CADERNOS DE MATEMÁTICA **15**, 269–287 October (2014)
ARTIGO NÚMERO SMA#407

Focal set of curves in the Minkowski space

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We study the geometry of curves in the Minkowski space and in the de Sitter space, specially at points where the tangent direction is lightlike (i.e. has length zero). At such points, the curvature and the Frenet-Serret frame are not defined. We define the focal sets of these curves and study the metric structure of them. We use the Singularity Theory technics to carry out our study. October, 2014 ICMC-USP

1. INTRODUCTION

We consider the geometry of a smooth and regular curve γ and of its focal set in the Minkowski space $\mathbb{R}^3_1$ which is captured by its contact with pseudo-spheres. This contact is studied using the family of distance squared functions on γ. See [1] and [11] for more on singularities of functions, contact and their applications to geometry of curves in $\mathbb{R}^3$ and $\mathbb{R}^2_1$, respectively.

In Section 2, we prove that the lightlike points of a generic curve γ are isolated. We define the bifurcation set (focal set) of the distance squared function on γ.

We consider in §3 the geometry of spacelike and timelike curves using the Frenet-Serret formulae. These formulae and the distance squared function on γ are the main tools for the study of the focal set. We study the geometry and metric structure of the focal sets of γ.

In Section 4, we study the focal set in the neighborhood of lightlike points. In this case we can not parametrise the curve by arc length then we can not use the Frenet-Serret formulae.

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In §5 and §6, we consider curves and their focal set in the pseudo-spheres S_1^2 and S_1^3. We study the metric structure of the focal sets of such curves locally at lightlike points of γ.

2. PRELIMINARIES

The *Minkowski space* $\mathbb{R}_1^{n+1}$ is the vector space $\mathbb{R}^{n+1}$ endowed with the pseudo-scalar product $\langle x, y\rangle = -x_1y_1 + x_2y_2 + \ldots + x_{n+1}y_{n+1}$, for any $x = (x_1, x_2, \ldots, x_{n+1})$ and $y = (y_1, y_2, \ldots, y_{n+1})$ in $\mathbb{R}_1^{n+1}$. We say that a non-zero vector $x \in \mathbb{R}_1^{n+1}$ is *spacelike* if $\langle x, x\rangle > 0$, *lightlike* if $\langle x, x\rangle = 0$ and *timelike* if $\langle x, x\rangle < 0$. The norm of a vector $x \in \mathbb{R}_1^{n+1}$ is defined by $\| x \| = \sqrt{| \langle x, x\rangle |}$. This is an example of Lorentzian metric. In $\mathbb{R}_1^3$, the pseudo vector product of $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is:

$$x \wedge y = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. For basic concepts and details of properties, see [10].

We have the following pseudo-spheres in $\mathbb{R}_1^{n+1}$ with centre 0 and radius $r > 0$,

$$H^n(-r) = \{x \in \mathbb{R}_1^{n+1} \mid \langle x, x\rangle = -r^2\};$$

$$S_1^n(r) = \{x \in \mathbb{R}_1^{n+1} \mid \langle x, x\rangle = r^2\};$$

$$LC^* = \{x \in \mathbb{R}_1^{n+1} \mid \langle x, x\rangle = 0\}.$$

The LC^* is called *Lightcone* of $\mathbb{R}_1^{n+1}$, $H^n(-r)$ is the *Hyperbolic* n*-space* and $S_1^n(r)$ is the *de Sitter* n*-space*. Instead of $S_1^n(1)$, we usually write S_1^n.

Let V be a vector subspace of $\mathbb{R}_1^{n+1}$. Then we say that V is *timelike* if and only if V has a timelike vector, *spacelike* if and only if every non-zero vector in V is spacelike, or *lightlike* otherwise.

We consider embeddings $\gamma : I \to \mathbb{R}_1^3$, where I is an open interval of $\mathbb{R}$. The set $Emb(I, \mathbb{R}_1^3)$ of such embeddings is endowed with the Whitney C^∞-topology. We say that a property is *generic* if it is satisfied by curves in a residual subset of $Emb(I, \mathbb{R}_1^3)$.

We say that γ is *spacelike* (resp. *timelike*) if $\gamma'(t)$ is a *spacelike* (resp. *timelike*) vector for all $t \in I$. A point $\gamma(t)$ is called a *lightlike point* if $\gamma'(t)$ is a *lightlike vector*.

We define the subset Ω of $Emb(I, \mathbb{R}_1^3)$ such that a curve γ is in Ω if and only if $\langle \gamma''(t), \gamma'(t)\rangle \neq 0$ whenever $\langle \gamma'(t), \gamma'(t)\rangle = 0$. One can show, using Thom's transversality results (see for example [2], Chapter 9), that Ω is a residual subset of $Emb(I, \mathbb{R}_1^3)$.

PROPOSITION 2.2.1. *Let $\gamma \in \Omega$. Then the lightlike points of γ are isolated points.*

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Proof. Let $\gamma \in \Omega$. We define $g(t) = \langle \gamma'(t), \gamma'(t) \rangle$, where $g : I \to \mathbb{R}$. Consider $A = \{t \in I \,|\, g(t) = 0\}$. Let $t \in A$, then $g(t) = 0$ and as $\gamma \in \Omega$, it follows that $g'(t) \neq 0$. Therefore 0 is a regular value of g, thus $g^{-1}(0) = A$ is a submanifold of codimension 1 in I. That is, the dimension of A is 0 and thus A has only isolated points. ∎

To study the local properties of γ at $\gamma(t_0)$, we use the germ $\gamma : \mathbb{R}, t_0 \to \mathbb{R}^3_1$ of γ at t_0. The family of distance squared functions $f : I \times \mathbb{R}^3_1 \to \mathbb{R}$ on γ is given by

$$f(t, v) = \langle \gamma(t) - v, \gamma(t) - v \rangle.$$

We denote by $f_v : I \to \mathbb{R}$ the function given by $f_v(t) = f(t, v)$, for any fixed $v \in \mathbb{R}^3_1$.

The distance squared function f_v has singularity of type A_k at t_0 if the derivatives $(f_v)^{(p)}(t_0) = 0$ for all $1 \leq p \leq k$, and $(f_v)^{(k+1)}(t_0) \neq 0$. We also say that f_v has singularity of type $A_{\geq k}$ at t_0 if $(f_v)^{(p)}(t_0) = 0$ for all $1 \leq p \leq k$. This is valid including if $\gamma(t_0)$ is a lightlike point of the curve. Now let $F : \mathbb{R}^3_1 \to \mathbb{R}$ a submersion and $\gamma : I \to \mathbb{R}^3_1$ be a regular curve. We say that γ and $F^{-1}(0)$ have contact of order k or k-point contact at $t = t_0$ if the function $g(t) = F \circ \gamma(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0$ and $g^{(k+1)}(t_0) \neq 0$, i.e, if g has singularity A_k at t_0. Then the singularity type of f_v at t_0 measures the contact of γ at $\gamma(t_0)$ with the pseudo-sphere of centre v and radius $\| \gamma(t_0) - v \|$. The type of pseudo-sphere is determined by the sign of $\langle \gamma(t_0) - v, \gamma(t_0) - v \rangle$. For a generic curve in $\mathbb{R}^3_1$, f_v has local singularities of type A_1, A_2, A_3 or A_4 (see [3]) and the singularities A_4 are isolated points of the curve. We define the *bifurcation set* of f by

$$\mathfrak{Bif}(f) = \{v \in \mathbb{R}^3_1 \mid f'_v(t) = f''_v(t) = 0 \text{ in } (t, v) \text{ for some } t\},$$

i.e., the directions where f_v at t has a degenerate (non-stable) singularity, that is, the singularity is of type $A_{\geq 2}$.

The *focal set* is defined as being the locus of centres of pseudo-spheres that has at least 2-point contact with the curve. Therefore the definitions of focal set and $\mathfrak{Bif}(f)$ coincide.

