CADERNOS DE MATEMÁTICA **16**, 123–141 May (2015)
ARTIGO NÚMERO SMA#8

Cheeger-Müller theorem for manifolds with isolated conical singularities: A survey

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We present the history of the Cheeger-Müller theorem and the recent advances for its extension for manifolds with isolated conical singularities. May, 2015 ICMC-USP

1. INTRODUCTION

The Atiyah-Singer index theorem [AS] affirms the equality between the topological index and the analytical index of an elliptic operator over a closed manifold. This remarkable theorem makes a connection between topology and analysis. Following this line, D. B. Ray and I. M. Singer [RS] defined the Analytic Torsion with the goal to obtain a description to Reidemeister Torsion using analytic terms. They proved some properties of Analytic Torsion that Reidemeister Torsion satisfies too, but they did not prove the equality between this two torsions. Only in 1979, J. Cheeger [Che-1] and W. Müller [Mul-1] showed that Reidemeister Torsion and Analytic torsion are equal in a closed manifold. The proof of J. Cheeger is different from the one presented by W. Müller, and this is the reason that the equality between this two torsions is called Cheeger-Müller theorem. This theorem has applications in knot theory, number theory, gauge theory and others.

If the manifold has boundary, we do not have the equality between Analytic Torsion and Reidemeister Torsion anymore, since some boundary terms arise. The first result in this direction was presented by W. Lück [Luc]. W. Luck extended the Cheeger-Müller theorem for manifolds with boundary but with the restriction that the metric near the boundary is product. In this case the boundary term depends only of the Euler characteristic of the boundary. The full picture was prove only recently by J. Brüning and X. Ma [BM-1, BM-2]. They showed that if we have a manifold with boundary then the boundary terms depends of the Euler characteristic of the boundary and of the second fundamental form of the boundary. So the Cheeger-Müller theorem for smooth manifolds is completely understood

* Partially supported by FAPESP 2012/24454-8 and CNPq, Brazil.

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and the next step is the extension to singular spaces. The simplest singular space is a manifold with isolated conical singularity, which is a pseudomanifold. In [Che-2, Che-3], J. Cheeger studied the L^2-theory for manifolds with isolated conical singularities and M. Goresky and R. MacPhershon [GM-1, GM-2] developed the intersection homology theory for pseudomanifolds. J. Cheeger still proved that the L^2-cohomology of a manifold with isolated conical singularity is isomorphic to its intersection homology with middle perversity. With the works of J. Cheeger, M. Goresky and R. MacPherson in hands, A. Dar [Dar-1, Dar-2] extend the definition of Analytic Torsion and Reidemeister Torsion to manifolds with isolated conical singularities then the question about the Cheeger-Müller theorem in this context makes sense. We observe that there are extensions of the Cheeger-Müller theorem for other types of singularities, see for example [ARS, Pfa, Ver-3].

2. SMOOTH MANIFOLDS

2.1. Reidemeister Torsion

The Reidemeister Torsion was defined by K. Reidemeister [Rei] during his study about Lens spaces. Reidemeister used the torsion to present a counter example to the Hurewicz conjecture:

"Two homotopic equivalent compact manifolds are homeomorphic".

This conjecture is true for dimensions one and two but it is false in dimension three. A counter example is the Lens space $L(7,1)$ and $L(7,2)$ [Coh]. The definition of Reidemeister Torsion has an algebraic nature, let V be a finite dimensional vector space, if $\mathbf{v} = \{v_1, \dots, v_{\dim V}\}$ and $\mathbf{w} = \{w_1, \dots, w_{\dim V}\}$ are two bases of V we will denote by $(\mathbf{w}/\mathbf{v})$ the matrix of change of basis from v to w, i.e, $w_i = \sum_{j=1}^{\dim V} (\mathbf{w}/\mathbf{v})_{ij} v_j$. We will denote by $[\mathbf{w}/\mathbf{v}]$ the module of the determinant of $(\mathbf{w}/\mathbf{v})$. Let

$$\mathfrak{C} : C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0. \tag{2.1}$$

be a finite chain complex of finite dimensional vector spaces. Denote by $Z_q = \ker \partial_q$, $B_q = \mathrm{Im}\partial_{\mathrm{q}+1}$ and $H_q(\mathfrak{C}) = Z_q/B_q$. Fix a preferred base $\mathbf{c}_q$ for each C_q and a base $\mathbf{h}_q$ for $H_q(\mathfrak{C})$. Choose a lift $\tilde{\mathbf{h}}_q$ for $\mathbf{h}_q$, i.e, $\tilde{\mathbf{h}}_q \subset C_q$, $\mathbf{h}_q$ is linear independent, such that $\pi(\tilde{\mathbf{h}}_q) = \mathbf{h}_q$, a base $\mathbf{b}_q$ for B_q and let $\tilde{\mathbf{b}}_{q-1}$ a linear independent set of C_q such that $\partial_q(\tilde{\mathbf{b}}_{q-1}) = \mathbf{b}_{q-1}$. Then $\mathbf{b}_q\mathbf{c}_q\tilde{\mathbf{b}}_{q-1}$ is a base to C_q and it makes sense to consider $[\mathbf{b}_q\tilde{\mathbf{h}}_q\tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q]$. We will denote this number by $[\mathbf{b}_q\mathbf{h}_q\mathbf{b}_{q-1}/\mathbf{c}_q]$ as it depends only on $\mathbf{b}_q$, $\mathbf{h}_q$ and $\mathbf{b}_{q-1}$.

Definition 2.1. The Reidemeister Torsion of a chain complex $\mathfrak{C}$ with relation to $\mathbf{h}$ and $\mathbf{c}$ is the positive real number

$$\tau(\mathfrak{C}; \mathbf{h}; \mathbf{c}) = \prod_{q=0}^{m} [\mathbf{b}_q\mathbf{h}_q\tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q]^{(-1)^q}, \tag{2.2}$$

where $\mathbf{c} = \{\mathbf{c}_q\}$ and $\mathbf{h} = \{h_q\}$.

The number $\tau(\mathfrak{C};\mathbf{h};\mathbf{c})$ does not depend on the choice of $\mathbf{b}_q$ and $\tilde{\mathbf{b}}_{q-1}$.

