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Singular levels and topological invariants of Morse Bott systems on surfaces

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We classify up to homeomorphisms closed curves and eights of saddle points on orientable closed surfaces. This classification is applied to Morse Bott foliations and Morse Bott integrable systems allowing us to define a complete invariant. We state also a realization Theorem based in two transformations and one generator (the foliation of the sphere with two centers). May, 2014
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1. INTRODUCTION

The research on topological invariants of flows $X(\Sigma)$ and foliations $\mathcal{F}(\Sigma)$ on surfaces Σ has a long history. Some basic references are: [3], [13], [17] [16] and the references cited in these papers. An introduction to the subject can be also found in the book [15] and the relation with C^*-algebras is analyzed in [22] and [7]. At present a lot of efforts are devoted to singular foliations and integrable flows on manifolds of larger dimensions, see for instance [20], [21], [8], [24], [25] and [12]. Nevertheless, as we try to show in this paper, the two dimensional case is surprisingly incomplete.

Recall that two systems $X_1(\Sigma)$ and $X_2(\Sigma)$ are *topologically equivalent* if there exists a homeomorphism $h : \Sigma \to \Sigma$ that sends orbits of $X_1(\Sigma)$ into orbits of $X_2(\Sigma)$ preserving

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the sense of the orbits. To describe the equivalence classes it is useful to define a set of topological invariant.

We try to sketch the usual method of construction of invariants. The strategy lies on several initial reductions of the system. The first one consists of the construction of the space of orbits Σ/X (or $\Sigma/\mathcal{F}$, space of leaves in the case of a foliation), that is to say: two points belong to the same class if and only if they lie on the same orbit. If some orbits can be related, for instance their union is a one dimensional manifold, a reduced space of orbits, $(\Sigma/X, \sim)$ could be defined. Usually $(\Sigma/X, \sim)$ is a singular foliation. In a second step, a finite number of open regions in $(\Sigma/X, \sim)$ are defined. In each region the orbits has a homogeneous kind of behavior. They have the same asymptotic properties or define a parallel flow. Some kind of graph Γ whose vertices or edges are the homogeneous regions encodes the relation between these regions. To recover the structure of the initial system additional information is added to the graph, basically specifications of local flows or order. The graph and the additional information define the invariant.

If the additional information it is not enough to reconstruct the flow one gets an invariant that is not complete. A consideration usually overlooked is that if two systems are equivalent the topological type of two corresponding invariant sets must be the same. In Example 4.1, Figures 4.2, 4.3 we have two foliations with isomorphic space of leaves, but the foliations are not topologically equivalent because some singular leaves are not topologically equivalent.

Following the described guidelines, in section three we classify the basic leaves, closed curves and saddles with their separatrices according to their topological type. In this case, the topological equivalence is similar to the equivalence used to define knots on S^3. The particular case of closed curves in the torus is analyzed in [19], page 25. This classification can be applied to almost all flows and foliations on a surface and in fact is an independent part of the paper. This classification closes a remarkable gap in the study of two-dimensional systems.

If Γ determines the surface Σ, for instance by the number of cycles, the topological type of the basic leaves can be implicitly included in the invariant. This is particularly true for Morse Bott foliations (see section 2) and Morse Bott integrable systems. But this is not always the case:

Example 1.1. The orbit space Σ/X of the system defined on the 2-sphere and represented in Figure 1.1 contains four cycles. The sphere is not contractible to Σ/X.

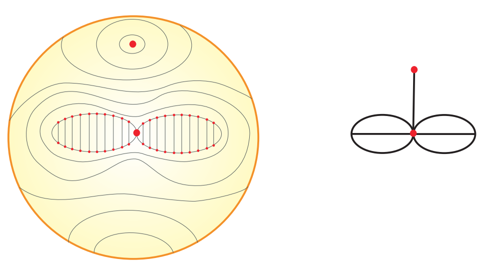

FIGURE 1.1: System with one center and its orbit space.

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The last two sections of the paper are devoted to Morse Bott systems. They are a natural generalization of Hamiltonian systems and Morse systems, see [3], and are present in many practical situations. In [6] all continuous flows without wandering points are classified, Morse Bott systems are systems without wandering points, without quasiminimal recurrence regions and with simple saddles.

Recall that for a vector field X on a manifold W, a *first integral* f is a real C^r, $r \geq 1$ function: $W \to R$ that is constant on orbits of X but not identically constant on open sets of W. If X has a first integrable f, all function functionally related with f are also first integrals; therefore, if needed we will denote by (X, f) the pair of a vector field and a particular first integral.

A vector field with a first integral that is a Morse Bott function has center equilibrium points associated to extreme values of f and saddle points associated to singularities of index 1 of f. Moreover it can have other equilibrium points. We restrict this possibility to a finite number of regular level sets filled by equilibrium points. The set of these systems will be denoted by $\psi_{\mathcal{MB}}(W)$.

In section four we define the invariants for Morse Bott foliations, see Definition 4.1 and Proposition 4.3 and prove that the invariant is complete, Theorem 4.1. In order to describe the invariant, we remind that in a saddle singularity, a family of pairs of regular circles, J_1, J_2 intersect and define an eight that is surrounded by other regular level sets J_3. These J_3 branch out in J_1, J_2. Roughly speaking, the invariant is ${}^{\Sigma}/_{\mathcal{F}}$ with the information of the sense of branch out. Then the complete invariant for an integrable Morse Bott system is the complete invariant for the Morse Bott foliation defined by the first integral with the information about the extra singularities of the flow, see Theorem 4.2. See again, Example 4.1 and Figures 4.2, 4.3. Consider the two systems obtained given a sense to the leaves, the invariant defined in [13] do not distinguish between them. On the other hand, the foliation in Figure 4.1 is not a Morse foliation.

The last section is devoted to a realization Theorem. The basic foliation will be the foliation on the sphere with only two center singularities. We define two transformations that allow us to construct all the systems from the basic foliation.

2. BACKGROUND

We consider a compact manifold W with the Riemannian metric induced by its standard immersion in $\mathbb{R}^k$ and let $f : W \to \mathbb{R}$, $f \in C^k$, $k \geq 2$.

Definition 2.1. A point $p \in W$ is called a *singular point* of f if the $rank(\mathrm{df}(p))$ is less than the maximum possible value. Otherwise p is a *regular point.* A value $b \in \mathbb{R}$ is called a *singular value* of f if the $f^{-1}(b)$ contains a singular point of f. Otherwise $b \in \mathbb{R}$ is called a *regular value.*

The collection of all singular points of f is called the *singular set* of f and is denoted by $Sing(f)$.

A *fiber* of f on W is each connected component of the level sets of f. A connected component of the level set that contains a singular point of f is called a *singular fiber*. Given a real valued twice continuously differentiable function on a compact manifold with isolated singularities $f : W \to \mathbb{R}$ the graph that as a set is the space obtained from W by contracting each fiber to a point and the vertices are the singular fibers of the function is called the Reeb graph of f, $R_G(f)$. See [18].

Definition 2.2. A smooth submanifold $S \subset Sing(f)$ is said to be a nondegenerate critical submanifold of f if the following hold.

- S is compact and connected
- $\forall s \in S$, we have $T_sS = \, ker\, Hess_s f$.

The function f is called a Morse Bott function ($\mathcal{MB}$ function from now on) if the set $Sing(f)$ consists of nondegenerate critical submanifolds. See [2], [4], [14].

Let $p \in S \subset Sing(f)$, then the Morse Bott Lemma says that there is a local chart of W around p and a local splitting of the normal bundle of S, $N_q(S) = N_q^+(S) \oplus N_q^-(S)$ so that if $p = (s, x, y), s \in S, x \in N_p^+(S)$, $y \in N_p^-(S)$:

$$T_p(\Sigma) = T_p(S) \oplus N_p^+(S) \oplus N_p^-(S)$$
$$f(p) = f(s) + |x|^2 - |y|^2.$$

The dimension of $N_p^-(S)$ is the index of S.

A $\mathcal{MB}$ foliation is a foliation defined by the level sets of a $\mathcal{MB}$ function g. We will denote such foliation as $\mathcal{F}(g)$ on the understanding that g is a $\mathcal{MB}$ function. In all the paper we will consider that $\mathcal{F}(g)$ is simple in the sense that there are not separatrix going from a saddle to another saddle point.

