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On reproducing kernel and density problems

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Positive definite or reproducing kernel on X, in its various formats, become common topics in many branches of mathematics. Many papers is devoted to study approximation problems of functions in a given space, being $C(X)$ and $L^p(X,\mu)$ the most common ones. In this case is important to know if we can use functions in a dense reproducing kernel Hilbert space (RKHS). That is why this paper is devoted to study embedding and denseness properties of RKHS, when X is a locally compact topological Hausdorff space or a locally compact topological Hausdorff abelian group. May, 2015 ICMC-USP

1. INTRODUCTION AND MOTIVATION

Let X being a non-empty set and $K : X \times X \to \mathbb{C}$ be a *positive definite kernel* on X, that is, a function satisfying the inequality

$$\sum_{i,j=1}^{n} \overline{c_i} c_j K(x_i, x_j) \geq 0, \tag{1.1}$$

whenever $n \geq 1$, $x_i \in X$ and $c_i \in \mathbb{C}$. This means that the quadratic form (1.1) defined in $\mathbb{C}^n$ is nonnegative, in other words, the hermitian matrix $[K(x_i, x_j)]_{n\times n}$ has only nonnegative eigenvalues. It follows that

$$0 \leq K(x,x), \quad |K(x,y)|^2 \leq K(x,x)K(y,y), \quad x,y \in X. \tag{1.2}$$

Also, if (1.1) is null iff $\{c_i\} = \{0\}$ then K is a strictly positive definite kernel. For $x \in X$, let us write K^x to denote the function $y \in X \to K(y,x) \in \mathbb{C}$. The unique Hilbert space $\mathcal{H}_K$ that complete the inner product space given by the linear span of the set $\{K^x : x \in X\}$, with inner product

$$\langle K^x, K^y \rangle_K = K(y,x), \quad x,y \in X,$$

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is called *reproducing kernel Hilbert space* (RKHS). This name comes from the *reproducing property*

$$f(x) = \langle f, K^x \rangle_K, \quad x \in X, \quad f \in \mathcal{H}_K. \tag{1.3}$$

Another way to talk about RKHS is to take a Hilbert space $\mathcal{H}$ of functions $f : X \to \mathbb{C}$ with bounded linear functional $\delta_x : \mathcal{H} \to \mathbb{C}$, given by

$$\delta_x(f) = f(x), \quad f \in \mathcal{H}, \quad x \in X.$$

This means there are constants $C_x \geq 0$ such that

$$|f(x)| = \|\delta_x(f)\|_{\mathcal{H}} \leq C_x \|f\|_{\mathcal{H}}, \quad f \in \mathcal{H}, \quad x \in X.$$

Clearly this agree with (1.3). Next result is called Moore-Aronszajn theorem and his proof follows from well known Riezs representation theorem.

Lemma 1.1. *A Hilbert space $\mathcal{H}$ of functions $f : X \to \mathbb{C}$ is a RKHS iff the evaluation functional δ_x, $x \in X$, is bounded. In particular, the reproducing kernel is unique.*

Versions of the next lemma are important in some approximation problems, as you can see in [1], and give us a way to handle with functions in a RKHS. We use $C(X)$ to denote the set of complex valued continuous functions on a topological space X.

Lemma 1.2. *If $\mathcal{H}_K \subset C(X)$ and Y is a dense set in a topological space X, then the linear span of $\{K^x\}_{x \in Y}$ is dense in $\mathcal{H}_K$. This set is always linearly independent iff K is strictly positive definite.*

Proof: Let $f \in \mathcal{H}_K$ being orthogonal to the set $\{K^x\}_{x \in Y}$. It follows that

$$f(x) = \langle f, K^x \rangle_K = 0, \quad x \in Y,$$

and hence $f = 0$ because it is continuous. To finish just note that

$$g = \sum_{i=1}^{n} c_i K^{x_i} = 0 \Longleftrightarrow \langle g, g \rangle_K = \sum_{i,j=1}^{n} c_i \overline{c_j} K(x_i, x_j) = 0 \Longleftrightarrow c_i = 0,\ i = 1,\ 2, \ldots,\ n,$$

iff K is strictly positive definite. □

Previous result tell us that in some context is possible to choose functions in a dense subset of $\mathcal{H}_K$ to approximate others, when it is a subset of $C(X)$. We may then ask if we can use this method to approxmate functions in spaces like $C(X)$ or $L^p(X, \nu)$. In this way this paper is to study embedding and denseness properties of reproducing kernel Hilbert spaces in another ones. The papers [10, 13] was a motivation to start this work. The paper [12] arises after we start to work on it but it is also a motivation.

Remark 1.3. Let $H : [0,1]^2 \to \mathbb{C}$ and $u : [0,1] \to \mathbb{C}$ being continuous functions. This means that the Volterra integral equation

$$f(t) + \int_0^x H(x,t) f(t) dt = u(t), \quad t,\, x \in [0,1],$$

has a unique solution in $C[0,1]$. If $u_j \to u$ in $C[0,1]$ then, for each j,

$$f_j(x) - f(x) = T_{u_j}(f_j)(x) - T_u(f)(x) = u_j(x) - u(x) - \int_0^x H(x,t)(f(t) - f_j(t))dt$$

and follows that $f_j(0) \to f(0)$. Also, $|f_j(x) - f(x)| \leq \|u_j - u\|_c + Mx\|f - f_j\|_c$, where $M = \max\{|H(x,y)|\}$ and $\|g\|_c = \max_{x\in[0,1]}\{|g(x)|\}$. As so, there is a $x \in (0,1]$ such that $0 \leq Mx < 1$ and follows that $f_j(t) \to f(t)$ uniformly in $[0,x]$. We can pick u_j in some special dense set of $C[0,1]$ to find approximation to f_j by using results from [1].

2. ON MERCER THEORY AND DENSENESS PROBLEMS

In this section X is also endowed with a convenient measure ν. In this way we may think about the integral operator $\mathcal{K} : L^2(X,\nu) \to L^2(X,\nu)$ given by the formula

$$\mathcal{K}(f)(x) = \int_X K(x,y)f(y)\,d\nu(y), \quad f \in L^2(X,\nu), \quad x \in X. \tag{2.1}$$

If this operator is well defined and

$$\langle \mathcal{K}f, f\rangle_2 \geq 0, \quad f \in L^2(X,\nu), \tag{2.2}$$

then K is *L^2-positive definite*. In our context, this implies that $\mathcal{K}$ is self-adjoint. If the inequality (2.2) is always strict when $f \neq 0$ then K is strictly L^2-positive definite.

Remark 2.1. With this in mind we may then ask if this concept of positive definiteness has something in common to the previous one? To answer it we need more restrictions.

A positive Borel measure ν on X is a Borel measure ([6]) for which all open sets has positive measure and all point $x \in X$ has an open neighbourhood with finite measure. In the case in which ν is a Radon measure, and here we need that X is a locally compact Hausdorff space, then we can show that a continuous kernel K is L^2-positive definite iff it is positive definite ([2, 5]), but this say nothing about strict positive definiteness.

An easy way to construct a kernel K under the previous conditions is to take a set $\{\phi_n\}_{n\in\mathbb{N}}$ being an orthonormal subset of $L^2(X,\nu)$ having continuous functions only and a sequence $\{\lambda_n\}$ decreasing to 0. The kernel

$$K(x,y) = \sum_{n=1}^{\infty} \lambda_n \phi_n(x)\overline{\phi_n(y)}, \quad x,y \in X, \tag{2.3}$$

is a continuous kernel with series representation absolute and uniform convergent on compact subsets in both variable. Clearly holds that

$$\mathcal{K}(f) = \sum_{n=1}^{\infty} \lambda_n \langle f, \phi_n\rangle_2 \phi_n, \quad f \in L^2(X,\nu),$$

is also a continuous function, with absolutely and uniform convergent on compact subsets.

We say that a continuous kernel K is a *Mercer's kernel* on (X,ν) when it possesses a series representation like (2.3) in which $\{\phi_n\}$ is an $L^2(X,\nu)$-orthonormal sequence of continuous functions on X and $\{\lambda_n\}$ decreases to 0. This motivate the next result. We denote by $L^2_c(X,\nu)$ the set of all functions in $L^2(X,\nu)$ having compact support. This set is dense in $L^2(X,\nu)$, when ν is a Radon measure (see [6, p.217]).

Theorem 2.2 (Mercer theorem)**.** *Let X be a topological space endowed with a positive Borel measure ν. If $\kappa \in L^1(X,\nu)$ or $L^2_c(X,\nu)$ is dense in $L^2(X,\nu)$, then all continuous L^2-positive definite kernel, with $\mathcal{K}$ compact, is a Mercer kernel. In particular, $\{\lambda_n^{1/2}\phi_n\}$ is an orthonormal basis to $\mathcal{H}_K \subset C(X)$ and*

$$\langle f, g\rangle_2 = \langle \mathcal{K}(f), g\rangle_K, \quad f \in L^2(X,\nu), \quad g \in \mathcal{H}_K.$$

Proof: Please take a look at [3]. □

Remark 2.3. Papers [2, 3] has much more result on Mercer theory. In the paper [4] you can find very interesting results concerning Mercer theorem and reproducing properties of derivatives of the kernel, when X is an open set of the euclidian space $\mathbb{R}^n$.

Corollary 2.4. *Let us assume the setting in Theorem 2.2. K is strictly L^2-positive definite, that is $\lambda_n > 0$, $n = 1, 2, \ldots$, iff $\mathcal{H}_K$ is dense in $L^2(X,\nu)$.*

Proof: Lemma 1.2 implies that $\{K(\cdot,x)\}_{x\in Y}$ is dense in $\mathcal{H}_K$ (for all dense set Y in X). Hence, if $f \in L^2(X,\nu)$ is orthogonal to $\overline{[\{K^x\}]}$ then

$$\mathcal{K}(f)(x) = \langle f, K^x\rangle_2 = \langle \mathcal{K}(f), K^x\rangle_K = 0, \quad x \in Y,$$

that is, $\mathcal{K}(f) = 0$. This means that $\mathcal{K}$ has no null eigenvalue iff $f = 0$. By spectral theorem to compact self-adjoint operator this happens iff $\mathcal{H}_K$ is dense in $L^2(X,\nu)$. □

Remark 2.5. The previous arguments implies that $\{K^x\}_{x\in Y}$ is dense in $L^2(X,\nu)$ or K is L^2-strictly positive definite, iff $\mathcal{K}$ has no null eigenvalue. But, if

$$0 = g = \sum_{i=1}^{j} c_i K^{x_i} = \sum_{n=1}^{\infty}\left(\lambda_n \sum_{i=1}^{j} c_i \overline{\phi_n(x_i)}\right)\phi_n = \sum_{n=1}^{\infty}\langle g, \phi_n\rangle_2 \phi_n,$$

It follows that $0 = \lambda_n \sum_{i=1}^{j} c_i \overline{\phi_n(x_i)}$. But this doesn't looks to be suffice to conclude that K is strictly positive definite. Please compare this with Corollary 2.7. Compare also Corollary 2.4 with Theorem 2.6 ii), to the case in which μ is not finite.

We are going to think about the last remarks in Theorem 2.6, but before that we need some results from Functional Analysis. We denote by $C_0(X)$ the uniform closure of the set $C_c(X)$ of all continuous and compact supported complex function on X. Of course we have $C_0(X) \subset C(X)$ and it is a Banach space, also $C_0(X) = C(X)$ when X is a compact set. The proof of the next theorem uses Riezs representation theorem to the dual set of

$C_0(X)$ (please see [6, p.233], for instance). This result tell us that, if X is a locally compact Hausdorff space, then all continuous linear functional $I : C_0(X) \to \mathbb{C}$ has the form

$$I(f) = \int_X f(x)\, d\mu(x), \quad f \in C_0(X),$$

where μ is a "finite" complex Radon measure on X. This set of measure is denoted by $M(X)$ and is endowed with the norm $\|\mu\| = |\mu|(X) < \infty$. If μ is a real measure then $\mu = \mu^+ - \mu^-$, $|\mu| = \mu^+ + \mu^-$ and μ^+, μ^- are finite positive Radon measures. As so, $M(X) = C_0(X)^*$ in a isometric point of view.

Theorem 2.6. *Let X being a locally compact Hausdorff space. If K is a bounded and continuous positive definite kernel with $K^x \in C_0(X)$, for all $x \in X$, then all items ahead are equivalent:*
i) $\mathcal{H}_K$ is dense in $C_0(X)$;
ii) If μ is a finite positive Radon measure on X, then K is (strictly) L^2-positive definite;
iii) For any finite real non null Radon measure μ, it holds

$$g(x) = \int_X K(x, y)\, d\mu(y) \neq 0, \ \ a.e..$$

Proof: First note that if $K^x \in C_0(X)$, for all $x \in X$, then results in [3] implies that $\mathcal{H}_K \subset C_0(X)$. Note also that K is a Mercer kernel on $(supp(\mu), \mu)$, when μ is a finite positive Radon measure.
$i) \Rightarrow ii)$ To see that $\mathcal{H}_K$ is dense in $L^2(X, \mu)$, we use the fact that $C_c(X)$ is dense there. If $f \in L^2(X, \mu)$ and $j \in \mathbb{N}$, there exists $f_j \in C_c(X)$, with compact support contained in $C_j \subset C_{j+1}$ such that (please see [6, p.47, 217])

$$\int_X |f(x) - f_j(x)|^2 d\mu(x) < \frac{1}{j}, \quad x \in X.$$

Of course we may assume that $\mu(C_j) > 0$. If $\mathcal{H}_K \subset L^2(X, \mu)$ and is dense in $C_0(X)$, there exists $g_j \in \mathcal{H}_K$ such that $\|g_j - f_j\|_c < 1/j$. It follows that

$$\int_X |g_j(x) - f_j(x)|^2 d\mu(x) < \frac{\mu(X)}{j}.$$

Hence

$$\int_X |f(x) - g_j(x)|^2 d\mu(x) \leq \int_X |f(x) - f_j(x)|^2 d\mu(x) + \int_X |f_j(x) - g_j(x)|^2 d\mu(x) \to 0.$$

As so, if

$$\langle \mathcal{K}(f), f \rangle = \sum_{n=1}^{\infty} \lambda_n |\langle \phi_n, f \rangle|^2 = 0, \quad 0 \neq f \in L^2(X, \mu),$$

then $\mathcal{K}(f) = 0$. It follows from Mercer theorem that $\mathcal{H}_K$ (restrict to $supp(\mu)$) is not dense in $L^2(supp(\mu), \mu)$. Also, $\mathcal{H}_K$ (restrict to $supp(\mu)$) is not dense in $C_0(supp(\mu))$ and $\mathcal{H}_K$ is not dense in $C_0(X)$.

ii) $\Rightarrow$ *iii)* If we assume that μ is real with $|\mu|(X) < \infty$, there are a set $A \subset X$ and positive measures μ^+ and μ^- such that, $\mu = \mu^+ - \mu^-$, $|\mu| = \mu^+ + \mu^-$, $\mu^+(A) = \mu^+(X)$ and $\mu^-(A) = 0$. If

$$g(x) = \int_X K(x, y)\, d\mu(y) = 0, \;\; a.e..$$

then

$$g(x) = \int_X K(x, y)(\chi_A(y) - \chi_{X \setminus A}(y))\, d|\mu|(y) = 0, \;\; a.e..$$

Hence, $\chi_A - \chi_{X\setminus A} \neq 0$ is an eigenfunction of $\mathcal{K}$, acting on $L^2(X, |\mu|)$, a contradiction.

iii) $\Rightarrow$ *i)* Note that any linear functional I in $C_0(X)^*$ may be written as a sum of two real functionals as $I = I_1 + iI_2$. Note also that if μ is a finite real Radon measure for which

$$g(x) = \int_X K(x, y)\, d\mu(y) \neq 0, \;\; a.e..$$

Then the associated bounded linear functional $T_\mu : C_0(X) \to \mathbb{C}$, given by $T_\mu(f) = \int_X f\, d\mu$, is non-null in $\mathcal{H}_K$. The proof follows from Riezs representation theorem, since if $\mathcal{H}_K$ is not dense in $C_0(X)$, Hahn-Banach theorem implies the existence of a real non null Radon measure μ such that T_μ is null $\mathcal{H}_K$, a contradiction. □

Corollary 2.7. *If $\mathcal{H}_K$ is dense in $C_0(X)$ then K is strictly positive definite.*

Proof: If $\mathcal{H}_K$ is dense in $C_0(X)$ and K is not strictly positive definite then there is a set $\{x_1, x_2, \ldots, x_n\}$ in X such that the matrix $[K(x_i, x_j)]$ has a null eigenvalue. Hence, the equation $[K(x_i, x_j)]c = 0$ has a non null solution in $\mathbb{C}^n$. This means that

$$\sum_{i=1}^{n} c_i K^{x_i} = 0.$$

If we define the Radon measure μ, with support in $\{x_1, x_2, \ldots, x_n\}$, given by $\mu(\{x_i\}) = c_i$, $i = 1, 2, \ldots, n$. It follows that

$$\int_X K(x, y) d\mu(y) = \sum_{i=1}^{n} \int_{\{x_i\}} K(x, x_i) d\mu(y) = \sum_{i=1}^{n} c_i K(x, x_i) = \sum_{i=1}^{n} c_i K^{x_i}(x) = 0, \quad x \in X,$$

a contradiction to the last theorem. □

Remark 2.8. An example in [13] shows that the strict positive definiteness of K is a necessary but not enough to ensure the denseness of $\mathcal{H}_K$ in $C_0(X)$. If K is strictly positive definite then, given a function $f : X \to \mathbb{C}$ and a subset $\{x_i\}_{i=1}^n \subset X$ with finite number of elements, we can find an interpolation function $g \in \mathcal{H}_K$ such that $f(x_i) = g(x_i)$, but it say nothing about how close $f(x)$ and $g(x)$ are when $x \neq x_i$ (see the paper [11]).

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Remark 2.9. Please take a look at Theorem 3.2 in [12] to a particular version of the last Theorem to compact subsets of a locally compact Hausdorff abeilan group X.

3. ON POSITIVE DEFINITE FUNCTIONS AND DENSENESS

We start this end section with some facts on Fourier transform. The notation and basic properties of the Fourier transform are those from [6, 8]. The book [7] is a more complete reference, but with different notation (see also [12]).

Firstly we talk about it in the euclidian space $\mathbb{R}^m$, the usual inner product of two points x, y will be written as $x \cdot y$. As so, the *Fourier transform* is the linear mapping $f \in L^1(\mathbb{R}^m) \mapsto \hat{f}$ given by the formula

$$\hat{f}(v) = \int_{\mathbb{R}^m} f(x)e^{-2\pi i x\cdot v}\, dx, \quad v \in \mathbb{R}^m.$$

Since

$$|\hat{f}(v)| \leq \int_{\mathbb{R}^m} |f(x)e^{-2\pi i x\cdot v}|\, dx = \int_{\mathbb{R}^m} |f(x)| dx, \quad v \in \mathbb{R}^m,$$

it is easily seen that the range of the Fourier transform is composed of bounded functions.

Now we can go to a more general setting. As so, using the same notation (and some isomorphisms like that given in Theorem 3 in [8]) we are going to work assuming that the set X is a locally compact Hausdorff abelian group, with group operation denoted by $x+y$ and $-x$ denote the inverse element of x. This group is endowed with Haar measure dx. It is important to say that we don't need the Hausdorff condition in some results ahead, but it is needed in another one because of Theorem 2.6, for instance. On the other side, as Folland says in [6, p. 340], this is not as much of a restriction. With this in mind, we can now define Fourier transform $f \in L^1(X) \mapsto \hat{f}$ by the formula

$$\hat{f}(v) = \int_X f(x)e^{-2\pi i x\cdot v}\, dx, \quad v \in X.$$

Here the expression $e^{-2\pi i x\cdot v}$ is just a notation that becomes clear in each particular context (see Remark 3.7). Under this conditions we have the next fundamental theorem on Fourier analysis.

Theorem 3.1 (Plancherel)**.** *If $f \in L^1(X) \cap L^2(X)$ then $\hat{f} \in L^2(X)$ and there is a unique isometric operator $\mathcal{F} : L^2(X) \to L^2(X)$ such that*

$$\mathcal{F}(f) = \hat{f}, \quad f \in L^1(X) \cap L^2(X).$$

Also, $\langle f, g\rangle_2 = \langle \hat{f}, \hat{g}\rangle_2$, $f, g \in L^2(X)$, and holds the inversion formula

$$f(v) = \int_X \hat{f}(x)e^{2\pi i x\cdot v}\, dx, \quad v \in X, \quad \hat{f} \in L^1(X) \cap L^2(X).$$

We continue with some technical results to be used ahead.

Lemma 3.2. *If $f, g \in L^1(X)$ then $\hat{h}(x) = \hat{f}(x)\hat{g}(x)$, $x \in X$, where*

$$h(x) = f * g(x) = \int_X f(x-y)g(y)dy, \quad x \in X.$$

Of course the formulas in the previous results on this section remain valid in spirit when we work with functions of $L^2(X)$. This means that the equalities are given in an almost everywhere sense. To functions in $L^1(X)$ the things is in fact more stronger.

Lemma 3.3 (Riemann-Lebesgue)**.** *If f belongs to $L^1(X)$ then $\hat{f}$ is an element of $C_0(X)$.*

If the reader is asking himself about what connections may exist between positive definite kernels and Fourier transforms, the classical result below may be suggestive. But we do not use it here. On the other side, the paper [12] uses this result as a fundamental tool to prove its main result which describes equivalent conditions to the denseness of $\mathcal{H}_K$, restrict to a compact set B, in $C(B)$. We believe that our result doesn't follows from that one and, from the applications viewpoint, it is more stronger and interesting.

A function $k : X \to \mathbb{C}$ is positive definite when $K(x,y) = k(x-y)$ is positive definite.

Theorem 3.4 (Bochner)**.** *A continuous function $k : X \to \mathbb{C}$ is positive definite iff it is the inverse Fourier transform of a finite positive Borel measure σ, that is*

$$k(x) = \int_X e^{2\pi i y\cdot x}\, d\sigma(y), \quad x \in X.$$

If you want to see more result on Fourier transforms on $\mathbb{R}^n$ and positive definite kernels please take a look at [3]. From now on we use the notation $g^*(x) = \overline{g(-x)}$, $x \in X$.

Lemma 3.5. *If $K(x,y) = k(x-y)$, $x, y \in X$, then*

$$\mathcal{K}(f) = k * f, \qquad \langle \mathcal{K}(f), f\rangle = k * f * f^*(0).$$

In particular, K is positive definite iff $\hat{k}(x) \geq 0$, for almost all $x \in X$. Also, the inequality is strict (a.e) iff K is strictly L^2-positive definite. Particularly, if $k \in L^1(X)$ then $\hat{k}(x) \geq 0$, for all $x \in X$.

Proof: A direct calculation shows that

$$\mathcal{K}(f)(x) = \int_X k(x-y)f(y)\, dy = k * f(x), \quad x \in X.$$

As so, the previous results implies that

$$\langle \mathcal{K}(f), f\rangle = \langle k * f, f\rangle = \int_X k * f(x)\overline{f(x-0)}\, dx = \int_X k * f(x) f^*(0-x)\, dx = k * f * f^*(0).$$

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Applying Lemma 3.2, it follows too that

$$\langle \mathcal{K}(f), f\rangle = \langle \hat{k}\hat{f}, \hat{f}\rangle = \int_X \hat{k}(x)|\hat{f}(x)|^2\,dx.$$

By applying Plancherel's theorem, we know that $\hat{k}(x) < 0$, for all $x \in U$, $U \subset X$ with positive measure iff K is not positive definite (see Remark 2.1). The same argument implies that K is strictly L^2-positive definite iff $\hat{k}(x) > 0$ a.e. The last fact follows from Riemann-Lebesgue lemma. □

The next result is like a corollary of Theorem 2.6.

Theorem 3.6. *Let $K(x,y) = k(x-y)$, $x, y \in X$, with k being a non constant continuous positive definite function in $L^2(X)$. Then $\mathcal{H}_K$ restrict to a every compact set B in X is dense in $C(B)$. Particularly, if $k \in C_0(X)$, then $\mathcal{H}_K$ is dense in $C_0(X)$.*

Proof: We know that $\hat{k}(x) \geq 0$ a.e.. If B is a compact subset of X then we are going to use Theorem 2.6 to prove this one. In this way, let μ being a finite real Radon measure on B such that

$$g(x) = \int_B K(x,y)\,d\mu(y) = 0, \quad x \in B.$$

It follows from the inverse Fourier transform that

$$\int_B \int_X \hat{k}(z)e^{2\pi i z\cdot(x-y)}\,dz\,d\mu(y) = 0, \quad x \in B.$$

Hence, integrating again we have

$$0 = \int_X \hat{k}(z)\int_B e^{-2\pi i z\cdot y}\,d\mu(y)\int_B e^{2\pi i z\cdot x}\,d\mu(x)\,dz = \int_X \hat{k}(z)|\hat{\mu}(z)|^2 dz = 0,$$

where

$$\hat{\mu}(z) = \int_B e^{-2\pi i z\cdot y}\,d\mu(y),$$

is the Fourier transform of μ (see [7, p.94]). This means that $|\hat{\mu}(z)| = 0$ a.e., and Plancherel's theorem implies that $\mu = 0$.

To finish, note that if $k \in C_0(X) \cap L^2(X)$, then $\mathcal{H}_K$ is a subset of $C_0(X) \cap L^2(X)$. If in the previous argument we take μ as a finite real Radon measure on X and repeat the calculation we find that $\mathcal{H}_K$ is dense in $C_0(X)$. □

Remark 3.7. To find some example of locally compact Hausdorff abelian group please see Theorem 4.5 in [7, p.89] and [8, 12]. It is important to say that the euclidian space $\mathbb{R}^n$ itself, the torus $\mathbb{T}^n$ and the euclidian spheres $\mathbb{S}^0$, $\mathbb{S}^1$ and $\mathbb{S}^3$ has this structure, but not $\mathbb{S}^2$ and $\mathbb{S}^n$, with $n > 3$ ([9]). See the paper [10] to some results to handle with denseness problems to kernels on euclidian spheres. For the sake of completeness, we note that

$$\mathbb{T}^1 = \mathbb{S}^1 = \{u \in \mathbb{C} : |u| = 1\} = \{e^{2\pi i t} : t \in [0,1]\}.$$

Here, if $u = e^{i\theta}$ and $v = e^{i\omega}$ then the group operation is given by $u + v = e^{i(\theta+\omega)}$ and $-u = e^{-i\theta}$. You can find in [6, p.248] basic facts about Fourier transform on $\mathbb{T}^n$.

Remark 3.8. It follows from Theorem 3.6 that $\mathcal{H}_K$ with Gaussian kernel

$$K(x, y) = f(x - y), \qquad f(x) = e^{-\sigma^2|x|^2}, \quad x, y \in X \subset \mathbb{R}^n, \quad \sigma > 0,$$

is a dense subset of $C_0(X)$, if X is closed. Hence, as is well known, K is strictly positive definite on $\mathbb{R}^n$ and strictly L^2-positive definite when X has finite positive measure.

Remark 3.9. If we denote by $W[0, 1]$ the space of absolutely continuous functions f : $[0, 1] \to \mathbb{C}$, with $f' \in L^2[0, 1]$ and inner product given by

$$\langle f, g\rangle_W = f(0)g(0) + \int_0^1 f'(x)g'(x)dx.$$

This space is very usefull in applications involving differential end integral equations (please see [1, 3]). It is a reproducing kernel Hilbert space with kernel $K(x, y) = 1 + min(x, y)$. Note that all polynomial function in $[0, 1]$ is in $W[0, 1]$. It follows that $W[0, 1]$ is dense in $C[0, 1]$. Hence, applying Theorem 2.6 we can see that K is strictly L^2-positive definite and strictly positive definite on $[0, 1]$.

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