The Reidemeister Torsion could be defined for a finite cell (or simplicial) complex, but we are interested in a more rigid structure, this is the context followed by Ray and Singer. For another approach see [Coh, Mil, Rei]. Let (W,g) be a compact, orientable, m-dimensional riemannian manifold with metric g with boundary ∂W. We assume that ∂W is the disjoint union of two closed submanifolds W_1 and W_2. It is possible that W_1, or W_2, or both are empty. Let K be a cellular decomposition (or a triangulation) of W with L_1, L_2 cellular decompositions of W_1 and W_2 which are subcomplexes of K. Let $(\tilde{K},\tilde{L}_1)$ be the universal covering cell complex pair of (K,L_1). Consider an orthogonal representation of $\pi_1(K)$, this means that $\rho:\pi_1(K)\to O_N(\mathbb{R})$ is a group homomorphism from $\pi_1(K)$ to the N dimensional real orthogonal group. Then define the chain complex of real vector spaces

$$\mathfrak{C}_q(K,L_1;E_\rho):=\mathbb{R}^N\otimes_{\pi_1(K)}C_q(\tilde{K},\tilde{L}_1;\mathbb{R}\pi_1(K)),$$

where E_ρ is the associated vector bundle to ρ. Choose a preferred base of $\mathfrak{C}_q(K,L_1)$ given by $\{\mathbf{e}\otimes\mathbf{c}_q\}$, where $\mathbf{e}$ is an orthonormal base of $\mathbb{R}^N$ and $\mathbf{c}_q=\{c_{q,j}\}$, $c_{q,j}$ runs over the preferred base of $C_q(\tilde{K};\pi_1(K))$ consisting of cells covering those in $K-L_1$.

Definition 2.2. [Reidemeister Torsion of a Riemannian Manifold] The Reidemeister Torsion of (W,g) for a orthonormal representation ρ and a homology base $\mathbf{h}=\{\mathbf{h}_q\}$ is

$$\tau(W,W_1;\mathbf{h};\rho):=\tau(\mathfrak{C}(K,L_1;E_\rho);\mathbf{h};\mathbf{e}\otimes\mathbf{c}),$$

where $\mathbf{e}\otimes\mathbf{c}=\{\mathbf{e}\otimes\mathbf{c}_q\}$.

Apparently this definition depends on the choice of the embedding K in $\tilde{K}$, the orientation of the cells of $K-L_1$ and the choice of the orthonormal base $\mathbf{e}$. But any other choice of any other three elements generates only a change of sign in the determinant. Then this definition depends only on $\mathbf{h}$ and the representation ρ. If the representation is acyclic J. Milnor [Mil] proved that the Reidemeister Torsion is a combinatorial invariant.

To compare the Reidemeister Torsion with the Analytic Torsion, Ray and Singer proposed a particular choice of the base $\mathbf{h}$. We will present this choice, but first, we need to define boundary conditions. Near one of the components of ∂W, suppose W_1 for example, we can decompose $\omega\in\Omega^*(W)$ in the form

$$\omega=\omega_1+dx\wedge\omega_2$$

where $\omega_1\in C^\infty(W)\otimes\Omega^*(W_1)$ is the *tangencial component* and $\omega_2\in C^\infty(W)\otimes\Omega^*(W_1)$ is the *normal component* defined by

$$*\omega_2=dx\wedge *\omega$$

and dx is the one form corresponding to the outward point unit normal vector to the boundary.

Definition 2.3. The absolute boundary conditions on W_1 is defined by

$$\mathcal{B}_{\rm abs}(\omega)=0\Leftrightarrow\omega_1=(\delta\omega)_1=0. \tag{2.3}$$

The relative boundary conditions on W_1 is defined by

$$\mathcal{B}_{\mathrm{rel}}(\omega) = 0 \Leftrightarrow \omega_2 = (d\omega)_2 = 0. \tag{2.4}$$

Let $\Omega^q(W, W_1; E_\rho)$ be the space of C^∞ q-forms on W with values in E_ρ, relative boundary conditions on W_1 and absolute boundary conditions on W_2. The riemannian metric g defines a duality $* : \Omega^q(W, W_1; E_\rho) \to \Omega^{m-q}(W, W_2; E_\rho)$, the operator $*$ is called *Hodge star*. Note that the Hodge Star permutes the boundary conditions. The Hodge star defines an inner product in $\Omega^q(W, W_1; E_\rho)$, for all q, given by

$$\langle \omega, \eta \rangle = \int_W \omega \wedge *\eta.$$

The *formal adjoint* of the exterior differential d is defined by $\delta = (-1)^{mq+m+1} * d *$. Define the *laplacian* by

$$\Delta^{(q)} = d^{(q-1)}\delta^{(q)} + \delta^{(q+1)}d^{(q)} \tag{2.5}$$

and the space of q-harmonic forms with relative boundary conditions on W_1 and absolute boundary conditions on W_2 by $\mathcal{H}^q(W, W_1; E_\rho) := \{\omega \in \Omega^q(W, W_1; E_\rho) : \Delta^{(q)}\omega = 0\}$. Consider the isomorphism

$$\begin{aligned} \mathcal{A}_q :& \mathcal{H}^q(W, W_1; E_\rho) \to H_q(W, W_1; E_\rho) \\ & \omega \to (-1)^{(m-1)q} \mathcal{P}_q^{-1} \mathcal{A}^{m-q} * \omega, \end{aligned}$$

where $\mathcal{A}^q$ is the de Rham map, $\mathcal{P}_q$ is the Poincaré duality and $*$ is the Hodge star. If we consider an orthonormal base $\mathfrak{h}_q$ to $\mathcal{H}^q(W, W_1; E_\rho)$ then $\mathcal{A}_q(\mathfrak{h}_q)$ is a base for $H_q(W, W_1; E_\rho)$.

Definition 2.4. The Reidemeister Torsion with the Ray and Singer base if defined by

$$\tau(W, W_1; \rho) := \tau(W, W_1; \mathcal{A}(\mathfrak{h}); \rho),$$

where $\mathcal{A}(\mathfrak{h}) = \{\mathcal{A}_q(\mathfrak{h}_q)\}$.

For now on this will be the Reidemeister Torsion that we will deal, this means that, if we refer to the Reidemeister Torsion then we are fixing the homology base using the harmonic forms. This torsion is a combinatorial invariant [RS, Proposition 3.7].

2.2. Motivation for the definition of Analytic Torsion

Before we define the Analytic torsion, we will present the motivation of Ray and Singer for its definition. This motivation is presented in [RS, Proposition 1.7] too, but they assumed that the complex is acyclic for simplicity. Consider the chain complex (2.1) of finite dimensional vector spaces. Fix an preferred base $\mathbf{c}_q$ for each q, and define an inner product such that $\mathbf{c}_q$ is an orthonormal base for each q. Then write

$$C_q = B_q \oplus H_q \oplus \partial_{q+1}(B_{q+1}),$$

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where $H_q = (\partial_{q+1}(B_{q+1}))^{\perp} \cap Z_q$ and $B_q = (Z_q)^{\perp}$. Note that, if $b \in B_q - \{0\}$ then $\partial_q(b) \neq 0$ and

$$\pi : H_q \to \frac{Z_q}{\partial_{q+1}(C_{q+1})} = H_q(\mathfrak{C})$$

is an isomorphism. Choose an orthonormal base to H_q, $\mathbf{h}_q = \{h_{q,1}, \ldots, h_{q,r_q}\}$. Define the combinatorial laplacian, $\Delta_c^{(q)} = \partial_{q-1}^* \partial_q + \partial_{q+1} \partial_q^*$, where ∂_{q-1}^* is the adjoint in relation the inner product of ∂_q, i.e., the transpose matrix of ∂_q. We claim that B_q is $\Delta_c^{(q)}$ - invariant. First, if $b \in B_q$ then $\partial_q^*(b) = 0$, since for all $c_{q+1} \in C_{q+1}$

$$\langle \partial_q^*(b), c_{q+1} \rangle = \langle b, \partial_{q+1}(c_{q+1}) \rangle = 0. \tag{2.6}$$

Now consider $z_q \in Z_q$, then

$$\langle \Delta_c^{(q)}(b), z_q \rangle = \langle \partial_{q-1}^* \partial_q(b), z_q \rangle = \langle \partial_q(b), \partial_q z_q \rangle = 0,$$

and this implies that $\Delta_c^{(q)}|_{B_q} \subset B_q$.

