CADERNOS DE MATEMÁTICA **16**, 143–160 May (2015)
ARTIGO NÚMERO SMA#9

On the essential spectrum of the Laplacian and drifted Laplacian on smooth metric measure spaces

Leonardo Silvares

Departamento de Matemática e Estatística, CCET, Universidade Federal do Estado do Rio de Janeiro, Av. Pasteur 458, CEP 22290-240, Rio de Janeiro RJ, Brazil.
E-mail: leo.silvares@uniriotec.br

In this paper, we study the L^2 essential spectra of the Laplacian Δ and the drifted Laplacian Δ_f on a complete smooth measure metric space $(M, g, e^{-f} d\,\mathrm{vol}_g)$. Assuming that the Bakry-Émery Ricci curvature tensor satisfies $\mathrm{Ric}_f \geqslant \frac{1}{2}g$ and $|\nabla f|^2 \leqslant f$ in an end E of M, we show that the essential spectrum of the Laplacian is $[0, \infty)$. In such conditions, we proved that the volume E does not grow nor decay exponentially. When Ric_f is nonnegative and f has sublinear growth we show that the essential spectrum of the Laplacian is also $[0, \infty)$. May, 2015 ICMC-USP

1. INTRODUCTION

Let M be a complete Riemmanian manifold. The Laplace-Beltrami operator $\Delta = \mathrm{div} \circ \nabla$ acting on the space of smooth spaces on M is essentially self-adjoint, thus it may be extended to an unbounded operator, also denoted by Δ, on the Hilbert space $L^2(M)$.

The study and computation of the spectra of the operator $-\Delta$ on complete non-compact manifolds has been an interest problem in geometric analysis, since these spectra may lead to important geometric and topological information of the manifold. Indeed, in the past two decades, that was a very active area of mathematical research, requiring the use of techniques that belongs to Geometric and Functional Analysis among others.

When the manifold has a soul whose exponential map is a diffeomorphism, supposing non-negative sectional curvature and some other additional conditions, Escobar [6] and Escobar-Freire [7] proved that the L^2 spectrum of the Laplacian is $[0, \infty)$. Subsequently, Zhou [16] proved this result with no need to suppose those aditional conditions.

Assuming that the manifold has a pole, and supposing the Ricci curvature non-negative, Li [8] proved that the essential spectrum of the Laplacian is $[0, \infty)$. Supposing that the sectional curvature in the radial directions are non-negative rather than the hypothesis of the Ricci curvature, Chen-Lu [4] proved the same result. This result was also proved by Donnelly [5] when the Ricci curvature is non-negative and the volume has Euclidean growth.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

J. Wang [14] removed the need to suppose the existence of a pole, proving that the essential spectrum L^p of the Laplacian is $[0, \infty)$, for $p \in [1, \infty)$, provided the Ricci curvature is bounded below by $-\delta/r^2$, where r is the distance from a fixed point on the manifold and $\delta > 0$ depends only on the dimension. Lu and Zhou [9] generalized this result, supposing that $\underline{\lim}_{r\to\infty} \operatorname{Ric} = 0$. An important argument in both works was a result proved by Sturm [13] which asserted that the L^p essential spectra of the Laplacian are the same, regardless of the value of $p \in [1, +\infty)$, when the volume has uniform sub-exponential growth and the Ricci curvature is bounded below.

By proving a generalization of the Classical Weyl's Criterion, Charalambous and Lu [3] extended the results previously obtained in L^2 to the case where Ric is asymptotically non-negative and $\overline{\lim}_{r\to\infty} \Delta r \leqslant 0$. This generalization of Weyl's Criterion will be used in the results of our paper.

Note that most of the above work relates the spectrum of the Laplacian with the Ricci curvature. This is somehow natural, since the Laplacian and the Ricci curvature are related by the Bochner formula

$$\frac{1}{2}\Delta \left|\nabla u\right|^2 = \left|\operatorname{Hess}\ u\right|^2 + \langle \nabla u, \nabla \Delta u\rangle + \operatorname{Ric}\left(\nabla u, \nabla u\right).$$

Indeed, much of the analysis of the Laplacian passes by this formula. There are also many relations between Laplacian and the standard measure of the manifold, that is, the one associated to the volume form. As exemple, we may cite the Green's formulae, the Laplacian as derivative of Dirichlet energy functional, among others.

In this paper, we generalized some of these previous results to the context of a smooth metric measure space, that is, a manifold M endowed with a weighted measure of the form $e^{-f} d\,\mathrm{vol}_g$, where f is a smooth function, called *weight function*. The operator Δ_f, defined by

$$\Delta_f u = \Delta u - \langle \nabla f, \nabla u\rangle,$$

is associated with volume form $e^{-f} d\,\mathrm{vol}_g$ the same way Δ is associated to $d\,\mathrm{vol}_g$. Moreover, Δ_f is a self-adjoint operator on the space L^2_f of square integrable functions on M with respect to the measure $e^{-f} d\,\mathrm{vol}_g$.

And, since

$$\frac{1}{2}\Delta_f \left|\nabla u\right|^2 = \left|\operatorname{Hess}\ u\right|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + \operatorname{Ric}\left(\nabla u, \nabla u\right) + \operatorname{Hess}\ f\left(\nabla u, \nabla u\right),$$

it is natural to define

$$\operatorname{Ric}_f = \operatorname{Ric} + \operatorname{Hess} f,$$

and obtain a correspondent Bochner formula

$$\frac{1}{2}\Delta_f \left|\nabla u\right|^2 = \left|\operatorname{Hess}\ u\right|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + \operatorname{Ric}_f\left(\nabla u, \nabla u\right).$$

The operator Δ_f is the *drifted Laplacian* (or *drifting Laplacian*), and Ric_f is the Bakry-Émery Ricci curvature tensor. They are extensions of Δ and Ric, since these ones are

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

obtained when f is a constant function. The drifted Laplacian is also related with diffusion processes, as studied by Bakry and Émery [1].

Many results obtained when the Ricci curvature is assumed to be bounded below, such as volume estimates and splitting theorems, have been generalized to the case where the Bakry-Émery Ricci curvature is bounded below. In all these results, some hypotheses are assumed on the growth of weight function f. Interesting generalizations may be found in [15] and [10].