A *singular leaf* of a foliation of codimension n is a leaf of dimension lower than n or a leaf that is not a manifold. A *singular foliation* is a foliation with singular leaves. In the case of a simple $\mathcal{MB}$ foliation the singularities are center points or eights. A circle can be a singular fiber for the $\mathcal{MB}$ function but not a singular leaf of the foliation.

We will say that two foliations on Σ are *topologically equivalent* if it exists a homeomorphism on Σ that sends the leaves of one foliation to the leaves of the other.

3. CLASSIFICATION OF PERIODIC ORBIT AND SEPARATRIX EIGHTS

In this section we give the topological classification of curves that can be singular levels of a $\mathcal{MB}$ function. This classification differs from the homotopic or isotopic one. $\Sigma(g, m)$ will be a compact connected orientable surface of genus g ($g \geq 0$) and m holes. Since it is orientable we can assume that $\Sigma(g, m)$ is a subset of R^3. The boundary of $\Sigma(g, m)$ is a collection of m disjoint Jordan curves ($J_1, \ldots, J_m$). Recall that the one dimensional

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homology groups are:

$$\mathbf{H}_1(\Sigma(g,0)) = 2g\mathbf{Z},$$
$$\mathbf{H}_1(\Sigma(g,m)) = (2g+m-1)\mathbf{Z}.$$

DEFINITION 3.1. ([23]) A concordant orientation of $(J_1,\dots,J_m)$ consists of an orientation on each Jordan curve, $J_1,\dots,J_m$, such that the orientation induced on $\Sigma(g,m)$ by the orientation on J_i is independent of $i=1,\dots,m$.

THEOREM 3.1 ([23]). *A homeomorphism $h:(J_1^1,\dots,J_m^1)\to(J_1^2,\dots,J_m^2)$ can be extended to a homeomorphism between $\Sigma^1(g,m)$ and $\Sigma^2(g,m)$ if, and only if, h carries a concordant orientation of $(J_1^1,\dots,J_m^1)$ into a concordant orientation of $(J_1^2,\dots,J_m^2)$.*

3.1. Classification of periodic orbits on Σ

DEFINITION 3.2. An embedded circle on Σ will be the image of an embedding $\phi : S^1 \to \Sigma$. An oriented embedded circle, or shortly, an oriented circle, will be an embedded circle with one of the two possible orientations. A periodic orbit of a flow on Σ is an embedded circle with the orientation induced by the flow.

DEFINITION 3.3. Two embeddings ϕ_i, $i=1,2$ of S^1 into Σ are topologically equivalent if there is a homeomorphism $h:\Sigma\to\Sigma$, such that $h\left(\phi_1(S^1)\right)=\phi_2(S^1)$.

Two embedded circles are equivalent if they can be defined by equivalent embeddings ϕ_i.

Two oriented circles are equivalent if the homeomorphism that conjugates the embedded circles preserves their orientations.

Given an embedding $\phi : S^1 \to \Sigma$, $\phi(\alpha)=\beta$, and given an orientation to S^1 then ϕ induces an orientation on its image; the embedding $-\phi : S^1\to\Sigma$, $-\phi(\alpha)=-\beta$, induces the opposite orientation.

If $\phi(S^1)=J$ is the image of an embedding, then $\Sigma(g,0)\setminus J$ could have one or two connected components. Let K, (K_i) be the compact surface with holes that is the closure of these components. Recall J is homotopic to zero if it bounds a disc in Σ, and we will say that J is of type l_0; if not it is essential. It is homologous to zero if Σ is divided by the curve into two surfaces K_i so that K_1 has genus g_1 and K_2 genus g_2 where $g=g_1+g_2$, $0\le g_1\le g_2$. We will say that the curve is of type l_i if it is homologous to zero but not homotopic to zero where the subscript i refers to the genus of K_1; of type l_K if the curve is not homologous to zero (see Figure 3.1).

Finally, $\mathcal{P}(K)$, $(\mathcal{P}(K_i))$ be the surface obtained by attaching discs to the holes of K (K_i).

Let $E\,(a)$ be the largest integer not greater than a and $C\,(a)$ is the smallest integer not less than a.

THEOREM 3.2. *Let $\Sigma(g,0)$ be an orientable closed surface. The number of non-equivalent embedding of S^1 on Σ is 1 if $g=0$ and $E\left(\frac{g}{2}\right)+2$ with representative curves $l_0, l_1, \ldots, l_{E\left(\frac{g}{2}\right)}, l_K$ if $g>0$ (see Figure 3.1)*

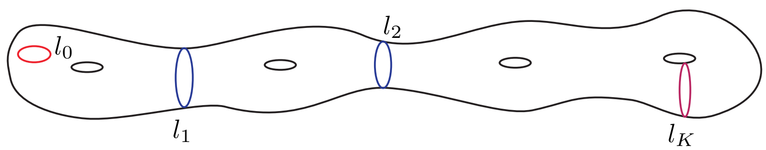

FIGURE 3.1

Proof. Let $\phi_i, i=1,2$ be embeddings of S^1 into $\Sigma(g,0)$ and let $\phi_i(S^1)=J_i$ be their images. If there exists a homeomorphism h from $\Sigma(g,0)$ into $\Sigma(g,0)$ such that $h(J_1)=J_2$ then the restriction of h to $\Sigma(g,0)\setminus J_1$ defines a homeomorphism from it into $\Sigma(g,0)\setminus J_2$.

In order to count the number of non-equivalent embedding on $\Sigma(g,0)$ we divide the set of all embeddings on $\Sigma(g,0)$ in classes such that two embeddings J_1 and J_2 belong to the same class if and only if $\Sigma(g,0)\setminus J_1$ is homeomorphic to $\Sigma(g,0)\setminus J_2$. In a second step we will show that these classes are unitary.

Step 1. (1a.) If $\Sigma(g,0)\setminus J$ is connected, it follows from [11] that J is not a null-homologous curve on Σ. As we can reconstruct $\Sigma(g,0)$ by gluing a handle to $\mathcal{P}(K)$, $\mathcal{P}(K)$ is a surface of genus $g-1$ and K is $\Sigma(g-1,2)$. This construction is not possible on the sphere.

(1b.) If $\Sigma(g,0)\setminus J$ has two connected components K_i then J is a null-homologous curve on Σ. Then $\Sigma(g,0)$ is the connected sum $\mathcal{P}(K_1)\#\mathcal{P}(K_2)$ and

$$\mathbf{H}_1\left(\Sigma(g,0)\right)=\mathbf{H}_1\left(\mathcal{P}(K_1)\right)\oplus\mathbf{H}_1\left(\mathcal{P}(K_2)\right). \tag{1}$$

As the dimension of $\mathbf{H}_1\left(\Sigma(g,0)\right)$ is $2g$ then there are $(g+1)$-possibilities for the pairs $\left(\mathbf{H}_1\left(\mathcal{P}(K_1)\right),\mathbf{H}_1\left(\mathcal{P}(K_2)\right)\right)$:

$$(0,2g),\ (1,2g-1),\ (2,2g-2),\ldots,(g,g).$$

Among these pairs we must eliminate those with odd values because they correspond to surfaces with two or more closed boundary curves. So we have $\mathbf{E}\left(\frac{g}{2}\right)+1$-possibilities.

Step 2. Let $J_1=\phi_1(S^1)$ and $J_2=\phi_2(S^1)$ two embedding of S^1 on $\Sigma(g,0)$ that belong to the same class. Consider the map $h: J_1\to J_2$ given by $h=\phi_1^{-1}\circ\phi_2$. We must proof that there is an extension of h to $\Sigma(g,0)$.

Suppose that $\Sigma(g,0)\setminus J_i$ has two components, each K_j^i, $j=1,2$ has only one closed boundary curve where h is defined. Hence the concordance of the orientation in the boundary

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curves is trivially verified, then we apply the Theorem 3.1 to each K^i_j to obtain extensions of h to K^i_j, say $f^i_j : K^i_j \to K^i_j$, $i, j = 1, 2$. As f^i_1 coincides with f^i_2 on J_1 by the Pasting Lemma there is an extension of h to Σ.

The case where $\Sigma(g, 0) \setminus J_i$ has one component is analogous. In this case we have two components on the boundary $(J_{i,1} = \mathrm{id}(J_i), J_{i,2} = \mathrm{id}(J_i))$ for each K^i. The homeomorphism relating $J_{1,1}$ with one curve $J_{2,k}$ and the homeomorphism that relates $J_{1,2}$ with the other curve $J_{2,3-k}$ are compositions of h with the identity. The hypothesis about the concordance of the orientation in Theorem 3.1 is again verified since the orientation of $(J_{i,1}, J_{i,2})$ is equally preserved or reversed for the two components. We have an extension of h to K^i that we defines on the entire $\Sigma(g, 0)$ identifying the curves $J_{i,1}$ with $J_{i,2}$. ∎

Corollary 3.1. *Two oriented circles are equivalent if and only if they are equivalent as embedded circles*

Proof. Assume that the orientation of each circle is the orientation induced by the embeddings ϕ_1 and $-\phi_2$ and also $\phi_2 = -\phi_1 \circ h$. With the notation used in Theorem 3.2, if h carries the orientation of J_1 into the orientation of J_2, then J_1, J_2 are equivalent. If h reverses the orientation, we consider that J_1 is the image of $-\phi_1$, then the new homeomorphism $h^* = -\phi_1^{-1} \circ \phi_2$ is the homeomorphism that can be extended to all the surface by Theorem 3.2. ∎

3.2. Classification of separatrix eights on Σ

Definition 3.4. A separatrix eight $\mathfrak{B}$, or in short an eight, is the image of an immersion of S^1 into Σ, $\psi : S^1 \to \Sigma$, homeomorphic to two circumferences glued by a point p. A component s_i, will be any of the two circumferences.

We have: $\mathfrak{B} = s_1 \cup_p s_2$

Definition 3.5. Two eights $\mathfrak{B}$ and $\mathfrak{B}'$ are topologically equivalent if there exists a homeomorphism $h : \Sigma \to \Sigma$, such that $h(\mathfrak{B}) = \mathfrak{B}'$.

Lemma 3.1. *Let $\mathfrak{B}$, $\mathfrak{B}'$ be eights on the surface $\Sigma(g, 0)$, $g \geq 0$, and $h : \mathfrak{B} \to \mathfrak{B}'$, a homeomorphism. Then h carries components s_i of $\mathfrak{B}$ into components s'_i of $\mathfrak{B}'$.*

Proof. Suppose that $\mathfrak{B} = s_1 \cup_p s_2$, $\mathfrak{B}' = s'_1 \cup_q s'_2$ with $h(p) \neq q$. Then the sets $\mathfrak{B} - \{p\}$ and $\mathfrak{B}' - \{h(p)\}$ will be homeomorphic. But the first one is disconnected and the second one is connected. Therefore $h(p) = q$ and consequently $h(s_i)$ is s'_1 or s'_2. ∎

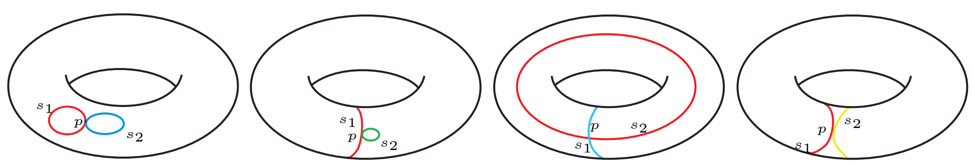
Lemma 3.2. *Let d be the dimension of the subgroup of the first group of homology of $\Sigma(g, 0)$ spanned by the components of $\mathfrak{B}$, s_1 and s_2. Then $\mathfrak{B}$ splits $\Sigma(g, 0)$ in $3-d$ connected regions.*

Proof. Let $\mathfrak{B}$ be an eight. From a topological viewpoint, $\mathfrak{B}$ is equivalent to a graph whose vertices are the vertices of two triangles glued by a common vertex and the edges are the edges of the two triangles. Thus by [9] page 181, we should get

$$\alpha_0(\mathfrak{B}) - \alpha_1(\mathfrak{B}) + r = k + 1 - d, \tag{2}$$

where d is the dimension of the image of $i_* : H_1(\mathfrak{B}, 2) \to H_1(\Sigma(g, 0), 2)$, r is the number of connected regions of $\Sigma(g, 0) \setminus \mathfrak{B}$, k is the number of components of $\mathfrak{B}$ and $\alpha_p = |p\text{-simplexes}|$ of $\mathfrak{B}$. Obviously $\alpha_0(\mathfrak{B}) = 5$, $\alpha_1(\mathfrak{B}) = 6$ and as $\mathfrak{B}$ is connected we have $k = 1$. From (2) we obtain $r = 3 - d$. ∎

Definition 3.6. We will say that an eight $\mathfrak{B}$ is a non-separating eight if $r = 1$. Otherwise, we will say that $\mathfrak{B}$ is a separating eight.



a) b) c) d)

FIGURE 3.2: Different ways to split $\Sigma(1, 0)$ by $\mathfrak{B}$.

Denote by $\mathfrak{N}\mathfrak{B}$ a closed regular neighborhood of $\mathfrak{B} = s_1 \underset{p}{\cup} s_2$. (for details about regular neighborhoods see [9], [10]), we have:

Lemma 3.3. *A closed regular neighborhood of $\mathfrak{B}$ is homeomorphic to $\Sigma(0, 3)$ or $\Sigma(1, 1)$. In the first case, $\Sigma(g, 0) \setminus \mathfrak{N}\mathfrak{B}$ can be:*

a) $\Sigma(g_1, 1)$, $\Sigma(g_2, 1)$, $\Sigma(g_3, 1)$, $g_1 + g_2 + g_3 = g$.
b) $\Sigma(g_1, 1)$, $\Sigma(g_2, 2)$, $g_1 + g_2 = g - 1$.
c) $\Sigma(g - 2, 3)$.
In the second case $\Sigma(g, 0) \setminus \mathfrak{N}\mathfrak{B}$ is homeomorphic to $\Sigma(g - 1, 1)$.

Proof. Consider on the surface a circle centered at the point p and small enough so that $\mathfrak{B}$ cuts the circle in exactly four points. Taking into account the component that corresponds to each points we have two cyclic orderings: s_1, s_1, s_2, s_2 and s_1, s_2, s_1, s_2. The first ordering corresponds to $\mathfrak{N}\mathfrak{B} \cong \Sigma(0, 3)$ and in the second $\mathfrak{N}\mathfrak{B} \cong \Sigma(1, 1)$. Considering the number of holes that each component of $\Sigma(g, 0) \setminus \mathfrak{N}\mathfrak{B}$ has we get the cases a), b), and c).

If $\mathfrak{N}\mathfrak{B} \cong \Sigma(1, 1)$, the border of $\Sigma(g, 0) \setminus \mathfrak{N}\mathfrak{B}$ is a circle and s_1, s_2 are not null homologous nor equivalent. If $\Sigma(g, 0) \setminus s_1$ is $\Sigma(g - 1, 2)$, s_2 can be a circle on $\Sigma(g, 0) \setminus s_1$ or not. If it is a

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circle we get case c) stated previously, if s_2 is not a circle it is a line joining the holes of $\Sigma(g-1,2)$. Therefore $\Sigma(g,0)\setminus\mathfrak{N}\mathfrak{B}\cong\Sigma(g-1,1)$. ▮

If $\mathfrak{B}$ is a separating eight, $\mathfrak{N}\mathfrak{B}$ is homeomorphic to $\Sigma(0,3)$ and has two boundary curves, J_1 and J_2, that are contractible to s_1 and s_2 respectively. We will note this type of curves by J_s. The third boundary curve, J_3 is contractible to $\mathfrak{B}$ and will be noted by $J_{\mathfrak{B}}$. Consider that s_1, s_2, are oriented then we can give to each J_s an orientation compatible with the contraction.

Definition 3.7. We will say that $\mathfrak{B}$ has a cyclic orientation if the orientation given to J_1, J_2 are concordant.

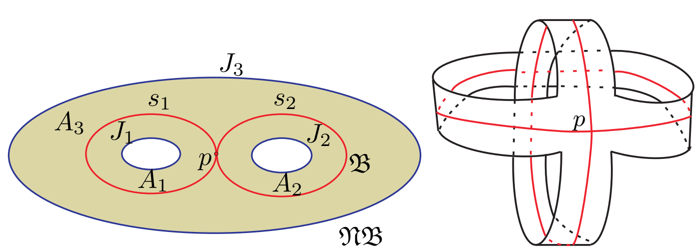
This is equivalent to say that we give to $J_{\mathfrak{B}}$ an orientation compatible with the contraction.

Assume now that $\mathfrak{B}$ is a non-separating eight and consider $\Sigma(g,0)\setminus s_1$. With the notation introduced after the Definition 3.3, s_2 can be a closed curve of K or a line. In the first possibility, $\mathfrak{N}\mathfrak{B}$ is again homeomorphic to $\Sigma(0,3)$ and all is similar to the separating case. In the second possibility, $\mathfrak{N}\mathfrak{B}$ is homeomorphic to $\Sigma(1,1)$.

Definition 3.8. We will say that $\mathfrak{B}$ is a toroidal eight if $\mathfrak{N}\mathfrak{B}$ is homeomorphic to $\Sigma(1,1)$ or a planar eight if $\mathfrak{N}\mathfrak{B}$ is homeomorphic to $\Sigma(0,3)$.

Lemma 3.4. *Let $\mathfrak{B}$ and $\mathfrak{B}'$ be planar eights on Σ. Then every homeomorphism $h:\mathfrak{B}\to\mathfrak{B}'$ that preserves the type of orientations of an eight can be extended to a homeomorphism $H:\mathfrak{N}\mathfrak{B}\to\mathfrak{N}\mathfrak{B}'$. If $\mathfrak{B}$ and $\mathfrak{B}'$ are toroidal eights every homeomorphism $h:\mathfrak{B}\to\mathfrak{B}'$ can be extended to a homeomorphism $H:\mathfrak{N}\mathfrak{B}\to\mathfrak{N}\mathfrak{B}'$.*

Proof. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be planar eights on Σ. We give to $\mathfrak{B}$ a cyclic orientation. With the notation just introduced $\mathfrak{N}\mathfrak{B}\setminus\mathfrak{B}$ and $\mathfrak{N}\mathfrak{B}'\setminus\mathfrak{B}'$ have three semi open annular components say A_j, A'_j, $j=1,2,3$, $J_j\in\partial A_j$ (see the Figure 3.3 (a)).



(a) $\mathfrak{N}\mathfrak{B}\cong\Sigma(0,3)$ (b) $\mathfrak{N}\mathfrak{B}\cong\Sigma(1,1)$

FIGURE 3.3: The closed regular neighborhood of $\mathfrak{B}$.

To extend h to A_1, A_2 is straightforward since J_1, J_2 are contractible to s_1, s_2.

In order to extend h to A_3 we consider two circles C, C', and two immersions $\psi_1(C) = \mathfrak{B}$ and $\psi_2(C') = \mathfrak{B}'$. Since h preserves the type of orientations of $\mathfrak{B}$ there exist a homeomorphism $\tilde{h} = \psi_2^{-1} \circ h \circ \psi_1$ from C to C'. We can identify A_3, A_3' with two semi open annulus limited by C, C'. The extension of $\tilde{h}$ to these annulus gives us an extension of h to A_3.

As h is defined from $\mathfrak{B}$ to $\mathfrak{B}'$ which are the boundaries of A_1, A_2 and A_3 we can apply the Pasting Lemma and obtain the extension H from $\mathfrak{N}\mathfrak{B}$ to $\mathfrak{N}\mathfrak{B}'$.

Consider now that $\mathfrak{B}$ and $\mathfrak{B}'$ are toroidal eights (see the Figure 3.3 (b)). A homeomorphism h of $\mathfrak{B}$ can be decomposed in two homeomorphism, h_i defined on the component s_i. As $\mathfrak{B}$ is a frame of $\mathfrak{N}\mathfrak{B}$ we can extend h to H defined on all the regular neighborhood considering that H is the product of h_1 and h_2. ∎

In the classification of the embedding of S^1 on Σ, the existence of a homeomorphism between $\Sigma \setminus \phi_i(S^1), i = 1, 2$ assures the equivalence of the embeddings $\phi_i(S^1)$. By Lemma 3.1, Figure 3.2 (b) and (d) give us an example where this condition is not enough in order to obtain the classification of the eights.

The main Theorem of this section is the following one:

THEOREM 3.3. *Let $\Sigma(g, 0)$ be an orientable closed surface with $g \geq 0$. Then the number of topological types of eights on $\Sigma(g, 0)$ is*
(1) $3g + 1$, if $g = 0, 1$,
(2) $E\left(\frac{g}{2}\right) C\left(\frac{g}{2}\right) + E\left(\frac{g}{2}\right) + 2g + 3$, if $g \geq 2$;

Proof. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be eights on $\Sigma(g, 0)$. We consider $\Sigma(g, 0) = \mathfrak{N}\mathfrak{B} \cup \mathfrak{N}\mathcal{C}$ where $\mathfrak{N}\mathcal{C}$ is a closed regular neighborhood of $\Sigma(g, 0) \setminus \mathfrak{N}\mathfrak{B}$. The proof will be achieved in two steps. In the first step we divide the set of all eights of $\Sigma(g, 0)$ in classes such that $\mathfrak{B}$ and $\mathfrak{B}'$ belongs to the same class if and only if each component of $\mathfrak{B}$ and it corresponding component in $\mathfrak{B}'$ have the same type of homology and $\Sigma(g, 0) \setminus \mathfrak{B}$ and $\Sigma(g, 0) \setminus \mathfrak{B}'$ are homeomorphics. In the second step we show that each class of the above relation has only one element.

Step 1.

If $g = 0$ all the closed curves on Σ are homotopic to zero then $\mathfrak{N}\mathcal{C}$ is the union of three disks, so there exists only one way of embedding an eight in the sphere. From now on we assume that $g > 0$.

Assume first that both components are homologous to zero with s_1 homotopic to zero. By Theorem 3.2 and Lemma 3.3 we have the following $1 + E\left(\frac{g}{2}\right)$ configurations:

$$\begin{aligned} K_1 \cong D^2, K_2 \cong \Sigma(g, 1), K_3 \cong D^2; \\ K_1 \cong D^2, K_2 \cong \Sigma(g - 1, 1), K_3 \cong \Sigma(1, 1); \\ K_1 \cong D^2, K_2 \cong \Sigma(g - 2, 1), K_3 \cong \Sigma(2, 1); \\ \vdots \\ K_1 \cong D^2, K_2 \cong \Sigma\left(E\left(\frac{g}{2}\right), 1\right), K_3 \cong \Sigma\left(E\left(\frac{g}{2}\right), 1\right) \end{aligned} \tag{3}$$

We will denote these eights by $\mathfrak{B}(l_0, l_i)$. See the cases 1 and 2 in the Table 3.1.

Consider now that the components of the eight are homologous to zero but not homotopic to zero, s_1 is of type l_i and s_2 is l_j. Choose s_1 in such a way that $i \leq j$. If the two components $\mathfrak{B} \setminus s_2$ are not homeomorphic, depending on which component contains s_1 we get two possibilities for K_1, K_2, K_3:

$$K_1 \cong \Sigma(i,1), K_2 \cong \Sigma(g-j,1), K_3 \cong \Sigma(j-i,1);$$
$$K_1 \cong \Sigma(i,1), K_2 \cong \Sigma(j,1), K_3 \cong \Sigma(g-(i+j),1); \tag{4}$$

They will be denoted by $\mathfrak{B}(l_i, l_j, j-i)$, $\mathfrak{B}(l_i, l_j, g-(i+j))$. See the cases 3 and 4 in the Table 3.1.

The number of non equivalent pairs (l_i, l_j), of non homotopic to zero curves is n^2, when g is even and $n(n+1)$, when g is odd. In other way, there exist $E\left(\frac{g}{2}\right).C\left(\frac{g}{2}\right)$ non equivalent pairs in this case.

The case with only one component homologous to zero, for instance s_1, corresponds to the case b) of Lemma 3.3. We have: $K_1 \cong \Sigma(g_i, 1)$ and $K_2 \cong \Sigma(g_j, 2)$. We will denote the eight by $\mathfrak{B}(l_n, g_i, g_j)$ assuming that the component with genus g_i is the component with one hole. One has g possibilities. See cases 5, 6 and 7 in the Table 3.2.

If the components s_i are essential $(i = 1, 2)$ and equivalent we are again in the case b) of Lemma 3.3. Therefore we have g possibilities and the eight will be denoted by $\mathfrak{B}(g_i, g_j)$ where $K_1 \cong \Sigma(g_i, 1)$ and $K_2 \cong \Sigma(g_j, 2)$. The regular neighborhood $\mathfrak{N}\mathcal{C}$ is the union of two surfaces K_i, $i = 1, 2$. See the cases 8, 9 and 10 in the Table 3.1.

Finally if the closed curves s_i are essential and not equivalent then $\mathfrak{N}\mathcal{C}$ has only one component K with $K \cong \Sigma(g-1, 1)$ or $K \cong \Sigma(g-2, 3)$ that will be noted $\mathfrak{B}(1)$ and $\mathfrak{B}(3)$ respectively. See the cases 11 and 12 in the Table 3.1.

Step 2.

Let $\mathfrak{B}$ and $\mathfrak{B}'$ two eights that belong to the same class. We will proof that they are equivalent. Consider first that $\mathfrak{B}$ is planar eight. It is always possible to define a homeomorphism $h : \mathfrak{B} \to \mathfrak{B}'$ that preserves the type of orientation. It follows from Lemma 3.4 that h can be extended to a homeomorphism $H : \mathfrak{N}\mathfrak{B} \to \mathfrak{N}\mathfrak{B}'$. In particular we have a homeomorphism $\tilde{H} : \partial(\mathfrak{N}\mathfrak{B}) \to \partial(\mathfrak{N}\mathfrak{B}')$ that preserves concordant orientations.

It can be extended to $\Sigma(g, 0) \setminus \mathfrak{N}\mathfrak{B}$ by Theorem 3.1 and by the Pasting Lemma find an extension of h on $\Sigma(g, 0)$.

If $\mathfrak{B}$ is a toroidal eight, we can extend $h : \mathfrak{B} \to \mathfrak{B}'$ in a similar way. In this case there not need to take into account orientability conditions on $\mathfrak{B}$. ∎

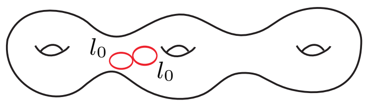
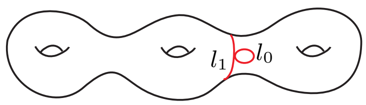
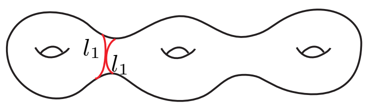
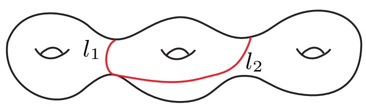
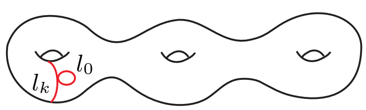
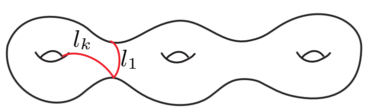
1		$K_1 \cong D^2$; $K_2 \cong D^2$; $K_3 \cong \Sigma(3,1)$	$\mathfrak{B}(l_0, l_0)$
2		$K_1 \cong D^2$; $K_2 \cong \Sigma(2,1)$; $K_3 \cong \Sigma(1,1)$	$\mathfrak{B}(l_0, l_1)$
3		$K_1 \cong \Sigma(1,1)$; $K_2 \cong \Sigma(2,1)$; $K_3 \cong \Sigma(0,1)$	$\mathfrak{B}(l_1, l_1, 0)$
4		$K_1 \cong \Sigma(1,1)$; $K_2 \cong \Sigma(1,1)$; $K_3 \cong \Sigma(1,1)$	$\mathfrak{B}(l_1, l_2, 0)$
5		$K_1 \cong \Sigma(0,1)$; $K_2 \cong \Sigma(2,2)$	$\mathfrak{B}(l_0, 0, 2)$
6		$K_1 \cong \Sigma(0,2)$; $K_2 \cong \Sigma(2,1)$	$\mathfrak{B}(l_1, 0, 2)$

TABLE 3.1: Eights on $\Sigma(3,0)$.

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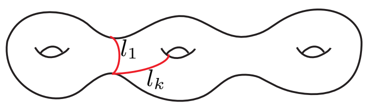
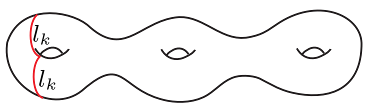
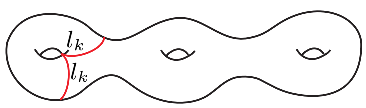
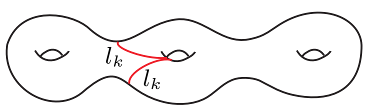
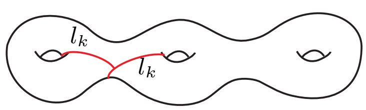
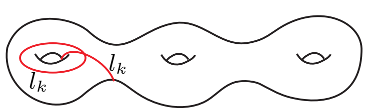
7		$K_1 \cong \Sigma(1,2)$; $K_2 \cong \Sigma(1,1)$	$\mathfrak{B}(l_1, 1, 1)$
8		$K_1 \cong \Sigma(0,1)$; $K_2 \cong \Sigma(2,2)$	$\mathfrak{B}(0, 2)$
9		$K_1 \cong \Sigma(2,1)$; $K_2 \cong \Sigma(0,2)$	$\mathfrak{B}(2, 0)$
10		$K_1 \cong \Sigma(1,1)$; $K_2 \cong \Sigma(1,2)$	$\mathfrak{B}(1, 1)$
11		$K \cong \Sigma(1,3)$	$\mathfrak{B}(3)$
12		$K \cong \Sigma(2,1)$	$\mathfrak{B}(1)$

TABLE 3.1: Eights on $\Sigma(3,0)$.

4. INVARIANT FOR MORSE BOTT SYSTEMS

The decomposition of the leaves of a foliation of codimension one on a surface in points, oriented lines or circles generates a flow on the surface. An invariant for the foliation is also an invariant for the flow; hence we begin with the invariant for $\mathcal{MB}$ foliations.

4.1. $\mathcal{MB}$ foliations

Proposition 4.1. *A toroidal eight is not admissible in an $\mathcal{MB}$ foliation.*

Proof. Suppose that $\mathfrak{B} = s_1 \cup_p s_2$ is a toroidal eight of a $\mathcal{MB}$ foliation $\mathcal{F}(g)$ on Σ and $g(\mathfrak{B}) = 0$. If p is a saddle point we can find two sectors in a neighborhood of p where g will be positive. In the complementary sectors, g will be negative. The interior $\mathfrak{N}\mathfrak{B}$ is filled by regular closed curves contractible to $\mathfrak{B}$. These curves connect positive and negative sectors, therefore the curves will be not level curves of g. ∎

Henceforth, we assume that all eights are planar. Next proposition shows that the structure of a $\mathcal{MB}$ foliation differs from the structure of a Morse foliation.

Proposition 4.2. *Given a Morse foliation, two components of an eight $\mathfrak{B}$ of $\mathcal{F}(g)$ cannot be connected by a family of closed curves. Two regular cylinders connecting two eights contain only J_s circles.*

Proof. Assume that two components of an eight $\mathfrak{B}$ are connected by a family of closed curves. This family can be parameterized by an open interval I. The limit circle of the family are the components of the eight. Since g takes the same value at each component of the boundary of the cylinder, g must have critical points on I. The critical level sets will be circles, that are not admissible singularities in Morse foliations.

In $\mathfrak{N}\mathfrak{B}$, the sign of $g(\mathfrak{B}) - g(J_s)$ is constant and opposite to the sign of $g(\mathfrak{B}) - g(J_{\mathfrak{B}})$. In a double connection between eights, the existence of a connecting cylinder with $J_{\mathfrak{B}}$ and without singular circles contradicts the rule of the sign. ∎

As an example, the foliation in Figure 4.3 is not a Morse foliation.

We will denote by $\mathcal{F}_c$ and $\mathcal{F}_s$ the foliation by circles of a neighborhood of a center and a saddle respectively. If the foliated surface has m holes bounded by regular levels, $s_1, \ldots, s_m$, the new foliation obtained from $\mathcal{F}$ attaching $\mathcal{F}_c$ to each hole will be denoted $\mathcal{P}_{s_1,\ldots,s_m}(\mathcal{F})$. Finally, $\mathcal{G}_c = \mathcal{P}(\mathcal{F}_c)$ and $\mathcal{G}_s = \mathcal{P}(\mathcal{F}_s)$. See the foliations on S^2 in the Table 5.1.

4.2. Reduced Graphs

The Reeb graph of a function g is a topological invariant widely used [1]. In our case $R_G(g)$, by itself, do not characterize the function and at the same time it contains unnecessary information about $\mathcal{F}(g)$. In Figure 4.1 we have an example of two equivalent foliations but with not isomorphic graphs. The $\mathcal{MB}$ function is in both cases the height function. The function on the left side has three singularities and the function on the right

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side, two. The associated Reeb graphs are shown bellow. Although we eliminate critical circumferences the topological type of the foliation does not change.

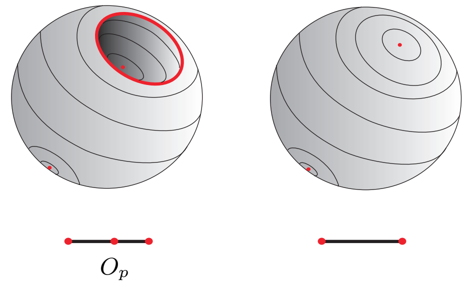

FIGURE 4.1: $\mathcal{F}(g_1) \cong \mathcal{F}(g_2)$, $R_G(g_1) \ncong R_G(g_2)$

Nevertheless, this reduction is not always possible, as in the case of the critical circumference in the foliation without saddle points on the torus in Table 5.1. These remarks leads us to the following definition

DEFINITION 4.1. [The Graph of the $\mathcal{MB}$ foliation] Let $\mathcal{F}$ be a $\mathcal{MB}$ foliation and f such that $\mathcal{F} = \mathcal{F}(f)$. Then the Graph $\Theta(\mathcal{F})$ of the $\mathcal{MB}$ foliation is:

a)A circle, in the case of a regular foliation by circles on the torus.

b)The graph obtained from the Reeb graph of f transforming the union of each vertex v associated to critical circumferences of f and the two incident edges in a new edge.

$\Theta(\mathcal{F})$ does not depend on the particular function f such that $\mathcal{F} = \mathcal{F}(f)$. Our construction is related to the construction described in [5], section 1.3 and page 13. This graph $\Theta(\mathcal{F})$ carries the information about the surface Σ since the number of independent cycles in $\Theta(\mathcal{F})$ is the genus of Σ. But, this graph is not a complete invariant, as the next example shows.

EXAMPLE 4.1. In the Figures 4.2 and 4.3 we show two non equivalent $\mathcal{MB}$ foliations on the torus, $\mathcal{F}_1$ and $\mathcal{F}_2$, which graphs and space of leaves are isomorphic. In fact, if there exist a homeomorphism h which sends the leaves of the first foliation to leaves of the second one, then the topological type of the leaves would be the same. But this does not happen for all the components of the saddle points and such homeomorphism does not exist.

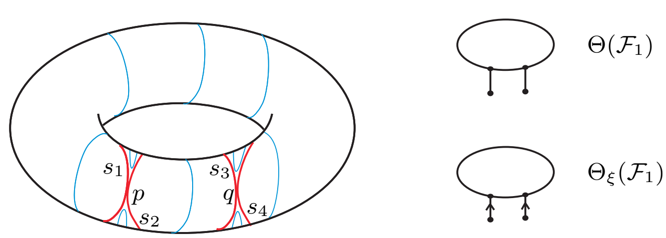

FIGURE 4.2

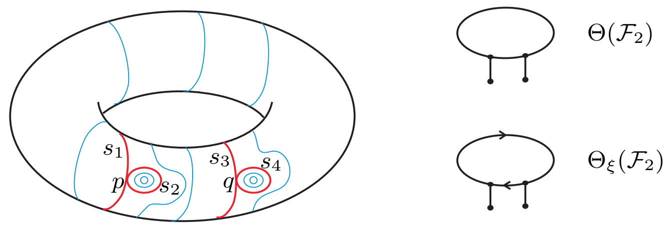

FIGURE 4.3

The topological type of an eight must be included in any complete invariant associated with $\mathcal{MB}$ foliations. Nevertheless, this topological type can be implicitly included in a graph adding some extra structure.

PROPOSITION 4.3. *Let ξ be a function on $\Theta(\mathcal{F})$ that associates to each saddle vertex the edge that contains the $J_{\mathfrak{B}}$ circles. The graph and ξ determines the topological type of the eights.*

Proof. The topological type of an eight contains information about the topology of the components s_i, if they are independent or not and finally on the components of $\Sigma(g,0)\setminus\mathfrak{N}\mathfrak{B}$. In $\Theta(\mathcal{F})$, the component J_s corresponds the point p_{J_s} and to $\mathfrak{B}$ corresponds the vertex $v_{\mathfrak{B}}$. Then $\Theta(\mathcal{F})\setminus p_{J_s}$ determines the type of J_s and therefore the type of the component of the eight that jointly with J_s bounds a cylinder. The remainder information can be deduced from the components of $\Theta(\mathcal{F})\setminus v_{\mathfrak{B}}$. ∎

An edge on $\Theta(\mathcal{F}(f))$ can have $0 \le n \le 2$ distinctions. We denote the edge by n-edge. In this paper, a 0-edge will be unmarked, a 1-edge will be an arrow directed towards the vertex that distinguishes, and a 2-edges will be represented by an left right arrow. The sense of the arrow indicates the sense of branching of the graph. With this representation, $\Theta(\mathcal{F})$ is a Dynkin's type graph.

4.3. Complete Invariant

We now present a complete invariant that solves the problem of the classification of $\mathcal{MB}$ foliations on orientable closed surfaces.

By $\Theta_\xi(\mathcal{F})$ will be denoted the pair formed by $\Theta(\mathcal{F})$ and the ξ as in the Proposition 4.3, and the foliation associated to Θ_ξ will be $\mathcal{F}ol(\Theta_\xi)$

We assume here that $\Theta_\xi(\mathcal{F}_1)$ and $\Theta_\xi(\mathcal{F}_2)$ are isomorphic if there exists an isomorphism from $\Theta(\mathcal{F}_1)$ onto $\Theta(\mathcal{F}_2)$ that preserves the assignments of the functions ξ.

THEOREM 4.1. *$\Theta_\xi(\mathcal{F})$ is a complete topological invariant for $\mathcal{MB}$ foliations on orientable closed surfaces.*

Proof. **Necessity.** $\Sigma(g,0)$ is contractible to $\Theta(\mathcal{F})$ considered as a cellular complex; the

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number of the different singularities and how they are connected by regular cylinders is the same for equivalent foliations. Therefore $\Theta(\mathcal{F})$ is a topological invariant of foliations. The topological type of the eights is by definition another invariant that is equivalent to the distinction between $J_{\mathfrak{B}}$ and J_s leaves.

Sufficiency. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two $\mathcal{MB}$ foliations on $\Sigma(g,0)$; assume that there exists an isomorphism $\theta : \Theta_\xi(\mathcal{F}_1) \to \Theta_\xi(\mathcal{F}_2)$. Denote by $S(\mathcal{F}_1)_i = f^{-1}(a_i)$ and $S(\mathcal{F}_2)_i = g^{-1}(b_i)$, $b_i = \theta(a_i)$ two related singular levels of the foliations; then there exist homeomorphisms $\theta_i^s : S(\mathcal{F}_1)_i \to S(\mathcal{F}_2)_i$ that in the case of eights can be chosen in such a way that it preserves the type of orientation. We are going to prove that it is possible to extend θ_i^s to a homeomorphism $h : \Sigma \to \Sigma$ sending leaves to leaves.

We assume that there exist saddle singularities in order to avoid trivial cases and we suppose also that the extension of θ_i^s to $\mathfrak{N}\mathfrak{B}_i$, denoted here θ_i^r and defined in Lemma 3.4 sends level curves to level curves. The restrictions of θ_i^r to the components of $\partial\mathfrak{N}\mathfrak{B}_i$ are concordant by construction. There exist extensions of θ_i^r to the entire $\Sigma(g,0)$ that we will denote $h_{\mathfrak{N}\mathfrak{B}_i}$.

Consider two connected eights $\mathfrak{B}_i$, $\mathfrak{B}_j$ and let $\mathcal{C}_{ij}^1$ be one connecting cylinder. We have a homeomorphism between each components of the border of $\mathcal{C}_{ij}^1$. If both homeomorphism preserve or reverse orientation, it is trivial to construct an extension, $\theta_{\mathcal{C}_{ij}^1}$, of the homeomorphisms on the border to the entire cylinder. If not, we can modify what kind of orientation preserves one of the θ_k^s in order to be in the previous case. If we have more cylinders $\mathcal{C}_{ij}^k$ connecting $\mathfrak{B}_i$, $\mathfrak{B}_j$ we follow the same process to define new $\theta_{\mathcal{C}_{ij}^k}$. The homeomorphism obtained pasting together θ_i^r, θ_j^r, and all $\theta_{\mathcal{C}_{ij}^k}$ can be extended to a homeomorphism $h_{\mathfrak{N}\mathfrak{B}_i,\mathfrak{N}\mathfrak{B}_j}$ on the entire $\Sigma(g,0)$ as in the case of one saddle. We iterate the extension to new saddles until all eights have been involved obtaining a homeomorphism $h_{\mathfrak{N}\mathfrak{B}} : \Sigma \to \Sigma$.

Finally, consider saturated neighborhoods $\mathfrak{N}c_k$ of the centers. They are disk trivially foliated by circles and $h_{\mathfrak{N}\mathfrak{B}}$ defines an homeomorphism between $\partial\mathfrak{N}c_k$ and $\partial\mathfrak{N}\theta(c_k)$. Since we can extend a homeomorphism between the borders of two disks fixing one point in the interior of one disk and its image on the other, we get finally the desired extension h. Note that the border of the basin of a center can be an eight or a component s_i. Therefore, in order to attach properly the disks the $J_{\mathfrak{B}}$ must be marked in $\Theta_\xi(\mathcal{F})$. ∎

4.4. Invariant for $\mathcal{MB}$ integrable systems

Given a $\psi_{\mathcal{MB}}(\Sigma)$ vector field, $\Theta_\xi(\mathcal{F})$ is an invariant for the system. Now we are going to adapt this system to the case of flows having into account the singularities of the system. Recall that, according to our definition, these singular leaves are filled by equilibrium points. We will say that this singular curve is an equatorial curve (e-curve) if the nearby periodic orbits have different sense of rotation; not equatorial (ne-curve) in the other case.

Definition 4.2. [The Graph of a system] Let $X \in \psi_{\mathcal{MB}}(\Sigma)$. Then the Graph $\Theta(X)$ associated to the system is the graph obtained from $\Theta(\mathcal{F})$, adding a vertex for each close curve filled with equilibrium points.

Definition 4.3. The invariant associated to $\mathcal{MB}$ systems X defined on a closed orientable surface $\Sigma(g,0)$ is $\Theta(X)$, the function ξ previously defined, and the function η that associates to each non periodic circle its type (e or ne). We denote this invariant by $\Theta_{\xi,\eta}(X)$.

Theorem 4.2. *$\Theta_{\xi,\eta}(X)$ is a complete invariant for $\psi_{\mathcal{MB}}(\Sigma)$ systems.*

Proof. Given an orbit of a $\psi_{\mathcal{MB}}(\Sigma)$ and knowing $\Theta_{\xi,\eta}(X)$, the sense of this orbit determines the sense of all the orbits. Therefore there exists at most two different systems with the same invariant. But in the case of $\mathcal{MB}$ systems we are going to see that the two systems are equivalent. Since the graph of the system can be immersed in the surface, the homeomorphism h between the surfaces generates a homeomorphism h_G between the graphs. Let x be a point in the graph associated with a regular level set l. On this level set h can be expressed : $h(x,\alpha) = (h_G(x), \phi_x(\alpha))$, $\alpha \in l$. This map is not defined on singular levels, nevertheless by continuity, this expression of h is enough to determine the homeomorphism in all points of the surface.

To finish the proof, if h is the homeomorphism that conjugates both systems and preserves the sense of the orbits, in suitable coordinates h is the identity: $h(x,\alpha) = (x,\alpha)$. If the sense of the orbits is not preserved, substitute h by $h^*(x,\alpha) = (x,-\alpha)$. ∎

5. REALIZATION OF THE INVARIANT

The aim of this section is to introduce the conditions that an abstract graph G with labels must fulfill to be realizable as the invariant of a $\mathcal{MB}$ foliation $\mathcal{F}$. We limit the study to $\mathcal{MB}$ foliations since realization for systems can be easily derived from the case of foliations. We will characterize the set of the graphs G which arise as a graph of a $\mathcal{MB}$ foliation in terms of one generator and two transformations.

The graph of a $\mathcal{MB}$ foliation has vertices with degree 1 or 3. Moreover if the foliation is defined on a simple closed orientable surface, $\Sigma(g,0)$, it is necessary that the number of cycles of the graph will be equal to the genus g of $\Sigma(g,0)$.

Definition 5.1. An abstract Θ_ξ graph is or a circle or a connected graph whose vertices has degree one or three, moreover, the vertices of degree three have one distinguished edge

We have three types of edges that we identify by a left right arrow, an arrow or a simple segment.

Fixing the surface $\Sigma(g,0)$, the foliations with a minimal number of centers will be called *minimal foliations on* $\Sigma(g,0)$.

The minimal foliation on $\Sigma(0,0)$ is $\mathcal{G}_c$. We will denote the invariant by G and call it the generator graph. Consists of an edge and two vertices.

Before to state the realization Theorem we need to define the union of two foliations. Let $\Sigma_i(g_i,0)$, $i=1,2$ be two surfaces and $\mathcal{F}_i$ a foliation on each surface. Consider the connected sum of both surfaces made over two discs D_i. On the connected sum of Σ_i we can define new foliations.

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Definition 5.2. Let each ∂D_i be a leaf of the basin of centers, $\mathcal{G}_i$ the restriction of $\mathcal{F}_i$ to $\Sigma_i(g_i, 0) \setminus \mathring{D}_i$, then the leaves of the new foliation $\mathcal{F}$ consists of the union of the leaves of $\mathcal{G}_1$ and $\mathcal{G}_2$. We call $\mathcal{F}$ the union of $\mathcal{F}_i$ and by extension the union of $\mathcal{G}_i$.

If c_1, c_2 are the centers involved in the union of two foliations, we will denote this union by: $\mathcal{F}_1 {}_{c_1}\!\bigsqcup_{c_2} \mathcal{F}_2$

The union of foliations can be easily generalized to the union with an arbitrary number of pairs of centers.

Next proposition can be easily proved.

Proposition 5.1. *The union of $\mathcal{MB}$ foliations are $\mathcal{MB}$ foliations.*

Next theorem characterize the realizable graphs using the generator G and two transformations defined on it.

Theorem 5.1. *An abstract graph obtained by applying one of the transformations defined bellow to the $\Theta_\xi(\mathcal{F})$ of a $\mathcal{MB}$ foliation is also the graph of a $\mathcal{MB}$ foliation.*

- *Transformation I: Replace a point in the interior of an n-edge by a vertex v of degree three with one distinguished edge and connected to a final vertex. The sum of the distinctions of the three edges adjacent to v must be $n+1$.*
- *Transformation II: Connect two final vertices of a graph or of a pair of graphs, eliminating the final vertices. The distinction of the new edge is the sum of the distinctions of the connected edges.*

Proof. Let $\mathcal{F}'$ be a $\mathcal{MB}$ foliation on $\Sigma(g, 0)$ and Θ'_ξ their associated graph. We are going to prove that the new foliation obtained applying a transformation to Θ'_ξ is again a $\mathcal{MB}$ foliation.

Let us start with the transformation I. Consider a regular circle s and assume first that this curve is represented by a point on a final edge. Then:

$$\mathcal{F} = \mathcal{F}' \bigsqcup_{s,J} \mathcal{G}_c$$

If s is not in a final edge, lets call s^1 and s^2 the boundary curves on $\Sigma(g, 2)$ obtained deleting the interior of a regular neighborhood of s and let $\mathcal{F}'_1$ be the associated foliation, then:

$$\mathcal{F} = \mathcal{P}_{s^1, s^2}\left(\mathcal{F}'_1\right) \bigsqcup_{s^i, J} \mathcal{G}_s$$

The type of J, (J_s or $J_{\mathfrak{B}}$) determines the labeling on the edges.

The transformations II is the union of two $\mathcal{MB}$ foliations or the union of a $\mathcal{MB}$ foliation and $\mathcal{G}_c$. ∎

The converse also holds.

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Theorem 5.2. *All $\mathcal{MB}$ foliations $\mathcal{F}$ on an orientable and compact surface $\Sigma(g,0)$ can be obtained applying a finite sequence of transformations applied to G.*

Proof. We will proof the Theorem by induction on the number of saddle points of $\mathcal{F}$.

Case $n = 0, 1$. In Table 5.1 are listed all $\mathcal{MB}$ foliation with $n \leq 1$ saddles. They are obtained considering all combinations of three-valent vertices and centers and adding the type of eight at each saddle vertex.

Case $n \geq 2$. We will suppose that the Theorem holds for foliations with n saddle points and prove that still holds for foliations with $n+1$ saddle points.

We will separate the proof by considering all types of eights on $\Theta_\xi(\mathcal{F})$, the foliation with $n+1$ saddles.

Case (a): Type $\mathfrak{B}(l_0, l_0)$. Let $\Sigma(g,0)$ be the foliated surface, $\mathcal{F}^{n+1}$ the foliation that contains $\mathfrak{B}(l_0, l_0)$ and J_1, J_2, J_3 as in the Definition 3.7. Let $D_{J_\mathfrak{B}}$ the disk bounded by $J_3 = J_\mathfrak{B}$. Consider the foliation $\mathcal{F}^n$ defined by $\mathcal{P}_{J_\mathfrak{B}}\left(\mathcal{F}^{n+1} \setminus D_{J_\mathfrak{B}}\right)$. Then $\mathcal{F}^{n+1} = \mathcal{F}ol(I(\Theta_\xi(\mathcal{F}^n))$

Case (b): The eight will be of type $\mathfrak{B}(l_X, l_0)$ (where $X = 1, ..., E\left(\frac{g}{2}\right)$) or $\mathfrak{B}(l_0, g-1, 0)$ if $X = K$. The foliation obtained by collapsing the disk bounded by the component of type l_0 to the saddle point is a $\mathcal{MB}$ foliation $\mathcal{F}^n$ with n saddles. Then $\mathcal{F} = \mathcal{F}ol(I(\Theta_\xi(\mathcal{F}^n))$.

Case (c): Type $\mathfrak{B}(l_X, l_X, 0)$ where $X = 1, ..., E\left(\frac{g}{2}\right)$. A regular neighborhood of the eight separates the surface in three surfaces, $\Sigma(X,1)$, $\Sigma(g-X,1)$ and $\Sigma(0,1)$. Let $\mathcal{G}_i$ the foliations obtained attaching J_c to these foliated surfaces. Then $\mathcal{F} = \mathcal{F}ol(I(\Theta_\xi(\mathcal{G}_1 \bigsqcup_{J_1,J_2} \mathcal{G}_2)))$.

Case (d): Type $\mathfrak{B}(l_i, l_j, g-(i+j))$ with $i \leq j$. Let us consider the closed regular neighborhood $\mathfrak{NB}$ of $\mathfrak{B}$; $\Sigma(g,0) \setminus \mathfrak{NB}$ is formed by the surfaces: $\Sigma(i,1)$ bounded by l_i, $\Sigma(j,1)$ bounded by l_j and $\Sigma(g-(i+j),1)$. The restriction of $\mathcal{F}$ to these surfaces and completed with J_c attached are $\mathcal{MB}$ foliations, $\mathcal{G}_i$. Then $\mathcal{F} = \mathcal{F}ol(I(\Theta_\xi(\mathcal{G}_1 \bigsqcup \mathcal{G}_2))) \bigsqcup \mathcal{G}_3$.

Case (e): If the eight is of type $\mathfrak{B}(l_i, l_j, j-i)$ with $i < j$ the induction method is similar.

Case (f): Type $\mathfrak{B}(l_n, g_i, g_j)$. The component of the eight of type l_n separates the surface $\Sigma(g,0)$ in two components $\Sigma(g_i,1)$, $\Sigma(g_j,1)$ with foliations $\mathcal{F}_1$, $\mathcal{F}_2$. Assume that the second one contains the component l_K of the eight. Let $\mathcal{G}_i = \mathcal{P}_{l_n}(\mathcal{F}_i)$, c_i the new centers. Let $\mathcal{G}_3$ the foliation on $\Sigma(g_j,0)$ obtained collapsing l_n to a point. The foliation $\mathcal{G}_2$ is equivalent to $I(\Theta_\xi(\mathcal{G}_3))$. Then: $\mathcal{F}^{n+1} = \mathcal{F}ol(II(\Theta_\xi(\mathcal{G}_1), I(\Theta_\xi(\mathcal{G}_3)))$. The transformation II has been applied over the centers c_1, c_2.

Case (g): Type $\mathfrak{B}(g_i, g_j)$. If $g_i = 0$, let $\mathcal{G}$ the foliation obtained by collapsing the closed disk bounded by the components of the eight to one component of type l_K. This new foliations has n saddle points and $\mathcal{F}^{n+1} = \mathcal{F}ol(I(\Theta_\xi(\mathcal{G})))$. If $g_i > 0$, $\Sigma(g,0) \setminus \mathfrak{NB}$ consists of one surface with two holes and another one with one hole. Let $\mathcal{G}_1$ ($\mathcal{G}_2$) the foliations obtained from the restrictions of $\mathcal{F}^{n+1}$ to the first (second) surface attaching J_c to the holes. Then $\mathcal{F}^{n+1} = \mathcal{F}ol(II\,(II(I(\Theta_\xi(\mathcal{G}_2)), \Theta_\xi(\mathcal{G}_1)), \Theta_\xi(\mathcal{G}_1)))$.

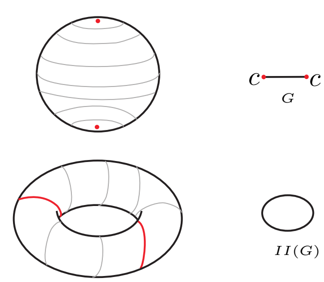
Case (h): The type of the eight $\mathfrak{B}$ formed by s_1, s_2, $s_1 \cap s_2 = p$ is $\mathfrak{B}(3)$. In this case the curves J_1 and J_2 are not trivial since they are contractible to s_1 and s_2. The curve J_3 is also trivial, if not, s_1 jointed with s_2 will bound a disk, therefore they will be homologically equivalent. The edges adjacent to the saddle point can not be connected to a center. Consider one saddle $\mathfrak{B}'$ connected to $\mathfrak{B}$.

The curve of type J_3 of $\mathfrak{B}$ is connected with another curves J of $\mathfrak{B}'$ though a cylinder. Let s be a regular circle of this cylinder , $\Sigma(g,0) \setminus s \cong \Sigma(g,2)$. Attach one J_c to each hole.

We have a new foliation with two centers c_1, c_2. Collapse the basin of one center to $\mathfrak{B}$ and we get a foliation $\mathcal{F}^n$ with n saddle points. Then $\mathcal{F}^{n+1} = \mathcal{F}ol(II\,(I(\Theta_\xi(\mathcal{F}^n))))$.

If $\mathfrak{B}$ and $\mathfrak{B}'$ are connected through J_s curves the reduction is similar, instead of collapsing the basin of a center to $\mathfrak{B}$, we collapse it to a component of $\mathfrak{B}$. ∎

Zero saddle points



One saddle points

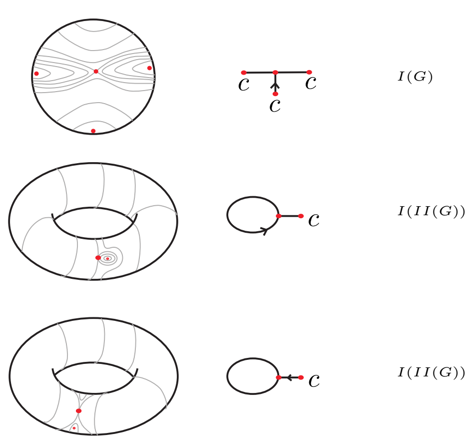


TABLE 5.1: $\mathcal{MB}$ foliation with $n \leq 1$ saddles.

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