We have the fundamental result of the unfolding theory:

Theorem 2.2.2. *[2] Let $G : (\mathbb{R} \times \mathbb{R}^3, (t_0, v_0)) \to \mathbb{R}$ be a 3-parameter unfolding of $g(t)$ which has the A_k-singularity at t_0. Suppose that G is an (p) versal unfolding.*

(a)If $k = 2$, then $\mathfrak{Bif}(G)$ is locally diffeomorphic to $\mathbb{R}^2$;

(b)If $k = 3$, then $\mathfrak{Bif}(G)$ is locally diffeomorphic to cuspidal edge $C \times \mathbb{R}$;

(c)If $k = 4$, then $\mathfrak{Bif}(G)$ is locally diffeomorphic to swallowtail SW.

Where, $C = \{(x_1, x_2) \,|\, x_1^2 = x_2^3\}$ is the ordinary cusp and $SW = \{(x_1, x_2, x_3) \,|\, x_1 = 3u^4 + u^2 v,\ x_2 = 4u^3 + 2uv,\ x_3 = v\}$ is the swallowtail.

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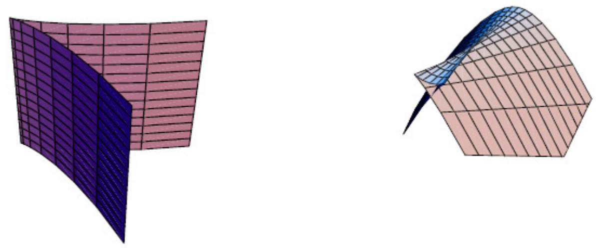

FIG. 1.

3. THE FOCAL SETS OF SPACELIKE AND TIMELIKE CURVES

Let $\gamma : I \to \mathbb{R}^3_1$ be a spacelike or a timelike curve and suppose that it is parametrised by arc length. This is possible because γ has no lightlike points.

In this section, we have the Frenet-Serret formulae of γ and we find the parametrisation of their focal surfaces. Furthermore, we study the metric structure of these focal surfaces.

We denote by $t(s) = \gamma'(s)$ the unit tangent vector to γ. Let $n(s)$ be the unit normal vector to γ given by $\gamma''(s) = k(s)n(s)$, where $k(s) = \|\gamma''(s)\|$ is defined as being the curvature of γ at s, and $b(s) = t(s) \wedge n(s)$ the unit binormal vector to γ. Then, we have the orthonormal basis $\{t(s), n(s), b(s)\}$ of $\mathbb{R}^3_1$ along γ. By the exactly same arguments as in the case for a curve in Euclidian 3-space, we have the following Frenet-Serret formulae (see [9], [6]):

$$\begin{cases} t'(s) = k(s)\, n(s) \\ n'(s) = -\varepsilon(\gamma(s))\, \delta(\gamma(s))\, k(s)\, t(s) + \varepsilon(\gamma(s))\, \tau(s)\, b(s) \\ b'(s) = \tau(s)\, n(s) \end{cases},$$

with $\tau(s)$ being the torsion of γ at s, $\varepsilon(\gamma(s)) = sign(t(s))$, $\delta(\gamma(s)) = sign(n(s))$, where $sign(v)$ is 1 if the vector v is spacelike or -1 if the vector v is timelike. We call them, ε and δ for short.

Observe that if γ is a spacelike or a timelike curve and $k(s) = 0$ for some point of γ, then $f''_v(s) = \varepsilon(\gamma(s)) \neq 0$ and we don't have singularities $A_{\geq 2}$, for any v. Now if $\tau(s) = 0$ for some point of the curve, we have that generically $f^{(3)}_v(s) = -\varepsilon(\gamma(s))k'(s) \neq 0$, that is, we don't have singularity $A_{\geq 3}$. This is the reason for which $k(s) \neq 0$ and $\tau(s) \neq 0$ in the following proposition.

Proposition 3.3.1. *[3] Let $\gamma : I \to \mathbb{R}^3_1$ be a spacelike or a timelike curve parametrised by arc length, with $k(s) \neq 0$ and $\tau(s) \neq 0$. Then*

(1)$f'_v(s_0) = 0$ if and only if there exist λ, $\mu \in \mathbb{R}$ such that $\gamma(s_0) - v = \lambda n(s_0) + \mu b(s_0)$.

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(2)$f_v'(s_0) = f_v''(s_0) = 0$ *if and only if* $v = \gamma(s_0) + \dfrac{\varepsilon(\gamma(s_0))}{\delta(\gamma(s_0))k(s_0)}n(s_0) + \mu b(s_0)$ *for some* $\mu \in \mathbb{R}$.

(3)$f_v'(s_0) = f_v''(s_0) = f_v^{(3)}(s_0) = 0$ *if and only if*

$$v = \gamma(s_0) + \frac{\varepsilon(\gamma(s_0))}{\delta(\gamma(s_0))k(s_0)}n(s_0) + \frac{k'(s_0)}{\varepsilon(\gamma(s_0))\delta(\gamma(s_0))k^2(s_0)\tau(s_0)}b(s_0).$$

Thus, for a spacelike or timelike curve γ parametrised by arc length with $k(s) \neq 0$, we have that the *focal surface of* γ is given by

$$\mathfrak{B}(s,\mu) = \gamma(s) + \frac{\varepsilon(\gamma(s))}{\delta(\gamma(s))k(s)}n(s) + \mu b(s), \tag{3.1}$$

with $\mu \in \mathbb{R}$. The *cuspidal curve* of the focal surface is given by

$$\mathfrak{B}(s) = \gamma(s) + \frac{\varepsilon(\gamma(s))}{\delta(\gamma(s))k(s)}n(s) + \mu(s)b(s), \tag{3.2}$$

with $\mu(s) = \dfrac{k'(s)}{\varepsilon(\gamma(s))\delta(\gamma(s))k^2(s)\tau(s)}$, that is, where the distance squared function has singularity $A_{\geq 3}$. We denote the cuspidal curve $\mathfrak{B}(s)$ by $\mathcal{C}$. Furthermore, we have that the focal surface is a ruled surface, where the base curve is given by the cuspidal curve and the director curve is the binormal vector $b(s)$. For more details see [4].

Proposition 3.3.2. *Let* γ *be a connected curve:*

(a)if γ *is timelike, then* γ *does not intersect its focal surface;*

(b)if γ *is spacelike, then* γ *intersects its focal surface at* $\gamma(s_2)$ *for some* $s_2 \in I$ *if and only if there exists* $s_1 \in I$ *such that* $\gamma(s_2)$ *belongs to a plane generated by* $n(s_1)$ *and* $b(s_1)$.

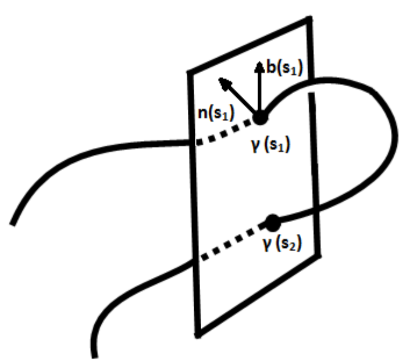

FIG. 2.

Proof. *(a) We suppose that γ intersects its focal surface, then there exists $s_1, s_2 \in I$ with $s_1 \neq s_2$ (for simplicity suppose that $s_2 < s_1$) such that,*

$$\gamma(s_1) - \frac{1}{k(s_1)}n(s_1) + \mu b(s_1) = \gamma(s_2).$$

By the mean value theorem, there exist $s_3 \in (s_2, s_1)$ such that $\gamma'(s_3)$ is parallel to $\gamma(s_2) - \gamma(s_1)$. Furthermore, we have

$$\left\langle -\frac{1}{k(s_1)}n(s_1) + \mu b(s_1), -\frac{1}{k(s_1)}n(s_1) + \mu b(s_1) \right\rangle = \frac{1}{k^2(s_1)} + \mu^2 > 0.$$

Thus $\gamma(s_2) - \gamma(s_1)$ is a spacelike vector parallel to the vector $\gamma'(s_3)$, that is a timelike vector. This is a contradiction. Therefore, γ does not intersect its focal surface.

(b) When γ intersects its focal surface at $\gamma(s_2)$ for some $s_2 \in I$, the result is clear. We suppose that there exists $s_1 \in I$ such that $\gamma(s_2)$ belongs to the plane generated by $n(s_1)$ and $b(s_1)$. Then we have that

$$\gamma(s_2) - \gamma(s_1) = \alpha n(s_1) + \beta b(s_1),$$

where $\alpha, \beta \in \mathbb{R}$. Therefore by (3.1), the focal surface intersects the curve at $\gamma(s_2)$ if $\alpha = \dfrac{1}{\delta(\gamma(s_1))k(s_1)}$.

