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Measure neutral functional differential equations as generalized ODEs

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In this paper, we introduce a class of measure neutral functional differential equations of type

$$\mathrm{D}[N(x_t,t)] = f(x_t,t)\mathrm{D}g(t)$$

through the relation with a certain class of generalized ordinary differential equations introduced in [4] (we write generalized ODEs), using similar ideas to those of [5]. By means of the correspondence with generalized ODEs, we state results on the existence, uniqueness and continuous dependences of solutions for our equation of neutral type. An example is given to illustrate the correspondence. October, 2013 ICMC-USP

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1. INTRODUCTION

During the last years, the interest in the theory of generalized ordinary differential equations (we write generalized ODEs, for short) has been increasing significantly. This interest lies on the fact that several kinds of differential equations such as ordinary differential equations, measure functional differential equations, impulsive differential equations and also dynamic equations on time scales can be regarded as generalized ODEs. See, for instance, [1, 5, 4, 6, 17, 18, 19, 22, 23, 24, 25]. This means that these types of differential equations can be treated via theory of abstract generalized ODEs which presents a much more simple and friendly environment to deal with than any of the above specific setting of differential equations.

While the theory of neutral functional differential equations is very well-known (see [2, 3, 9, 11, 12, 13, 14, 15, 16, 21, 20, 26], for instance), the literature concerning measure neutral functional differential equations is new.

In the present paper, we introduce a class of equations called measure neutral functional differential equations, which we refer to simply as measure NFDEs and which encompasses classic classes of NFDEs. Our main result (namely Theorems 4.1 and 4.2) states that, similarly to other kinds of differential equations, measure NFDEs can also be regarded as abstract generalized ODEs. Then, using the relation between measure NFDEs and generalized ODEs, we prove results on the existence and uniqueness of solutions and continuous dependence of solutions on parameters for our class of measure NFDEs.

The present paper is organized as follows. In the second section, we introduce some notation and terminology involving measure NFDEs. The third section is devoted to a short description of the basis of the theory of generalized ODEs. In the fourth section, we describe the framework of measure NFDEs and we establish and prove a one-to-one correspondence between a solution of a measure NFDE and a solution of a special class of generalized ODEs. The fifth section contains an existence and uniqueness result for measure NFDEs, using the correspondence presented in the previous section. In the sixth section, we establish a result on the continuous dependence on the initial data of solutions of measure NFDEs. In the last section, we provide an example of a measure NFDE, evaluate its corresponding generalized ODE as well as its solution as present the relation between the solutions of the two equations.

2. MEASURE NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

Let t_0, σ, r be given real numbers, with $\sigma, r > 0$. The theory of neutral functional differential equations is usually concerned with equations of type

$$\frac{\mathrm{d}}{\mathrm{d}t}N(y_t, t) = f(y_t, t), \quad t \in [t_0, t_0 + \sigma],$$

where $y_t(\theta) = y(t + \theta)$, for $\theta \in [-r, 0]$. The book [12] is a basic reference for this theory.

Because we would like to model real-world problems undergoing jumps or discontinuities, we will consider the space of regulated functions from $[t_0 - r, t_0 + \sigma]$ to $\mathbb{R}^n$ as our phase space.

Let X be a Banach space and $[a, b] \in \mathbb{R}$ be a compact interval. Recall that a function $f : [a, b] \to X$ is regulated, provided the one-sided limits

$$\lim_{s\to t-} f(s) = f(t-), \quad t \in (a, b], \quad \text{and} \quad \lim_{s\to t+} f(s) = f(t+), \quad t \in [a, b)$$

exist. We denote by $G([a, b], X)$ the space of all regulated functions $f : [a, b] \to X$. When endowed with the usual supremum norm

$$\|f\|_\infty = \sup_{a\leqslant t\leqslant b} \|f(t)\|,$$

$G([a, b], X)$ is a Banach space.

The first result we mention says that if a given function $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ is regulated, then the mapping $s \mapsto \|y_s\|_\infty$, $s \in [t_0, t_0 + \sigma]$, is also regulated. For a proof of such result, the reader may want to consult [5, Lemma 3.5].

Proposition 2.1. *If $y \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, then the function $s \mapsto \|y_s\|_\infty$, $s \in [t_0, t_0 + \sigma]$, is an element of $G([t_0, t_0 + \sigma], \mathbb{R})$.*

In the present paper, we focus our attention on equations of the form

$$\mathrm{D}\big[N(x_t, t)\big] = f(x_t, t)\mathrm{D}g, \tag{1}$$

where $\mathrm{D}[N(x_t, t)]$ and $\mathrm{D}g(t)$ are the distributional derivatives of $N(x_t, t)$ and $g(t)$ respectively in the sense of L. Schwartz (see the references [10, 27]). We call equation (1) a *measure neutral functional differential equation* or simply *measure NFDE*.

The setting of functions involved in equation (1) is described next.

Let $O \subset G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ be open and consider the set

$$P = \{y_t : y \in O,\ t \in [t_0, t_0 + \sigma]\} \subset G([-r, 0], \mathbb{R}^n).$$

Assume that $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ is a function such that, for each $y \in O$, the mapping $t \mapsto f(y_t, t)$ is integrable (in a sense that we will specify later) on $[t_0, t_0 + \sigma]$ with respect to a nondecreasing function $g : [t_0, t_0 + \sigma] \to \mathbb{R}$.

We assume that N is a linear and autonomous operator which means that $N(x_t, t) = N(t)x_t$. Therefore equation (1) can be rewritten as

$$\mathrm{D}[N(t)x_t] = f(x_t, t)\mathrm{D}g. \tag{2}$$

Moreover, we suppose that there is a matrix $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$, which is a mensurable and normalized function satisfying

$$\mu(t, \theta) = 0,\ \theta \geqslant 0; \quad \mu(t, \theta) = \mu(t, -r),\ \theta \leqslant -r. \tag{3}$$

Also, μ is a left-continuous function in $\theta \in (-r, 0)$, of bounded variation on $\theta \in [-r, 0]$, and the variation of the μ on $[s, 0]$, $\operatorname{var}_{[s,0]} \mu$, tends to zero as $s \to 0$, such that the operator N is given by

$$N(t)\varphi = \varphi(0) - \int_{-r}^{0} \mathrm{d}_\theta[\mu(t, \theta)]\varphi(\theta), \tag{4}$$

where $\varphi \in G([-r, 0], \mathbb{R}^n)$.