The Bakry-Émery Ricci curvature plays an important role in the study of the Ricci flow. *Gradient Ricci solitons* are defined to be complete manifolds (M, g) where $\mathrm{Ric}_f = \lambda g$ for some $\lambda \in \mathbb{R}$, and they are singularities of the Ricci flow. Note that Einstein manifolds are special cases of gradient Ricci solitons, obtained for $f \equiv c \in \mathbb{R}$. Gradient Ricci solitons may be *shrinking* if $\lambda > 0$, *steady* if $\lambda = 0$, or *expanding* if $\lambda < 0$; and function f is called *potential function*. The study of such manifolds has been an important motivation to the research on the smooth metric spaces and their related drifted Laplacian and Bakry-Émery Ricci curvature.

A remarkable result on shrinking solitons is that the potential function f has quadratic growth. This was proved by Cao and Zhou in [2] and has many consequences on the study of such solitons. In this work, we obtain a similar result in the hypotheses of Theorems 1.2 and 1.3 bellow.

In the doctoral thesis [12] that led to this work, we studied the L^2 and L^2_f essential spectra of $-\Delta$ and $-\Delta_f$ respectively, supposing some lower bounds for Ric_f and imposing conditions on f. Two of the results we obtained and which were publised in [11] are:

THEOREM 1.1. *Let M be a non-compact complete manifold. If $f : M \to \mathbb{R}$ is a smooth function such that $\mathrm{Ric}_f \geqslant 0$ and $\lim_{r\to\infty} \frac{|f|}{r} = 0$, the $L^2_f(M)$ essential spectrum of $-\Delta_f$ is $[0, \infty)$. (We are denoting by r the distance $r(x) = d(p, x)$ to a fixed point $p \in M$.)*

THEOREM 1.2. *Let M be a non-compact complete manifold. If $f : M \to \mathbb{R}$ is a smooth function such that $\mathrm{Ric}_f \geqslant \frac{1}{2}g$ and $|\nabla f|^2 \leqslant f$, the $L^2(M)$ essential spectrum of $-\Delta$ is $[0, \infty)$.*

If $\mathrm{Ric}_f = \frac{1}{2}g$ in Theorem 1.2, we have a gradient shrinking Ricci soliton, since the equation

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}, \rho > 0,$$

can be normalized to $\rho = \frac{1}{2}$. In this case, Lu - Zhou [9] and Charalambous - Lu [3] proved that the $L^2(M)$ essential spectrum of $-\Delta$ is $[0, \infty)$.

In an extension to Theorem 1.2, also present in [12], we proved that its hypotheses need only to be verified in an end of the manifold:

THEOREM 1.3. *Let M be a non-compact complete manifold. If $f : M \to \mathbb{R}$ is a smooth function such that $\mathrm{Ric}_f \geqslant \frac{1}{2}g$ and $|\nabla f|^2 \leqslant f$ hold on some end of M, the $L^2(M)$ essential spectrum of $-\Delta$ is $[0, +\infty)$.*

To prove these results, we needed to obtain some estimates on the f-volume growth. Theses estimates may be of independent interest and are stated in section 4.

2. NOTATION AND BASIC FACTS

Let M be a complete noncompact Riemmanian manifold. Given $R \subset M$ and $f \in C^\infty(M)$, we denote denote $L^p(R) := L^p(R, g, d\,\mathrm{vol}_g)$ and $L^p_f(R) := L^p(R, g, e^{-f}d\,\mathrm{vol}_g)$ and define the norms

$$\|u\|_{L^p(R)} := \|u\|_{L^p(R,g,d\,\mathrm{vol}_g)} := \left(\int_R |u|^p \; d\,\mathrm{vol}_g\right)^{\frac{1}{p}},$$

$$\|u\|_{L^p_f(R)} := \|u\|_{L^p(R,g,e^{-f}d\,\mathrm{vol}_g)} := \left(\int_R |u|^p \; e^{-f}d\,\mathrm{vol}_g\right)^{\frac{1}{p}}.$$

The spaces $L^2(M)$ and $L^2_f(M)$ provided of the inner products $(u,v) = \int_M u\bar{v}d\,\mathrm{vol}_g$ and $(u,v)_f = \int_M u\bar{v}e^{-f}d\,\mathrm{vol}_g$, respectively, are Hilbert spaces. The operators $-\Delta$ and $-\Delta_f$ defined in 1 may be uniquely extended to (unbounded) self-adjoint operators in $L^2(M)$ and $L^2_f(M)$, respectively.

A point $\lambda \in \mathbb{C}$ is called a *regular point* of Δ if $(\Delta - \lambda)^{-1}$ exists and is a bounded linear operator. If $\lambda \in \mathbb{C}$ is not a regular point, λ is said to be a *spectrum point*, and we define *the spectrum of* Δ, denoted by $\sigma(\Delta)$, the set of all spectrum points of Δ. We say λ is in the *discrete spectrum* of Δ if it is an isolated point in $\sigma(\Delta)$ and it is an eigenvalue of Δ of finite multiplicity. We will denote by $\sigma_{disc}(\Delta)$ the discrete spectrum of Δ. The *essential spectrum* of Δ, denoted by $\sigma_{ess}(\Delta)$, is the complementary of $\sigma_{disc}(\Delta)$ in $\sigma(\Delta)$, that is,

$$\sigma_{ess}(\Delta) = \sigma(\Delta) \setminus \sigma_{disc}(\Delta).$$

The definitions of $\sigma(\Delta_f)$, $\sigma_{disc}(\Delta_f)$ and $\sigma_{ess}(\Delta_f)$ are analogous.

Notice that, since $-\Delta$ and $-\Delta_f$ are positive and symmetric, their spectra are contained in $[0, \infty)$.

Given the manifold M and the function f as above, $\sigma_{ess}(\Delta)$ and $\sigma_{ess}(\Delta_f)$ may be essentially different, as we may see in the following example.

Example 2.1. Consider the Gaussian soliton $\left(R^n, \delta_{ij}, e^{-f(x)}d\,\mathrm{vol}_g\right)$, where $f(x) = -\frac{|x|^2}{4}$.