Indeed, differentiating the above equation with relation to s_1, we have

$$-\gamma'(s_1) = \alpha n'(s_1) + \beta b'(s_1) = \alpha(-\delta(\gamma(s_1))k(s_1)t(s_1) + \tau(s_1)b(s_1)) + \beta\tau(s_1)n(s_1).$$

Thus,

$$\langle \alpha\delta(\gamma(s_1))k(s_1)t(s_1), \gamma'(s_1)\rangle = \langle \beta\tau(s_1)n(s_1) + \alpha\tau(s_1)b(s_1) + \gamma'(s_1), \gamma'(s_1)\rangle,$$

that implies $\alpha = \dfrac{1}{\delta(\gamma(s_1))k(s_1)}$. ∎

For study the metric structure of the focal surface $\mathfrak{B}$, we need some concepts. A *spacelike surface* is a surface for which the tangent plane at any point, is a spacelike plane (i.e., consists only of spacelike vectors). A *timelike surface* is a surface for which the tangent plane at any point is a timelike plane (i.e., consists of spacelike, timelike and lightlike vectors).

The pseudo scalar product in $\mathbb{R}^3_1$ induces a metric on the focal surface $\mathfrak{B}$ that may be degenerated at some points of $\mathfrak{B}$. We label the locus of such points the *Locus of Degeneracy* and we denote by LD (see [12] for Locus of Degeneracy of caustics of surfaces in $\mathbb{R}^3_1$). At a point $p \in LD$, the tangent plane to the focal surface at p is lightlike. The LD may be empty or a smooth curve that splits the focal surface locally into a Riemannian and a Lorentzian region. It is interesting to study what happens at points where the metric is degenerated and explain the changes in the geometry from a Riemannian region to a Lorentzian region of the submanifold (see §4).

We consider the focal surface of a spacelike or a timelike curve γ , that is,

$$\mathfrak{B}(s,\mu) = \gamma(s) + \frac{\varepsilon(\gamma(s))}{\delta(\gamma(s))k(s)}n(s) + \mu b(s), \ \ \mu \in \mathbb{R}.$$

We prove in the next result that the tangent plane of the focal surface is not defined at the points of the cuspidal curve $\mathcal{C}$ given by the Equation (3.2).

Proposition 3.3.3. *At the points of the bifurcation set on the cuspidal curve $\mathcal{C}$, the tangent planes of the bifurcation set are not defined.*

Proof. The tangent plane at a point of the focal surface is generated by the vectors

$$\frac{\partial \mathfrak{B}}{\partial s}(s,\mu) = \mathfrak{B}_s = \left(\mu\tau(s) - \frac{\varepsilon(\gamma(s))k'(s)}{\delta(\gamma(s))k^2(s)}\right)n(s) + \frac{\tau(s)}{\delta(\gamma(s))k(s)}b(s) \ \text{ and}$$
$$\frac{\partial \mathfrak{B}}{\partial \mu}(s,\mu) = \mathfrak{B}_\mu = b(s).$$

Observe that $\mathfrak{B}_s$ is parallel to $\mathfrak{B}_\mu$ if and only if $\mu(s) = \dfrac{k'(s)}{\varepsilon(\gamma(s))\delta(\gamma(s))k^2(s)\tau(s)}$, that is the parametrisation of the curve where f_v has singularities of type $A_{\geq 3}$. ∎

Supposing

$$\mu(s) \neq \frac{k'(s)}{\varepsilon(\gamma(s))\delta(\gamma(s))k^2(s)\tau(s)},$$

then $\mathfrak{B}_s$ and $\mathfrak{B}_\mu$ generate the tangent planes of the surface $\mathfrak{B}$, and

$$\langle \lambda_1\mathfrak{B}_s+\lambda_2\mathfrak{B}_\mu, \lambda_1\mathfrak{B}_s+\lambda_2\mathfrak{B}_\mu\rangle = \lambda_1^2\left(\frac{\tau^2}{k^2}\langle b,b\rangle + \frac{k'^2\delta}{k^4} - 2\frac{\varepsilon k'\mu\tau}{k^2} + \mu^2\tau^2\delta\right) + 2\lambda_1\lambda_2\left(\frac{\tau}{\delta k}\langle b,b\rangle\right) + \lambda_2^2\langle b,b\rangle.$$

We use this expression in the following proposition.

Proposition 3.3.4. *Away from the cuspidal curve $\mathcal{C}$:*

(a)the focal surface of a timelike generic curve is spacelike;

(b)the focal surface of a spacelike generic curve is timelike.

Proof. *(a) Let γ be a timelike curve, then $n(s)$ and $b(s)$ are spacelike. Therefore,*

$$\langle \lambda_1\mathfrak{B}_s + \lambda_2\mathfrak{B}_\mu, \lambda_1\mathfrak{B}_s + \lambda_2\mathfrak{B}_\mu\rangle = \lambda_1^2\left(\left(\frac{k'}{k^2} + \mu\tau\right)^2 + \frac{\tau^2}{k^2}\right)(s) + 2\lambda_1\lambda_2\left(\frac{\tau}{k}\right)(s) + \lambda_2^2. \qquad (*)$$

Making $\langle \lambda_1 \mathfrak{B}_s + \lambda_2 \mathfrak{B}_\mu, \lambda_1 \mathfrak{B}_s + \lambda_2 \mathfrak{B}_\mu \rangle = 0$, *we can think in the above equation as a quadratic equation of* λ_1, *thus* $\Delta = -4\lambda_2^2 \left(\frac{k'}{k^2} + \mu\tau \right)^2 (s) \leq 0$.

Since $\mu(s) \neq -\frac{k'}{k^2\tau}(s)$ *at the regular points of the focal surface, then* $\Delta = 0 \Leftrightarrow \lambda_2 = 0$. *Replacing* $\lambda_2 = 0$ *in* $(*)$, *we have a lightlike direction if* $\tau(s_0) = 0$ *and* $k'(s_0) = 0$, *that generically does not occur to a curve in* $\mathbb{R}^3_1$. *Thus, generically, we have not lightlike directions in this plane, and therefore the tangent planes are spacelike, that is, the LD is empty.*

(b) Let γ *be a spacelike curve then we have two cases:* $n(s)$ *timelike and* $b(s)$ *spacelike or* $n(s)$ *spacelike and* $b(s)$ *timelike. In both cases, the prove follows analogous to the case (a).* ▮

4. THE FOCAL SET IN THE NEIGHBORHOOD OF LIGHTLIKE POINTS

We study until now what is happening with the focal surface of a spacelike or a timelike curve. In this section we are also interested in know what is happening with the focal surface at the neighborhood of lightlike points of a regular curve. The principal result of this section is given by Theorem 4.4.2.

To study the focal surface at lightlike points $\gamma(t_0)$, we can not parametrise the curve by arc length since $\langle \gamma'(t_0), \gamma'(t_0) \rangle = 0$. We consider then a regular curve $\gamma : I \to \mathbb{R}^3_1$ not parametrised by arc length. We saw that the distance squared function is given by $f_v(t) = \langle \gamma(t) - v, \gamma(t) - v \rangle$. Thus

$$\frac{1}{2} f'_v(t) = \langle \gamma(t) - v, \gamma'(t) \rangle.$$

It follows that f_v is singular at t if and only if $\langle \gamma(t) - v, \gamma'(t) \rangle = 0$, equivalently, $\gamma(t) - v = \mu N(t) + \lambda B(t)$, where $N(t)$ and $B(t)$ are vectors that generate the orthogonal plane to the vector $\gamma'(t)$. (This condition includes the lightlike points.)