Combining (2) and (4), we obtain

$$N(t)x_t - N(0)x_0 = \int_0^t f(x_s, s)\mathrm{d}g(s),$$

which implies

$$x(t) - \int_{-r}^{0} \mathrm{d}_\theta[\mu(t, \theta)]x(t + \theta) - x(0) + \int_{-r}^{0} \mathrm{d}_\theta[\mu(0, \theta)]\varphi(\theta) = \int_0^t f(x_s, s)\mathrm{d}g(s)$$

where the integral on the right-hand side can be understood is in the sense of Riemann-Stieltjes, Lebesgue-Stieltjes or even Kurzweil-Henstock-Stieltjes. Therefore, the integral form of equation (2) can be written as

$$x(t) - x(0) = \int_0^t f(x_s, s)\mathrm{d}g(s) + \int_{-r}^{0} d[\mu(t, \theta)]x(t + \theta) - \int_{-r}^{0} d[\mu(0, \theta)]\varphi(\theta).$$

3. GENERALIZED ODES

Throughout this paper, we use the following definition of integral introduced by J. Kurzweil in [18].

Consider a function $\delta : [a, b] \to \mathbb{R}^+$ (called a gauge on $[a, b]$). A tagged partition of the interval $[a, b]$ with division points $a = s_0 \leqslant s_1 \leqslant \ldots \leqslant s_k = b$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i = 1, \ldots, k$, is called δ-fine if

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)), \quad i = 1, \ldots, k.$$

Definition 3.1. Let X be a Banach space. A function $U(\tau, t) : [a, b] \times [a, b] \to X$ is called Kurzweil integrable over $[a, b]$, if there is an element $I \in X$ such that given $\varepsilon > 0$, there is a gauge δ on $[a, b]$ such that

$$\left\| \sum_{i=1}^{k} \Big[U(\tau_i, s_i) - U(\tau_i, s_{i-1}) \Big] - I \right\| < \varepsilon,$$

for every δ-fine tagged partition of $[a, b]$. In this case, I is called the *Kurzweil integral* of U over $[a, b]$ and it will be denoted by $\int_a^b \mathrm{D}U(\tau, t)$.

The Kurzweil integral has the usual properties of linearity, additivity with respect to adjacent intervals and integrability on subintervals. See [23], for these and other interesting properties.

The above definition generalizes the well-known Kurzweil-Henstock or Perron integral of a function $f : [a, b] \to X$, which is obtained by setting $U(\tau, t) = f(\tau)t$. The Perron-Siteltjes or Kurzweil-Henstock-Stieljtes integral of a function $f : [a, b] \to X$ with respect to a function $g : [a, b] \to \mathbb{R}$, which appears in the definition of a measure functional differential equation, corresponds to the choice $U(\tau, t) = f(\tau)g(t)$ and will be denoted by $\int_a^b f(s)\,\mathrm{d}g(s)$.

The first result we describe in this section concerns the Kurzweil-Henstock-Stieltjes integral. Such result is essential to our purposes; it is a special case of Theorem 1.16 in [23].

Theorem 3.1. *Let $f : [a, b] \to \mathbb{R}^n$ and $g : [a, b] \to \mathbb{R}$ be a pair of functions such that g is regulated and $\int_a^b f(t)\,\mathrm{d}g(t)$ exists. Then the function*

$$h(t) = \int_a^t f(s)\,\mathrm{d}g(s), \quad t \in [a, b],$$

is regulated and satisfies

$$\begin{aligned} h(t+) &= h(t) + f(t)\Delta^+ g(t), \quad t \in [a, b), \\ h(t-) &= h(t) - f(t)\Delta^- g(t), \quad t \in (a, b], \end{aligned}$$

where $\Delta^+ g(t) = g(t+) - g(t)$ and $\Delta^- g(t) = g(t) - g(t-)$.

The next result shows us a case when the Kurzweil-Henstock-Stieltjes integral exists. A proof of it can be found in [23, Corollary 1.34]. The inequalities follow directly from the definition of the Kurzweil-Henstock-Stieljtes integral.

Theorem 3.2. *If $f : [a, b] \to \mathbb{R}^n$ is a regulated function and $g : [a, b] \to \mathbb{R}$ is a nondecreasing function, then the integral $\int_a^b f(t)\,\mathrm{d}g(t)$ exists and*

$$\left| \int_a^b f(s)\,\mathrm{d}g(s) \right| \leqslant \int_a^b |f(s)|\,\mathrm{d}g(s) \leqslant \|f\|_\infty [g(b) - g(a)].$$

As we proceed, we present the concept of a generalized ordinary differential equation defined via Kurzweil integral. See [18, 19].

Definition 3.2. Let X be a Banach space. Consider a subset $O \subset X$, a compact interval $[a, b] \subset \mathbb{R}$ and a function $G : O \times [a, b] \to X$. Any function $x : [a, b] \to O$ is called a solution of the generalized ordinary differential equation (we write simply generalized ODEs)

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}G(x, t) \tag{5}$$

on the interval $[a,b]$, provided

$$x(d)-x(c)=\int_c^d \mathrm{D}G(x(\tau),t),$$

for every $c,d\in[a,b]$, where the integral is obtained by setting $U(\tau,t)=G(x(\tau),t)$ in the definition of the Kurzweil integral (Definition 3.1).

In order to obtain a good theory of generalized ODEs, we restrict our attention to equations whose right-hand sides satisfy the conditions described in the next definition. See [23].

Definition 3.3. Let X be a Banach space. Consider a set $O\subset X$ and an interval $[a,b]\subset\mathbb{R}$. If $h:[a,b]\to\mathbb{R}$ is a nondecreasing function, we say that a function $G: O\times[a,b]\to X$ belongs to the class $\mathcal{F}(O\times[a,b],h)$, if

$$\|G(x,s_2)-G(x,s_1)\|\leqslant|h(s_2)-h(s_1)|$$

for all (x,s_2), $(x,s_1)\in O\times[a,b]$ and

$$\|G(x,s_2)-G(x,s_1)-G(y,s_2)+G(y,s_1)\|\leqslant\|x-y\|\cdot|h(s_2)-h(s_1)|$$

for all (x,s_2), (x,s_1), (y,s_2), $(y,s_1)\in O\times[a,b]$.

When the right-hand side of the generalized ODE (33) satisfies the above mentioned conditions, we have the following information about its solutions. See [23, Lemma 3.12] for a proof.

Proposition 3.1. *Let X be a Banach space. Consider an open set $O\subset X$, an interval $[a,b]\subset\mathbb{R}$ and a function $G:O\times[a,b]\to X$. If $x:[a,b]\to O$ is a solution of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=\mathrm{D}G(x,t)$$

and $G\in\mathcal{F}(O\times[a,b],h)$, then x is a regulated function.

4. MEASURE NFDE AND GENERALIZED ODES

In this section, our goal is to establish a one-to-one correspondence between solutions of a measure NFDE of type

$$\mathrm{D}[N(t)x_t]=f(x_t,t)\mathrm{D}g$$

and solutions of a class of generalized ODEs.

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In what follows, we will show that, under certain assumptions, a measure NFDE with the following integral form

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s) + \int_{-r}^{0} \mathrm{d}_\theta[\mu(t,\theta)]y(t+\theta) - \int_{-r}^{0} \mathrm{d}_\theta[\mu(t_0,\theta)]y(t_0+\theta), \quad (6)$$

with a regulated solution $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$, can be converted into a generalized ODE of the form

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}G(x,t), \quad (7)$$

where $x : [t_0, t_0 + \sigma] \to O$.

We introduce the notation $[\cdot,\cdot,\cdot]$ where

$$[a,b,c] = \begin{cases} b, & \text{if } b \in [a,c] \\ a, & \text{if } b \leqslant a \\ c, & \text{if } b \geqslant c. \end{cases}$$

The function $G : O \times [t_0, t_0+\sigma] \to X$ on the right-hand side of the generalized ODE (7) is defined by

$$G(y,t)(\vartheta) = F(y,t)(\vartheta) + J(y,t)(\vartheta), \quad (8)$$

where for every $y \in O$ and $t \in [t_0, t_0 + \sigma]$ the functions F and J are given by

$$F(y,t)(\vartheta) = \int_{t_0}^{[t_0,\vartheta.t]} f(y_s,s)\,\mathrm{d}g(s) \quad (9)$$

and

$$J(y,t)(\vartheta) = \int_{-r}^{0} \mathrm{d}_\theta[\mu([t_0,\vartheta,t],\theta)]y([t_0,\vartheta,t]+\theta) - \int_{-r}^{0} \mathrm{d}_\theta[\mu(t_0,\theta)]y(t_0+\theta). \quad (10)$$

As we will verify, the relation between a solution x of the generalized ODE (7) and a solution y of the measure NFDE (6) is described by

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in [t_0 - r, t], \\ y(t), & \vartheta \in [t, t_0+\sigma], \end{cases}$$

where $t \in [t_0, t_0+\sigma]$. As a matter of fact, we need to relate initial value problems involving the two equations (7) and (6), their initial data and their unique solutions.

Because we need to ensure that if $y \in O$, then $x(t) \in O$ for every $t \in [t_0, t_0+\sigma]$, we have to assume a prolongation property introduced in the papers [17, 22].

DEFINITION 4.1. Let O be a subset of $G([t_0-r, t_0+\sigma], \mathbb{R}^n)$. We say that O has the prolongation property, if for every $y \in O$ and every $\bar{t} \in [t_0-r, t_0+\sigma]$, the function $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), & t_0 - r \leqslant t \leqslant \bar{t}, \\ y(\bar{t}), & \bar{t} < t \leqslant t_0 + \sigma \end{cases}$$

is also an element of O.

Here, we consider the sets $O \subset G([t_0-r, t_0+\sigma], \mathbb{R}^n)$ satisfying the prolongation property and $P = \{y_t;\ y \in O,\ t \in [t_0, t_0+\sigma]\} \subset G([-r, 0], \mathbb{R}^n)$, and the functions $f : P \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ and $g : [t_0, t_0+\sigma] \to \mathbb{R}$ with g being nondecreasing. Furthermore, we consider the following conditions:

(H1) The Kurzweil-Henstock-Stieltjes integral $\int_{t_0}^{t_0+\sigma} f(y_t, t)\,\mathrm{d}g(t)$ exists for every $y \in O$.

(H2) There exists a function $M : [t_0, t_0+\sigma] \to \mathbb{R}$ which is Lebesgue-Stieltjes integrable with respect to g such that

$$\left| \int_{t_0}^{t} f(y, s)\mathrm{d}g(s) \right| \leqslant \int_{t_0}^{t} M(s)\mathrm{d}g(s)$$

for every $y \in P$ and every $t \in [t_0, t_0+\sigma]$.

(H3) There exists a function $L : [t_0, t_0+\sigma] \to \mathbb{R}$ which is Lebesgue-Stieltjes integrable with respect to g such that

$$\left| \int_{t_0}^{t} [f(y, s) - f(z, s)]\mathrm{d}g(s) \right| \leqslant \int_{t_0}^{t} L(s)\|y - z\|_\infty \mathrm{d}g(s)$$

for every $y, z \in P$ and every $t \in [t_0, t_0+\sigma]$.

We also assume the following conditions on the normalized function $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$:

(H4) $\mu(t, \cdot)$ is left-continuous on $(-r, 0)$, of bounded variation on $[-r, 0]$ and the variation of $\mu(t, \cdot)$, $\operatorname{var}_{[s,0]} \mu(t, \cdot)$, on $[s, 0]$ tends to zero as $s \to 0$.

(H5) There exists a Lebesgue integrable function $C : [t_0, t_0+\sigma] \to \mathbb{R}$ such that for every $s_1, s_2 \in [t_0, t_0+\sigma]$ and $z \in O$

$$\left| \int_{-r}^{0} \mathrm{d}_\theta \mu(s_2, \theta) z(s_2+\theta) - \int_{-r}^{0} \mathrm{d}_\theta \mu(s_1, \theta) z(s_1+\theta) \right| \leqslant \int_{s_1}^{s_2} C(s) \int_{-r}^{0} \mathrm{d}_\theta \mu(s, \theta) \|z(s+\theta)\| \mathrm{d}s,$$

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In what follows, we consider an arbitrary element $\tilde{x} \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ and we restrict our attention to the case when

$$O = B_c = \{z_t \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n),\ \|z_t - \tilde{x}\| < c\},$$

and

$$P = P_c = \{y_t;\ y \in B_c,\ t \in [t_0, t_0 + \sigma]\},\ \text{for } c \geqslant 1.$$

The next lemma gives sufficient conditions for the function G given by (8) to belong to the class $\mathcal{F}(\Omega, h)$.

LEMMA 4.1. *Let $B_c = \{z_t \in P; \|z_t - \tilde{x}\| < c\}$, where $c \geqslant 1$, and $P_c = \{y_t;\ y \in B_c,\ t \in [t_0, t_0 + \sigma]\}$. Assume that $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a nondecreasing function and $f : B_c \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions* (H1), (H2), (H3)*. Moreover, suppose the normalized function $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ satisfies conditions* (H4) *and* (H5)*. Then the function $G : B_c \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ given by* (8) *belongs to the class $\mathcal{F}(B_c \times [t_0, t_0 + \sigma], h)$, where $h = h_1 + h_2$ with $h_1, h_2 : [t_0, t_0 + \sigma] \to \mathbb{R}$ given by*

$$h_1(t) = \int_{t_0}^{t} [L(s) + M(s)]\mathrm{d}g(s)$$

and

$$h_2(t) = \int_{t_0}^{t} C(s)\, \mathrm{var}_{\theta\in[-r,0]}\, \mu(s, \theta)\mathrm{d}s\, (\|\tilde{x}\|_\infty + c)\,.$$

Proof. At first, we will proof that $F \in \mathcal{F}(B_c \times [t_0, t_0 + h], h_1)$.

Condition (H1) implies that the integrals in the definition of F exist. Given $y \in B_c$ and $t_0 \leqslant s_1 < s_2 \leqslant t_0 + \sigma$, we see that

$$F(y, s_2)(\vartheta) - F(y, s_1)(\vartheta) = \int_{s_1}^{[s_1,\vartheta,s_2]} f(y_s, s)\, \mathrm{d}g(s). \tag{11}$$

Hence, for an arbitrary $y \in B_c$ and for $t_0 \leqslant s_1 < s_2 \leqslant t_0 + \sigma$, condition (H2) implies

$$\begin{aligned}
\|F(y, s_2) - F(y, s_1)\|_\infty &= \sup_{t_0 - r\leqslant\vartheta\leqslant t_0+\sigma} |F(y, s_2)(\vartheta) - F(y, s_1)(\vartheta)| \\
&= \sup_{s_1\leqslant\vartheta\leqslant s_2} |F(y, s_2)(\vartheta) - F(y, s_1)(\vartheta)| \\
&= \sup_{s_1\leqslant\vartheta\leqslant s_2} \left| \int_{s_1}^{\vartheta} f(y_s, s)\, \mathrm{d}g(s) \right| \\
&\leqslant \int_{s_1}^{s_2} M(s)\, \mathrm{d}g(s) \leqslant h_1(s_2) - h_1(s_1).
\end{aligned}$$

Similarly, by condition (H3), if $y, z \in B_c$ and $t_0 \leqslant s_1 \leqslant s_2 \leqslant t_0 + \sigma$, then

$$\begin{aligned}
\|F(y,s_2)-&F(y,s_1) - F(z,s_2) + F(z,s_1)\|_\infty \\
&= \sup_{s_1 \leqslant \vartheta \leqslant s_2} |F(y,s_2)(\vartheta) - F(y,s_1)(\vartheta) - F(z,s_2)(\vartheta) + F(z,s_1)(\vartheta)| \\
&= \sup_{s_1 \leqslant \vartheta \leqslant s_2} \left| \int_{s_1}^{\vartheta} [f(y_s,s) - f(z_s,s)] \, \mathrm{d}g(s) \right| \\
&\leqslant \sup_{s_1 \leqslant \vartheta \leqslant s_2} \int_{s_1}^{\tau} L(s) \|y_s - z_s\|_\infty \, \mathrm{d}g(s) \\
&\leqslant \|y - z\|_\infty \int_{s_1}^{s_2} L(s) \, \mathrm{d}g(s) \leqslant \|y - z\|_\infty (h_1(s_2) - h_1(s_1))
\end{aligned}$$

(note that the function $s \mapsto \|y_s - z_s\|_\infty$ is regulated according to Proposition 2.1, and therefore the integral $\int_{s_1}^{\vartheta} L(s)\|y_s - z_s\|_\infty \, \mathrm{d}g(s)$ exists). Thus $F \in \mathcal{F}(B_c \times [t_0, t_0 + \sigma], h_1)$.

Now, we will prove that $J \in \mathcal{F}(B_c \times [t_0, t_0 + \sigma], h_2)$. Given $x \in B_c$ and for $t_0 \leqslant s_1 < s_2 < t_0 + \sigma$, by the definition of the function $J : B_c \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ given in (10), we have

$$\begin{aligned}
&J(y,s_2)(\vartheta) - J(y,s_1)(\vartheta) = \\
&\qquad \int_{-r}^{0} d_\theta[\mu([s_1,\vartheta,s_2],\theta)] y([s_1,\vartheta,s_2] + \theta) - \int_{-r}^{0} d_\theta[\mu(s_1,\theta)] y(s_1 + \theta),
\end{aligned} \tag{12}$$

which implies

$$\begin{aligned}
&J(z,s_2)(\vartheta) - J(z,s_1)(\vartheta) - J(y,s_2)(\vartheta) + J(y,s_1)(\vartheta) = \\
&\qquad \int_{-r}^{0} d_\theta[\mu([s_1,\vartheta,s_2],\theta)][z([s_1,\vartheta,s_2] + \theta) - y([s_1,\vartheta,s_2] + \theta)] \\
&\qquad\qquad - \int_{-r}^{0} d_\theta[\mu(s_1,\theta)][z(s_1 + \theta) - y(s_1 + \theta)],
\end{aligned} \tag{13}$$

for $z, y \in B_c$ and $t_0 \leqslant s_1 < s_2 < t_0 + \sigma$.