Let $\mathbf{b'}_q = \{b_{q,1}, \ldots, b_{q,j_q}\}$ be an orthonormal base of eigenvectors from $\Delta_c^{(q)}$ in B_q. Then

$$\langle \partial_q(b_{q,k}), \partial_q(b_{q,j}) \rangle = \langle \partial_{q-1}^* \partial_q(b_{q,k}), b_{q,j} \rangle = \lambda_{q,k} \delta_{kj},$$

and $\partial \mathbf{b}_{q+1} = \{(\lambda_{q+1,1})^{-\frac{1}{2}} \partial_{q+1}(b_{q+1,1}), \ldots, (\lambda_{q+1,j_{q+1}})^{-\frac{1}{2}} \partial_{q+1}(b_{q+1,j_{q+1}})\}$ is an orthonormal base of $\partial_q(B_q)$. Since $\partial \mathbf{b}_{q+1} \cup \mathbf{b'}_q \cup \mathbf{h}_q$ is an orthonormal base of C_q, the determinant of the change of base matrix in relation to the base c_q is ± 1. Therefore, if we choose the base $\mathbf{b}_q = \{\lambda_{q,1}^{-\frac{1}{2}} b_{q,1}, \ldots, \lambda_{q,j_q}^{-\frac{1}{2}} b_{q,j_q}\}$, we obtain that

$$[\partial \mathbf{b}_{q+1} \cup \mathbf{b}_q \cup \mathbf{h}_q / \mathbf{c}_q)] = \prod_{j=1}^{j_q} \lambda_{q,j}^{-\frac{1}{2}}.$$

Note that,

$$\Delta_c^{(q)}(\partial_{q+1}(b_{q+1,k})) = \partial_{q+1} \partial_q^* \partial_{q+1}(b_{q+1,k}) = \lambda_{q,k} \partial_{q+1}(b_{q+1,k}),$$

then the base $\{\partial_{q+1}(b_{q+1,1}), \ldots, \partial_{q+1}(b_{q+1,j_{q+1}})\} \cup \mathbf{b'}_q$ is the base of eigenvectors of Δ_q. Define the zeta function associated to Δ_q by

$$\zeta\left(s, \Delta_c^{(q)}\right) = \sum_{j=1}^{j_q} \lambda_{q,j}^{-s} + \sum_{j=1}^{j_{q+1}} \lambda_{q+1,j}^{-s}. \tag{2.7}$$

If we derive $\zeta\left(s, \Delta_c^{(q)}\right)$ in $s = 0$ we obtain

$$-\zeta'\left(0, \Delta_c^{(q)}\right) = \log \left(\prod_{j=1}^{j_q} \lambda_{q,j}\right) \left(\prod_{j=1}^{j_{q+1}} \lambda_{q+1,j}\right) = \log \det \Delta_c^{(q)}.$$

But

$$\log[\partial \mathbf{b}_{q+1} \cup \mathbf{b}_q \cup \mathbf{h}_q/\mathbf{c}_q)] = \sum_{k=q}^{m} (-1)^{k-q} \log\det \Delta_c^{(q)},$$

and the Reidemeister Torsion of the chain complex $\mathfrak{C}$ with relation of the bases $\mathbf{h} = \{\mathbf{h}_q\}$ and $\mathbf{c} = \{\mathbf{c}_q\}$ is

$$\log \tau(\mathfrak{C}; \mathbf{h}; \mathbf{c}) = \frac{1}{2} \sum_{q=0}^{m} (-1)^q q \zeta' \left(0, \Delta_c^{(q)}\right).$$

2.3. Analytic Torsion

In the last section, we proved that is possible to write the Reidemeister Torsion in function of the zeta function of the combinatorial laplacian. This was the motivation of Ray and Singer for the definition of Analytic Torsion. For simplicity, assume for the present that W is closed. We know that the laplacian (2.5) is a symmetric and positive-definite operator on $\Omega^q(W; E_\rho)$ and it has a pure point spectrum. Then we define,

Definition 2.5. The zeta function associated to $\Delta^{(q)}$ is defined by

$$\zeta(s, \Delta^{(q)}) = \sum_{\lambda \in \mathrm{Sp}_+ \Delta^{(q)}} \lambda^{-s}, \tag{2.8}$$

when $\mathrm{Re}(s) > \frac{m}{2}$, where $\mathrm{Sp}_+ \Delta^{(q)}$ denotes the positive parte of the spectrum of $\Delta^{(q)}$.

The above series is uniformly convergent for $\mathrm{Re}(s) > \frac{m}{2}$ and R. T. Seeley [See] proved that $\zeta(s, \Delta^{(q)})$ extends to a meromorphic function of s in the hole complex plane which is analytic in $s = 0$. This makes possible the definition of the Analytic Torsion.

Definition 2.6. The Analytic Torsion of $(W; g)$ with an orthogonal representation ρ of the fundamental group of W is

$$\log T((W, g); \rho) = \frac{1}{2} \sum_{q=0}^{m} (-1)^q q \zeta(s, \Delta^{(q)}). \tag{2.9}$$

We will omit g or ρ if the context is clear.

If W is not closed, then we need to impose boundary conditions on ∂W and all the previous definitions are valid. We will denote by $\log T_{\mathrm{abs}}((W, g); \rho)$ the Analytic Torsion of W with absolute boundary conditions on ∂W, by $\log T_{\mathrm{rel}}((W, g); \rho)$ if we impose relative boundary condition on ∂W and by $\log T_{\mathrm{rel/abs}}((W, g); \rho)$ if we impose relative boundary condition on W_1 and absolute boundary condition on W_2.

This was the candidate of an analytic representation for Reidemeister Torsion. Ray and Singer proved that if the representation ρ generates an acyclic complex, then the Analytic Torsion does not depend on the metric g ([RS, Theorem 2.1]). They proved some other properties for Analytic Torsion that Reidemeister Torsion does, for example, if $\dim W$ is

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even, then $\log T(W;\rho) = 0$ ([RS, Theorem 2.3]) and they conjecture the equality between Analytic Torsion and Reidemeister Torsion when W is closed.

2.4. Cheeger-Müller Theorem

At the end of seventies, Cheeger[Che-1] and Müller[Mul-1] proved the conjecture of Ray and Singer independently, and this theorem is called Cheeger-Müller Theorem.

THEOREM 2.1. *If (W,g) is a closed orientable riemannian manifold and ρ is an orthogonal representation of the fundamental group of W, then*

$$\tau(W;\rho) = T(W;\rho).$$

The two proofs of the Cheeger-Müller theorem used different methods but have similar objectives and each has his own interest. The proof of Cheeger uses surgeries techniques and the proof of Müller uses combinatorial Hodges theory, both of objetives are to reduce to the case of spheres. After that they use [Ray], where Ray proved the equality between the two torsions when W is a Lens space, and this proves the theorem.

After the Cheeger-Müller theorem, Analytic Torsion had his own interesse in many fields of Mathematics. We remember here the extension of Analytic Torsion to the notion of determinant line in cohomology made by D. Quillen [Qui]. This notion was important in [Mul-2, BZ, BM-1, BM-2]. In particular, W. Müller [Mul-2] extends Cheeger-Müller theorem to an unimodular representation when the dimension of W is odd and the associated fiber bundle E_ρ is flat. By a *unimodular representation* we understand a representation $\rho : \pi_1(W) \to GL(E)$ on a finite-dimensional real or complex vector space E such that $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(W)$. The full picture was proved by J. -M. Bismut and W. Zhang [BZ], i.e., for an unimodular representation with no restriction in the dimension of W or E_ρ. In this case the equality holds only in dimension odd and in dimension even the difference between $\log T(W;\rho)$ and $\log \tau(W;\rho)$ can determined [BZ, Theorem 0.2].

2.4.1. Cheeger-Müller theorem for manifolds with boundary

Here we assume that $\partial W \neq \emptyset$. First, as we observed in the Subsection 2.1, the Hodge Star permutes the boundary conditions and this is fundamental for the proof of the Poincaré duality for Analytic Torsion [Luc, Proposition 2.10].

PROPOSITION 2.1 (Poincaré duality). *Let (W,g) be a compact orientable riemannian manifold with boundary. Then*

$$\log T_{\text{abs}}(W;\rho) = (-1)^{\dim W+1} \log T_{\text{rel}}(W;\rho)$$

By the Poincaré duality we can concentrate only in one type of boundary conditions, so we choose absolute boundary condition. The extension of the Cheeger-Müller theorem for manifolds with boundary is not simple. Cheeger in [Che-1] mentioned that

$$\log T(W;\rho) = \log \tau(W;\rho) + f(\partial W),$$

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but the first result that determines $f(\partial W)$ was from W. Lück [Luc]. Lück proved an extension to Cheeger-Müller theorem in the case that the metric has product structure near the boundary. More precisely,

DEFINITION 2.7. We say that the metric g is product near the boundary if exists a collar $f : \partial W \times [0,1) \to U \subset W$, where U is a neighborhood of W with $\partial W \subset U$, such that f is an isometry if, $[0,1)$ has the usual metric, U and ∂W has the induced metric by W and $\partial W \times [0,1)$ has the product metric.

With this hypothesis the term $f(\partial W)$ is

THEOREM 2.2. *Let (W,g) be a compact orientable riemannian manifold with boundary ∂W and let ρ be an orthogonal representation of $\pi_1(W)$. If the metric near ∂W has a product structure, then*

$$\log T_{\text{abs}}(W;\rho) = \log \tau(W;\rho) + \frac{\text{rk}(\rho)}{4}\chi(\partial W)\log 2,$$

where $\text{rk}(\rho)$ *is the rank of the representation.*

The general case was recently study by X. Dai and H. Fang in [DF] and by J. Brüning and X. Ma in [BM-1, BM-2]. In [HMS, Section 4], the author, T. de Melo and M. Spreafico showed that the formula of the article [DF] has a problem, so we will describe the formula from [BM-1, BM-2]. In [BM-1] the main objective is the comparison of Analytic Torsion of W with two different metrics, this creates a new term in the comparison between Analytic Torsion and Reideimeister Torsion, we will define this term. We identify an antisymmetric endomorphism ϕ of a finite dimensional vector space V (over a field of characteristic zero) with the element $\hat{\phi} = \frac{1}{2}\sum_{j,k=1}^{\dim V}\langle\phi(v_j),v_k\rangle\hat{v}_j\wedge\hat{v}_k$, of $\widehat{\Lambda^2 V}$, for $\mathbb{Z}_2$ graded algebras. The elements $\langle\phi(v_j),v_k\rangle$ are the entries of the tensor representing ϕ in the base $\{v_k\}$, and this is an antisymmetric matrix. Now assume that r is an antisymmetric endomorphism of V with values in $\Lambda^2 V$. Then, $(R_{jk} = \langle r(v_j),v_k\rangle)$ is a tensor of two forms in $\Lambda^2 V$. Then we identify R with the element

$$\hat{R} = \frac{1}{2}\sum_{j,k=1}^{\dim V}\langle r(v_j),v_k\rangle\wedge\hat{v}_j\wedge\hat{v}_k,$$

of $\Lambda^2 V\wedge\widehat{\Lambda^2 V}$. In particular, all the construction can be done taking the dual V^* instead of V. Let g_0 be a suitable deformation of g that is a product near the boundary, we define

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the following forms (where $i : \partial W \to W$ denotes the inclusion)

$$\mathcal{S} = \frac{1}{2} \sum_{k=1}^{m-1} (i^*\omega - i^*\omega_0)_{0k} \wedge \hat{e}_k^*$$

$$\widehat{i^*\Omega} = \frac{1}{2} \sum_{k,h=1}^{m-1} i^*\Omega_{k,h} \wedge \hat{e}_k^* \wedge \hat{e}_h^*, \qquad \hat{\Theta} = \frac{1}{2} \sum_{k,h=1}^{m-1} \Theta_{k,h} \wedge \hat{e}_k^* \wedge \hat{e}_h^*,$$

where ω and ω_0 are the connection one forms associated to the metrics g and g_0, respectively, Ω is the curvature two form of g, Θ is the curvature two form of the boundary (with the metric induced by the inclusion), and $\{e_k\}_{k=0}^{m-1}$ is an orthonormal base of TW (with respect to the metric g). Then, setting

$$B = \frac{1}{2} \int_0^1 \int^B \mathrm{e}^{-\frac{1}{2}\hat{\Theta} - u^2 \mathcal{S}^2} \sum_{k=1}^{\infty} \frac{1}{\Gamma\left(\frac{k}{2}+1\right)} u^{k-1} \mathcal{S}^k du,$$

we define,

Definition 2.8. The Anomaly Boundary term is

$$A_{\mathrm{BM,abs}}(\partial W) = (-1)^{m+1} A_{\mathrm{BM,rel}}(\partial W) = \frac{1}{2} \int_{\partial W} B,$$

where the subscript abs/rel means the boundary conditions on ∂W.

Then we have the complete extension for Cheeger-Müller theorem for manifolds with boundary and an orthogonal representation of $\pi_1(W)$.

Theorem 2.3. *Let (W, g) be a compact orientable riemannian manifold with boundary ∂W and let ρ be an orthogonal representation of $\pi_1(W)$. Then,*

$$\log T_{\mathrm{abs}}(W;\rho) = \log \tau(W;\rho) + \frac{\mathrm{rk}(\rho)}{4} \chi(\partial W) \log 2 + \mathrm{rk}(\rho) A_{\mathrm{BM,abs}}(\partial W).$$

3. MANIFOLDS WITH ISOLATED CONICAL SINGULARITIES

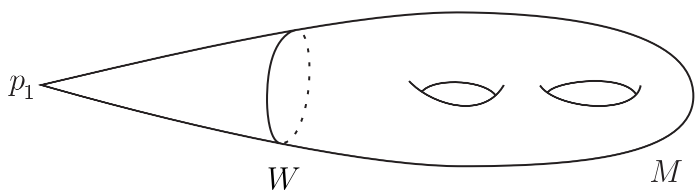
The next step of investigation is the extension of Cheeger-Müller theorem for singular manifolds. It was A. Dar [Dar-1, Dar-2], a student of J. Cheeger, who extended Reidemeister Torsion and Analytic Torsion for manifolds with isolated conical singularities. In fact, Dar extended Reidemeister Torsion to pseudomanifolds but the extension of Analytic Torsion was only for a manifold with isolated conical singularities, so we will discuss only this case.

Definition 3.1. Let $(W, \tilde{g})$ be a closed riemannian manifold of dimension m with metric $\tilde{g}$. A metric cone, denoted by CW, is the space $(0, \infty) \times W$, with the metric

$$g = dx \otimes dx + x^2 \tilde{g}. \tag{3.1}$$

A finite metric cone, denoted by $C_l W$, is the space $[0, l] \times W$ with the metric g for $x > 0$.

Definition 3.2. X is manifold with isolated conical singularities and dimension $m+1$ if exists $p_j \in X$, $j = 1, \ldots, k$, such that $X - \cup_{j=1}^{k}\{p_j\}$ is an open riemannian manifold (possible with boundary) and each p_j has a neighborhood U_j such that $U_j - \{p_j\}$ is isometric to $C_{l_j} W_j^m - \{v_j\}$, where v_j is the tip of the cone. The metric in the open riemannian manifold is denoted by g_X.



If $k = 1$ then $X = C_{l_1} W \cup M$, where M is a compact oriented riemannian manifold with boundary and dimension $m+1$, $\partial M = W$, W is a closed riemannian manifold of dimension m and the union is along the boundary.

A manifold with isolated conical singularities is a pseudomanifold [Dar-1, Dar-2, Che-3, HS-4] with a trivial stratification $\sigma = X_j = \{p_1, \ldots, p_k\}$, for $j = 0, \ldots, m$, $X_{m+1} = X$ and $X_{-1} = \emptyset$. We will assume that every pseudomanifold here is compact, orientable and has a pl-structure. We defined Reideimeister Torsion for a chain complex fixing special bases, looking for this side, we could define Reidemiester Torsion for a singular manifold. But in this situation, the Reidemeister Torsion of a cone depends only of the zero cells, loosing all the information in higher dimension. Besides that, the Poincaré duality does not work for a cone over a manifold in general. So the idea of Dar was to use the intersection homology theory in the place of cell (simplicial) theory. We will need some ingredients to define the intersection homology. A *perversity* is a sequence of integers $\overline{p} = \{p_j\}_{j=2}^{m+1}$ such that $p_2 = 0$ and $p_{j+1} = p_j$ or $p_j + 1$. The *null perversity* is $0_j = 0$, and the *top perversity* is $t_j = j - 2$. Given a perversity $\overline{p}$, the *complementary perversity* $\overline{p}^c$ is $p_j^c = t_j - p_j = j - p_j - 2$. We will use two particular perversities, *lower middle perversity* and *upper middle perversity*. The lower middle perversity is denoted by $\overline{m}$, and $\overline{m} = \{m_j = [j/2] - 1\}_{j=2}^{m+1}$, and $\overline{m}^c$ is the upper middle perversity. Now let X be manifold with isolated conical singularity and the previous stratification. If j is an integer and $\overline{p}$ a perversity, a subspace Y of X is said

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$(\overline{p}, j)$-*allowable* if

$$\dim(Y) \leq j, \qquad \dim(Y \cap X_{m+1-k}) \leq j - k + p_k, \quad \forall k \geq 2.$$

Let T a triangulation for X compatible with the trivial stratification such that ∂X is triangulated by a subcomplex $L = \partial T$ of T, following [GM-1]. Let $C^T(X) = C(T)$ denote the chain complex of simplicial chains of X with respect to T. Let $C(X)$ denote the direct limit chain complexes under refinement of the $C^T(X)$ over all triangulations of X compatible with the pl-structure. Since ∂X is pl-subspace of X, $C^T(\partial X) = C(L)$ is defined, and is a sub complex of $C^T(X)$, and the relative chain complex is also defined $C^T(X, \partial X) = C(T, L) = C(T)/C(L)$. The construction commutes with direct limit, and hence $C(X)$ and $C(X, \partial X)$ are defined. The *intersection chain group* of perversity $\overline{p}$, is the subgroup

$$I^{\overline{p}}C_q(X) = \{c \in C_q(X) : |c| \text{ is } (\overline{p}, q) - \text{allowable and } |\partial c| \text{ is } (\overline{p}, q-1) - \text{allowable}\}.$$

The *relative intersection chain group* of perversity $\overline{p}$, is $I^{\overline{p}}C_q(X, \partial X) = I^{\overline{p}}C_q(X)/I^{\overline{p}}C_q(\partial X)$, where $I^{\overline{p}}C_q(\partial X) = C_q(\partial X)$ for each $\overline{p}$, since ∂X is actually a manifold. The intersection chain groups of perversity $\overline{p}$with the usual boundary form a chain complex called *intersection chain complex* of the perversity $\overline{p}$, and his homology is called *intersection homology* of perversity $\overline{p}$. The algebraic dual of intersection homology is the intersection cohomology. Poincaré duality is recovered for pseudomanifolds using intersection homology

$$I\mathcal{P}_q : I^{\overline{p}}H_q(X) \to I^{\overline{p}^c}H^{m-q}(X, \partial X). \tag{3.2}$$

3.1. Intersection Reidemeister Torsion

To define the Intersection Reidemeister Torsion we need to work with finitely generated chain groups then we use the basic R-sets [GM-1]. Let $R_q^{\overline{p}}$ be the subcomplex of the first barycentric subdivision T' of T consisting of all simplices which are $(\overline{p}, q)$-allowable. Then, $R_q^{\overline{p}}$ is a subcomplex of the q-skeleton of T' and it is a subcomplex of $R_{q+1}^{\overline{p}}$. Define the complex $C^{\overline{p}}(X)$ by setting

$$C_q^{\overline{p}}(X) = H_q(R_q^{\overline{p}}, R_{q-1}^{\overline{p}}),$$

and boundary defined by the homology long exact sequence of the pair $(R_q^{\overline{p}}, R_{q-1}^{\overline{p}})$. This is a free abelian group generated by finitely many chains with contractible support. The homology of $C^{\overline{p}}(X)$ is isomorphic to $I^{\overline{p}}H_q(X)$. Let $P_q^{\overline{p}} = R_{q+1}^{\overline{p}} \cap L'$. Then, $P_q^{\overline{p}}$ is an R-set $R_q^{\overline{p}}$ of ∂X, and $\dim(R_q^{\overline{p}}) = q - 1$. Since ∂X is a manifold $P_q^{\overline{p}} = L'_{(q)}$ is the q-skeleton of L'. Define the chain complex $C^{\overline{p}}(\partial X)$ as above. Then, the homology of $C^{\overline{p}}(\partial X)$ is isomorphic $H_q(\partial X)$. We define the complex $C^{\overline{p}}(X, \partial X)$ by setting

$$C_q^{\overline{p}}(X, \partial X) = H_q(R_q^{\overline{p}} \cup L', R_{q-1}^{\overline{p}} \cup L'),$$

and boundary defined by the homology long exact sequence of the pair $(R^{\overline{p}}_q \cup L', R^{\overline{p}}_{q-1} \cup L')$. This is a free abelian group generated by finitely many chains with contractible support. The homology of $C^{\overline{p}}(X, \partial X)$ is isomorphic to the relative intersection homology of the pair $(X, \partial X)$. We can proceed in a very similar way as in the smooth case and define,

$$\mathfrak{C}^{\overline{p}}_q(X; E_\rho) := \mathbb{R}^N \otimes_{\pi_1(X)} C^{\overline{p}}_q(\tilde{T}; \mathbb{R}\pi_1(X))$$
$$\mathfrak{C}^{\overline{p}}_q(X, \partial X; E_\rho) := \mathbb{R}^N \otimes_{\pi_1(X)} C^{\overline{p}}_q(\tilde{T}, \tilde{L}; \mathbb{R}\pi_1(X)).$$

Consider g_X the riemannian structure of the non-singular part of X. With this structure we can define L^2 forms in $X - \Sigma$, in [Che-3, Section 6], Cheeger proved that the intersection homology of X with middle perversity is isomorphic to L^2 cohomology. This and the Poincaré Duality (3.2) give to us the extension the isomorphism $\mathcal{A}_q$ to $I^{\overline{m}}\mathcal{A}_q$. For simplicity, we put the additional hypothesis that when $\dim X$ is even, X satisfy the Witt condition, see [MV]. With the Witt condition the intersection homology with lower middle perversity and upper middle perversity coincide. Then we define,

Definition 3.3. The Intersection Reidemeister Torsion of X with respect to the representation ρ is defined by

$$I\tau((X, g_X); \rho) = \tau(\mathfrak{C}^{\overline{m}}(X; I^{\overline{m}}\mathcal{A}(\mathfrak{h}); \rho)),$$

where $I^{\overline{m}}\mathcal{A}(\mathfrak{h}) = \{I^{\overline{m}}\mathcal{A}_q(\mathfrak{h}_\mathfrak{q})\}$, $\mathfrak{h} = \{\mathbf{h}_q\}$ and $\mathbf{h}_q$ is an orthonormal base of the L^2 harmonic q-forms. The Relative Intersection Reidemeister Torsion of $(X, \partial X)$ is defined similary.

3.2. Analytic Torision

Let X be a manifold with isolated conical singularity of dimension $m+1$, as above. Without loss of generality, suppose $k = 1$, then $X = C_1 W \cup M$, where M is a compact oriented riemannian manifold with boundary $\partial M = W$, W is a closed riemannian manifold of dimension and the union is along the boundary. Let $\rho : \pi_1(X) \to O_N(\mathbb{R})$ be an orthonormal representation and consider E_ρ the associated vector bundle, as previously. Cheeger [Che-2] made an extensive study of functional calculus on CW and Dar [Dar-1] used Cheeger's work to define the Analytic Torsion of X.

Let $\Omega^*(X; E_\rho)$ be the space of the differential forms with values in E_ρ (remember that the analysis is made in the open space $X - \{p_1\}$). Consider the laplacian $\Delta^{(q)}_X$ as an unbounded operator in $L^2\Omega^q(X; E_\rho)$ with domain in the space of the q-differential forms with compact support $\Omega^q_c(X; E_\rho)$. Choose the Friedrichs extension for $\Delta^{(q)}_X$, so $\Delta^{(q)}_X$ has pure point spectrum and for $\mathrm{Re}(s) > \frac{m+1}{2}$, $\zeta(s, \Delta^{(q)}_X)$ is defined as in (2.8). In [Che-2] is presented the asymptotic expansion of the trace of heat kernel on $C_1(W)$, this permits to obtain a meromorphic extension of $\zeta(s, \Delta^{(q)}_X)$, but this extension possible has a pole in $s = 0$. Dar [Dar-1, Theorem 4.4] proved that these poles cancels in the alternated sum of the definition of the Analytic Torsion, then we can define $\log T((X,g_X);\rho)$ as in Definition 2.9. More recently, Müller and Vertman [MV] showed that the extension of $\zeta(s,\Delta_X^{(q)})$ is regular at $s=0$.

3.3. Cheeger-Müller theorem for manifolds with isolated conical singularities

The last two subsections take us to the natural question: Is it possible to present a version to Cheeger-Müller theorem for manifolds with conical singularities? The answer, we do not know yet but some progress are made in this yet. By Poincaré duality and Hodge star, A. Dar [Dar-1, Theorem 4.5] proved that,

Theorem 3.1. *If X is a closed manifold with isolated conical singularity with dimension even then* $\log T(X;\rho)=\log I\tau(X;\rho)=0$.

In [Les-1], M. Lesch propose a line of investigation for the Cheeger-Müller theorem for manifolds with isolated conical singularities. The result of Vishik [Vis] motivated Lesch [Les-1, Problem 5.3] to propose the problem to determine $\log T(C_1W;\rho)$, because if we know the Cheeger-Müller theorem for C_1W then with a gluing formula we can prove the Cheeger-Müller theorem for X. More precisely, consider X as in the last subsection, $X=C_1W\cup M$. Since X is a union of two space along his boundaries, we try to understand the extension in each part and then we "glue" the two space to obtain the extension to X.

In the Intersection Reidemeister Torsion situation we have the gluing formula proved by Milnor [Mil, Theorem 3.2] (the result of Milnor applies to a chain complex) that shows that

$$\log I\tau(X;\rho)=\log I\tau((C_1W,g);\rho)+\log\tau(M,W;\rho)+\log\tau(\mathcal{H}),$$

where $\log\tau(\mathcal{H})$ is the Reidemeister Torsion of the exact homology sequence induced by the exact sequence $0\longrightarrow C_1W\hookrightarrow X\twoheadrightarrow M\longrightarrow 0$.

For the analytic side, Vishik [Vis] proved a gluing formula for the Analytic Torsion for smooth manifolds but with the restriction that the metric near the gluing is product (see Definition 2.7). Only in 2013, Brüning and Ma [BM-2] presented a gluing formula without any restriction on the metric in the smooth case.

Theorem 3.2 ([BM-2]). *Let M be a closed Riemannian manifold such that $M=M_1\cup_W M_2$, where M_1 and M_2 are compact Riemannian manifolds with boundary $\partial M_1=\partial M_2=W$, W is closed. Then,*

$$\log T(M)=\log T(M_1)+\log T(M_2,W)+\log\tau(\mathcal{H})+\frac{1}{2}\chi(W)\log 2+(-1)^{\dim M}A_{\rm BM}(W),$$

where $\mathcal{H}$ is the exact sequence of the pair (M,M_1), in the harmonic forms.

In the singular situation, Lesch extended the result of Vishik to manifolds with isolated conical singularities with the restriction of the metric is product near the gluing. In fact the formula applies to others types of singularities, see [Les-2].

Now we will focus on case to extend the Cheeger-Müller theorem to C_1W. Remember that W is a closed orientable riemannian manifold of dimension m and metric $\tilde{g}$. Since $\pi_1(C_1W)$ is trivial we have only the trivial representation in this case, so we consider $N = 1$. In [HS-1, Theorem 1], we proved a particular case when $W = S^p$, with $p = 1, 2, 3$ we identified a topological interpretation for some terms.

THEOREM 3.3. *The Analytic Torsion of the cone $C_\alpha S^p_{\sin\alpha}$ of angle α, and length 1, over the sphere $S^p_{\sin\alpha}$ (of radius $\sin\alpha$), with the standard metric induced by the immersion in $\mathbb{R}^{p+2}$, and absolute boundary condition is, for $p = 1, 2, 3$.*

$$\begin{aligned}
\log T(C_\alpha S^1_{\sin\alpha}) &= \frac{1}{2}\log \mathrm{Vol}(C_\alpha S^1_{\sin\alpha}) + \frac{1}{2}\sin\alpha, \\
\log T(C_\alpha S^2_{\sin\alpha}) &= \frac{1}{2}\log \mathrm{Vol}(C_\alpha S^2_{\sin\alpha}) - \frac{1}{2}f(\csc\alpha) + \frac{1}{4}\sin^2\alpha \\
\log T(C_\alpha S^3_{\sin\alpha}) &= \frac{1}{2}\log \mathrm{Vol}(C_\alpha S^3_{\sin\alpha}) + \frac{3}{4}\sin\alpha - \frac{1}{12}\sin^3\alpha,
\end{aligned}$$

where the function $f(\nu)$ is

$$f(\nu) = \log\frac{\nu^2}{\pi} + \zeta(\frac{1}{2}, \mathrm{Sp}_+\Delta^{(0)}_{S^2})\frac{1}{\nu} + \sum_{\substack{j,k=0,\\ j+k\neq 0}}^{\infty} \frac{1}{(2k+1)2^{2k}}\frac{1}{2^{2j}}\binom{-k-\frac{1}{2}}{j}\frac{\zeta(k+j+\frac{1}{2}, \mathrm{Sp}_+\Delta^{(0)}_{S^2})}{\nu^{2k+2j+1}}.$$

This result and the results in [HMS] presented some ideas to understand all the terms in the general formula. It was B. Vertman [Ver, Ver-1], the author and M. Spreafico [Har, HS-2] that presented the first formulas for $\log T(C_1W)$. Vertman present the general formula for $\log T(C_1W)$, but he did not interpret the topological parts of the formula, and the author and M. Spreafico worked in the particular case of the sphere S^n and interpreted the topological parts. We showed that the anomaly boundary term (Definition 2.8) appears in the Analytic Torsion of spheres, and we proved an extension of Cheeger-Müller theorem for odd dimensional sphere.

PROPOSITION 3.1 ([Har, HS-2]). *If $S^{2p-1}_{\sin\alpha}$ is the odd dimensional sphere (of radius $\sin\alpha$), with the standard induced Euclidean metric, then the Anomaly Boundary contribution in the Analytic Torsion of $C_1S^{2p-1}_{\sin\alpha}$ is the Anomaly Boundary term of Brüning and Ma, namely $A_{\mathrm{BM}}(\partial C_l S^{2p-1}_{\sin\alpha})$. In this case, the formula for the Analytic Torsion reads*

$$\log T_{\mathrm{abs}}(C_1S^{2p-1}_{\sin\alpha}) = \frac{1}{2}\log \mathrm{Vol}(C_1S^{2p-1}_{\sin\alpha}) + A_{\mathrm{BM}}(\partial C_1 S^{2p-1}_{\sin\alpha}),$$

where

$$A_{\mathrm{BM}}(\partial C_1 S^{2p-1}_{\sin\alpha}) = \frac{(2p-1)!}{4^p(p-1)!}\sum_{k=0}^{p-1}\frac{1}{(p-1-k)!(2k+1)}\sum_{j=0}^{k}\frac{(-1)^{k-j}2^{j+1}}{(k-j)!(2j+1)!!}\sin^{2k+1}\alpha.$$

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For even spheres the proof that the Anomaly Boundary contribution in the Analytic Torsion of $C_1 S^{2p}_{\sin\alpha}$ is the Anomaly Boundary term of Brüning and Ma is similar, see [Har-1]. Then using [HMS, Proposition 2] and the calculation in [HS-3], we obtain

COROLLARY 3.1. *The Cheeger-Müller theorem for $C_1 S^{2p-1}_{\sin\alpha}$ reads*

$$\log T_{\text{abs}}(C_1 S^{2p-1}_{\sin\alpha}) = \log I\tau(C_1 S^{2p-1}_{\sin\alpha}) + A_{\text{BM, abs}}(\partial C_1 S^{2p-1}_{\sin\alpha}).$$

This motivated the topological interpretation of the general case. To present this formula, we need some definition. Let $\lambda_{q,n}$ denotes the eigenvalue of a coexact eigenform, $m_{\text{cex},q,n}$ its multiplicity, $\alpha_q = \frac{1+2q-m}{2}$, $\mu_{q,n} = \sqrt{\lambda_{q,n} + \alpha_q^2}$ and

$$\zeta_{\text{cex}}(s, \tilde{\Delta}^{(q)} + \alpha_q^2) = \sum_{n=1}^{\infty} \frac{m_{\text{cex},q,n}}{(\lambda_{q,n} + \alpha_q^2)^{-s}}.$$

THEOREM 3.4 ([Ver-2, HS-2]). *If dimension of W is odd and equal to $2p-1$ ($p \geq 1$) then the Analytic torsion of $C_1 W$ is*

$$\log T(C_1 W) = \frac{1}{2}\log T(W, \tilde{g}) + \frac{1}{2}\sum_{q=0}^{p-1}(-1)^q r_q \log\frac{1}{2(p-q)} + \frac{1}{2}\sum_{q=0}^{p-1}(-1)^q \sum_{j=1}^{p-1} \operatorname*{Res}_{s=0}{}_0\, \Phi^{\text{odd}}_{2j+1,q}(s) \operatorname*{Res}_{s=j+\frac{1}{2}}{}_1\, \zeta_{\text{cex}}\left(s, \tilde{\Delta}^{(q)} + \alpha_q^2\right)$$

where the functions $\Phi^{\text{odd}}_{2j+1,q}(s)$ are some universal functions explicitly known by some recursive relations, and $\tilde{\Delta}$ is the Laplace operator on forms on the section of the cone.

THEOREM 3.5 ([MV, HS-5]). *If dimension of W is even and equal to $2p$ ($p \geq 1$) then the Analytic torsion of $C_1 W$ is*

$$\begin{aligned}\log T(C_1 W) &= \sum_{q=0}^{p-1}(-1)^q \frac{r_q}{2}\log\frac{1}{2p-2q+1} + \frac{1}{2}\chi(W)\log 2 \\ &+ \sum_{q=0}^{p-1}(-1)^{q+1} r_q \log(2p-2q-1)!! + \frac{1}{2}\sum_{q=0}^{p-1}(-1)^{q+1}\mathcal{A}_{0,0,q}(0) \\ &+ \frac{1}{2}\sum_{q=0}^{p-1}(-1)^q \sum_{j=1}^{p} \operatorname*{Res}_{s=0}{}_0\, \Phi^{\text{even}}_{2j,q}(s) \operatorname*{Res}_{s=j}{}_1\, \zeta_{\text{cex}}\left(s, \tilde{\Delta}^{(q)} + \alpha_q^2\right),\end{aligned}$$

where the functions $\Phi^{\rm even}_{2j,q}(s)$ *are some universal functions explicitly known by some recursive relations,* $\tilde\Delta$ *is the Laplace operator on forms on the section of the cone and*

$$\mathcal{A}_{0,0,q}(s) = \sum_{n=1}^{\infty}\left(\log\left(1-\frac{\alpha_q}{\mu_{q,n}}\right) - \log\left(1+\frac{\alpha_q}{\mu_{q,n}}\right)\right)\frac{m_{q,n}}{\mu_{q,n}^{2s}}.$$

It is convenient to define,

Definition 3.4. The Anomaly Boundary contribution in the analytic torsion of a cone over a closed manifold W, denoted by $\log T_{\rm AB}(C_1W)$, is

$$\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q\sum_{j=1}^{p-1}\operatorname*{Res_0}_{s=0}\Phi^{\rm odd}_{2j+1}(s)\operatorname*{Res_1}_{s=j+\frac{1}{2}}\zeta_{\rm cex}\left(s,\tilde\Delta^{(q)}+\alpha_q^2\right),$$

if $\dim W = 2p-1$ and

$$\frac{1}{2}\sum_{q=0}^{p-1}(-1)^q\sum_{j=1}^{p}\operatorname*{Res_0}_{s=0}\Phi^{\rm even}_{2j}(s)\operatorname*{Res_1}_{s=j}\zeta_{\rm cex}\left(s,\tilde\Delta^{(q)}+\alpha_q^2\right),$$

if $\dim W = 2p$.

The proof that $\log_{\rm AB}(C_lW)$ is equal to $A_{\rm BM}(\partial C_1W)$ is indirect. To describe the idea of the proof, consider the *conical frustum*, i.e., the space $[l_1,1]\times W$, $0<l_1<1$, with the metric g in (3.1). We denote this the conical frustum by $C_{[l_1,1]}W$. Note that $\partial C_{[l_1,1]}W$ is a disjoint union of two copies of W, one with the metric $l_1^2\tilde g$ and the other with the metric $\tilde g$. Denote by W_1 the riemannian manifold $(W, l_1^2\tilde g)$. If we calculate the Analytic Torsion of C_1W, by definition, we obtain

$$\begin{aligned}\log T_{\rm rel/abs}(C_{[l_1,1]}W) &= 2\log T_{\rm AB}(C_1W), \qquad \dim W = 2p-1,\\ \log T_{\rm abs}(C_{[l_1,1]}W) &= 2\log T_{\rm AB}(C_1W), \qquad \dim W = 2p.\end{aligned}$$

If we calculate using [BM-1, BM-2], we obtain

$$\begin{aligned}\log T_{\rm rel/abs}(C_{[l_1,1]}W) &= 2A_{\rm BM,abs}(\partial C_1W), \qquad \dim W = 2p-1,\\ \log T_{\rm abs}(C_{[l_1,1]}W) &= 2A_{\rm BM,abs}(\partial C_1W), \qquad \dim W = 2p.\end{aligned}$$

Then we have,

Theorem 3.6 ([HS-2, HS-5, MV, Ver-2]). *The analytic torsion* $T_{\rm abs}(C_1W)$ *of the cone over* W *with absolute and ideal boundary conditions is as follows, where* $r_q = {\rm rk}H_q(W)$ *and* $p\geq 1$*: If* $m=2p-1$ *is odd we have*

$$\log T_{\rm abs}(C_1W) = \frac{1}{2}\log T(W,\tilde g) + \frac{1}{2}\sum_{q=0}^{p-1}(-1)^{q+1}r_q\log(2p-2q) + A_{\rm BM,abs}(\partial C_1W),$$

and if $m = 2p$ *is even*

$$\log T_{\mathrm{abs}}(C_1W) = \sum_{q=0}^{p-1}(-1)^{q+1}\frac{r_q}{2}\log(2p-2q+1) + \frac{1}{2}\chi(W)\log 2 + A_{\mathrm{BM,abs}}(\partial C_1 W) + \sum_{q=0}^{p-1}(-1)^{q+1} r_q \log(2p-2q-1)!! + \frac{1}{2}\sum_{q=0}^{p-1}(-1)^{q+1}\mathcal{A}_{0,0,q}(0).$$

If $\dim W = 2p-1$, the author and Spreafico [HS-4, Proposition 4.1] proved, using the gluing formula of Milnor [Mil], that

$$\log I\tau(C_1W) = \frac{1}{2}\log\tau(W,\tilde{g}) + \frac{1}{2}\sum_{q=0}^{p-1}(-1)^{q+1} r_q \log(2p-2q),$$

then we obtain a Cheeger-Müller theorem for C_1W.

THEOREM 3.7. *If the dimension of* W *is odd then the Cheeger-Müller Theorem for* C_1W *reads*

$$\log T_{\mathrm{abs}}(C_1W) = \log I\tau(C_1W) + A_{\mathrm{BM,abs}}(\partial C_1W).$$

If $\dim W = 2p$ then the question about the extension of Cheeger-Müller theorem is still open. The great difference between the odd case and the even case is that we can explicit all the terms in the Analytic Torsion. More precisely, it is still unknown a good representation for the term

$$\frac{1}{2}\sum_{q=0}^{p-1}(-1)^{q+1}\mathcal{A}_{0,0,q}(0),$$

even in the case that $W = S^2$. We observe that this terms appears in the Analytic Torsion of a disc [HMS], but a disc is a smooth manifold and in this case it is possible determine this term explicitly. An interesting result of W. Müller and B. Vertman [MV] shows the asymptotic behavior of the term $\frac{1}{2}\sum_{q=0}^{p-1}(-1)^{q+1}\mathcal{A}_{0,0,q}(0)$ when we deform the metric of W.

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