For $k = 0, 1, \ldots$, and $x_i \in \mathbb{R}$, we define the Hermite polynomial H_k by

$$H_k(x_i) = (-2)^k \, e^{\frac{x_i^2}{4}} \frac{d^k}{dx_i^k} e^{-\frac{x_i^2}{4}}.$$

Writing $x = (x_1, ..., x_n) \in \mathbb{R}^n$, and given $k_1, ..., k_n$ in $\{0, 1, 2, ...\}$, we define

$$H_{k_1,\ldots,k_n}(x) = H_{k_1}(x_1) \cdot \ldots \cdot H_{k_n}(x_n).$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

These polynomials are, up to multiplication by a constant, the classical Hermite polynomials of probability. It may be proved that $\{H_{k_1,\ldots,k_n}\}$ is an orthogonal basis of $L^2(\mathbb{R}^n, e^{-x^2/4})$. Moreover, a direct calculation shows that

$$\Delta_f H_{k_1,\ldots,k_n}(x) = -\left(\frac{\sum_{i=1}^n k_i}{2}\right) H_{k_1,\ldots,k_n}(x).$$

Thus, each $H_{k_1,\ldots,k_n}$ is an eigenfunction of $-\Delta_f$, associated with the eigenvalue

$$\lambda_{k_1,\ldots,k_n} = \frac{\sum_{i=1}^n k_i}{2}.$$

Since $\{H_{k_1,\ldots,k_n}\}$ is an orthogonal basis of $L^2(\mathbb{R}^n, \gamma_n)$, the real positive numbers $\lambda_{k_1,\ldots,k_n}$ will be all the possible eigenvalues, each one of then having finite multiplicity. Therefore,

$$\sigma_{ess}(\Delta_f) \subset [0,\infty) \setminus \{i/2, i \in \mathbb{N}\}.$$

By a direct calculation, we have

$$\mathrm{Ric}_f = \mathrm{Ric}_{\mathbb{R}^n} + \mathrm{Hess}\, f = \frac{1}{2},$$

and

$$|\nabla f|^2 = \left|\frac{|x|}{2}\nabla x\right|^2 = \frac{|x|^2}{4} = f.$$

Therefore, using Theorem 1.2, we have $\sigma_{ess}(\Delta) = [0, +\infty)$.

3. WEYL'S CRITERIA FOR THE ESSENTIAL SPECTRUM

As tool for studying the essential spectrum, we will use a Weyl's Criteria. The following two criteria are corollaries of the classical one, and their proofs may be found in [11].

Proposition 3.1. *If there exists a sequence $\{u_i\} \subset C_0^\infty(M)$ such that*

1. $\dfrac{\|u_i\|_{L^\infty(M)}\ \|(-\Delta-\lambda)u_i\|_{L^1(M)}}{\|u_i\|^2_{L^2(M)}} \to 0$ *as $i \to \infty$;*
2. *for any compact $K \subset M$, there exists i_0 such that the support of u_i is outside K for $i > i_0$; and*
3. *$\partial(\mathrm{supp}(u_i))$ is a C^∞ $(n-1)$-submanifold of M,*

then $\lambda \in \sigma_{\mathrm{ess}}(-\Delta)$.

PROPOSITION 3.2. *If there exists a sequence* $\{u_i\} \subset C_0^\infty(M)$ *such that*

1. $\dfrac{\|u_i\|_{L^\infty(M)}\ \|(-\Delta_f-\lambda)u_i\|_{L^1_f(M)}}{\|u_i\|^2_{L^2_f(M)}} \to 0$ *as* $i\to\infty$*;*
2. *for any compact* $K\subset M$*, there exists* i_0 *such that the support of* u_i *is outside* K *for* $i>i_0$*; and*
3. $\partial(\mathrm{supp}(u_i))$ *is a* C^∞ $(n-1)$*-submanifold of* M*,*

then $\lambda\in\sigma_{\mathrm{ess}}(-\Delta_f)$.

These Propositions are key arguments to the proofs of Theorems 1.1, 1.2 and 1.3. For each $\lambda>0$, we will construct a sequence of functions satisfying conditions (1) to (3). The first attempt is to define each function u_i in such way that $u_i(x)$ deppends only on the distance $r(x)=d(x,p)$ of x to a fixed point $p\in M$. This approach, which seens very natural, fails because the distance $r(x)$ may be not smooth and therefore $u_i(r(x))$ may not belong to $C_0^\infty(M)$, as required. The solution is to build an smooth approximation to the distance, as done in [9], [3] and [11], and stated below.

PROPOSITION 3.3. *Given* $p\in M$*,* $f\in C^\infty(M)$ *and a decreasing continuous function* $\delta:\mathbb{R}^+\to\mathbb{R}^+$*, such that* $\lim_{r\to\infty}\delta(r)=0$*, there exists functions* b *and* $\tilde r$ *in* $C^\infty(M)$ *satisfying*

1. $\|b\|_{L^1_f(M\setminus B_p(r))}\leqslant\delta(r)$*;*
2. $\|\nabla\tilde r-\nabla r\|_{L^1_f(M\setminus B_r(p))}\leqslant\delta(r)$*;*
3. $|\tilde r(x)-r(x)|\leqslant\delta(r(x))$ *and* $|\nabla\tilde r(x)|\leqslant 2, \forall x\in M, r(x)>2$*, and*
4. $\Delta_f\tilde r(x)\leqslant\sup\limits_{y\in B_x(1)}\{\Delta_f r(y)\}+\delta(r(x))+|b(x)|, \forall x\in M, r(x)>2$ *in the sense of distributions.*

Conditions 1, 2 and 4 say that the approximation is good in an "average" sense, and condition 3 gives a pontwise criterion of the approximation. If we define the $\tilde r$-balls

$$\widetilde B_p(t):=\{x\in M,\tilde r(x)<t\},$$

given $a>1$, the construction of $\tilde r$ may be made [11] so that, if $\mathrm{vol}_f(M)=\infty$,

$$a^{-1}\,\mathrm{vol}_f(B_p(t))\leqslant\mathrm{vol}_f(\widetilde B_p(t))\leqslant a\,\mathrm{vol}_f(B_p(t)),$$

and, if $\mathrm{vol}_f(M)<\infty$,

$$a^{-1}\left[\mathrm{vol}_f(M)-\mathrm{vol}_f(\widetilde B_p(t))\right]\leqslant\mathrm{vol}_f(M)-\mathrm{vol}_f(B_p(t))\leqslant a\left[\mathrm{vol}_f(M)-\mathrm{vol}_f(B_p(t))\right], \quad (3.1)$$

for all $t > 1$.

These comparisons between the volumes of r-balls and $\tilde{r}$-balls will be usefull to the calculations of the norms in condition 1 of Propositions 3.1 and 3.2.

4. ESTIMATES FOR VOLUME AND LAPLACIANS OF DISTANCE

Before we discuss the proofs of main Theorems, we present some results concerning the volume and the f-volume of $B_p(r)$. There are many classic results in Geometry which establishes some control on the growth and the decay of the volume, provided Ricci curvature has a lower bound. In this Section, we we obtain some similar results, using our hypotheses on the Bakry-Émery-Rici curvature tensor Ric_f.

If $\mathrm{vol}(M) > \infty$, we say that vol *grows exponentially* if there is $C > 0$ such that

$$\mathrm{vol}(B_p(r)) > e^{Cr},$$

and, if $\mathrm{vol}_f(M) < \infty$, we say that the volume of M *decays exponentially at* p if there exists $C > 0$ such that

$$\mathrm{vol}(M) - \mathrm{vol}(B_p(r)) < e^{-Cr},$$

for all r large enough. This definitions are naturally extended to vol_f.

In the scenario studied in Theorem 1.1, we have the following estimate, from [12]:

Proposition 4.1. *Let M be a non-compact complete manifold. If $f : M \to \mathbb{R}$ is a smooth function such that* $\mathrm{Ric}_f \geqslant 0$ *and* $\lim_{r\to\infty} \frac{|f|}{r} = 0$*, then* vol_f *does not grow nor decay exponentially.*

In the hypotheses of Theorem 1.2, Munteanu and Wang [10] proved that the volume growth is at most Euclidean, and at least linear. In other words, fixed $p \in M$, there exists $c, C \in \mathbb{R}$ such that

$$c\, r \leqslant \mathrm{vol}(B_p(r)) \leqslant C\, r^n.$$

In the hypotheses of Theorem 1.3, i.e. if the hypotheses of Theorem 1.2 are verified only in an end E of M, the volume growth remains at most Euclidean (see [12, Lemma 4.5], an adaptation of [10]), but there is no guarantee that is at least linear. We have, however, the following estimate:

Proposition 4.2. *If* $\mathrm{Ric}_f \geqslant \frac{1}{2} g$ *and* $|\nabla f|^2 \leqslant f$ *in some end E of M and* $\mathrm{vol}(E) < \infty$*, then* $\mathrm{vol}(E \setminus B_p(r))$ *does not decay exponentially.*

In order to prove the Proposition above, we need a deeper understanding of weight function f. The next Lemma, which is an adaptation of [10, Proposition 4.2], shows that f has quadratic growth.

LEMMA 4.1. *In the Hypotheses of Theorem 1.3, given $p \in M$, there exists a constant $c_p > 0$ such that*

$$\frac{1}{4}(d(p,x) - c_p)^2 \leqslant f(x) \leqslant \frac{1}{4}(d(p,x) + c_p)^2,$$

for any $x \in E$.

Proof: Let γ be a minimizing geodesic from p to a point of E.

Since $|\nabla f|^2 \leqslant f$, we have

$$\left|\nabla(2\sqrt{f})\right| \leqslant 1. \tag{4.1}$$

Integrating along γ, we get

$$\sqrt{f(t)} \leqslant \frac{1}{2}(t - r_0) + \sqrt{f(t)}.$$

By taking $c_0 = \sup_{y \in S_p(r_0)}\{\sqrt{f(y)} - \frac{r_0}{2}\}$, we have

$$\sqrt{f(t)} \leqslant \frac{1}{2}t + c_0,$$

which proves the upper bound.

By the second variation formula of arc-length, along γ we have

$$\int_\gamma \mathrm{Ric}(\gamma', \gamma')\phi^2 \leqslant (n-1)\int_\gamma (\phi')^2$$

for any ϕ with compact support on γ. Now set

$$\phi = \begin{cases} 1, & r_0 + 1 \leqslant t \leqslant r - 1 \\ t - r_0, & r_0 \leqslant t \leqslant r_0 + 1 \\ r - t, & r - 1 \leqslant t \leqslant r \\ 0, & t > r. \end{cases}$$

Using that $\mathrm{Ric}_f = \mathrm{Ric} + f'' \geqslant 1/2$, it follows that

$$\begin{array}{rcl} (n-1)\int_\gamma (\phi')^2 & \geqslant & \frac{1}{2}\int_\gamma \phi^2 - \int_\gamma f''(t)\phi^2(t)dt \\ & = & \frac{1}{2}(r - r_0) + c_1 + 2\int_\gamma f'(t)\phi(t)\phi'(t)dt, \end{array}$$

and so

$$\frac{1}{2}(r - r_0) - c_2 \leqslant -2\int_\gamma f'(t)\phi(t)\phi'(t)dt, \tag{4.2}$$

where $c_2 = c_2(n)$. By (4.1), $2\left|\sqrt{f(s)} - \sqrt{f(t)}\right| \leqslant |s - t|$, so

$$\left|\int_{r-1}^r f'(t)\phi(t)dt\right| \leqslant \left|\int_{r-1}^r \sqrt{f(t)}\phi(t)dt\right| \leqslant (\sqrt{f(r)} + 1/2)\int_{r-1}^r \phi(t)dt = \frac{1}{2}\sqrt{f(r)} + \frac{1}{4}.$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Hence, by (4.2),

$$\begin{aligned}\tfrac{1}{2}(r-r_0)-c_2 &\leqslant -2\int_\gamma f'(t)\phi(t)\phi'(t)dt\\ &= -2\int_{r_0}^{r_0+1} f'(t)\phi(t)dt + 2\int_{r-1}^{r} f'(t)\phi(t)dt\\ &\leqslant c_3 + 2\left|\int_{r-1}^{r} f'(t)\phi(t)\phi'(t)dt\right|\\ &\leqslant c_3+\sqrt{f(r)}+\tfrac{1}{4},\end{aligned}$$

where c_3 is the supremum of $\left(-2\int_{r_0}^{r_0+1} f'(t)\phi(t)dt\right)$ on all geodesics γ as above. So

$$\sqrt{f(r)} \geqslant \frac{1}{2}r - c_4.$$

□

Proof of Proposition 4.1:

Suppose by contradiction the existence of $C>0$ such that

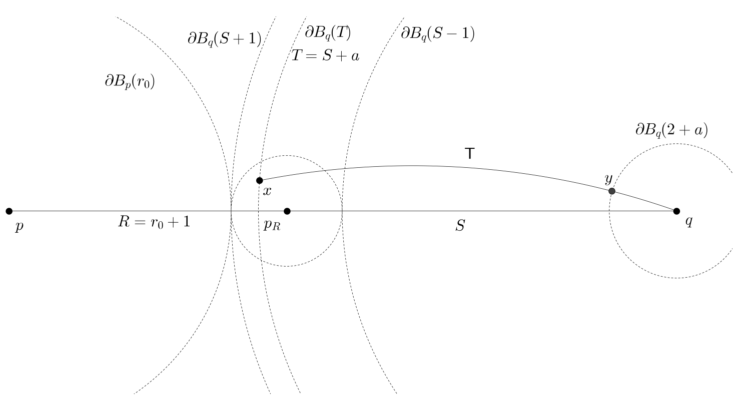
$$\mathrm{vol}(E\setminus B_p(r)) \leqslant e^{-Cr}. \tag{4.3}$$

Now let $R=r_0+1$, take $q\in E$ such that $d(p,q)=R+1+S$, for arbitrary $S>0$, and denote

$$J(t,\xi)\;dt\,d\xi = d\,\mathrm{vol}\,|_{\exp(t\xi)},$$

the volume form in geodesic polar coordinates from q.

Let p_R be a point in the minimizing geodesic from p to q, and such that $d(p_R,p)=R$. Taking a, $-1<a<1$, let x be an arbitrary point of $\partial B_q(S+a)\cap B_{p_R}(1)$ and γ be the minimizing geodesic such that $\gamma(0)=q$ and $\gamma(S+a)=x$. Notice that, since γ is a minimizing geodesic, if we define $y:=\gamma(2+a)$, each x will determine, in an injective way, an unique $y\in B_q(2+a)$. Moreover, we have $y=\exp_q(a_y\,\xi_y)$, where $a_y=a$ and $\xi_y=\gamma'(0)$.



By Lemma 4.1,

$$\frac{1}{4}(d(p,x)-c_p)^2 \leqslant f(x) \leqslant \frac{1}{4}(d(p,x)+c_p)^2$$

for some $c_p \in \mathbb{R}$ and for all $x \in E$. Denoting $f(t) = f(\gamma(t))$, for $0 \leqslant t \leqslant T := S + a$, and remembering $\gamma(0) = q$, $\gamma(T) = x$ we have

$$|f'(t)| \leqslant \sqrt{f(t)} \leqslant \frac{1}{2}(d(\gamma(t), p) - c_p) \leqslant \frac{1}{2}((T - t + R + 1) + c_p) = \frac{1}{2}(T - t + c), \tag{4.4}$$

where c depends only on n, r_0 and p. Similarly, adjusting c if necessary,

$$f(t) \geqslant \frac{1}{4}(T - t - c)^2 \tag{4.5}$$

$$f(t) \leqslant \frac{1}{4}(T - t + c)^2 \tag{4.6}$$

By the Bochner formula,

$$(\Delta t)'(t) + \frac{1}{n-1}\Delta(f) = -\operatorname{Ric}(\nabla t, \nabla t) \leqslant -\frac{1}{2} + f''(t).$$

Multiplying by t^k, $k \geqslant 2$, integrating by parts and rearranging terms from 0 to $s < T$,

$$\Delta t(s) \leqslant \frac{(n-1)k^2}{4(k-1)s} - \frac{1}{2(k+1)}s + f'(s) - \frac{k}{s^k}\int_0^s f'(t)t^{k-1}dt.$$

Integrating from $b := a + 2$ to T, we have

$$\begin{aligned} \log(J(T,\xi)) - \log(J(b,\xi)) &= \int_b^T \frac{d}{ds}\log(J(s,\xi))\, ds \\ &= \int_b^T \Delta t(s)\, ds \\ &\leqslant \frac{(n-1)k^2}{4(k-1)}\log T - \frac{1}{4(k+1)}T^2 + \int_b^T \left(\frac{k}{s^k}\int_0^s f'(t)t^{k-1}dt\right) ds. \end{aligned}$$

Some simple calculations lead to

$$\int_b^T \left(\frac{k}{s^k}\int_0^s f'(t)t^{k-1}dt\right) ds = -\frac{1}{k-1}(f(T) - f(b)) + \frac{k}{k-1}\left(\frac{1}{t^{k-1}}\int_0^s f'(t)t^{k-1}dt\right)\Bigg|_{s=b}^{s=T}. \tag{7}$$

Using (4.5) and (4.6), we have now

$$
\begin{aligned}
-\frac{1}{k-1}(f(T)-f(b)) &\leqslant -\frac{1}{k-1}\left(\frac{1}{4}c^2-\frac{1}{4}(T-b+c)^2\right)\\
&= \frac{1}{4(k-1)}\left((T-b)^2+2(T-b)c\right)\\
&\leqslant -\frac{1}{k-1}\left(\frac{1}{4}c^2-\frac{1}{4}(T-b+c)^2\right)\\
&= \frac{1}{4(k-1)}T^2+\frac{c_1}{k}T, \qquad (8)
\end{aligned}
$$

for some c_1 depending only on n, r_0 and p.

By (4.5), we have

$$
\begin{aligned}
\frac{k}{k-1}\frac{1}{T^{k-1}}\int_0^T f'(t)t^{k-1}dt &= \frac{k}{k-1}f(T)-\frac{k}{T^{k-1}}\int_0^T f(t)t^{k-2}dt\\
&\leqslant c_2-\frac{k}{T^{k-1}}\int_0^T (T-t-c)^2t^{k-2}dt\\
&= c_2-\frac{k}{4T^{k-1}}\int_0^T t^k dt+\frac{2k(T-c)}{4T^{k-1}}\int_0^T t^{k-1}dt\\
&\quad -\frac{k(T-c)^2}{4T^{k-1}}\int_0^T t^{k-2}dt\\
&= c_2-\frac{k}{4(k+1)}T^2+\frac{1}{2}(T-c)T-\frac{k(T-c)^2}{4(k-1)}\\
&= c_2-\frac{1}{2(k^2-1)}T^2+\frac{c}{2(k-1)}T \qquad (9)
\end{aligned}
$$

where $c_2=\sup_{y\in E\cap B_p(r_0+2)}\{f(y)\}$ depends only on n, r_0 and p. Now using (4.4) we get

$$
\begin{aligned}
-\frac{k}{k-1}\frac{1}{b^{k-1}}\int_0^b f'(t)t^{k-1}dt &\leqslant \frac{k}{2(k-1)}\frac{1}{b^{k-1}}\int_0^b (T-t+c)t^{k-1}dt\\
&= \frac{T+c}{2(k-1)}\,b-\frac{k}{2(k-1)(k+1)}\,b^2\\
&= \frac{T}{2(k-1)}\,b+c_3 \qquad (10)
\end{aligned}
$$

where c_3 depends only on n, r_0 and p.

Putting (7), (8) and (9) into (7), we get

$$
\begin{aligned}
\int_b^T\left(\frac{k}{s^k}\int_0^s f'(t)t^{k-1}dt\right)ds &\leqslant \frac{1}{4(k-1)}T^2-\frac{1}{2(k^2-1)}T^2+\frac{c_4}{k-1}T\\
&= \frac{1}{4(k+1)}T^2+\frac{c_4}{k-1}T, \qquad (11)
\end{aligned}
$$

where $c_4 = c_4(n, r_0, p)$.

Replacing this in (4.6), we get

$$\begin{aligned}\log(J(T,\xi)) - \log(J(b,\xi)) &= \tfrac{(n-1)k^2}{4(k-1)}\log T - \tfrac{1}{4(k+1)}T^2 + \tfrac{c_4}{4(k+1)}T^2 + \tfrac{c_6}{k-1}T \\ &\leqslant c_5 k \log T - \tfrac{c_6}{k-1}T \\ &\leqslant c_5 k \log T - \tfrac{c_6}{k}T.\end{aligned}$$

If now we take

$$k = \sqrt{\frac{T}{\log T}} + 1$$

in (10), we get

$$\log(J(T,\xi)) - \log(J(b,\xi)) \leqslant c_7\sqrt{T\ \log T},$$

which implies

$$J(b,\xi) \geqslant e^{-c_7\sqrt{T\ \log T}} J(T,\xi).$$

Since $S-1 \leqslant T \leqslant S+1$ and $-1 \leqslant a \leqslant 1$, there exists c_8 and c_9 constants and independent from a, ξ, S, q such that

$$J(b,\xi) \geqslant c_8 e^{-c_9\sqrt{S\ \log S}} J(S+a,\xi).$$

Now, let $U \subset (B_q(3))$ and $V \subset T_qM$ be, respectively, the set of all y and (a_y, ξ_y) constructed as above. Notice that each $x \in B_q(1)$ determines an unique y in U and (a_y, ξ_y) in V. Therefore

$$\begin{aligned}e^{-C(R+S-3)} &\geqslant \mathrm{vol}(M \setminus B_p(R+S-3)) \\ &\geqslant \mathrm{vol}(B_q(3)) \\ &\geqslant \mathrm{vol}(U) \\ &\geqslant \textstyle\int_V J(a,\xi)\, dr\, d\xi \\ &\geqslant c_8 e^{-c_9\sqrt{S\ \log S}} \textstyle\int_V J(S+a,\xi)\, dr\, d\xi \\ &\geqslant c_8 e^{-c_9\sqrt{S\ \log S}} \textstyle\int_{\exp_q(V)} d\,\mathrm{vol}_g \\ &\geqslant c_8 e^{-c_9\sqrt{S\ \log S}} \mathrm{vol}(B_{p_R}(1)).\end{aligned}$$

Since M is complete, we take $K = \min_{p_R} \mathrm{vol}(B_{p_R}(1))$ and so

$$\left(e^{-C(R-3)}\right)\ e^{-CS} = e^{-C(R+S-3)} \geqslant K c_8 e^{-c_9\sqrt{S\ \log S}}.$$

Therefore

$$e^{c_9\sqrt{S\ \log S}} \geqslant \left[\frac{K c_8}{e^{-C(R-3)}}\right] e^{CS},$$

which is an absurd for large S.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

□

EXAMPLE 4.1. If the end E is given by $E = [2, +\infty) \times S^2$, $g|_E = dr^2 + \frac{1}{r} d\theta^2$ and $f|_E = \frac{r^2}{2}$, we have $\mathrm{Ric}_f \leqslant \frac{1}{2}$ and $\mathrm{vol}(E) < \infty$. This example shows that, unlike what happens in Theorem 1.2, in the hypotheses of Theorem 1.3 the volume of E may be finite.

In addition to estimates of the growth and decay volume, we need some estimates on Δr and $\Delta \tilde{r}$ to prove our main Theorems.

LEMMA 4.2. *In the hypotheses of Theorem 1.1,* $\lim_{r\to\infty} \Delta_f r \leqslant 0$, *and, in the hypotheses of Theorems 1.2 or 1.3,* $\lim_{r\to\infty} \Delta r \leqslant 0$.

This Lemma gives us the following three Lemmas.

LEMMA 4.3. *For all* $\varepsilon > 0$ *and* t_0 *large enough, there exists* $R = R(\varepsilon, t_0)$ *such that, for* $t > R$,

- *in the hypotheses of Theorem 1.1 and assuming* $\mathrm{vol}_f(M) = \infty$,

$$\int_{B_p(t)\setminus B_p(t_0)} |\Delta_f \tilde{r}|\ e^{-f} d\,\mathrm{vol}_g \leqslant \varepsilon\,\mathrm{vol}_f(B_p(t+1))\ ;$$

- *in the hypotheses of Theorem 1.2 (remember that* $\mathrm{vol}(M) = \infty$*),*

$$\int_{B_p(t)\setminus B_p(t_0)} |\Delta \tilde{r}|\ d\,\mathrm{vol}_g \leqslant \varepsilon\,\mathrm{vol}(B_p(t+1))\ ;$$

- *in the hypotheses of Theorem 1.3 and assuming* $\mathrm{vol}(M) = \infty$,

$$\int_{(B_p(t)\setminus B_p(t_0))\cap E} |\Delta \tilde{r}|\ d\,\mathrm{vol}_g \leqslant \varepsilon\,\mathrm{vol}\left(E \cap B_p(t+1)\right).$$

LEMMA 4.4. *Given* $\varepsilon > 0$*, the construction of* $\tilde{r}$ *in Proposition 3.3 may be made so that, for* $R > 0$ *large enough, we have, for all* $t > R$,

- *in the hypotheses of Theorem 1.1 and assuming* $\mathrm{vol}_f(M) < \infty$,

$$\int_{M\setminus B_p(t)} |\Delta_f \tilde{r}|\ e^{-f} d\,\mathrm{vol}_g \leqslant \varepsilon(\mathrm{vol}_f(M) - \mathrm{vol}_f(B_p(t))) + 2\,\mathrm{vol}_f(\partial B_p(t))\ ;$$

- *in the hypotheses of Theorem 1.3 and assuming* $\mathrm{vol}(M) < \infty$,

$$\int_{E\setminus B_p(t)} |\Delta \tilde{r}|\ d\,\mathrm{vol}_g \leqslant \varepsilon(\mathrm{vol}(E \setminus B_p(t))) + 2\,\mathrm{vol}(\partial B_p(t)).$$

for $t > R$.

Lemma 4.5. *Let M and f be as in the hypotheses of Theorem 1.3 and suppose $\mathrm{vol}(M) < \infty$. Given $\epsilon > 0$, $C > 0$, there exist $R > 0$ large and a sequence of real numbers $r_k \to \infty$ such that*

$$\epsilon[\mathrm{vol}(M) - \mathrm{vol}(B_p(r_k - R))] + C\ \mathrm{vol}(\partial B(r_k - R)) \leqslant 2\epsilon\ [\mathrm{vol}(M) - \mathrm{vol}(B(r_k))].$$

The four previous Lemmas are proved in [12] and [11] (hypotheses of Theorems 1.1 and 1.2 only).

5. PROOF OF THEOREM 1.3

According to Proposition 3.1, in order to prove that $\sigma_{\mathrm{ess}}(\Delta) = [0, \infty)$ in L^2, we need only to construct, for all $\lambda > 0$, a sequence $\{u_i\} \subset C_0^\infty(M)$ satisfying

1. $\dfrac{\|u_i\|_{L^\infty(M)}\ \|(-\Delta - \lambda)u_i\|_{L^1(M)}}{\|u_i\|_{L^2(M)}} \to 0$ as $i \to \infty$;
2. $u_i \to 0$ weakly; and
3. $\partial(\mathrm{supp}(u_i))$ is a C^∞ $(n-1)$-submanifold of M.

The proof will be divided in two cases, $\mathrm{vol}(E) = \infty$ and $\mathrm{vol}(E) < \infty$. For the sake of simplicity of the notation, we will write $V(r)$ instead of $\mathrm{vol}(E \cap B_p(r))$, and $\widetilde{V}(r)$ instead of $\mathrm{vol}\left(E \cap \widetilde{B}_p(r)\right)$.

Proof of Theorem 1.3, case $\mathrm{vol}(E) = \infty$ **:**

Given $p \in M$, let $\tilde{r}$ be the C^∞ approximation for r we have constructed in Proposition 3.3. Let x, y, R be such that $0 < R < x < y$ and whose values will be chosen later. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a cut-off function satisfying

$$\psi(r) = \begin{cases} 1, & r \in [x/R, y/R] \\ 0, & r \notin [x/R - 1, y/R + 1] \end{cases}$$

so that $|\psi|$ and $|\psi'|$ are bounded.

Given $\lambda > 0$, we define

$$\phi(x) = \psi\left(\frac{\tilde{r}(x)}{R}\right) e^{i\sqrt{\lambda}\tilde{r}}.$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Hence, $|\phi| \leqslant 1$ and

$$\begin{aligned}\Delta\phi + \lambda\phi \;=\; & \left[\frac{1}{R}\left(\frac{\psi''}{R} + 2i\sqrt{\lambda}\psi'\right)|\nabla\tilde{r}|^2 + \left(i\sqrt{\lambda}\psi + \frac{\psi'}{R}\right)\Delta\tilde{r}\right]e^{i\sqrt{\lambda}\tilde{r}} \\ & -\lambda\left(1 - |\nabla\tilde{r}|^2\right).\end{aligned}$$

So we have

$$|\Delta\phi + \lambda\phi| \leqslant \frac{C}{R} + C\,|\Delta\tilde{r}| + C\,|\nabla\tilde{r} - \nabla r|$$

and therefore,

$$\begin{aligned}\|\Delta\phi + \lambda\phi\|_{L^1(M)} \;\leqslant\; & \frac{C}{R}(\widetilde{V}(y+R) - \widetilde{V}(x-R)) + C\delta(x-R) \\ & + C\int_{(B_p(y+R)\setminus B_p(x-R))\cap E} |\Delta\tilde{r}|\;d\,\mathrm{vol}_g\,. \end{aligned} \tag{1}$$

By Lemma 4.3 we choose y large enough such that

$$\int_{(B_p(y+R)\setminus B_p(x-R))\cap E} |\Delta\tilde{r}|\;d\,\mathrm{vol}_g \leqslant \varepsilon V(y+R+1). \tag{2}$$

On the other hand, fixing x and R with R and $x-R$ large enough, we have

$$\begin{aligned}\frac{C}{R}(\widetilde{V}(y+R) - \widetilde{V}(x-R)) \;<\; & \varepsilon(\widetilde{V}(y+R) - \widetilde{V}(x-R)) \\ <\; & \varepsilon\widetilde{V}(y+R+1) \\ <\; & 2\varepsilon V(y+R+1)\end{aligned} \tag{3}$$

(in the last inequality above we have inequality (3.1)) and

$$C\delta(x-R) < \varepsilon V(y+R+1). \tag{4}$$

Using (2), (3) and (4) in (1), we get

$$\|\Delta\phi + \lambda\phi\|_{L^1(M)} < 4\varepsilon V(y+R+1).$$

Since $|\phi| \equiv 1$ in $\widetilde{B}_p(y) \setminus \widetilde{B}_p(x)$, we have $\|\phi\|^2_{L^2(M)} \geqslant \widetilde{V}(y) - \widetilde{V}(x)$. Fixing x and taking y large enough, $\|\phi\|^2_{L^2(M)} \geqslant \frac{1}{2}\widetilde{V}(y)$ and, by (3.1), $\|\phi\|^2_{L^2(M)} \geqslant \frac{1}{4}V(y)$. As a consequence of Proposition 4.2, taking y even larger, we have $V(y+R+1) \leqslant 2V(y)$, so,

$$\frac{\|\phi\|_{L^\infty(M)}\;\|(-\Delta-\lambda)\phi\|_{L^1(M)}}{\|\phi\|_{L^2(M)}} < \frac{4\varepsilon V(y+R+1)}{\frac{1}{4}V(y)} \leqslant \frac{8\varepsilon V(y)}{\frac{1}{4}V(y)} < 32\varepsilon.$$

Thus, since $\|\phi\|_{L^\infty(M)} = 1$, we have constructed a compact supported function ϕ such that

$$\frac{\|\phi\|_{L^\infty(M)} \ \|(-\Delta-\lambda)\phi\|_{L^1(M)}}{\|\phi\|_{L^2(M)}} < 32\varepsilon$$

for an arbitrarily small $\varepsilon > 0$.

In order to obtain the desired sequence u_n, we consider, in the above construction, $\varepsilon = 1/n$ and $u_n = \phi$, with x greater than the o value of $y + R$ of the construction of u_{n-1}.

Proof of Theorem 1.3, case $\mathrm{vol}(E) < \infty$ **:**

In the construction of $\tilde{r}$ in Proposition 3.3, we take δ such that

$$\delta(r) \leqslant \frac{1}{r}(\mathrm{vol}(M) - V(r)). \tag{5}$$

By Lemma 4.4, we choose x large enough so that

$$\begin{aligned}\int_{(B_p(x+R)\setminus B_p(x-R))\cap E} |\Delta\tilde{r}| \ d\,\mathrm{vol}_g &\leqslant \int_{E\setminus B_p(x-R)} |\Delta\tilde{r}| \ d\,\mathrm{vol}_g \\ &\leqslant \varepsilon(\mathrm{vol}(M) - V(x-R)) \\ &\quad +2\,\mathrm{vol}(\partial B_p(x-R)).\end{aligned}$$

By using this in (1) of case $\mathrm{vol}(E) = \infty$, we have

$$\begin{aligned}\|\Delta\phi + \lambda\phi\|_{L^1(M)} &\leqslant \tfrac{C}{R}(\widetilde{V}(y+R) - \widetilde{V}(x-R)) + C\delta(x-R) + \\ &\quad + C\varepsilon(\mathrm{vol}(M) - V(x-R)) + 2C \ \mathrm{vol}(\partial B_p(x-R)) \\ &\leqslant \tfrac{C}{R}(\mathrm{vol}(M) - \widetilde{V}(x-R)) + C\delta(x-R) + \\ &\quad + C\varepsilon(\mathrm{vol}(M) - V(x-R)) + 2C \ \mathrm{vol}(\partial B_p(x-R)) \\ &\leqslant 2\tfrac{C}{R}(\mathrm{vol}(M) - V(x-R)) + C\delta(x-R) + \\ &\quad + C\varepsilon(\mathrm{vol}(M) - V(x-R)) + 2C \ \mathrm{vol}(\partial B_p(x-R)) \\ &\leqslant C\left(\tfrac{2}{R} + \varepsilon\right)(\mathrm{vol}(M) - V(x-R)) + C\delta(x-R) + \\ &\quad + 2C\,\mathrm{vol}(\partial B_p(x-R))\end{aligned}$$

(in the third inequality above we have used inequality (3.1)).

Since M has finite volume, and by (5), we may choose R and x so that

$$\frac{2}{R} \leqslant \varepsilon,$$

and

$$\delta(x-R) \leqslant \varepsilon(\mathrm{vol}(M) - V(x-R)).$$

Thus

$$\|\Delta\phi + \lambda\phi\|_{L^1(M)} \leqslant 3C\varepsilon(\mathrm{vol}(M) - V(x-R)) + 2C \ \mathrm{vol}(\partial B_p(x-R)).$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

By Lemma 4.5, taking x even larger,

$$3\varepsilon(\mathrm{vol}(M) - V(x-R)) + 2\ \mathrm{vol}(\partial B_p(x-R)) \leqslant 6\varepsilon\ (\mathrm{vol}(M) - V(x)),$$

therefore,

$$\|\Delta\phi + \lambda\phi\|_{L^1(M)} \leqslant 6\varepsilon\ C(\mathrm{vol}(M) - V(x)). \tag{6}$$

Using again that the volume of M is finite, making y large enough, we get

$$\widetilde{V}(y) - \widetilde{V}(x) \geqslant \frac{1}{2}(\mathrm{vol}(M) - \widetilde{V}(x)) \geqslant \frac{1}{4}(\mathrm{vol}(M) - V(x)),$$

and, by (6), we have

$$\begin{aligned} \|\Delta\phi + \lambda\phi\|_{L^1(M)} &\leqslant 6\varepsilon\ C(\mathrm{vol}(M) - V(x)) \\ &\leqslant 24\varepsilon\ C(V(y) - V(x)) \\ &\leqslant 24\varepsilon\ C\ \ \|\phi\|^2_{L^2(M)}. \end{aligned}$$

The proof follows now the case of infinite volume, as above.

□

The proof of Theorem 1.1 is analogous to the proof of Theorem 1.3, since all Lemmas and Propositions used in the proof above are equally valid in the hypotheses of Theorem 1.1. The same may be applied to Theorem 1.2 and, even it is a particular case of Theorem 1.3, an independent proof could also be obtained.

REFERENCES

1. D. Bakry and M. Émery. Diffusions hypercontractives. *Lecture Notes in Math.*, 1123:177–206, Springer, 1985.
2. H.-D. Cao and D. Zhou. On complete gradient shrinking Ricci solitons. *J. Differential Geom.*, 85(2):175–186, 2010.
3. N. Charalambous and Z. Lu. The essential spectrum of the Laplacian. Available at arXiv:1211.3225, 2012.
4. Z. H. Chen and Z. Q. Lu. Essential spectrum of complete Riemannian manifolds. *Sci. China Ser. A*, 35(3):477–501, 1992.
5. H. Donnelly. Exhaustion functions and the spectrum of a complete Riemannian manifold. *Indiana Univ. Math. J.*, 46(2):505–527, 1997.
6. J. F. Escobar. On the spectrum of the Laplacian on complete Riemannian manifolds. *Comm. Partial Differential Equations 11*, 65(1):63–85, 1986.
7. J. F. Escobar and A. Freire. The spectrum of the Laplacian of manifolds of positive curvature. *Duke Math. J.*, 65(1):1–21, 1992.
8. J. Y. Li. Spectrum of the Laplacian on a complete Riemannian manifolds with nonnegative Ricci curvature which possess a pole. *J. Math. Soc. Japan*, 46(2):213–216, 1994.
9. Z. Lu and D. Zhou. On the essential spectrum of complete non-compact manifolds. *J. Funct. Anal.*, 260(11):3283–3298, 2011.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

10. O. Munteanu and J. Wang. Geometry of manifolds with densities. *Advances in Mathematics*, 259(0): 269–305, 2014.

11. L. Silvares. On the essential spectrum of the Laplacian and the drifted Laplacian. *J. Funct. anal.*, 266:3906–3936, 2014.

12. L. Silvares. *On the essential spectrum of the Laplacian and the drifted Laplacian.* PhD thesis, Universidade Federal Fluminense, April 2013.

13. K-T. Sturm. On the L^p-spectrum of uniformly elliptic operators on Riemannian manifolds. *J. Funct. anal.*, 118:442–453, 1993.

14. J. Wang. The spectrum of the Laplacian on a manifold of nonnegative Ricci curvature. *Math. Res. Lett.*, 4(4):473–479, 1997.

15. G. Wei and W. Wylie. Comparison geometry for the Bakry-Émery tensor. *J. Differential Geom.*, 2(83):337–405, 2009.

16. D. Zhou. Essential spectrum of the Laplacian on manifolds of nonnegative curvature. *Int. Math. Res. Not.*, 5, 1994.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC