. The answer to this question is known [5] to be positive in the case of generic foliations in a pencil of foliations.

The work developed in my PhD thesis was the first to consider this problem from a *local* point of view. We considered the alternative for complex hyperbolic foliations on $(\mathbb{C}^3, 0)$. A germ $\mathcal{F}$ of singular holomorphic foliation of codimension one in $(\mathbb{C}^n, 0)$ is a *Complex Hyperbolic foliation* (for short, a *CH-foliation*) if for every holomorphic map germ $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)$ generically transversal to $\mathcal{F}$, the transformed foliation $\phi^*\mathcal{F}$ is such that there are no saddle nodes in its reduction of singularities.

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The work of my thesis originated the paper [3]. A second paper [4], which is a continuation of the work of the thesis developed during a two year Post-Doctoral stage, was also produced. Below we briefly describe the most important results of both works.

The first result of [3] is the following:

Theorem 1.1. *Let $\mathcal{F}$ be a CH-foliation on $(\mathbb{C}^3, 0)$ without germ of invariant surface. Assume that there is a reduction of singularities of $\mathcal{F}$ without nodal components. There is a neighborhood U of the origin $0 \in \mathbb{C}^3$ such that, for each leaf $L \subset U$ of $\mathcal{F}$ in U there is a germ of analytic curve γ at the origin such that $\gamma \subset L \cup \{0\}$.*

We know by [1] that there is a reduction of singularities for any codimension one foliation $\mathcal{F}$ on $(\mathbb{C}^3, 0)$. That is, there is a morphism

$$\pi : (M, \pi^{-1}(0)) \to (\mathbb{C}^3, 0)$$

which is a composition of blow-ups with invariant centers that produces a normal crossings exceptional divisor $E \subset M$, in such a way that all the points $p \in \pi^{-1}(0)$ are simple points for the pair $\pi^*\mathcal{F}, E$. The singular locus $\operatorname{Sing} \pi^*\mathcal{F}$ is a union of nonsingular curves.

A key remark for the understanding of germs of foliations without an invariant germ of surface is that they must be *dicritical*. A germ of foliation $\mathcal{F}$ is dicritical if there is a holomorphic map germ

$$\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0); \quad (x, y) \mapsto \phi(x, y) = (\phi_1(x, y), \phi_2(x, y), \phi_3(x, y))$$

such that $\phi(\{y = 0\})$ is invariant by $\mathcal{F}$ and the pullback $\phi^*\mathcal{F}$ coincides with the foliation $dx = 0$ in $(\mathbb{C}^2, 0)$. In [2] it is proved that any non dicritical foliation in $(\mathbb{C}^3, 0)$ has an invariant germ of analytic hypersurface. Furthermore, the arguments of [2] may be extended to the case where all compact components of the exceptional divisor are invariant (an irreducible component D of E is compact if and only if $D \subset \pi^{-1}(0)$). Therefore, if $\mathcal{F}$ does not admit invariant surfaces, there is at least one compact component D of E that is generically transversal (i.e. dicritical).

The main idea for Theorem 1.1 is that all the leaves of $\pi^*\mathcal{F}$ must intersect the union of compact dicritical components. At the intersection points we detect a germ of analytic curve contained in the leaf, that projects over the desired germ of analytic curve in $(\mathbb{C}^3, 0)$. The obstruction to having this property is the possible existence of *nodal components*, which could "attract and lock the leaves". We will describe these objects below.

A *nodal point* for a codimension one foliation in $(\mathbb{C}^n, 0)$ is a point where the foliation is given in local coordinates $x_1, x_2, \ldots, x_n$, by $\omega = 0$ where

$$\omega = \sum_{i=1}^{\tau} \lambda_i \frac{dx_i}{x_i}; \quad \lambda_i \in \mathbb{C}^*, \ i = 1, 2, \ldots, \tau$$

with $\lambda_i/\lambda_j \in \mathbb{R}$, for any i, j, and $\lambda_s/\lambda_j \in \mathbb{R}_{<0}$ for at least two indices s, j. The number τ is called the dimensional type of the point and corresponds to the number of variables needed

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to locally describe the foliation. A curve in the singular locus $\operatorname{Sing}\pi^*\mathcal{F}$ is *generically nodal* if its generic point is a nodal point. A *nodal component* $\mathcal{N}$ of the pair $\pi^*\mathcal{F}, E$ is a connected component of the union of generically nodal curves such that *all* the points in $\mathcal{N}$ are nodal (and not only the generic points of the curves).

Our work is the first to describe these objects in dimension three. After reduction of singularities, the exceptional divisor E of π is a normal crossings divisor and the singular locus $\operatorname{Sing}\pi^*\mathcal{F}$ is a finite union of irreducible nonsingular curves having normal crossings with E. Any point $p \in \operatorname{Sing}\pi^*\mathcal{F}$ has dimensional type $\tau_p \in \{2, 3\}$. If $\tau_p = 2$, there are local coordinates (x, y, z) at p such that $\pi^*\mathcal{F}$ is given by

$$\frac{dy}{y} - (\lambda + \phi(x, y))\frac{dx}{x} = 0, \quad \phi(0, 0) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{Q}_{\geq 0} \tag{1.1}$$

and moreover $(x = 0) \subset E_{\text{inv}} \subset (xy = 0)$, where E_{inv} is the union of the invariant irreducible components of E. Note that $xy = 0$ are invariant surfaces for $\mathcal{F}$ and that the singular locus $\operatorname{Sing}\pi^*\mathcal{F}$ is $(x = y = 0)$ locally at p. The *transversal type* of $\pi^*\mathcal{F}$ at p is the germ of foliation $\mathcal{T}_p$ in $(\mathbb{C}^2, 0)$ given by Equation 1.1.

Let Γ be the only irreducible curve of $\operatorname{Sing}\pi^*\mathcal{F}$ passing through p. We know that $\mathcal{T}_p = \mathcal{T}_q$ for any $q \in \Gamma$ with $\tau_q = 2$. Thus $\mathcal{T}_p = \mathcal{T}_\Gamma$ is the *transversal type* of Γ. We say that Γ is *nodal* if $\lambda \in \mathbb{R}_{>0}$; in this case the transversal type is linearizable of the form $d(y/x^\lambda) = 0$. If $\lambda \in \mathbb{R}_{<0}$, we say that Γ is a *real saddle* and if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we say that Γ is a *complex saddle*.

At a point q of dimensional type three, the foliation $\pi^*\mathcal{F}$ is locally given by

$$\frac{dx}{x} + (\lambda + \phi(x, y, z))\frac{dy}{y} + (\mu + \psi(x, y, z))\frac{dz}{z} = 0$$

where $\phi(0, 0, 0) = \psi(0, 0, 0) = 0$ and $\lambda, \mu \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$, $\lambda/\mu \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$. Moreover

$$(xy = 0) \subset E_{\text{inv}} \subset (xyz = 0).$$

Note that the coordinate planes $xyz = 0$ are invariant surfaces and

$$\operatorname{Sing}\pi^*\mathcal{F} = (x = y = 0) \cup (x = z = 0) \cup (y = z = 0).$$

Thus there are exactly three curves Γ_1, Γ_2, Γ_3 of $\operatorname{Sing}\pi^*\mathcal{F}$ arriving at q. Up to reordering, we have the following five possibilities:

1. Γ_1, Γ_2 are nodal curves and Γ_3 is a real saddle.
2. Γ_1 is a nodal curve and Γ_2, Γ_3 are complex saddles.
3. Γ_1, Γ_2 and Γ_3 are real saddles.
4. Γ_1 is a real saddle and Γ_2, Γ_3 are complex saddles.
5. Γ_1, Γ_2 and Γ_3 are complex saddles.

We define an *uninterrupted nodal component* $\mathcal{N} \subset \operatorname{Sing}\pi^*\mathcal{F}$ as any connected union of nodal curves such that at each point q of dimensional type three there are exactly two curves $\Gamma_1, \Gamma_2 \subset \mathcal{N}$ through q; that is, we have the first case above.

The second result of [3] concerns the structure of the nodal components for a particular type of foliations that we call RICH-foliations. The idea is that we will be able to detect the possible existence of a nodal component $\mathcal{N}$ before doing the reduction of singularities, in the sense that $\mathcal{N}$ should project onto at least one of the curves $\Gamma \subset (\mathbb{C}^3, 0)$ of the singular locus Sing $\mathcal{F}$ and the transversally generic behavior of Γ is either dicritical or has a nodal separator. We prove the following result:

THEOREM 1.2. *Let $\mathcal{F}$ be a RICH-foliation in $(\mathbb{C}^3, 0)$. Assume that there is no germ of invariant analytic surface for $\mathcal{F}$. Then one of the two properties holds*

(i) There is a neighborhood U of the origin $0 \in \mathbb{C}^3$ such that, for each leaf $L \subset U$ of $\mathcal{F}$ in U there is an analytic curve $\gamma \subset L$ with $0 \in \gamma$.

(ii) There is an analytic curve Γ contained in the singular locus Sing $\mathcal{F}$ *such that, $\mathcal{F}$ is generically dicritical or it has a nodal separator along Γ.*

Let us explain the concepts appearing in Theorem 1.2. The term *RICH-foliation* stands for *Relatively Isolated Complex Hyperbolic foliation*. A germ $\mathcal{F}$ of singular holomorphic foliation of codimension one in $(\mathbb{C}^3, 0)$ is a *RICH-foliation* if it is a CH-foliation and admits a *Relatively Isolated* reduction of singularities (for details, see [3]).

We say that a germ of foliation $\mathcal{G}$ on $(\mathbb{C}^2, 0)$ contains a *nodal separator* if, in the reduction of singularities, there is a singularity analytically equivalent to $xdy - \lambda ydx = 0$ where λ is a non rational positive real number. Now, consider a germ of curve Γ contained in the singular locus of a foliation $\mathcal{F}$ in $(\mathbb{C}^3, 0)$. We say that $\mathcal{F}$ is *generically dicritical along* Γ if it is dicritical at a generic point of Γ. If $\mathcal{F}$ is not generically dicritical along Γ, it is known [1] that the equireduction along Γ is given by the (non-dicritical) reduction of singularities of the restriction $\mathcal{G}$ of $\mathcal{F}$ to a plane section transversal to Γ at a generic point. In this case, we say that $\mathcal{F}$ *has a nodal separator along* Γ if this is true for such plane transversal sections $\mathcal{G}$.

Finally, condition (ii) of Theorem 1.2 is equivalent to the fact that any nodal component intersects the dicritical components or it contains a non compact curve. To be precise, Theorem 1.2 is a consequence of Theorem 1.1 and the following result of structure for the nodal components [3]:

THEOREM 1.3. *Let $\mathcal{F}$ be a RICH-foliation in $(\mathbb{C}^3, 0)$ and let $\pi : (M, \pi^{-1}(0)) \to (\mathbb{C}^3, 0)$ be an RI-reduction of singularities for $\mathcal{F}$ with total exceptional divisor $E \subset M$. Any compact nodal component $\mathcal{N}$ of $\pi^*\mathcal{F}, E$ intersects the union of the dicritical components of E.*

2. FURTHER RESULTS - PARTIAL SEPARATRICES

Following Theorems 1.1 and 1.2, we naturally asked whether any germ of codimension one foliation $\mathcal{F}$ over $(\mathbb{C}^3, 0)$ without invariant germ of surface satisfies the following property:

($\star$) *There is an open neighborhood U of $0 \in \mathbb{C}^3$ such that any leaf of $\mathcal{F}|_U$ contains a germ of analytic curve at the origin.*

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Although in [4] we only consider a particular class of codimension one foliations, we believe that there are enough reasons to state the following conjecture:

"Any germ $\mathcal{F}$ of CH-foliation on $(\mathbb{C}^3, 0)$ without germ of invariant analytic surface satisfies $(\star)$".

As stated in the previous section, our general strategy to prove the conjecture is to show that all the leaves "go" to a compact dicritical component after reduction of singularities. In fact, if L is a leaf of $\pi^*\mathcal{F}$ intersecting a compact dicritical component D at a point p, we can find a germ of analytic curve $(\tilde{\gamma}, p) \subset L$ and the image $(\pi(\tilde{\gamma}), 0)$ is the desired germ of analytic curve. As seen in Theorems 1.1 and 1.2, the only obstruction to following this strategy is the existence of *nodal components*.

Now, the natural procedure is to prove that any nodal component also goes to a compact dicritical component, carrying the leaves with it, and thus it does not produce an obstruction to property $(\star)$. It is indeed necessary to assume that the foliation has no invariant germ of surface. We interpret this fact after reduction of singularities by observing that all the *partial separatrices* − objects also described for the first time − go to a compact dicritical component as well. The relationship between nodal components and partial separatrices is the main argument we use to obtain a proof of the conjecture for a particular class of CH-foliations on $(\mathbb{C}^3, 0)$.

Let us explain the concept of *partial separatrix*. We say that a curve $\Gamma \subset \operatorname{Sing} \pi^*\mathcal{F}$ is a *trace curve* if it is contained in only one invariant irreducible component of the exceptional divisor E. Otherwise, the curve is the intersection of two invariant irreducible components of E and it is a *corner curve*. By definition, a *partial separatrix* C is any connected component of the union of trace curves. We say that C is *complete* if it does not intersect the compact dicritical part of E, otherwise, we say it is *incomplete*.

More precisely, let C be a partial separatrix and take a point $p \in C \cap \pi^{-1}(0)$. Since the final singularities are complex hyperbolic, depending on the dimensional type $\tau = \tau(\pi^*\mathcal{F})$ we find two situations:

1. If $\tau = 2$, there are coordinates (x, y, z) at p such that $E_{\text{inv}} = (x = 0)$, $E_{\text{dic}} \subset (z = 0)$ (here E_{dic} is the union of dicritical components of E) and $C = (x = y = z) = \operatorname{Sing} \pi^*\mathcal{F}$. Moreover $S = (y = 0)$ is the only invariant germ of surface for $\pi^*\mathcal{F}$ not contained in E.
2. If $\tau = 3$, there are coordinates (x, y, z) at p such that $E = E_{\text{inv}} = (xy = 0)$,

$$C = (x = z = 0) \cup (y = z = 0),$$

and $\operatorname{Sing} \pi^*\mathcal{F} = C \cup (x = y = 0)$. Moreover $S = (z = 0)$ is the only invariant germ of surface for $\pi^*\mathcal{F}$ at p not contained in E.

Following Cano-Cerveau's argumentations as in [2], given a partial separatrix C we find a germ of invariant surface

$$(S, C \cap \pi^{-1}(0)) \subset (M, \pi^{-1}(0))$$

supported by C. The inclusion above is closed if and only if $S_C \cap \pi^{-1}(0) = C \cap \pi^{-1}(0)$. On the other hand, we have

$$S_C \cap \pi^{-1}(0) = C \cap \pi^{-1}(0) \Leftrightarrow C \cap E_{\mathrm{c,dic}} = \emptyset$$

where $E_{\mathrm{c,dic}}$ is the union of compact dicritical components of E. In other words, we obtain a closed immersion exactly when C is a complete partial separatrix. In this case we find by direct image a germ of surface $(\pi(S), 0)$ invariant for $\mathcal{F}$. Hence, we conclude:

Proposition 2.1. *If $\mathcal{F}$ has no invariant germ of analytic surface, all the partial separatrices are incomplete.*

The incomplete partial separatrices are the "guides" we use to take the nodal components to a compact dicritical component of the exceptional divisor. To do this, we need an accurate control of the transitions of the Camacho-Sad indices along the curves in the singular locus from one component of the exceptional divisor to another. This quantitative analysis focused on the partial separatrices is in contrast with the qualitative and combinatorial arguments we used in [3] to obtain the first results concerning the conjecture.

In [4] we prove the conjecture for the case of *special relatively isolated Complex Hyperbolic* germs $\mathcal{F}$ of codimension one foliations in $(\mathbb{C}^3, 0)$. Roughly speaking, this means that we can perform a reduction of singularities by blowing-up points until we reach a situation of equireduction along non compact curves, which we resolve by blowing-up only curves.

The main result of [4] is:

Theorem 2.1. *Any special relatively isolated CH-foliation $\mathcal{F}$ in $(\mathbb{C}^3, 0)$ without germ of invariant analytic surface satisfies property $(\star)$.*

In some sense the global alternative of M. Brunella may be interpreted as a property concerning the "concentration−diffusion" of the non-transcendency of the leaves of a foliation: either we concentrate the non-transcendency in an algebraic leaf, or all the leaves are not completely transcendent in the sense that they are foliated by algebraic curves. In our local situation we have an analogous of this phenomenon based on the concept of "end of a leaf". In forthcoming works we intend to study the ends of the leaves for CH-foliations without invariant surface. All these ends will be "semi-transcendental" in the sense that either they contain an analytic curve, or they are of a "valuative type" that admits bifurcation at all the accumulation points after blow-up. Moreover, the leaves in a neighborhood will have at least one end and in this sense we can reformulate a local version of Brunella's alternative by saying that, either we have an invariant germ of surface, or there is a neighborhood of the origin such that all the leaves are "semi-transcendental".

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