Differentiating again we get

$$\frac{1}{2} f''_v(t) = \langle \gamma(t) - v, \gamma''(t) \rangle + \langle \gamma'(t), \gamma'(t) \rangle.$$

The singularity of f_v is degenerate if and only if $f'_v(t) = f''_v(t) = 0$, equivalently, $\gamma(t) - v = \mu N(t) + \lambda B(t)$ and

$$\mu \langle N(t), \gamma''(t) \rangle + \lambda \langle B(t), \gamma''(t) \rangle + \langle \gamma'(t), \gamma'(t) \rangle = 0. \tag{4.1}$$

It follows that the bifurcation set of f is given by

$$\mathfrak{Bif}(f) = \{ \gamma(t) - \mu N(t) - \lambda B(t) \mid (\mu, \lambda) \text{ is solution of } (4.1) \}.$$

Away from the isolated lightlike points of γ, the bifurcation set is precisely the focal surface of the spacelike and timelike components of γ.

Our aim now, is to find the general expression of the focal surface to analyse what is happening with the focal surface when the curve γ has lightlike points. We observe that since $\gamma \in \Omega$, then near a lightlike point, γ changes from a spacelike curve to a timelike curve.

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Taking $\mathbf{t}(t) = \gamma'(t)$, $N(t) = \gamma'(t) \wedge \gamma''(t)$ and $B(t) = \gamma'(t) \wedge (\gamma'(t) \wedge \gamma''(t))$ and replacing in (4.1), we have

$$\lambda\langle \gamma'(t) \wedge \gamma''(t), \gamma'(t) \wedge \gamma''(t)\rangle - \langle \gamma'(t), \gamma'(t)\rangle = 0. \tag{4.2}$$

Thus the bifurcation set of f is given by

$$\mathfrak{Bif}(f) = \{\gamma(t) - \mu\left(\gamma'(t) \wedge \gamma''(t)\right) - \lambda\left(\gamma'(t) \wedge (\gamma'(t) \wedge \gamma''(t))\right) \mid \mu \in \mathbb{R} \text{ and } \lambda \text{ is solution of (4.2)}\}$$

Since $\gamma \in \Omega$, at a lightlike point $\gamma(t_0)$ of γ, the vector $N(t_0) = \gamma'(t_0) \wedge \gamma''(t_0)$ is not lightlike, thus the bifurcation set above is well defined. Furthermore, $B(t_0)$ is parallel to $\gamma'(t_0)$, and the vectors $t(t_0)$, $N(t_0)$ and $B(t_0)$, generate a plane.

PROPOSITION 4.4.1. *Let $\gamma \in \Omega$. If $\gamma(t_0)$ is a lightlike point of γ then the distance squared function f_v at t_0, has singularity of type $A_{\leq 2}$.*

Proof. *Consider $f_v(t) = \langle \gamma(t) - v, \gamma(t) - v\rangle$ the distance squared function on γ.*

Therefore, at a lightlike point $\gamma(t_0)$, $f_v^{(3)}(t_0) = 6\langle \gamma'(t_0), \gamma''(t_0)\rangle \neq 0$ since $\gamma \in \Omega$. Thus, $f_v(t_0)$ is not a singularity of type $A_{\geq 3}$. ∎

THEOREM 4.4.2. *Let $\gamma \in \Omega$. Then,*

(1)the focal surface $\mathfrak{B}$, of γ, intersects the curve γ only at lightlike points, $\gamma(t_0)$.

(2)the focal surface $\mathfrak{B}$ is regular at $\gamma(t_0)$.

(3)the tangent line to the curve at $\gamma(t_0)$ is contained in the tangent plane to $\mathfrak{B}$ at such point, that is, the unique lightlike direction of the tangent plane of $\mathfrak{B}$ at $\gamma(t_0)$ is the tangent line of γ at $\gamma(t_0)$.

(4)the LD set of the focal surface is the normal line to the curve passing by $\gamma(t_0)$, that is a smooth curve, and splits the focal surface into a Riemannian and a Lorentzian region.

Proof. *(1) Since $\gamma \in \Omega$, we can solve the equation (4.2) at a lightlike point $\gamma(t_0)$, to obtain*

$$\lambda(t) = \frac{\langle \gamma'(t), \gamma'(t)\rangle}{\langle \gamma'(t) \wedge \gamma''(t), \gamma'(t) \wedge \gamma''(t)\rangle}$$

at t near of t_0. Then $\lambda(t_0) = 0$ and $\mathfrak{B}(t_0, 0) = \gamma(t_0)$.

(2) From the Proposition 4.4.1, we have that at $\gamma(t_0)$ the curve has only singularity of type A_2. Thus by the Theorem 2.2.2 the focal surface is locally diffeomorphic to a plane, i.e, the focal surface is regular.

(3) It is enough to show that $\gamma'(t_0)$ belongs to the tangent plane of the surface at $\gamma(t_0)$, that is generated by $\dfrac{\partial \mathfrak{B}}{\partial t}(t_0, 0) = 3\gamma'(t_0)$ and $\dfrac{\partial \mathfrak{B}}{\partial \mu}(t_0, 0) = -\left(\gamma'(t_0) \wedge \gamma''(t_0)\right)$.

Then, the vectors of the tangent plane to the surface at $\gamma(t_0)$, are given by:

$$v = 3\lambda_1\gamma'(t_0) - \lambda_2\left(\gamma'(t_0) \wedge \gamma''(t_0)\right),$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. Taking $\lambda_1 = \dfrac{1}{3}$ and $\lambda_2 = 0$, we have that $\gamma'(t_0)$ belongs to tangent plane to the surface at $\gamma(t_0)$ and furthermore we have that $\langle v, v\rangle = \lambda_2^2\langle \gamma'(t_0), \gamma''(t_0)\rangle^2 \geq 0$. Thus $\gamma'(t_0)$ is the unique lightlike direction of the tangent plane, i.e. the tangent plane at $\gamma(t_0)$ is lightlike.

(4) As $\mathfrak{B}(t_0,\mu)=\gamma(t_0)-\mu N(t_0)$*, then the normal line of* γ *is contained in the focal surface. Furthermore, by the Proposition 3.3.4 the focal surface concern to the spacelike (timelike) side of the curve is timelike (spacelike, resp.) Thus the LD is contained in the normal line,* $\gamma(t_0)-\mu N(t_0)$.

Being thus, we are interested in knowing the induced metric along this normal line. We have that, $\dfrac{\partial\mathfrak{B}}{\partial t}(t_0,\mu)=3\gamma'(t_0)-\mu\left(\gamma'(t_0)\wedge\gamma'''(t_0)\right)$ *and* $\dfrac{\partial\mathfrak{B}}{\partial\mu}(t_0,\mu)=-\left(\gamma'(t_0)\wedge\gamma''(t_0)\right)$.

Therefore, the vectors of the tangent planes at points of normal line are given by:

$$v=\lambda_1(3\gamma'(t_0)-\mu\left(\gamma'(t_0)\wedge\gamma'''(t_0)\right))-\lambda_2\left(\gamma'(t_0)\wedge\gamma''(t_0)\right).$$

For $\mu=0$ *we have by Proposition 4.4.2 (3) that the tangent plane at* $\gamma(t_0)$ *is lightlike.*

Now we suppose that $\mu\neq 0$*, then*

$$\langle v,v\rangle=\lambda_1^2\mu^2\langle\gamma'(t_0)\wedge\gamma'''(t_0),\gamma'(t_0)\wedge\gamma'''(t_0)\rangle+2\lambda_1\lambda_2\mu\langle\gamma'(t_0)\wedge\gamma'''(t_0),\gamma'(t_0)\wedge\gamma''(t_0)\rangle+ \\ \lambda_2^2\langle\gamma'(t_0)\wedge\gamma''(t_0),\gamma'(t_0)\wedge\gamma''(t_0)\rangle.$$

Making $\langle v,v\rangle=0$*, we think in the above equation as a quadratic equation of* λ_1*, then* $\Delta=4\mu^2\lambda_2^2\langle\gamma'(t_0),\gamma''(t_0)\rangle^2\langle\gamma'(t_0),\gamma'''(t_0)\rangle^2-4\mu^2\lambda_2^2\langle\gamma'(t_0),\gamma''(t_0)\rangle^2\langle\gamma'(t_0),\gamma'''(t_0)\rangle^2$.

Thus, all tangent plane has a unique lightlike direction, given by

$$(\lambda_1,\lambda_2)=(-\frac{\lambda_2\langle\gamma'(t_0)\wedge\gamma'''(t_0),\gamma'(t_0)\wedge\gamma''(t_0)\rangle}{\mu\langle\gamma'(t_0)\wedge\gamma'''(t_0),\gamma'(t_0)\wedge\gamma'''(t_0)\rangle},\lambda_2)=(-\lambda_2\frac{\langle\gamma'(t_0),\gamma''(t_0)\rangle}{\mu\langle\gamma'(t_0),\gamma'''(t_0)\rangle},\lambda_2),$$

with $\dfrac{\langle\gamma'(t_0),\gamma''(t_0)\rangle}{\langle\gamma'(t_0),\gamma'''(t_0)\rangle}\neq 0$ *for* $\gamma\in\Omega$*. Therefore the induced metric on these planes is degenerated and the normal line is the LD set of the focal surface.* ∎

Remark 4.4.3. *Along the LD set, we have*

$$\frac{\partial\mathfrak{B}}{\partial t}(t_0,\mu)\wedge\frac{\partial\mathfrak{B}}{\partial\mu}(t_0,\mu)=\mu(-\langle\gamma'(t_0)\wedge\gamma'''(t_0),\gamma''(t_0)\rangle)\gamma'(t_0),$$

then the tangent only is not defined along the LD if $\dfrac{\partial\mathfrak{B}}{\partial t}(t_0,\mu)\wedge\dfrac{\partial\mathfrak{B}}{\partial\mu}(t_0,\mu)=0$*, i.e, if* $\langle\gamma'(t_0)\wedge\gamma'''(t_0),\gamma''(t_0)\rangle=0$*, that generically does not occur for a curve in* Ω*. Thus, the LD does not intersect the cuspidal curve of the focal surface.*

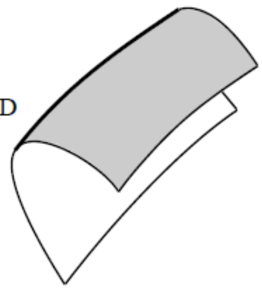

FIG. 3. Metric structure of the focal surface locally at a lightlike point of γ.

5. FOCAL SET OF CURVES IN S_1^2

In this section, we study the focal set in the *de Sitter space* $S_1^2 \subset \mathbb{R}_1^3$, which we call spherical focal curve. For obtain the results for curves in the de Sitter space, we have as motivation the Section 4.

Let $\gamma : I \to S_1^2$ be a spacelike or a timelike smooth and regular curve in S_1^2 parametrised by arc length. For this curve, consider the orthonormal basis $\{\gamma(s), t(s) = \gamma'(s), n(s) = \gamma(s) \wedge t(s)\}$ of $\mathbb{R}_1^3$ along γ. By standard arguments, we have the following Frenet-Serret type formulae:

$$\begin{cases} \gamma'(s) = t(s) \\ t'(s) = -\varepsilon(\gamma(s))\,\gamma(s) + \delta(\gamma(s))\,k_g(s)\,n(s) \\ n'(s) = -\varepsilon(\gamma(s))\,k_g(s)\,t(s) \end{cases},$$

where $\varepsilon(\gamma(s)) = sign(t(s))$, $\delta(\gamma(s)) = sign(n(s))$ and $k_g(s) = \langle \gamma''(s), n(s)\rangle$ is the geodesic curvature of γ at s.

As we saw previously, consider the family of distance squared functions $f : I \times S_1^2 \to \mathbb{R}$ on γ, given by

$$f(s, v) = \langle \gamma(s) - v, \gamma(s) - v\rangle.$$

We denote by $f_v : I \to \mathbb{R}$ the function given by $f_v(s) = f(s, v)$, for some $v \in S_1^2$ fixed.

The *spherical bifurcation set* of f is given by

$$\mathfrak{Bif}(f) = \{v \in S_1^2 \mid f_v'(s) = f_v''(s) = 0 \text{ in } (s, v) \text{ for some } s\},$$

i.e, the directions where the singularity of f_v at s is at least A_2. The *spherical focal curve of* γ is given by the spherical bifurcation set of f. Furthermore, for curves in $S_1^2 \subset \mathbb{R}_1^3$, the spherical focal curve is the intersection of the focal surface in $\mathbb{R}_1^3$ with the de Sitter space S_1^2. Observe that if $v \in S_1^2$ then $-\dfrac{1}{2} f_v(s) = \langle \gamma(s), v\rangle - 1$ and the singularities of the distance squared function and of the height function are the same. Then the evolute of a curve in S_1^2, coincide with the spherical focal curve of γ. In [8] the authors study the evolutes of hyperbolic plane curves, that is, a curve in $H^2(-1)$. The evolutes there also coincide with the bifurcation set in $H^2(-1)$.

For a spacelike or a timelike curve γ parametrised by arc length with $k_g(s) \neq 0$, we have that the spherical focal curve of γ is given by

$$\alpha^{\pm}(s) = \mathfrak{B}(s, \mu(s)) = \pm \frac{k_g(s)}{\sqrt{k_g^2(s) + \delta(\gamma(s))}} \gamma(s) \pm \frac{\varepsilon(\gamma(s))}{\sqrt{k_g^2(s) + \delta(\gamma(s))}} n(s).$$

REMARK 5.5.1. *To define the spherical focal curve, we must have $k_g^2(s) + \delta(\gamma(s)) > 0$. Then in the case that γ is spacelike, we must have $k_g(s) < -1$ or $k_g(s) > 1$ and in the case that γ is timelike, the spherical focal curve is always defined. Furthermore, as $\alpha^-(s) = -\alpha^+(s)$ we work only with $\alpha^+(s)$.*

PROPOSITION 5.5.2. *The singular points of α^+ are given by $S_1^2 \cap \mathcal{C}$, where $\mathcal{C}$ is the cuspidal curve.*

Proof. Observe that f_v has singularity $A_{\geq 3}$ at s_0 if and only if $k_g'(s_0) = 0$ if and only if s_0 is a singular point of the curve α^+. ▮

On the next proposition we study the metric structure of the spherical focal curve of a spacelike curve and of a timelike curve.

PROPOSITION 5.5.3. *Away from the singular points,*

(a)the spherical focal curve of a spacelike curve is timelike;

(b)the spherical focal curve of a timelike curve is spacelike.

Proof.
(a) Let γ be a spacelike curve, then $n(s)$ is a timelike vector. Thus the spherical focal curve is given by

$$\alpha^+(s) = \frac{k_g(s)}{\sqrt{k_g^2(s) - 1}} \gamma(s) + \frac{1}{\sqrt{k_g^2(s) - 1}} n(s).$$

Now,

$$(\alpha^+)'(s) = \frac{-k_g'(s)}{(k_g^2(s) - 1)\sqrt{k_g^2(s) - 1}} \gamma(s) - \frac{k_g(s) k_g'(s)}{(k_g^2(s) - 1)\sqrt{k_g^2(s) - 1}} n(s).$$

Observe that at the points $s_0 \in I$ where $k_g'(s_0) = 0$, we have $(\alpha^+)'(s_0) = 0$, i.e, s_0 is a singular point of α^+. Then away from the singular points of α^+, i.e, where $k_g'(s) \neq 0$ we have that α^+ is a timelike curve because

$$\langle (\alpha^+)'(s), (\alpha^+)'(s) \rangle = \frac{-(k_g')^2(s)}{(k_g^2(s) - 1)^2} < 0.$$

(b) This prove is analogous to the case (a). ▮

We want to know what is happening at the lightlike points of this curve. For this, let us find a expression of the bifurcation set. Here we can not consider $\gamma : I \to S_1^2$ parametrised by arc length.

Consider the orthogonal basis $\gamma(t), \mathbf{t}(t) = \gamma'(t), N(t) = \gamma(t) \wedge \mathbf{t}(t)$ for $\mathbb{R}_1^3$ along of γ and consider still the family of distance squared functions $f : I \times S_1^2 \to \mathbb{R}$ on γ. For definition, we have that the bifurcation set of f is given by

$$\mathfrak{Bif}^{\pm}(f) = \{\pm\sqrt{1+\mu^2\langle\gamma'(t),\gamma'(t)\rangle}\gamma(t) + \mu N(t)\,|\,\mu \text{ is solution of the equation } (1^{\pm})\},$$

where $\mu\langle\gamma(t)\wedge\gamma'(t),\gamma''(t)\rangle \pm \sqrt{1+\mu^2\langle\gamma'(t),\gamma'(t)\rangle}\langle\gamma(t),\gamma''(t)\rangle = 0. \quad (1^{\pm})$

If $\mu(t)$ is solution of (1^+) then, $-\mu(t)$ is solution of (1^-) or vice-versa. We denote by

$$\alpha^+(t) = \sqrt{1+\mu^2(t)\langle\gamma'(t),\gamma'(t)\rangle}\gamma(t) + \mu(t)N(t) \text{ and}$$
$$\alpha^-(t) = -\sqrt{1+\mu^2(t)\langle\gamma'(t),\gamma'(t)\rangle}\gamma(t) - \mu(t)N(t).$$

The spherical focal curve $\mathfrak{Bif}^{\pm}(f) = \alpha^+ \cup \alpha^-$, where α^+ and α^- are symmetric.

Proposition 5.5.4. *The spherical focal curve α^+ intersects the curve γ at a lightlike point of γ and have the same tangent line at this point, but α^+ is not regular at this point. The curve α^- does not intersect the curve γ, but it has the same geometry of α^+, by symmetry.*

Proof. Let $\gamma(t_0)$ be a lightlike point of γ. The parametrisation of the spherical focal curve locally at t_0, is $\alpha^+(t) = \sqrt{1+\mu^2(t)\langle\gamma'(t),\gamma'(t)\rangle}\gamma(t) + \mu(t)n(t)$.

Solving the equation (1^+) and using the fact that $\langle\gamma(t),\gamma''(t)\rangle = -\langle\gamma'(t),\gamma'(t)\rangle$, it follows that $\mu(t)$ is

$$\frac{\sqrt{\langle\gamma'(t),\gamma'(t)\rangle^2}}{\sqrt{\langle\gamma(t)\wedge\gamma'(t),\gamma''(t)\rangle^2 - \langle\gamma'(t),\gamma'(t)\rangle^3}} \quad \text{or} \quad -\frac{\sqrt{\langle\gamma'(t),\gamma'(t)\rangle^2}}{\sqrt{\langle\gamma(t)\wedge\gamma'(t),\gamma''(t)\rangle^2 - \langle\gamma'(t),\gamma'(t)\rangle^3}}$$

Thus $\mu(t_0) = 0$, since $\langle\gamma'(t_0),\gamma'(t_0)\rangle = 0$ and then $\alpha^+(t_0) = \gamma(t_0)$. Therefore, γ and α^+ intersect at $\gamma(t_0)$.

We prove now that α^+ and γ have the same tangent line at $\gamma(t_0)$. Suppose without loss of generality that γ is spacelike for $t > t_0$ and γ is timelike for $t < t_0$. Observe that we can write $\mu(t)$ (or $-\mu(t)$) equal to

$$\begin{cases} \mu_1(t) = \dfrac{\langle\gamma'(t),\gamma'(t)\rangle}{\sqrt{\langle\gamma(t)\wedge\gamma'(t),\gamma''(t)\rangle^2 - \langle\gamma'(t),\gamma'(t)\rangle^3}}, & t \geq t_0; \\[2ex] \mu_2(t) = \dfrac{-\langle\gamma'(t),\gamma'(t)\rangle}{\sqrt{\langle\gamma(t)\wedge\gamma'(t),\gamma''(t)\rangle^2 - \langle\gamma'(t),\gamma'(t)\rangle^3}}, & t < t_0. \end{cases}$$

Thus,

$$a = \lim_{t\to t_0{}^+} \mu'(t) = \lim_{t\to t_0{}^+} \mu_1'(t) = \frac{2\langle\gamma'(t_0),\gamma''(t_0)\rangle}{\sqrt{\langle\gamma(t_0)\wedge\gamma'(t_0),\gamma''(t_0)\rangle}^2} \quad \textit{and}$$
$$b = \lim_{t\to t_0{}^-} \mu'(t) = \lim_{t\to t_0{}^-} \mu_2'(t) = \frac{-2\langle\gamma'(t_0),\gamma''(t_0)\rangle}{\sqrt{\langle\gamma(t_0)\wedge\gamma'(t_0),\gamma''(t_0)\rangle}^2}.$$

Since the curve γ is in Ω, we have $\langle\gamma'(t_0),\gamma''(t_0)\rangle \neq 0$ and then $a \neq b$. Thus, we have that $\lim\limits_{t\to t_0^+} (\alpha^+)'(t) = \gamma'(t_0) + aN(t_0)$ and $\lim\limits_{t\to t_0^-} (\alpha^+)'(t) = \gamma'(t_0) + bN(t_0)$. Since $\gamma'(t_0)$ and $n(t_0)$ are parallel to the same lightlike vector, we have that the tangent vector to the curve α^+ when t tends to t_0 is tending to the same line whose vector direction is $\gamma'(t_0)$. We have then that $\alpha^+(t)$ is the union of two parts of regular curves at $\alpha(t_0)$:

$\sqrt{1+\mu_1^2(t)\langle\gamma'(t),\gamma'(t)\rangle}\gamma(t)+\mu_1(t)N(t)$ and $\sqrt{1+\mu_2^2(t)\langle\gamma'(t),\gamma'(t)\rangle}\gamma(t)+\mu_2(t)N(t)$. ∎

Here, we have an example of a spherical focal curve α^+, smooth for pieces, generated by $\gamma(t) = (t^2-t, t^2+t, \sqrt{1-4t^3})$. We use the software Maple to get the expression of α^+ (that we omit here) and the bellow figure.

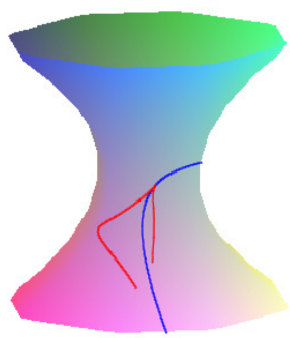

FIG. 4. Example of a spherical focal curve α^+ (The curve γ is blue and α^+ is red).

REMARK 5.5.5. *The curve α^- does not intersect the curve γ, but α^- intersects $-\gamma$ at the lightlike point $-\gamma(t_0)$. The bifurcation set of γ and of $-\gamma$ are the same.*

6. FOCAL SET OF CURVES IN S_1^3

In this section, we consider curves in *de Sitter space* $S_1^3 \subset \mathbb{R}_1^4$. Furthermore, we study the focal set of these curves. Let $\gamma : I \to S_1^3$ be a smooth and regular curve in S_1^3. In the case that the curve is spacelike or timelike, we can parametrise it by arc length s. Thus, for spacelike curve, we take the unit tangent vector $t(s) = \gamma'(s)$. Suppose that $\langle t'(s), t'(s)\rangle \neq 1$, then $\| t'(s) + \gamma(s) \| \neq 0$, and we have other unit vector $n(s) = \dfrac{t'(s)+\gamma(s)}{\| t'(s)+\gamma(s) \|}$. We define also an unit vector by $e(s) = \gamma(s) \wedge t(s) \wedge n(s)$, then we have an orthonormal basis $\{\gamma(s), t(s), n(s), e(s)\}$ of $\mathbb{R}_1^4$ along γ. The Frenet-Serret type formulae is given by

$$\begin{cases} \gamma'(s) = t(s) \\ t'(s) = -\gamma(s) + k_g(s)\, n(s) \\ n'(s) = -\delta(\gamma(s))\, k_g(s)\, t(s) + \tau_g(s)\, e(s) \\ e'(s) = \tau_g(s) n(s) \end{cases},$$

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where $\delta(\gamma(s)) = sign(n(s))$, $k_g(s) = \parallel t'(s) + \gamma(s) \parallel$ and $\tau_g(s) = \dfrac{\delta(\gamma(s))}{k_g^2(s)} \det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))$, where det is the determinant of the 4×4 matrix. Here k_g is called geodesic curvature and τ_g geodesic torsion of γ (see [5]).

Since $\langle t'(s) + \gamma(s), t'(s) + \gamma(s) \rangle = \langle t'(s), t'(s) \rangle - 1$, the condition $\langle t'(s), t'(s) \rangle \neq 1$ is equivalent to the condition $k_g(s) \neq 0$.

If the curve is timelike, we take the unit tangent vector $t(s) = \gamma'(s)$. By Supposing a generic condition $\langle t'(s), t'(s) \rangle \neq 1$, then $\parallel t'(s) - \gamma(s) \parallel \neq 0$, and we have other unit vector $n(s) = \dfrac{t'(s) - \gamma(s)}{\parallel t'(s) - \gamma(s) \parallel}$. We also define an unit vector by $e(s) = \gamma(s) \wedge t(s) \wedge n(s)$, then we have an orthonormal basis $\{\gamma(s), t(s), n(s), e(s)\}$ of $\mathbb{R}^4_1$ along γ. Thus, the Frenet-Serret type formulae is given by

$$\begin{cases} \gamma'(s) = t(s) \\ t'(s) = \gamma(s) + k_h(s)\, n(s) \\ n'(s) = k_h(s)\, t(s) + \tau_h(s)\, e(s) \\ e'(s) = -\tau_h(s)\, n(s) \end{cases},$$

where $k_h(s) = \parallel t'(s) - \gamma(s) \parallel$ and $\tau_g(s) = -\dfrac{1}{k_h^2(s)} \det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))$. Here k_h is called hyperbolic curvature and τ_h hyperbolic torsion of γ (see [7]).

Since $\langle t'(s) - \gamma(s), t'(s) - \gamma(s) \rangle = \langle t'(s), t'(s) \rangle - 1$, the condition $\langle t'(s), t'(s) \rangle \neq 1$ is equivalent to the condition $k_h(s) \neq 0$.

Consider the family of distance squared functions, $f : I \times S^3_1 \to \mathbb{R}$, on γ given by

$$f(s, v) = \langle \gamma(s) - v, \gamma(s) - v \rangle.$$

We denote by $f_v : I \to \mathbb{R}$ the function given by $f_v(s) = f(s, v)$, for some $v \in S^3_1$ fixed. Observe that if $v \in S^3_1$, then $-\dfrac{1}{2} f_v(s) = \langle \gamma(s), v \rangle - 1$ and the singularities of the distance squared function and the height function are the same.

The *spherical bifurcation set* of f is given by

$$\mathfrak{Bif}(f) = \{v \in S^3_1 \mid f'_v(s) = f''_v(s) = 0 \text{ at } (s, v) \text{ for some } s\},$$

i.e, the directions where the singularity of f_v at s is $A_{\geq 2}$.

The spherical focal surface of γ coincide with the spherical bifurcation set of f. Furthermore, for curves in $S^3_1 \subset \mathbb{R}^4_1$ the spherical focal surface is the intersection of the focal hypersurface in $\mathbb{R}^4_1$ with the de Sitter space S^3_1.

For a spacelike curve γ parametrised by arc length with $k_g(s) \neq 0$, we have that the spherical focal of γ is given by

$$\mathfrak{B}^{\pm}(s, \mu) = \mu\gamma(s) + \frac{\mu}{\delta(\gamma(s)) k_g(s)} n(s) \pm \frac{\sqrt{-\delta(\gamma(s)) k_g^2(s) + \delta(\gamma(s)) \mu^2 (k_g^2(s) + \delta(\gamma(s))}}{k_g(s)} e(s),$$

with $\mu \in \mathbb{R}$. The *g-spherical cuspidal curve* is given by $\mathfrak{B}^{\pm}(s, \mu(s)) = \mathfrak{B}^{\pm}(s)$, where

$$\mu(s) = \frac{\pm \tau_g(s) k_g^2(s)}{\sqrt{\tau_g^2(s) k_g^4(s) - k_g'^2(s) \delta(\gamma(s)) - \tau_g^2(s) k_g^2(s) \delta(\gamma(s))}}.$$

For a timelike curve γ parametrised by arc length with $k_h(s) \neq 0$ the spherical focal surface is given by

$$\mathfrak{B}^{\pm}(s,\mu) = \mu\gamma(s) - \frac{\mu}{k_h(s)}n(s) \pm \frac{\sqrt{k_h^2(s) - \mu^2(k_h^2(s)+1)}}{k_h(s)}e(s),$$

with $\mu \in \mathbb{R}$. The *h-spherical cuspidal curve* is given by $\mathfrak{B}^{\pm}(s,\mu(s)) = \mathfrak{B}^{\pm}(s)$, where

$$\mu(s) = \frac{\pm\tau_h(s)k_h^2(s)}{\sqrt{\tau_h^2(s)k_h^4(s) + k_h'^2(s) + \tau_h^2(s)k_h^2(s)}}.$$

REMARK 6.6.1. *The spherical focal surface of a spacelike curve is defined if* $-\delta(\gamma(s))k_g^2(s) + \delta(\gamma(s))\mu^2(k_g^2(s) + \delta(\gamma(s)) \geq 0$. *As* $k_g(s) \neq 0$, *we have that* $n(s)$ *is spacelike or timelike. In the case that* $n(s)$ *is spacelike, the spherical focal surface is defined for*

$$\mu \leq -\frac{k_g(s)}{\sqrt{k_g^2(s)+1}} \ \text{or} \ \mu \geq \frac{k_g(s)}{\sqrt{k_g^2(s)+1}},$$

otherwise in the case that $n(s)$ *is timelike, the spherical focal surface is defined for*

$$-\frac{k_g(s)}{\sqrt{k_g^2(s)-1}} \leq \mu \leq \frac{k_g(s)}{\sqrt{k_g^2(s)-1}}.$$

The spherical focal surface of a timelike curve is defined if

$$-\frac{k_h(s)}{\sqrt{k_h^2(s)+1}} \leq \mu \leq \frac{k_h(s)}{\sqrt{k_h^2(s)+1}}.$$

Furthermore, in both cases $\mathfrak{B}^+(s,\mu)$ *and* $\mathfrak{B}^-(s,\mu)$ *are symmetric then we study only* $\mathfrak{B}^+(s,\mu)$.

We prove in the next results that the tangent plane of the spherical focal surface is not defined at the points of the g-spherical cuspidal curve and of the h-spherical cuspidal curve. Away from the g-spherical cuspidal curve,

$$\mu(s) \neq \frac{\pm\tau_g(s)k_g^2(s)}{\sqrt{\tau_g^2(s)k_g^4(s) - k_g'^2(s)\delta(\gamma(s)) - \tau_g^2(s)k_g^2(s)\delta(\gamma(s))}},$$

or away from the h-spherical cuspidal curve, $\mu(s) \neq \frac{\pm\tau_h(s)k_h^2(s)}{\sqrt{\tau_h^2(s)k_h^4(s) + k_h'^2(s) + \tau_h^2(s)k_h^2(s)}}$, $v = \lambda_1\mathfrak{B}_s^+ + \lambda_2\mathfrak{B}_\mu^+$, with $\lambda_1, \lambda_2 \in \mathbb{R}$, are the vectors of the tangent plane of the spherical focal surface at $\mathfrak{B}^+(s,\mu)$ and $\langle v,v\rangle = \lambda_1^2\langle\mathfrak{B}_s^+,\mathfrak{B}_s^+\rangle + 2\lambda_1\lambda_2\langle\mathfrak{B}_s^+,\mathfrak{B}_\mu^+\rangle + \lambda_2^2\langle\mathfrak{B}_\mu^+,\mathfrak{B}_\mu^+\rangle$, by using the respective parametrisation of $\mathfrak{B}(s,\mu)$ for spacelike or timelike γ. Away from these curves we analyse the metric structure of the spherical focal surface.

PROPOSITION 6.6.2. *Let* γ *be a spacelike curve.*

(a)The tangent plane of the spherical focal surface of γ *is not defined at the g-spherical cuspidal curve.*

(b)Away from the g-spherical cuspidal curve, the spherical focal surface of γ *is timelike.*

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Proof. *(a) Considering a spacelike curve, the tangent plane at the points of the spherical focal surface is generated by the vectors*

$$\mathfrak{B}_s^+(s,\mu) = \left(\frac{-\mu k_g'(s) + \delta(\gamma(s))\tau_g(s)k_g(s)\sqrt{-\delta(\gamma(s))k_g^2(s) + \delta(\gamma(s))\mu^2(k_g^2(s) + \delta(\gamma(s))}}{\delta k_g^2(s)}\right) n(s)$$
$$+ \left(\frac{\mu\tau_g(s)k_g(s)\sqrt{-\delta(\gamma(s))k_g^2(s) + \delta(\gamma(s))\mu^2(k_g^2(s) + \delta(\gamma(s)))} - \delta(\gamma(s)\mu^2 k_g'(s))}{\delta(\gamma(s))k_g^2(s)\sqrt{-\delta(\gamma(s))k_g^2(s) + \delta(\gamma(s))\mu^2(k_g^2(s) + \delta(\gamma(s)))}}\right) e(s)$$

and

$$\mathfrak{B}_\mu^+(s,\mu) = \gamma(s) + \frac{1}{\delta(\gamma(s))k_g(s)} n(s) + \left(\frac{\delta(\gamma(s))\mu(k_g^2(s) + \delta(\gamma(s)))}{k_g(s)\sqrt{-\delta(\gamma(s))k_g^2(s) + \delta(\gamma(s))\mu^2(k_g^2(s) + \delta(\gamma(s))}}\right) e(s).$$

The vectors $\{\mathfrak{B}_s^+, \mathfrak{B}_\mu^+\}$ *are linearly dependent if and only if*

$$\mu(s) = \frac{\pm\tau_g(s)k_g^2(s)}{\sqrt{\tau_g^2(s)k_g^4(s) - k_g'^2(s)\delta(\gamma(s)) - \tau_g^2(s)k_g^2(s)\delta(\gamma(s))}}$$

that is precisely where the tangent plane is not defined and furthermore is where f_v *has singularities of type* $A_{\geq 3}$*, that is the g-spherical cuspidal curve.*

(b) Let γ *be a spacelike curve. Let us suppose that* $n(s)$ *is spacelike and* $e(s)$ *is timelike, thus making* $\langle v, v\rangle = 0$*, and thinking in this equation as a quadratic equation, then*

$$\Delta = 4\lambda_2^2 \frac{(\tau_g k_g \sqrt{-k_g^2 + \mu^2(k_g^2+1)} - \mu k_g')^2}{k_g^2(-k_g^2 + \mu^2(k_g^2+1))}.$$

The tangent plane generated by $\mathfrak{B}_s^+$ *and* $\mathfrak{B}_\mu^+$ *can be lightlike, if* $\Delta = 0$*. As we are supposing* $\tau_g(s)k_g(s)\sqrt{-k_g^2(s) + \mu^2(k_g^2(s)+1)} - \mu k_g'(s) \neq 0$*, then we have* $\Delta = 0$ *if and only if* $\lambda_2 = 0$*, that is, if* $\mathfrak{B}_s^+$ *is lightlike. But,* $\mathfrak{B}_s^+(s,\pm 1)$ *are the lightlike vectors of the plane. Besides* $\mathfrak{B}_\mu^+(s,\pm 1)$ *are timelike vectors, i.e, the tangent plane is timelike. Therefore, we have* $\Delta > 0$ *and thus the spherical focal surface is timelike.* ∎

Proposition 6.6.3. *Let* γ *be a timelike curve.*

(a)The tangent plane of the spherical focal surface of γ *is not defined at the h-spherical cuspidal curve.*

(b)Away from the h-spherical cuspidal curve, the spherical focal surface of γ *is spacelike.*

Proof. *The proves are analogous to Proposition 6.6.2. In case (b),* $\mathfrak{B}_s^+(s,\pm 1)$ *are not defined then* $\Delta < 0$*.* ∎

Our aim now is to find the general expression of the spherical focal surface to know what is happening with the surface at a lightlike point of γ. For this, consider the curve γ not parametrised by arc length and suppose that there exist a vector $N(t)$ such that $\{\gamma(t), \mathbf{t}(t) = \gamma'(t), N(t), E(t) = \gamma(t) \wedge \mathbf{t}(t) \wedge N(t)\}$ is an orthogonal basis. For definition, we have that the spherical focal surface of γ is given by

$$\mathfrak{B}^{\pm}(t, \mu) = \mu\gamma(t) + \beta N(t) + \lambda(t) E(t),$$

where $\mu \in \mathbb{R}$, β and λ satisfies the bellow equations:

$$\lambda = \frac{\mu\langle\gamma'(t), \gamma'(t)\rangle - \beta\langle\gamma''(t), N(t)\rangle}{\langle\gamma''(t), E(t)\rangle} \qquad \textit{and}$$

$$\beta = \left(\frac{\mu\langle\gamma',\gamma'\rangle\langle\gamma'',N\rangle\langle E,E\rangle \pm \sqrt{\langle\gamma'',N\rangle^2\langle E,E\rangle\langle\gamma'',E\rangle^2(1-\mu^2) + \langle N,N\rangle\langle\gamma'',E\rangle^4(1-\mu^2) - \langle N,N\rangle\langle\gamma'',E\rangle^2\langle\gamma',\gamma'\rangle^2\langle E,E\rangle\mu^2}}{\langle\gamma'',E\rangle^2\langle N,N\rangle^2 + \langle E,E\rangle\langle\gamma'',N\rangle^2} \right)(t)$$

REMARK 6.6.4. *The spherical focal surface is well defined near a lightlike point $\gamma(t_0)$. Let $R(t, \mu) = A(t)\mu^2 + B(t)$, the term inside the squared root of the above β, where*

$$A(t) = (-\langle\gamma'', N\rangle^2\langle E, E\rangle\langle\gamma'', E\rangle^2 - \langle N, N\rangle\langle\gamma'', E\rangle^4 - \langle N, N\rangle\langle\gamma'', E\rangle^2\langle\gamma', \gamma'\rangle^2\langle E, E\rangle)(t),$$

$B(t) = (\langle\gamma'', E\rangle^2\langle\gamma'', N\rangle^2\langle E, E\rangle + \langle N, N\rangle\langle\gamma'', E\rangle^4)(t)$. *Then for the spherical focal surface be defined, we must have $R(t, \mu) \geq 0$. Let $\gamma(t_0)$ be a lightlike point of γ, then at the timelike part in the neighborhood of this lightlike point, we show that $A(t) < 0$ and $B(t) > 0$; at the lightlike point $\gamma(t_0)$ we have $A(t_0) < 0$ and $B(t_0) > 0$, and in this case the spherical focal surface is defined only for $-1 \leq \mu \leq 1$. Thus for continuity, at the spacelike part near the $\gamma(t_0)$ also we will have $A(t) < 0$ and $B(t) > 0$.*

PROPOSITION 6.6.5. *The spherical focal surface intersects the curve γ at a lightlike point and furthermore, the tangent plane to the spherical focal surface is not defined in this point.*

Proof.
Let $\gamma(t_0)$ be a lightlike point of γ. Then, $\beta(t_0, 1) = 0$ and thus $\lambda(t_0, 1) = 0$. Therefore, $\mathfrak{B}^{\pm}(t_0, 1) = \gamma(t_0)$. Since $R(t_0, 1) = 0$, then the tangent plane to the spherical focal surface at $\mathfrak{B}^{\pm}(t_0, 1)$ are not defined. ∎

We observe that $\mathfrak{B}^{+}(t_0, -1) = \mathfrak{B}^{-}(t_0, -1) = -\gamma(t_0)$ and the bifurcation set of γ and $-\gamma$ are the same. Furthermore $R(t_0, -1) = 0$, then the tangent planes to the spherical focal surface also are not defined at these points. Then, we have the next result.

PROPOSITION 6.6.6. *The LD set of the spherical focal surface of γ are the curves $\mathfrak{B}^{\pm}(t_0, \mu)$, $-1 < \mu < 1$.*

Proof. The tangent planes at $\mathfrak{B}^{\pm}(t_0, \mu)$ exist for $-1 < \mu < 1$. For Propositions 6.6.2 and 6.6.3, the metric structure of the spherical focal surface is given by a spacelike part and a timelike part separated by the curves $\mathfrak{B}^{\pm}(t_0, \mu)$, $-1 < \mu < 1$. ∎

ACKNOWLEDGEMENTS

The authors would like to thank Farid Tari for suggesting the problem investigated here and for useful discussions and comments.

The first author would like to thank very much the FAPESP grant 2013/02794-4. The second author would like to thank very much all FAPESP support given during the development of the PhD project, grant number 2010/20301-7.

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Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC