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Pseudo-spherical evolutes of curves on a timelike surface in three dimensional Lorentz-Minkowski space

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In this paper we introduce the notion of pseudo-spherical evolutes of curves on a timelike surface in three dimensional Lorentz-Minkowski space. We investigate the singularities and geometric properties of these pseudo-spherical evolutes. Furthermore, we investigate the relation of the de Sitter (hyperbolic) evolute of a spacelike curve in S^2_1 with the lightlike surface along this spacelike curve. October, 2014 ICMC-USP

1. INTRODUCTION

We introduce the notion of pseudo-spherical evolutes of curves on a timelike surface in the Minkowski space $\mathbb{R}^3_1$ and investigate their geometric properties. The study of submanifolds in Minkowski space is of interest in relativity theory. See [2] and [7] for more on evolutes. The principal tools for the study of evolutes are the Frenet-Serret formula and height functions along a curve on a timelike surface. We explain in §2 the basic notions of Lorentz-Minkowski space and introduce Lorentzian Darboux frame that will be used throughout the paper. In §3, we define two families of functions on a curve, which are a spacelike height function H^S and a timelike height function H^T. By differentiating these functions, we obtain new invariants σ_D and σ_H whose properties are characterized by some conditions on the derivatives of H^S and H^T. We also define two important curves d_γ in de Sitter space

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and h_γ in the hyperbolic space by observing the conditions of first and second derivatives of H^S and H^T, respectively. We call d_γ a de Sitter evolute of γ relative to M and h_γ a hyperbolic evolute of γ relative to M. We show that the de Sitter evolute d_γ is constant if and only if $\sigma_D \equiv 0$. In this case the curve γ is a special curve on the surface M, which is called a de Sitter-slice (or an D-slice) of M. We also show that the hyperbolic evolute h_γ is constant if and only if $\sigma_H \equiv 0$ and define a special curve on the surface M called a hyperbolic-slice (or an H-slice) of M. The D-slice and H-slice on M can be consider as the model curve on the surface M. We show that H-slice is always non-singular, but we have the case that D-slice has a singular point (see §3). In §4, as an application of the theory of unfoldings of functions in [1], we give a classification of singularities of both the de Sitter evolute and the hiperbolic evolute in Theorems 4.4.4 and 4.4.5, which are some of the main results in this paper. In §5, we consider curves on a timelike plane, $\mathbb{R}^2_1$, and on the de Sitter space, as special cases of curves on timelike surface. Finally in §6, we give a relation of the de Sitter evolute and of the hyperbolic evolute of a spacelike curve γ in S^2_1 with the lightlike surface along γ.

2. PRELIMINARIES

The *Minkowski space* $\mathbb{R}^3_1$ is the vector space $\mathbb{R}^3$ endowed with the pseudo-scalar product $\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2$, for any $x = (x_0, x_1, x_2)$ and $y = (y_0, y_1, y_2)$ in $\mathbb{R}^3_1$. We say that a non-zero vector $x \in \mathbb{R}^3_1$ is *spacelike* if $\langle x, x\rangle > 0$, *lightlike* if $\langle x, x\rangle = 0$ and *timelike* if $\langle x, x\rangle < 0$. We say that $\gamma : I \to \mathbb{R}^3_1$ is *spacelike* (resp. *timelike*) if $\gamma'(t)$ is a *spacelike* (resp. *timelike*) vector for all $t \in I$. A point $\gamma(t)$ is called a *lightlike point* if $\gamma'(t)$ is a lightlike vector. The norm of a vector $x \in \mathbb{R}^3_1$ is defined by $\| x \| = \sqrt{| \langle x, x\rangle |}$. For a non-zero vector $v \in \mathbb{R}^3_1$ and a real number c, we define a *plane* with *pseudo-normal* v by

$$P(v, c) = \{x \in \mathbb{R}^3_1 \mid \langle x, v\rangle = c\}.$$

We call $P(v, c)$ a spacelike plane, a timelike plane or lightlike plane if v is timelike, spacelike or lightlike, respectively. We now define *Hyperbolic plane* by

$$H^2_+(-1) = \{x \in \mathbb{R}^3_1 \mid \langle x, x\rangle = -1, x_0 > 0\}$$

and de *Sitter space* by

$$S^2_1 = \{x \in \mathbb{R}^3_1 \mid \langle x, x\rangle = 1\}.$$

For any $x = (x_0, x_1, x_2)$, $y = (y_0, y_1, y_2) \in \mathbb{R}^3_1$, the pseudo vector product of x and y is defined as follows:

$$x \wedge y = \begin{vmatrix} -e_0 & e_1 & e_2 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix},$$

where $\{e_0, e_1, e_2\}$ is the canonical basis of $\mathbb{R}^3$.

We consider a timelike embedding $X : U \to \mathbb{R}^3_1$ from an open subset $U \subset \mathbb{R}^2$. We write $M = X(U)$ and identify M and U through the embedding X. We say that X is a *timelike* *embedding* if the tangent space T_pM is a timelike plane at any $p = X(u)$. Let $\bar{\gamma} : I \to U$ be a regular curve and a curve $\gamma : I \to M \subset \mathbb{R}^3_1$ defined by $\gamma(s) = X(\bar{\gamma}(s))$. We say that γ is a curve on the timelike surface M.

Observe that the curve γ can be spacelike, timelike or the curve can have lightlike points. In the case that γ is spacelike or timelike, we can reparameterize it by the arc-length s. So we have the unit tangent vector $t(s) = \gamma'(s)$ of $\gamma(s)$. Since X is a timelike embedding, we have a unit spacelike normal vector field n along $M = X(U)$ defined

$$n(p) = \frac{X_{u_1}(u) \wedge X_{u_2}(u)}{\| X_{u_1}(u) \wedge X_{u_2}(u) \|},$$

for $p = X(u)$. We define $n_\gamma(s) = n \circ \gamma(s)$, so that we have a unit spacelike normal vector field n_γ along γ. Therefore we can construct the binormal vectors $b(s)$ given by $b(s) = n_\gamma(s) \wedge t(s)$. We say that a vector v is *future directed* if $\langle v, e_0 \rangle < 0$. We choose the orientation of M such that b (resp. t) is future directed when γ is spacelike (resp. timelike). We have also that $\langle t(s), t(s) \rangle = \varepsilon(\gamma(s))$, $\langle n_\gamma(s), n_\gamma(s) \rangle = 1$, $\langle n_\gamma(s), b(s) \rangle = 0$ and $\langle b(s), b(s) \rangle = -\varepsilon(\gamma(s))$, where $\varepsilon(\gamma(s)) = sign(t(s))$, that can be 1 if γ is spacelike or -1 if γ is timelike. Then we have the pseudo-orthonormal frames $\{b(s), n_\gamma(s), t(s)\}$ if γ is spacelike and $\{t(s), b(s), n_\gamma(s)\}$ if γ is timelike, which are called the Lorentzian Darboux frames along γ. By standard arguments, we have the following Frenet-Serret type formulae:

$$\begin{cases} b'(s) = \tau_g(s)\, n_\gamma(s) - \varepsilon(\gamma(s))\, k_g(s)\, t(s) \\ n'_\gamma(s) = \varepsilon(\gamma(s))\, \tau_g(s)\, b(s) - \varepsilon(\gamma(s))\, k_n(s)\, t(s) \\ t'(s) = -\varepsilon(\gamma(s))\, k_g(s)\, b(s) + k_n(s)\, n_\gamma(s) \end{cases}$$

where $k_n(s) = \langle n_\gamma(s), t'(s) \rangle$, $k_g(s) = \langle b(s), t'(s) \rangle$, $\tau_g(s) = \langle n_\gamma(s), b'(s) \rangle$ and $\varepsilon(\gamma(s)) = sign(t(s))$.

Here, we have the following properties of γ characterized by the conditions of k_g, k_n, τ_g.

$$\gamma \text{ is } \begin{cases} \text{a geodesic curve if and only if} & k_g \equiv 0 \\ \text{an asymptotic curve if and only if} & k_n \equiv 0 \\ \text{a principal curve if and only} & \tau_g \equiv 0 \end{cases}$$

Observe that $t'(s) = 0$ means that $k_n(s) = 0$ and $k_g(s) = 0$. We suppose then $t'(s) \neq 0$ to define, for example, the pseudo-spherical evolutes.

3. HEIGHT FUNCTIONS

In this section, we introduce two families of functions on a curve on a timelike surface M: the timelike height function and the spacelike height function. Furthermore, we define the pseudo-spherical evolutes.

We define the family of height functions on a curve, $\gamma : I \to M \subset \mathbb{R}^3_1$, on a timelike surface M as follows:

$$H^S : I \times S^2_1 \to \mathbb{R}; \quad (s, v) \mapsto \langle \gamma(s), v \rangle.$$

We call H^S the spacelike height function of γ on M. We denote $h^S_v(s) = H^S(s, v)$ for any fixed $v \in S^2_1$.

Proposition 3.3.1. *Suppose that $t'(s) \neq 0$. Then for any $(s, v) \in I \times S^2_1$, we have the following:*

(1)$(h^S_v)'(s) = 0$ if and only if $v = \mu b(s) + \lambda n_\gamma(s)$, where μ, $\lambda \in \mathbb{R}$ such that $-\varepsilon(\gamma(s))\mu^2 + \lambda^2 = 1$.

(2)$(h^S_v)'(s) = (h^S_v)''(s) = 0$ if and only if

$$v = \pm \frac{1}{\sqrt{k_g^2(s) - \varepsilon(\gamma(s))k_n^2(s)}}(k_n(s)b(s) - k_g(s)n_\gamma(s))$$

and $k_g^2(s) > \varepsilon(\gamma(s))k_n^2(s)$.

(3)$(h^S_v)'(s) = (h^S_v)''(s) = (h^S_v)'''(s) = 0$ if and only if

$$v = \pm \frac{1}{\sqrt{k_g^2(s) - \varepsilon(\gamma(s))k_n^2(s)}}(k_n(s)b(s) - k_g(s)n_\gamma(s)),$$

$k_g^2(s) > \varepsilon(\gamma(s))k_n^2(s)$ and $\sigma_D(s) = 0$, where $\sigma_D(s) = (k_g' k_n + \varepsilon k_g^2 \tau_g - k_g k_n' - k_n^2 \tau_g)(s)$.

(4)$(h^S_v)'(s) = (h^S_v)''(s) = (h^S_v)'''(s) = (h^S_v)^{(4)}(s) = 0$ if and only if

$$v = \pm \frac{1}{\sqrt{k_g^2(s) - \varepsilon(\gamma(s))k_n^2(s)}}(k_n(s)b(s) - k_g(s)n_\gamma(s)),$$

$k_g^2(s) > \varepsilon(\gamma(s))k_n^2(s)$, $\sigma_D(s) = 0$ and $(\sigma_D)'(s) = 0$.

Proof. *(1)* In order to show the proposition, we use the Frenet-Serret type formulae. Then,

$$(h^S_v)'(s) = \langle \gamma'(s), v \rangle = \langle t(s), v \rangle = 0,$$

that is, there are μ, $\lambda \in \mathbb{R}$ such that $v = \mu b(s) + \lambda n_\gamma(s)$ and as $v \in S^2_1$ we have that $-\varepsilon(\gamma(s))\mu^2 + \lambda^2 = 1$.

(2) $(h^S_v)'(s) = (h^S_v)''(s) = 0$ if and only if $\langle t'(s), \mu b(s) + \lambda n_\gamma(s) \rangle = 0$ with $-\varepsilon(\gamma(s))\mu^2 + \lambda^2 = 1$. This is equivalent to $k_g(s)\mu + k_n(s)\lambda = 0$ with $-\varepsilon(\gamma(s))\mu^2 + \lambda^2 = 1$. This means that $\mu^2(k_g^2(s) - \varepsilon(\gamma(s))k_n^2(s)) = k_n^2(s)$. Therefore, considering the condition that $k_g^2(s) > \varepsilon(\gamma(s))k_n^2(s)$ we have

$$v = \pm \frac{1}{\sqrt{k_g^2(s) - \varepsilon(\gamma(s))k_n^2(s)}}(k_n(s)b(s) - k_g(s)n_\gamma(s)).$$

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For *(3)*, we have that $(h_v^S)'(s) = (h_v^S)''(s) = (h_v^S)'''(s) = 0$ if and only if $(k_g'k_n + \varepsilon k_g^2\tau_g - k_gk_n' - k_n^2\tau_g)(s) = 0$. So, we define $\sigma_D(s) = (k_g'k_n + \varepsilon k_g^2\tau_g - k_gk_n' - k_n^2\tau_g)(s)$. Therefore, $(\sigma_D)'(s) = (k_g''k_n + 2\varepsilon k_gk_g'\tau_g + \varepsilon k_g^2\tau_g' - k_gk_n'' - 2k_nk_n'\tau_g - k_n^2\tau_g')(s)$. But, we have that $(h^S{}_v)'(s) = (h^S{}_v)''(s) = (h^S{}_v)'''(s) = (h^S{}_v)^{(4)}(s) = 0$ if and only if $(k_g''k_n + 2\varepsilon k_gk_g'\tau_g + \varepsilon k_g^2\tau_g' - k_gk_n'' - 2k_nk_n'\tau_g - k_n^2\tau_g')(s) = 0$, i.e, the item *(4)* is proved. ∎

The above proposition induces an invariant σ_D. Motivated by the above calculations we define a curve $d_\gamma : I \to S_1^2$ by

$$d_\gamma(s) = \frac{k_g(s)}{\sqrt{k_g^2(s) - \varepsilon(\gamma(s))k_n^2(s)}}n_\gamma(s) - \frac{k_n(s)}{\sqrt{k_g^2(s) - \varepsilon(\gamma(s))k_n^2(s)}}b(s).$$

We call d_γ a *de Sitter evolute* of γ relative to M.

Lemma 3.3.2. *$d_\gamma'(s) = 0$ if and only if $\sigma_D(s) = 0$.*

Proof. We have

$$d_\gamma'(s) = \left(\frac{-k_n'k_g + k_nk_g' + \varepsilon k_g^2\tau_g - k_n^2\tau_g}{(k_g^2 - \varepsilon k_n^2)\sqrt{k_g^2 - \varepsilon k_n^2}}\right)(s)(k_g(s)b(s) - \varepsilon k_n(s)n_\gamma(s)).$$

Therefore,

$$d_\gamma'(s) = 0 \text{ if and only if } \sigma_D(s) = (-k_n'k_g + k_nk_g' + \varepsilon k_g^2\tau_g - k_n^2\tau_g)(s) = 0.$$

∎

We define also the family of height functions on a curve, $\gamma : I \to M \subset \mathbb{R}^3_1$, on a timelike surface M as follows:

$$H^T : I \times H_+^2(-1) \to \mathbb{R}; \quad (s,v) \mapsto \langle \gamma(s), v\rangle,$$

We call H^T the timelike height function of γ on M. We denote $h_v^T(s) = H^T(s,v)$ for any fixed $v \in H_+^2(-1)$.

For any $(s,v) \in I \times H^2(-1)$, we have that $(h_v^T)'(s) = 0$ if and only if $v = \mu b(s) + \lambda n_\gamma(s)$ where $\mu, \lambda \in \mathbb{R}$ such that $-\varepsilon(\gamma(s))\mu^2 + \lambda^2 = -1$.

Remark 3.3.3. *In the case that γ is a timelike curve, i.e, $\varepsilon(\gamma(s)) = -1$, there is no $v \in H^2(-1)$ such that $(h_v^T)'(s) = 0$ for some $s \in I$. Thus, we have that the bifurcation set of H^T for a timelike curve is empty. Then, in this case we consider only a spacelike curve γ on the timelike surface M.*

We have the following proposition.

Proposition 3.3.4. *Suppose that $t'(s) \neq 0$. Then for any $(s,v) \in I \times H^2(-1)$, we have the following:*

(1)$(h_v^T)'(s)=0$ *if and only if* $v=\mu b(s)+\lambda n_\gamma(s)$ *where* $\mu,\ \lambda\in\mathbb{R}$ *such that* $-\mu^2+\lambda^2=-1$.
(2)$(h_v^T)'(s)=(h_v^T)''(s)=0$ *if and only if*

$$v=\pm\frac{1}{\sqrt{k_n^2(s)-k_g^2(s)}}(k_n(s)b(s)-k_g(s)n_\gamma(s))$$

and $k_n^2(s)>k_g^2(s)$.
(3)$(h_v^T)'(s)=(h_v^T)''(s)=(h_v^T)'''(s)=0$ *if and only if*

$$v=\pm\frac{1}{\sqrt{k_n^2(s)-k_g^2(s)}}(k_n(s)b(s)-k_g(s)n_\gamma(s)),$$

$k_n^2(s)>k_g^2(s)$ *and* $\sigma_H(s)=0$, *where* $\sigma_H(s)=(k_gk_n'+k_n^2\tau_g-k_g'k_n-k_g^2\tau_g)(s)$.
(4)$(h_v^T)'(s)=(h_v^T)''(s)=(h_v^T)'''(s)=(h_v^T)^{(4)}(s)=0$ *if and only if*

$$v=\pm\frac{1}{\sqrt{k_n^2(s)-k_g^2(s)}}(k_n(s)b(s)-k_g(s)n_\gamma(s)),$$

$k_n^2(s)>k_g^2(s)$, $\sigma_H(s)=0$ *and* $(\sigma_H)'(s)=0$.

Proof. *(1)* In order to show the proposition, we use the Frenet-Serret type formulae in the case that γ is spacelike. Then,

$$(h_v^T)'(s)=\langle\gamma'(s),v\rangle=\langle t(s),v\rangle=0,$$

that is, there are $\mu,\ \lambda\in\mathbb{R}$ such that $v=\mu b(s)+\lambda n_\gamma(s)$ and as $v\in H^2(-1)$ we have $-\mu^2+\lambda^2=-1$.

(2) $(h_v^T)'(s)=(h_v^T)''(s)=0$ if and only if $\langle t'(s),\mu b(s)+\lambda n_\gamma(s)\rangle=0$ with $-\mu^2+\lambda^2=-1$. This is equivalent to $k_g(s)\mu+k_n(s)\lambda=0$ with $-\mu^2+\lambda^2=-1$. This means that $\mu^2(k_n^2(s)-k_g^2(s))=k_n^2(s)$. Therefore, considering the condition that $k_n^2(s)>k_g^2(s)$ we have

$$v=\pm\frac{1}{\sqrt{k_n^2(s)-k_g^2(s)}}(k_n(s)b(s)-k_g(s)n_\gamma(s)).$$

For *(3)*, we have that $(h_v^T)'(s)=(h_v^T)''(s)=(h_v^T)'''(s)=0$ if and only if $(k_gk_n'+k_n^2\tau_g-k_g'k_n-k_g^2\tau_g)(s)=0$. Then we define $\sigma_H(s)=(k_gk_n'+k_n^2\tau_g-k_g'k_n-k_g^2\tau_g)(s)$. Thus, follows that $(\sigma_H)'(s)=(k_gk_n''-k_g''k_n+2k_nk_n'\tau_g+k_n^2\tau_g'-2k_gk_g'\tau_g-k_g^2\tau_g')(s)$. But we have that $({h^T}_v)'(s)=({h^T}_v)''(s)=({h^T}_v)'''(s)=({h^T}_v)^{(4)}(s)=0$ if and only if $(k_gk_n''-k_g''k_n+2k_nk_n'\tau_g+k_n^2\tau_g'-2k_gk_g'\tau_g-k_g^2\tau_g')(s)=0$, i.e, the item *(4)* is proved. ∎

Similar to Proposition 3.3.1, the above proposition induces an invariant σ_H and motivated by the above calculations we define a curve $h_\gamma : I \to H^2_+(-1)$ by

$$h_\gamma(s) = -\frac{k_g(s)}{\sqrt{k_n^2(s) - k_g^2(s)}} n_\gamma(s) + \frac{k_n(s)}{\sqrt{k_n^2(s) - k_g^2(s)}} b(s).$$

We call h_γ a *hyperbolic evolute* of γ relative to M. Furthermore, we have the following result analogous to Lemma 3.3.2.

LEMMA 3.3.5. *$h'_\gamma(s) = 0$ if and only if $\sigma_H(s) = 0$.*

Proof. We have

$$h'_\gamma(s) = \left(\frac{-k_n k'_g + k'_n k_g - k_g^2 \tau_g + k_n^2 \tau_g}{(k_n^2 - k_g^2)\sqrt{k_n^2 - k_g^2}} \right)(s)(\varepsilon k_n(s) n_\gamma(s) - k_g(s) b(s)).$$

Therefore,

$$h'_\gamma(s) = 0 \text{ if and only if } \sigma_H(s) = (k'_n k_g - k_n k'_g - k_g^2 \tau_g + k_n^2 \tau_g)(s) = 0.$$

∎

We also call d_γ and h_γ a *pseudo-spherical evolute* of γ relative to M. By Lemma 3.3.2, $d_\gamma(s) = v_0$ is constant if and only if $\sigma_D(s) \equiv 0$. In this case, by Proposition 3.3.1 *(2)*, $h^S_{v_0}$ is constant, that is, there is a real number $c \in \mathbb{R}$ such that $\langle \gamma(s), v_0 \rangle = c$. It means that $\mathrm{Im}\gamma = P(v_0, c) \cap M$. It suggests that curves of the form $P(v, c) \cap M$ for $v \in S^2_1$ are the candidates of model curves on M. We call it a *de Sitter-slice* (or, a D-*slice*) of M. Here we remark that we can consider the D-slice under the condition $k_n \neq 0$. If $k_n(s_0) = 0$, we have $d_\gamma(s_0) = n_\gamma(s_0)$ and thus $P(n_\gamma(s_0), c_0)$ is the tangent plane $T_{\gamma(s_0)}M$ of M, where $c_0 = h^S_{n_\gamma(s_0)}(s_0)$ and $P(n_\gamma(s_0), c_0) \cap M$ has a singular point.

By the same way, by Lemma 3.3.5, we can also define *hyperbolic-slice* (or, an H-*slice*) of M by $P(v, c) \cap M$ for $v \in H^2_+(-1)$. Since $P(v, c)$ for $v \in H^2_+(-1)$ is a spacelike plane and M is a timelike surface, an H-slice is always a regular curve

Let us study the geometry of the invariants σ_D and σ_H. For this purpose, we define the order of contact between curves and surfaces.

DEFINITION 3.3.6. *Let $F : \mathbb{R}^3_1 \to \mathbb{R}$ (respectively, $F\mid_M : M \to \mathbb{R}$) be a submersion and $\gamma : I \to M$ be a regular curve. We say that γ and $F^{-1}(0)$ (respectively $F^{-1}(0) \cap M$) have contact of order k at s_0 if the function $f(s) = F \circ \gamma(s)$ satisfies $f(s_0) = f'(s_0) = \cdots = f^{(k)}(s_0) = 0$ and $f^{(k+1)}(t_0) \neq 0$, i.e, f has A_k-type singularity at t_0.*

We now introduce the following another family of function:

$$\mathcal{H}^S : \mathbb{R}^3_1 \times S^2_1 \to \mathbb{R}; \quad (x, v) \mapsto \langle x, v \rangle.$$

We denote $\mathfrak{h}^S_{v_0}(x) = \mathcal{H}^S(x, v_0)$ for any fixed $v_0 \in S^2_1$, then, we have

$$h^S_{v_0}(s) = \langle \gamma(s), v_0 \rangle = \mathcal{H}^S(\gamma(s), v_0) = \mathfrak{h}^S_{v_0}(\gamma(s)).$$

Moreover, for any $s_0 \in \mathbb{R}$ and $v_0 = d_\gamma(s_0)$, $(\mathfrak{h}^S_{v_0}|_M)^{-1}(c)$ is a D-slice of M.

Observe that by Proposition 3.3.1, $(\mathfrak{h}^S_{v_0})^{-1}(c_0) = P(v_0, c_0)$ is tangent to γ at $\gamma(s_0)$, where $c_0 = h^S_{v_0}(s_0)$. We denote $TP^T_{v_0,\gamma(s_0)} = P(v_0, c_0)$ which is called a *timelike tangent plane of γ at $\gamma(s_0)$ with respect to $v_0 = d_\gamma(s_0)$.*

LEMMA 3.3.7. *If $k_n(s_0) \neq 0$, i.e, the D-slice $(\mathfrak{h}^S_{v_0}|_M)^{-1}(c_0)$ is non-singular at $\gamma(s_0)$, and $P(v_0, c_0)$ is tangent to γ at $\gamma(s_0)$, where $c_0 = h^S_{v_0}(s_0)$. Then, the D-slice is a curve of M, tangent to γ at $\gamma(s_0)$.*

Proof. We suppose that the D-slice $(\mathfrak{h}^S_{v_0}|_M)^{-1}(c_0)$ and γ intersects transversely at $\gamma(s_0)$. As $P(v_0, c_0)$ is tangent to γ at $\gamma(s_0)$ and the D-slice is contained in $P(v_0, c_0)$, follow that the $\gamma'(s_0)$ and the tangent vector to the D-slice at $\gamma(s_0)$ generate the tangent plane to M at $\gamma(s_0)$. Therefore, we conclude that $P(v_0, c_0)$ is precisely this plane, i.e, the D-slice is singular and therefore we have a contradiction. ∎

We call the D-slice $(\mathfrak{h}^S_{v_0}|_M)^{-1}(c_0)$ of a *tangent D-slice of γ at $\gamma(s_0)$ relative to M* and we denoted by $T^D_{M,\gamma(s_0)}$. By Proposition 3.3.1, we conclude that γ and $TP^T_{v_0,\gamma(s_0)}$ have contact of order three at $\gamma(s_0)$ if and only if $\sigma_D(s_0) = 0$ and $\sigma'_D(s_0) \neq 0$. Under the assumption that $k_n(s_0) \neq 0$ the above conditions are equivalent to the condition that γ and $T^D_{M,\gamma(s_0)}$ have contact of order three at $\gamma(s_0)$. Therefore, we have the following proposition:

PROPOSITION 3.3.8. *Let $\gamma : I \to M$ be a regular curve on M. Then the following conditions are equivalent:*

(1)γ and the timelike tangent plane $TP^T_{v_0,\gamma(s_0)}$ have contact of order three, where $v_0 = d_\gamma(s_0)$,
(2)$\sigma_D(s_0) = 0$ and $\sigma'_D(s_0) \neq 0$.

If $k_n(s_0) \neq 0$, then the tangent D-slice $T^D_{M,\gamma(s_0)}$ of γ at $\gamma(s_0)$ is non-singular and the above two conditions are equivalent to the following condition:
(3)γ and the tangent D-slice $T^D_{M,\gamma(s_0)}$ have contact of order three.

Proof. For $v_0 = d_\gamma(s_0)$ and $c_0 = h^S_{v_0}(s_0)$, we define $F = \widetilde{\mathfrak{h}^S_{v_0}} : \mathbb{R}^3_1 \to \mathbb{R}$ by $F(x) = \widetilde{\mathfrak{h}^S_{v_0}}(x) = \langle x, v_0 \rangle - c_0$ and consider $f = F \circ \gamma$. Thus the prove follows from Definition 3.3.6 and Proposition 3.3.1. ∎

Remember that an H-slice is always a regular curve. We also define

$$\mathcal{H}^T : \mathbb{R}^3_1 \times H^2_+(-1) \to \mathbb{R}; \quad (x, v) \mapsto \langle x, v \rangle.$$

By exactly the same arguments as the above case, we have the notions of *spacelike tangent plane of γ at $\gamma(s_0)$ with respect to $v_0 = h_\gamma(s_0)$* and *tangent H-slice of γ at $\gamma(s_0)$ relative* *to* M. We respectively denote these as $TP^S_{v_0,\gamma(s_0)}$ and $T^H_{M,\gamma(s_0)}$. In this case the tangent H-slice is always non-singular at $\gamma(s_0)$. It also follows from Proposition 3.3.4 that γ and $T^H_{M,\gamma(s_0)}$ have contact of order three at $\gamma(s_0)$ if and only if $\sigma_H(s_0)=0$ and $\sigma'_H(s_0)\neq 0$.

By the Remark 3.3.3, in the next result we only need to consider spacelike curve.

PROPOSITION 3.3.9. *Let $\gamma : I \to M$ be a spacelike regular curve on M. The following conditions are equivalent:*

(1)γ and the spacelike tangent plane $TP^S_{v_0,\gamma(s_0)}$ have contact of order three, where $v_0 = h_\gamma(s_0)$,
(2)$\sigma_H(s_0)=0$ and $\sigma'_H(s_0)\neq 0$,
(3)γ and the tangent H-slice $T^H_{M,\gamma(s_0)}$ have contact of order three.

As a consequence, we have that γ is a model curve on M if and only if its pseudo-spherical evolutes are constant, that is, $h'_\gamma \equiv 0$ or $d'_\gamma \equiv 0$, or equivalently the invariants are zero functions.

4. UNFOLDING OF FUNCTIONS

In this section, we investigate the singularities of pseudo-spherical evolutes and we use well known theorem in the unfolding theory for obtaining new results.

Let $F : \mathbb{R}\times\mathbb{R}^r, (s_0, x_0) \to \mathbb{R}$ be a function germ. We call F an r-parameter unfolding of f, where $f(s) = F_{x_0}(s)$. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{k-1}(\frac{\partial F}{\partial x_i}(s,x_0))(s_0) = \sum_{j=1}^{k-1}\alpha_{ji}s^j$ for $i = 1,\ldots,r$. Then F is called a (p)-versal unfolding if the $(k-1)\times r$ matrix of coefficients (α_{ji}) has rank $k-1$ $(k-1\le r)$. The *bifurcation set* of F is defined to be

$$B_F = \{x\in\mathbb{R}^r \mid \frac{\partial F}{\partial s}(s,x) = \frac{\partial^2 F}{\partial s^2}(s,x) = 0 \; at \; (s,x) \; for \; some \; s\}.$$

Then we have the following fundamental result of the unfolding theory (see [1]).

THEOREM 4.4.1. *Let $F : \mathbb{R}\times\mathbb{R}^r, (s_0,x_0) \to \mathbb{R}$ be an r-parameter unfolding of f which has the type A_k at s_0. If F is a (p)-versal unfolding and $k = 3$, then the germ of B_F at x_0 is diffeomorphic to $(C\times\mathbb{R}^{r-2})$ as set germs, where $C = \{(x_1,x_2) \mid x_1^2 = x_2^3)\}$.*

By Propositions 3.3.1 and 3.3.4, we have the following.

PROPOSITION 4.4.2. *(1) For a curve $\gamma : I \to M$ with the arc-length parameter s and $t'(s)\neq 0$, the bifurcation set of the height function H^S is $B_{H^S} = \{d_\gamma(s) \mid s\in I\}$.*

(2) For a timelike curve $\gamma : I \to M$, the bifurcation set of the height function H^T is empty. Moreover, for a spacelike curve with the arc-length parameter s and $t'(s)\neq 0$, the bifurcation set of H^T is $B_{H^T} = \{h_\gamma(s) \mid s\in I\}$.

We have the following proposition.

PROPOSITION 4.4.3. *Let $\gamma : I \to M$ be a curve with the arc-length parameter s and $t'(s_0) \neq 0$.*

(1)If $h^S_{\mathbf{v}_0}(s)$ has type A_3 at s_0, then H^S is a (p)-versal unfolding of $h^S_{\mathbf{v}_0}(s)$.

(2)If $h^T_{\mathbf{v}_0}(s)$ has type A_3 at s_0, then H^T is a (p)-versal unfolding of $h^T_{\mathbf{v}_0}(s)$.

Proof. (1) *We denote* $\gamma(s) = (x_0(s), x_1(s), x_2(s))$, $\mathbf{v} = (v_0, v_1, \sqrt{1+v_0^2-v_1^2}) \in S^2_1$. *Therefore we have*

$$H^S(s,\mathbf{v}) = -x_0(s)v_0 + x_1(s)v_1 + x_2(s)\sqrt{1+v_0^2-v_1^2},$$

and

$$\frac{\partial H^S}{\partial v_0} = -x_0(s) + \frac{v_0}{\sqrt{1+v_0^2-v_1^2}}x_2(s), \quad \frac{\partial H^S}{\partial v_1} = x_1(s) - \frac{v_1}{\sqrt{1+v_0^2-v_1^2}}x_2(s)$$

$$\frac{\partial^2 H^S}{\partial s \partial v_0} = -x_0'(s) + \frac{v_0}{\sqrt{1+v_0^2-v_1^2}}x_2'(s), \quad \frac{\partial^2 H^S}{\partial s \partial v_1} = x_1'(s) - \frac{v_1}{\sqrt{1+v_0^2-v_1^2}}x_2'(s)$$

$$\frac{\partial^3 H^S}{\partial s \partial v_0} = -x_0''(s) + \frac{v_0}{\sqrt{1+v_0^2-v_1^2}}x_2''(s), \quad \frac{\partial^3 H^S}{\partial s \partial v_1} = x_1''(s) - \frac{v_1}{\sqrt{1+v_0^2-v_1^2}}x_2''(s)$$

So, we have the following matrix

$$A = \begin{pmatrix} -x_0'(s) + \frac{v_0}{\sqrt{1+v_0^2-v_1^2}}x_2'(s) & x_1'(s) - \frac{v_1}{\sqrt{1+v_0^2-v_1^2}}x_2'(s) \\ -x_0''(s) + \frac{v_0}{\sqrt{1+v_0^2-v_1^2}}x_2''(s) & x_1''(s) - \frac{v_1}{\sqrt{1+v_0^2-v_1^2}}x_2''(s) \end{pmatrix}.$$

By Proposition 3.3.1, we have that $h^S_{\mathbf{v}}$ has type A_3 at s if and only if

$$\mathbf{v} = \pm\frac{1}{\sqrt{k_g^2(s) - \varepsilon(\gamma(s))k_n^2(s)}}(k_n(s)b(s) - k_g(s)n_\gamma(s)),$$

$k_g^2(s) > \varepsilon(\gamma(s))k_n^2(s)$, $\sigma_D(s) = 0$ *and* $\sigma_D'(s) \neq 0$.

For prove the assertion (1), *we have to show that the matrix is non-singular, i.e,* $\det A \neq 0$. *Therefore we calculate the determinant of this matrix.*

$$\det A = ((x_0', x_1', x_2') \wedge (x_0'', x_1'', x_2'')) \begin{pmatrix} \dfrac{v_0}{\sqrt{1+v_0^2-v_1^2}} \\ -\dfrac{v_1}{\sqrt{1+v_0^2-v_1^2}} \\ -1 \end{pmatrix}$$

$$= -\frac{1}{\sqrt{1+v_0^2-v_1^2}}(t \wedge (-\varepsilon(\gamma(s))k_g(s)b(s) + k_n(s)n_\gamma(s))) \begin{pmatrix} -v_0 \\ v_1 \\ \sqrt{1+v_0^2-v_1^2} \end{pmatrix}$$

$$= \pm\frac{\sqrt{k_g^2(s) - \varepsilon(\gamma(s))k_n^2(s)}}{\sqrt{1+v_0^2-v_1^2}} \neq 0.$$

By the same way, if we consider the timelike height function H^T, we can prove (2). ∎

As a consequence, we have the following theorems:

THEOREM 4.4.4. *Let $\gamma : I \to M$ be a regular curve such that $t'(s) \neq 0$. Then we have the following assertions:*

(1)The de Sitter evolute at $d_\gamma(s_0)$ is regular if $\sigma_D(s_0) \neq 0$.
(2)The following conditions are equivalent:

(i)the germ of the de Sitter evolute at $d_\gamma(s_0)$ is diffeomorphic to the ordinary cusp C, where $C = \{(x_1, x_2) \mid x_1^2 = x_2^2\}$;

(ii)$\sigma_D(s_0) = 0$ and $\sigma_D'(s_0) \neq 0$;

(iii)γ and the timelike tangent plane $TP^T_{v_0,\gamma(s_0)}$ have contact of order three;

(iv)if $k_n(s_0) \neq 0$, then the tangent D-slice $T^D_{M,\gamma(s_0)}$ of γ at $\gamma(s_0)$ is non-singular, and γ and the tangent D-slice $T^D_{M,\gamma(s_0)}$ have contact of order three.

Proof. *(1)* By Lemma 3.3.2, we have $d_\gamma'(s) = 0$ if and only if $\sigma_D(s) = 0$. It means that the de Sitter evolute at $d_\gamma(s_0)$ is regular if $\sigma_D(s_0) \neq 0$.

(2) By Proposition 3.3.1, the bifurcation set of H^S is

$$B_{H^S} = \left\{ v = \pm\frac{1}{\sqrt{k_g^2(s) - \varepsilon(\gamma(s))k_n^2(s)}}(k_n(s)b(s) - k_g(s)n_\gamma(s)) \;\middle|\; k_g^2(s) > \varepsilon(\gamma(s))k_n^2(s) \right\}.$$

By Theorem 4.4.1 and Proposition 4.4.3, the germ of the bifurcation set is diffeomorphic to the ordinary cusp if $\sigma_D(s_0) = 0$ and $\sigma_D'(s_0) \neq 0$. Moreover we have the other equivalences by Proposition 3.3.8. This complete the proof for *(1)* and *(2)*. ∎

THEOREM 4.4.5. *Let $\gamma : I \to M$ be a spacelike regular curve such that $t'(s) \neq 0$. Then we have the following assertions:*

(1)The hyperbolic evolute at $h_\gamma(s_0)$ is regular if $\sigma_H(s_0) \neq 0$.
(2)The following conditions are equivalent:

(i)the germ of the hyperbolic evolute at $h_\gamma(s_0)$ is diffeomorphic to the ordinary cusp C;

(ii)$\sigma_H(s_0) = 0$ and $\sigma'_H(s_0) \neq 0$;

(iii)γ and the spacelike tangent plane $TP^S_{v_0,\gamma(s_0)}$ have contact order three;

(iv)γ and the tangent H-slice $T^H_{M,\gamma(s_0)}$ have contact of order three.

Proof. The prove of this proposition is analogous the prove of the above proposition. ▮

5. EXAMPLES

Now, we consider two examples of curves on a timelike surface: curves on a timelike plane, $\mathbb{R}^2_1$, and curves on the de Sitter space, S^2_1.

EXAMPLE 5.5.1. *Suppose that $M = \mathbb{R}^2_1 = \{x = (x_0, x_1, x_2) \mid x_2 = 0\}$. We consider a plane curve $\gamma : I \to \mathbb{R}^2_1$. In this case we have $n_\gamma = e_2$, $t(s) = \gamma'(s)$ and $b(s) = e_2 \wedge t(s)$. It follows that $k_n(s) \equiv \tau_g(s) \equiv 0$ and $k_g = \langle b(s), t'(s)\rangle = -\varepsilon(\gamma(s))k(s)$. Then we have the following Frenet-Serret formulae on Minkowski plane:*

$$\begin{cases} b'(s) = & k(s)t(s) \\ t'(s) = & k(s)b(s) \end{cases}$$

Here we have $\sigma_D \equiv 0$ and the constant de Sitter evolute $d_\gamma(s) = n_\gamma = e_2$ of γ relative to M. It means that the D-slice is $M = P(v, c) \cap M$. Moreover, we do not have hyperbolic evolutes. We observe that in [6] the authors study the evolute of γ in $\mathbb{R}^2_1$. In all above cases the evolutes are given by the intersection of the focal surface, of γ, in $\mathbb{R}^3_1$ (see [4], [5]) with the pseudo-spheres (de Sitter and Hyperbolic evolutes) and with $\mathbb{R}^2_1$.

In the above example observe that $k_n \equiv 0$, for this we have $M = P(v, c) \cap M$ and γ is not considered a model curve on M even with the constant de Sitter evolute.

EXAMPLE 5.5.2. *Suppose that $M = S^2_1$. In this case, we have $n_\gamma(s) = \gamma(s)$, $t(s) = \gamma'(s)$ with $\| t(s) \| = 1$ and $b(s) = n_\gamma(s) \wedge t(s)$. Therefore, we have $\tau_g(s) = 0$, $k_n(s) = -\varepsilon(\gamma(s))$, where $\varepsilon(\gamma(s)) = 1$ if γ is spacelike and $\varepsilon(\gamma(s)) = -1$ if γ is timelike. By the Frenet-Serret type formulae, we have the following formulae (see [4], [5]):*

$$\begin{cases} t'(s) = & -\varepsilon(\gamma(s))\gamma(s) - \varepsilon(\gamma(s))k_g(s)b(s) \\ \gamma'(s) = & t(s) \\ b'(s) = & -\varepsilon(\gamma(s))k_g(s)t(s) \end{cases}$$

Here, we have $\sigma_D = -\varepsilon k_g'$, and $\sigma_H = k_g'$, and we have the de Sitter evolute of γ relative to M,

$$d_\gamma(s) = \frac{k_g(s)}{\sqrt{k_g^2(s) - \varepsilon(\gamma(s))}} n_\gamma(s) + \frac{\varepsilon(\gamma(s))}{\sqrt{k_g^2(s) - \varepsilon(\gamma(s))}} b(s)$$

for $k_g^2(s) > \varepsilon(\gamma(s))$. The hyperbolic evolute of γ relative to M, is defined only if γ is spacelike as

$$h_\gamma(s) = -\frac{k_g(s)}{\sqrt{1 - k_g^2(s)}} n_\gamma(s) - \frac{1}{\sqrt{1 - k_g^2(s)}} b(s)$$

for $k_g^2(s) < 1$. The study of the de Sitter evolute was made in [4] and [5], where we conclude that the singular points of the de Sitter evolute are the points where $k_g' = 0$. By Theorem 4.4.4, the evolute at $d_\gamma(s_0)$ is regular if $\sigma_D = -\varepsilon k_g' \neq 0$, and is a ordinary cusp locally if $\sigma_D = -\varepsilon k_g' = 0$ and $\sigma_D' = -\varepsilon k_g'' \neq 0$. Moreover the hyperbolic evolute at $h_\gamma(s_0)$ is regular if $\sigma_H = k_g' \neq 0$, and is a ordinary cusp locally if $\sigma_H = k_g' = 0$ and $\sigma_H' = k_g'' \neq 0$. Therefore the cusps of the pseudo-spherical evolute corresponds to the points $\gamma(s)$ with $k_g'(s) = 0$ and $k_g''(s) \neq 0$.

6. LIGHTLIKE SURFACE IN THE MINKOWSKI SPACE ALONG DE SITTER SPACELIKE PLANE CURVES

In this section we investigate the relation of the de Sitter (hyperbolic) evolute of a spacelike curve γ in S_1^2 with the lightlike surface along γ. Let $\gamma : I \to S_1^2$ be a parametrised by arc length spacelike curve, then we have that the Frenet-Serret type formulae of γ is given by

$$\begin{cases} \gamma'(s) = t(s) \\ t'(s) = -\gamma(s) - k_g(s)\, n(s) \\ n'(s) = -k_g(s)\, t(s) \end{cases},$$

where $k_g(s) = \langle \gamma''(s), n(s) \rangle$ is the geodesic curvature of γ at s (see [5]). We say that a surface is lightlike if each tangent plane at the regular points of the surface is lightlike. Following the definition of lightlike hypersurface along spacelike submanifolds in [3], the *lightlike surfaces along* γ are given by the maps $\mathbb{LS}_\gamma^\pm : I \times \mathbb{R} \to \mathbb{R}^3_1$ defined by

$$\mathbb{LS}_\gamma^\pm(s, \mu) = \gamma(s) + \mu(n(s) \pm \gamma(s)).$$

We only consider $\mathbb{LS}_\gamma^+$, so that we denote that by $\mathbb{LS}_\gamma$.

We have the following:

$$\frac{\partial \mathbb{LS}_\gamma}{\partial s} = (1 - \mu k_g(s) + \mu)t(s)$$
$$\frac{\partial \mathbb{LS}_\gamma}{\partial \mu} = n(s) + \gamma(s).$$

Under the condition that $k_g(s) \neq 1$, $\left\{ \frac{\partial \mathbb{LS}_\gamma}{\partial s}, \frac{\partial \mathbb{LS}_\gamma}{\partial \mu} \right\}$ is linearly dependent if and only if $1-\mu k_g(s)+\mu = 0$. Therefore, (s,μ) is a singular point of $\mathbb{LS}_\gamma$ if and only if $\mu = \dfrac{1}{k_g(s)-1}$. In [3], the *lightlike focal set* of the submanifold is defined as being the critical value set of the lightlike hypersurface along a spacelike submanifold, then the lightlike focal set of γ is given by the curve

$$\beta(s) = \mathbb{LS}_\gamma(s, \mu(s)) = \left(\frac{k_g(s)}{k_g(s)-1} \right) \gamma(s) + \left(\frac{1}{k_g(s)-1} \right) n(s).$$

Since $\langle \beta(s), \beta(s) \rangle = \dfrac{k_g^2(s)-1}{(k_g(s)-1)^2}$, we have

$$\beta(s) \text{ is } \begin{cases} \text{spacelike if} & k_g^2(s) - 1 > 0 \\ \text{lightlike if} & k_g^2(s) - 1 = 0, \\ \text{timelike if} & k_g(s)^2 - 1 < 0. \end{cases}$$

We define a mapping

$$\Phi : \mathbb{R}^3_1 \setminus LC^* \to H^2(-1) \cup S^2_1$$

by $\Phi(x) = \dfrac{x}{\parallel x \parallel}$. We have $\mathbb{R}^3_1 \setminus LC^* = S \cup T$, where $S = \{x \in \mathbb{R}^3_1 \mid \langle x, x \rangle > 0\}$ and $T = \{x \in \mathbb{R}^3_1 \mid \langle x, x \rangle < 0\}$. Therefore, we have projections $\Phi^S = \Phi|_S : S \to S^2_1$ and $\Phi^T = \Phi|_T : T \to H^2(-1)$. Suppose that $k_g^2(s) - 1 > 0$. Then

$$\Phi^S \circ \beta(s) = \frac{k_g(s)}{\sqrt{k_g^2(s)-1}} \gamma(s) + \frac{1}{\sqrt{k_g^2(s)-1}} n(s) = d_\gamma(s).$$

On the other hand, suppose that $k_g^2(s) - 1 < 0$. Then

$$\Phi^T \circ \beta(s) = -\frac{k_g(s)}{\sqrt{1-k_g^2(s)}} \gamma(s) - \frac{1}{\sqrt{1-k_g^2(s)}} n(s) = h_\gamma(s).$$

We now define

$$\beta^S_\gamma = \{\beta(s) \mid s \in I, k_g^2(s) > 1\},$$

$$\beta^T_\gamma = \{\beta(s) \mid s \in I, k_g^2(s) < 1\}.$$

We call β^S_γ the *spacelike part of the lightlike focal set of* γ and β^T_γ the *timelike part of the lightlike focal set of* γ. We show that the projection of β^S_γ (β^T_γ) is the de Sitter (hyperbolic) evolute. Then we have the following result.

THEOREM 6.6.1. *Let $\gamma : I \to S^2_1$ be a spacelike curve with the arc-length parameter s and $k_g(s) \neq 1$. Then we have*

$$\Phi^S(\beta^S_\gamma) = d_\gamma(I_{>1}) \text{ and } \Phi^T(\beta^T_\gamma) = h_\gamma(I_{<1}),$$

where $I_{>1} = \{s \in I \mid k_g^2(s) > 1\}$ and $I_{<1} = \{s \in I \mid k_g^2(s) < 1\}$.

Now we consider a family of functions on spacelike curve in de Sitter space in order have a relation with the lightlike surface of the spacelike curve. Let $\gamma : I \to S^2_1$ be a parametrised by arc length spacelike curve. We define a family of distance squared functions

$$D : I \times \mathbb{R}^3_1 \to \mathbb{R}$$

by $D(s, v) = \langle \gamma(s) - v, \gamma(s) - v\rangle$ and denote $D_v(s) = D(s, v)$.

PROPOSITION 6.6.2. *For a spacelike curve $\gamma : I \to S^2_1$ with the arc-length parameter s and $\langle t'(s), t'(s)\rangle \neq 0$, we have the following:*

(1)$D_v(s) = D'_v(s) = 0$ if and only if there exists $\mu \in \mathbb{R}$ such that $v = \gamma(s)+\mu(n(s)\pm\gamma(s))$.
(2)$D_v(s) = D'_v(s) = D''_v(s) = 0$ if and only if

$$v = \gamma(s) + \frac{1}{k_g(s) \mp 1}(n(s) \pm \gamma(s)).$$

(3)$D_v(s) = D'_v(s) = D''_v(s) = D'''_v(s) = 0$ if and only if

$$v = \gamma(s) + \frac{1}{k_g(s) \mp 1}(n(s) \pm \gamma(s)) \text{ and } k'_g(s) = 0.$$

Proof. *(1)* We have $D'_v(s) = \langle\gamma(s) - v, t(s)\rangle = 0$ if and only if there exist $\bar\mu, \lambda \in \mathbb{R}$ such that $\gamma(s) - v = \lambda\gamma(s) + \bar\mu n(s)$. Then, $D_v(s) = D'_v(s) = 0$ if and only if $\lambda = \pm\bar\mu$, that is, $v = \gamma(s)-\bar\mu(n(s)\pm\gamma(s))$. As $\bar\mu \in \mathbb{R}$, we can call $\mu = -\bar\mu$ and then $v = \gamma(s)+\mu(n(s)\pm\gamma(s))$.

(2) Since $\frac{1}{2}D''_v(s) = 1 + \langle\gamma(s) - v, -\gamma(s) - k_g(s)n(s)\rangle$, we have that $D_v(s) = D'_v(s) = D''_v(s) = 0$ if and only if $\mu = \frac{1}{k_g(s) \mp 1}$ and therefore

$$v = \gamma(s) + \frac{1}{k_g(s) \mp 1}(n(s) \pm \gamma(s)).$$

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(3) Since $\frac{1}{2}D_v'''(s) = \langle \gamma(s) - v, -\gamma'(s) - k_g'(s)n(s) + k_g^2(s)t(s)\rangle$, we have that $D_v(s) = D_v'(s) = D_v''(s) = D_v'''(s) = 0$ if and only if

$$v = \gamma(s) + \frac{1}{k_g(s) \mp 1}(n(s) \pm \gamma(s)) \text{ and } k_g'(s) = 0.$$

∎

Let $F : \mathbb{R} \times \mathbb{R}^r, (s_0, v_0) \to \mathbb{R}$, be a function germ. We call F an r-parameter unfolding of f, where $f(s) = F_{v_0}(s)$. We introduce an important set concerning the unfolding. The *discriminant set* of F is

$$\mathfrak{D}_F = \{v \in \mathbb{R}^r \mid F(s, v) = \frac{\partial F}{\partial s}(s, v) = 0 \text{ for some } s \in \mathbb{R}\}.$$

By Proposition 6.6.2, the discriminant set of the distance squared function $D(s, v)$ is given by

$$\mathfrak{D}_D = \{\gamma(s) + \mu(n(s) \pm \gamma(s)) \mid s \in I,\ \mu \in \mathbb{R}\},$$

which is the image of the lightlike surfaces along a spacelike curve $\gamma : I \to S_1^2$.

For a spacelike or timelike curve $\gamma : I \to \mathbb{R}_1^3$ parametrised by arc length with $k(s) \neq 0$, we have that the *focal surface of* γ is given by

$$\mathfrak{B}(s, \mu) = \gamma(s) + \frac{\varepsilon(\gamma(s))}{\delta(\gamma(s))k(s)}n(s) + \mu b(s),$$

with $\mu \in \mathbb{R}$. The *cuspidal curve* of the focal surface is given by

$$\mathfrak{B}(s) = \gamma(s) + \frac{\varepsilon(\gamma(s))}{\delta(\gamma(s))k(s)}n(s) + \mu(s)b(s),$$

with $\mu(s) = \dfrac{k'(s)}{\varepsilon(\gamma(s))\delta(\gamma(s))k^2(s)\tau(s)}$, that is, where the distance squared function has singularity $A_{\geq 3}$. We denote the cuspidal curve $\mathfrak{B}(s)$ by $\mathcal{C}$. (For more details see [4] and [5].)

So, if $\gamma : I \to \mathbb{R}_1^3$ is a spacelike or timelike curve, by [2], [4] and [5] we know that:

(a) the focal surface $\mathfrak{B}$ of a spacelike curve γ is a timelike surface in $\mathbb{R}_1^3$, and $\mathfrak{B}$ is a spacelike surface if γ is a timelike curve;
(b) the de Sitter evolute of γ is the curve $\mathfrak{B} \cap S_1^2$;
(c) the hyperbolic evolute of γ is the curve $\mathfrak{B} \cap H^2(-1)$;
(d) the singular curve of the focal surface $\mathfrak{B}$ is the cuspidal curve $\mathcal{C}$.

We observe that (a) to (d) is true for $\gamma : I \to S_1^2$. Furthermore, for a spacelike curve $\gamma : I \to S_1^2$, by Theorem 6.6.1, we have the following:

(e) the projection of β to S_1^2 is the de Sitter evolute of γ;

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(f) the projection of β to $H^2(-1)$ is the hyperbolic evolute of γ, where β is the lightlike focal curve of γ.

By the above calculations, we have the following result.

PROPOSITION 6.6.3. *For a spacelike curve $\gamma : I \to S^2_1$ with the arc-length parameter s and $\langle t'(s), t'(s)\rangle \neq 0$. We have the following:*

(i)the lightlike focal set β of γ is the curve $\mathfrak{B} \cap \mathfrak{D}_D$;

(ii)the curve β is regular at s_0 if and only if $k'_g(s_0) \neq 0$ and the regular part of β is contained in the regular part of $\mathfrak{B}$;

(iii)the singular points of β are isolated points given by $\beta \cap \mathcal{C}$, that is, where $k'_g(s) = 0$. More specifically, $\mathfrak{D}_D$ is locally diffeomorphic to the swallowtail (see [1]) that intercepts the cuspidal curve of $\mathfrak{B}$ exactly at the singular points of β (see Figure 1);

(iv)the singular points of the spacelike part β^S_γ, of β, are projected to the singular points of the de Sitter evolute;

(v)the singular points of the timelike part β^T_γ, of β, are projected to the singular points of the hyperbolic evolute, where $\mathfrak{B}$ is the focal surface of γ and $\mathfrak{D}_D$ is the discriminant set of the distance squared function D, that is, $\mathfrak{D}_D$ is the lightlike surface $\mathbb{LS}_\gamma$.

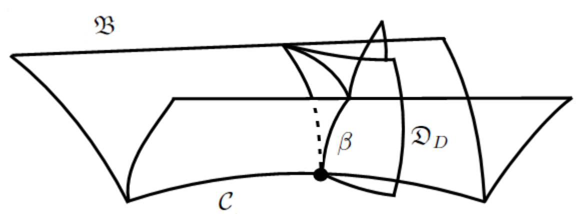

FIG. 1.

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