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Hence, using (12) and condition (H5), we obtain

$$\begin{aligned}\|J(x,s_2)-J(x,s_1)\| &= \sup_{\vartheta\in[t_0-r,t_0+\sigma]}|J(x,s_2)(\vartheta)-J(x,s_1)(\vartheta)|\\ &= \sup_{\vartheta\in[s_1,s_2]}|J(x,s_2)(\vartheta)-J(x,s_1)(\vartheta)|\\ &= \sup_{\vartheta\in[s_1,s_2]}\left|\int_{-r}^{0}\mathrm{d}_\theta[\mu(\vartheta,\theta)]x(\vartheta+\theta)-\int_{-r}^{0}\mathrm{d}_\theta[\mu(s_1,\theta)]x(s_1+\theta)\right|\\ &\leqslant \int_{s_1}^{s_2}C(s)\int_{-r}^{0}\mathrm{d}_\theta[\mu(s,\theta)]\|x(s+\theta)\|\mathrm{d}s\\ &\leqslant \int_{s_1}^{s_2}C(s)\int_{-r}^{0}\mathrm{d}_\theta[\mu(s,\theta)]\mathrm{d}s\|x\|_\infty\\ &\leqslant \int_{s_1}^{s_2}C(s)\int_{-r}^{0}\mathrm{d}_\theta[\mu(s,\theta)]\mathrm{d}s\,(\|\tilde{x}\|_\infty+c)\leqslant h_2(s_2)-h_2(s_1).\end{aligned}$$

Similarly, (13) implies

$$\begin{aligned}&\|J(x,s_2)-J(x,s_1)-J(y,s_2)+J(y,s_1)\|\\ &\quad= \sup_{\vartheta\in[t_0-r,t_0+\sigma]}|J(x,s_2)(\vartheta)-J(x,s_1)(\vartheta)-J(y,s_2)(\vartheta)+J(y,s_1)(\vartheta)|\\ &\quad= \sup_{\vartheta\in[s_1,s_2]}|J(x,s_2)(\vartheta)-J(x,s_1)(\vartheta)-J(y,s_2)(\vartheta)+J(y,s_1)(\vartheta)|\\ &\quad\leqslant \sup_{\vartheta\in[s_1,s_2]}\Bigg|\int_{-r}^{0}\mathrm{d}_\theta[\mu(\vartheta,\theta)][x(\vartheta+\theta)-y(\vartheta+\theta)]\\ &\qquad\qquad -\int_{-r}^{0}\mathrm{d}_\theta[\mu(s_1,\theta)][x(s_1+\theta)-y(s_1+\theta)]\Bigg|\\ &\quad\leqslant \int_{s_1}^{s_2}C(s)\int_{-r}^{0}\mathrm{d}_\theta[\mu(s,\theta)]|x(s+\theta)-y(s+\theta)|\mathrm{d}s\\ &\quad\leqslant \int_{s_1}^{s_2}C(s)\,\mathrm{var}_{\theta\in[-r,0]}\,\mu(s,\theta)\|x_s-y_s\|_\infty\mathrm{d}s\\ &\quad\leqslant \int_{s_1}^{s_2}C(s)\,\mathrm{var}_{\theta\in[-r,0]}\,\mu(s,\theta)ds\left(\frac{\|x-y\|_\infty}{\|\tilde{x}\|+c}(\|\tilde{x}\|+c)\right)\\ &\quad\leqslant \frac{\|x-y\|_\infty}{\|\tilde{x}\|+c}[h_2(s_2)-h_2(s_1)]\leqslant\|x-y\|_\infty[h_2(s_2)-h_2(s_1)].\end{aligned}$$

Therefore $J\in\mathcal{F}(B_c\times[t_0,t_0+\sigma],h_2)$.

Finally, using the fact that

$$G(x,t)(\vartheta)=F(x,t)(\vartheta)+J(x,t)(\vartheta),$$

it is clear that $G\in\mathcal{F}(B_c\times[t_0,t_0+\sigma],h)$. ∎

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The following statement is a slightly modified version of Lemma 3.3 from [4], which is concerned with impulsive functional differential equations. The proof from [4] can be carried out without any changes. Thus we omit its proof here.

Lemma 4.2. *Let $B_c = \{z_t \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n); \|z_t - \tilde{x}\| < c\}$, with $c \geqslant 1$ and $P_c = \{y_t;\ y \in B_c,\ t \in [t_0, t_0 + \sigma]\}$. Assume that $\phi \in P_c$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a nondecreasing function, and $f : P_c \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ is such that the integral $\int_{t_0}^{t_0+\sigma} f(y_t, t)\,\mathrm{d}g(t)$ exists for every $y \in P_c$. Moreover, suppose $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is a normalized function which satisfies conditions (H4) and (H5). Consider G given by (8) and assume that $x : [t_0, t_0 + \sigma] \to B_c$ is a solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}G(x, t)$$

with initial condition $x(t_0)(\vartheta) = \phi(\vartheta)$ for $\vartheta \in [t_0 - r, t_0]$, and $x(t_0)(\vartheta) = x(t_0)(t_0)$ for $\vartheta \in [t_0, t_0 + \sigma]$. If $v \in [t_0, t_0 + \sigma]$ and $\vartheta \in [t_0 - r, t_0 + \sigma]$, then

$$x(v)(\vartheta) = x(v)(v), \quad \vartheta \geqslant v,$$

and

$$x(v)(\vartheta) = x(\vartheta)(\vartheta), \quad v \geqslant \vartheta.$$

The proofs of the following two theorems are inspired by similar proofs from papers [5, 25].

Theorem 4.1. *Let $B_c = \{z \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n); \|z - \tilde{x}\| < c\}$, with $c \geqslant 1$, $P_c = \{x_t;\ x \in B_c,\ t \in [t_0, t_0 + \sigma]\}$ $\phi \in P_c$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a nondecreasing function, $f : P_c \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions (H1)-(H3). Moreover, suppose the normalized function $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ satisfies conditions (H4) and (H5). Let $G : B_c \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ be given by (8) and assume that $G(x, t) \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ for every $x \in B_c$, $t \in [t_0, t_0 + \sigma]$. Let $y \in P_c$ be a solution of the measure neutral functional differential equation*

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s) + \int_{-r}^{0} \mathrm{d}_\theta[\mu(t, \theta)]y(t + \theta) - \int_{-r}^{0} \mathrm{d}_\theta[\mu(t_0, \theta)]y(t_0 + \theta) \quad (14)$$

on $[t_0, t_0 + \sigma]$ subjected to the initial condition $y_{t_0} = \phi$. For every $t \in [t_0 - r, t_0 + \sigma]$, let

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in [t_0 - r, t], \\ y(t), & \vartheta \in [t, t_0 + \sigma]. \end{cases}$$

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Then the function $x : [t_0, t_0+\sigma] \to B_c$ is a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}G(x,t).$$

Proof. We will show that, for every $v \in [t_0, t_0+\sigma]$, the integral $\int_{t_0}^{v} \mathrm{D}G(x(\tau),t)$ exists and

$$x(v) - x(t_0) = \int_{t_0}^{v} \mathrm{D}G(x(\tau),t).$$

Let an arbitrary $\varepsilon > 0$ be given. Since g is nondecreasing, it can have only a finite number of points $t \in [t_0, v]$ such that $\Delta^+ g(t) \geqslant \varepsilon$. Denote these points by $t_1, \ldots, t_m$. Consider a gauge $\delta : [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\delta(\tau) < \min\left\{\frac{t_k - t_{k-1}}{2},\ k = 2, \ldots, m\right\},\quad \tau \in [t_0, t_0+\sigma]$$

and

$$\delta(\tau) < \min\left\{|\tau - t_k|, |\tau - t_{k-1}|;\ \ \tau \in (t_{k-1}, t_k),\ k = 1, \ldots, m\right\}.$$

These conditions assure that if a point-interval pair $(\tau, [c,d])$ is δ-fine, then $[c,d]$ contains at most one of the points $t_1, \ldots, t_m$, and, moreover, $\tau = t_k$ whenever $t_k \in [c,d]$.

Since $y_{t_k} = x(t_k)_{t_k}$, it follows from Theorem 3.1 that

$$\lim_{s \to t_k+} \int_{t_k}^{s} L(s)\|y_s - x(t_k)_s\|_\infty \,\mathrm{d}g(s) = L(t_k)\|y_{t_k} - x(t_k)_{t_k}\|_\infty \Delta^+ g(t_k) = 0$$

for every $k \in \{1, \ldots, m\}$. Thus the gauge δ can be chosen in such a way that

$$\int_{t_k}^{t_k+\delta(t_k)} L(s)\|y_s - x(t_k)_s\|_\infty \,\mathrm{d}g(s) < \frac{\varepsilon}{4m+1},\ \ k \in \{1, \ldots, m\}.$$

and, also,

$$\int_{t_k}^{t_k+\delta(t_k)} C(s)\,\mathrm{var}_{\theta\in[-r,0]}\,\mu(s,\theta)\|y_s - x(t_k)_s\|_\infty ds < \frac{\varepsilon}{4m+1},\quad k \in \{1, \ldots, m\}.$$

Using Theorem 3.1 again, we obtain

$$\|y(\tau+t) - y(\tau)\| \leqslant h(t+\tau) - h(\tau),$$

and, therefore,

$$\|y(\tau+) - y(\tau)\| \leqslant \Delta^+ h(\tau) < \varepsilon,\quad \tau \in [t_0, t_0+\sigma] \backslash \{t_1, \ldots, t_m\}.$$

Thus, we can assume that the gauge δ is such that

$$\|y(\rho) - y(\tau)\| \leqslant \varepsilon$$

for every $\tau \in [t_0, t_0+\sigma] \backslash \{t_1, \dots, t_m\}$ and $\rho \in [\tau, \tau+\delta(\tau))$.

Now, assume that $\{(\tau_i, [s_{i-1}, s_i]),\ i = 1, \dots, l\}$ is a δ-fine tagged partition of the interval $[t_0, v]$. Using the definition of x, it can be easily shown that

$$\begin{aligned}\big(x(s_i) - x(s_{i-1})\big)(\vartheta) &= \int_{s_{i-1}}^{[s_{i-1},\vartheta,s_i]} f(y_s, s)\,\mathrm{d}g(s)\\ &+ \int_{-r}^{0} \mathrm{d}_\theta[\mu([s_{i-1},\vartheta,s_i],\theta)]y([s_{i-1},\vartheta,s_i]+\theta) - \int_{-r}^{0} \mathrm{d}_\theta[\mu(s_{i-1},\theta)]y(s_{i-1}+\theta),\end{aligned}$$

Similarly, it follows from the definition of G that

$$\begin{aligned}&\big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})\big](\vartheta) =\\ &\qquad \big[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big](\vartheta) + \big[J(x(\tau_i), s_i) - J(x(\tau_i), s_{i-1})\big](\vartheta),\end{aligned}$$

where

$$\big[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})\big](\vartheta) = \int_{s_{i-1}}^{[s_{i-1},\vartheta,s_1]} f(x(\tau_i)_s, s)\,\mathrm{d}g(s).$$

Moreover,

$$\begin{aligned}&\big[J(x(\tau_i), s_i) - J(x(\tau_i), s_{i-1})\big](\vartheta) =\\ &\qquad \int_{-r}^{0} \mathrm{d}_\theta[\mu([s_{i-1},\vartheta,s_i],\theta)]x(\tau_i)([s_{i-1},\vartheta,s_i]+\theta)\\ &\qquad\qquad - \int_{-r}^{0} \mathrm{d}_\theta[\mu(s_{i-1},\theta)]x(\tau_i)(s_{i-1}+\theta).\end{aligned}$$

By combination of the previous equalities, we obtain

$$\begin{aligned}&\big[x(s_i) - x(s_{i-1})\big](\vartheta) - \big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})\big](\vartheta)\\ &= \int_{s_{i-1}}^{[s_{i-1},\vartheta,s_i]} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\,\mathrm{d}g(s)\\ &+ \int_{-r}^{0} \mathrm{d}_\theta[\mu([s_{i-1},\vartheta,s_i],\theta)](y([s_{i-1},\vartheta,s_i]+\theta) - x(\tau_i)(\vartheta+\theta))\\ &- \int_{-r}^{0} \mathrm{d}_\theta[\mu(s_{i-1},\theta)](y(s_{i-1}+\theta) - x(\tau_i)(s_{i-1}+\theta)).\end{aligned}$$

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Consequently,

$$\begin{aligned}
&\|x(s_i) - x(s_{i-1}) - \big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})\big]\|_\infty \\
&\quad = \sup_{\vartheta\in[t_0-r,t_0+\sigma]} \Big|\big[x(s_i) - x(s_{i-1})\big](\vartheta) - \big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})\big](\vartheta)\Big| \\
&\quad = \sup_{\vartheta\in[s_{i-1},s_i]} \bigg|\int_{s_{i-1}}^{\vartheta} \big[f(y_s, s) - f(x(\tau_i)_s, s)\big]\, \mathrm{d}g(s) \\
&\qquad\qquad + \int_{-r}^{0} \mathrm{d}_\theta[\mu(\vartheta,\theta)](y(\vartheta+\theta) - x(\tau_i)(\vartheta+\theta)) \\
&\qquad\qquad - \int_{-r}^{0} \mathrm{d}_\theta[\mu(s_{i-1},\theta)](y(s_{i-1}+\theta) - x(\tau_i)(s_{i-1}+\theta))\bigg| \\
&\quad \leqslant \sup_{\vartheta\in[s_{i-1},s_i]} \bigg|\int_{s_{i-1}}^{\vartheta} \big[f(y_s, s) - f(x(\tau_i)_s, s)\big]\, \mathrm{d}g(s)\bigg| \\
&\qquad + \sup_{\vartheta\in[s_{i-1},s_i]} \bigg|\int_{-r}^{0} \mathrm{d}_\theta[\mu(\vartheta,\theta)](y(\vartheta+\theta) - x(\tau_i)(\vartheta+\theta)) \\
&\qquad\qquad - \int_{-r}^{0} \mathrm{d}_\theta[\mu(s_{i-1},\theta)](y(s_{i-1}+\theta) - x(\tau_i)(s_{i-1}+\theta))\bigg|.
\end{aligned}$$

By the definition of x, $x(\tau_i)_s = y_s$ whenever $s \leqslant \tau_i$. Thus,

$$\int_{s_{i-1}}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\, \mathrm{d}g(s) = \begin{cases} 0, & \vartheta \in [s_{i-1}, \tau_i], \\ \displaystyle\int_{\tau_i}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\, \mathrm{d}g(s), & \vartheta \in [\tau_i, s_i]. \end{cases} \tag{15}$$

Then, by condition (H3), we obtain

$$\bigg|\int_{\tau_i}^{\vartheta} \big(f(y_s, s) - f(x(\tau_i)_s, s)\big)\, \mathrm{d}g(s)\bigg| \leqslant \int_{\tau_i}^{\vartheta} L(s)\|y_s - x(\tau_i)_s\|_\infty\, \mathrm{d}g(s) \leqslant \int_{\tau_i}^{s_i} L(s)\|y_s - x(\tau_i)_s\|_\infty\, \mathrm{d}g(s)$$

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and, analogously, condition (H5) implies

$$\int_{s_{i-1}}^{\vartheta} C(s)\operatorname{var}_{\theta\in[-r,0]}\mu(s,\theta)\|y_s - x(\tau_i)_s\|_\infty \mathrm{d}s \\ = \begin{cases} 0, & \vartheta\in[s_{i-1},\tau_i] \\ \displaystyle\int_{\tau_i}^{\vartheta} C(s)\operatorname{var}_{\theta\in[-r,0]}\mu(s,\theta)\|y_s - x(\tau_i)_s\|_\infty \mathrm{d}s, & \vartheta\in[\tau_i,s_i] \end{cases} \tag{16}$$

Given a particular point-interval pair $(\tau_i, [s_{i-1}, s_i])$, there are two possibilities:

(i) The intersection of $[s_{i-1}, s_i]$ and $\{t_1,\dots,t_m\}$ contains a single point $t_k = \tau_i$.
(ii) The intersection of $[s_{i-1}, s_i]$ and $\{t_1,\dots,t_m\}$ is empty.

Consider case (i). As it was explained before, it follows from the definition of the gauge δ that

$$\int_{\tau_i}^{s_i} L(s)\|y_s - x(\tau_i)_s\|_\infty \,\mathrm{d}g(s) \leqslant \frac{\varepsilon}{4m+1},$$

$$\int_{\tau_i}^{s_i} C(s)\operatorname{var}_{\theta\in[-r,0]}\mu(s,\theta)\|y_s - x(\tau_i)_s\|_\infty \mathrm{d}s < \frac{\varepsilon}{4m+1}.$$

In case (ii), we have

$$\|y_s - x(\tau_i)_s\|_\infty = \sup_{\rho\in[\tau_i,s]}\|y(\rho)-y(\tau_i)\| \leqslant \varepsilon, \quad s\in[\tau_i,s_i],$$

by the definition of the gauge δ.

Combining cases (i) and (ii) and using the fact that case (i) occurs at most $2m$ times, we obtain

$$\begin{aligned} &\left\| x(v) - x(t_0) - \sum_{i=1}^{l}\big[G(x(\tau_i),s_i) - G(x(\tau_i),s_{i-1})\big]\right\|_\infty \\ &< \varepsilon\int_{t_0}^{t_0+\sigma} L(s)\,\mathrm{d}g(s) + \varepsilon\int_{t_0}^{t_0+\sigma} C(s)\operatorname{var}_{\theta\in[-r,0]}\mu(s,\theta)\mathrm{d}s + \frac{4m\varepsilon}{4m+1} \\ &< \varepsilon\left(\int_{t_0}^{t_0+\sigma} L(s)\,\mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} C(s)\operatorname{var}_{\theta\in[-r,0]}\mu(s,\theta)\mathrm{d}s + 1\right). \end{aligned}$$

Since ε is arbitrary, it follows that

$$x(v) - x(t_0) = \int_{t_0}^{v} \mathrm{D}G(x(\tau),t)$$

and we obtain the desired result. ∎

Now, we proof the reciprocal result.

THEOREM 4.2. *Let* $B_c = \{z \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n); \|z - \tilde{x}\| < c\}$, *with* $c \geqslant 1$, $P_c = \{z_t; z \in B_c,\ t \in [t_0, t_0 + \sigma]\}$, $\phi \in P_c$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ *be a nondecreasing function and let* $f : P_c \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ *satisfy conditions* (H1)-(H3). *Assume that the normalized function* $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ *satisfies conditions*(H4) *and* (H5). *Let* $G : B_c \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ *be given by* (8) *and assume that* $G(x, t) \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ *for every* $x \in B_c$ *and* $t \in [t_0, t_0 + \sigma]$. *Let* $x : [t_0, t_0 + \sigma] \to B_c$ *be a solution of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}G(x, t),$$

with the initial condition

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & t_0 - r \leqslant \vartheta \leqslant t_0, \\ x(t_0)(t_0), & t_0 \leqslant \vartheta \leqslant t_0 + \sigma. \end{cases}$$

Then, the function $y \in B_c$ *defined by*

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \leqslant \vartheta \leqslant t_0, \\ x(\vartheta)(\vartheta), & t_0 \leqslant \vartheta \leqslant t_0 + \sigma. \end{cases}$$

is a solution of the measure neutral functional differential equation

$$\begin{cases} y(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s) \\ \qquad\qquad + \displaystyle\int_{-r}^{0} \mathrm{d}_\theta \mu(t, \theta) y(t + \theta) - \int_{-r}^{0} \mathrm{d}_\theta \mu(t_0, \theta) y(t_0 + \theta), \\ y_{t_0} = \phi \end{cases} \tag{17}$$

on $t \in [t_0 - r, t_0 + \sigma]$.

Proof. The equality $y_{t_0} = \phi$ follows easily from the definitions of y and $x(t_0)$. It remains to prove that, if $v \in [t_0, t_0 + \sigma]$, then

$$y(v) - y(t_0) = \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s) + \int_{-r}^{0} \mathrm{d}_\theta \mu(v, \theta) y(v + \theta) - \int_{-r}^{0} \mathrm{d}_\theta \mu(t_0, \theta) y(t_0 + \theta).$$

But, using Lemma 4.2, we obtain

$$y(v) - y(t_0) = x(v)(v) - x(t_0)(v) = \left(\int_{t_0}^{v} \mathrm{D}G(x(\tau), t) \right)(v).$$

Thus

$$\begin{aligned} y(v)-y(t_0)-\int_{t_0}^{v} f(y_s,s)\,\mathrm{d}g(s)-\int_{-r}^{0}\mathrm{d}_\theta\mu(t,\theta)y(t+\theta)+\int_{-r}^{0}\mathrm{d}_\theta\mu(t_0,\theta)y(t_0+\theta) \\ =\left(\int_{t_0}^{v}\mathrm{D}G(x(\tau),t)\right)(v)-\int_{t_0}^{v} f(y_s,s)\,\mathrm{d}g(s) \\ -\int_{-r}^{0}\mathrm{d}_\theta\mu(v,\theta)y(t+\theta)+\int_{-r}^{0}\mathrm{d}_\theta\mu(t_0,\theta)y(t_0+\theta). \end{aligned} \tag{18}$$

Let an arbitrary $\varepsilon > 0$ be given. Since g is nondecreasing, it has only a finite number of points $t \in [t_0, v]$ such that $\Delta^+ g(t) \geqslant \varepsilon$. Denote these points by $t_1, \ldots, t_m$.

Consider a gauge $\delta : [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\delta(\tau) < \min\left\{\frac{t_k - t_{k-1}}{2},\ k = 2, \ldots, m\right\},\quad \tau \in [t_0, t_0 + \sigma]$$

and

$$\delta(\tau) < \min\{|\tau - t_k|,\ |\tau - t_{k-1}|;\quad \tau \in (t_{k-1}, t_k),\ k = 1, \ldots, m\}.$$

As in the proof of Theorem 4.1, these conditions assure that if a point-interval pair $(\tau, [c, d])$ is δ-fine, then $[c, d]$ contains at most one of the points $t_1, \ldots, t_m$, and, moreover, $\tau = t_k$ whenever $t_k \in [c, d]$.

Again, the gauge δ might be chosen in such a way that

$$\int_{t_k}^{t_k+\delta(t_k)} L(s)\|y_s - x(t_k)_s\|_\infty\,\mathrm{d}g(s) < \frac{\varepsilon}{4m+1},\quad k \in \{1, \ldots, m\} \tag{19}$$

and, also,

$$\int_{t_k}^{t_k+\delta(t_k)} C(s)\,\mathrm{var}_{\theta\in[-r,0]}\,\mu(s,\theta)\|y_s - x(t_k)_s\|_\infty ds < \frac{\varepsilon}{4m+1},\quad k \in \{1, \ldots, m\}. \tag{20}$$

According to Lemma 4.1, the function G given by (8) belongs to the class $\mathcal{F}(B_c \times [t_0, t_0 + \sigma], h)$, where

$$h(t) = \int_{t_0}^{t} [L(s) + M(s)]\mathrm{d}g(s) + \int_{t_0}^{t} C(s)\,\mathrm{var}_{\theta\in[-r,0]}\,\mu(s,\theta)\mathrm{d}s\,(\|\tilde{x}\|_\infty + c)\,. \tag{21}$$

By the definition of h given by (21), for every $\tau \in [t_0, t_0 + \sigma]\backslash\{t_1, \ldots, t_m\}$, we have

$$\|h(\tau + t) - h(\tau)\| < \varepsilon.$$

Thus, we can assume that the gauge δ satisfies

$$\|h(\rho) - h(\tau)\| \leqslant \varepsilon \ \text{ for every } \ \rho \in [\tau, \tau + \delta(\tau)).$$

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Finally, the gauge δ should be such that

$$\left\| \int_{t_0}^{v} \mathrm{D}G(x(\tau), t) - \sum_{i=1}^{l} \big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})\big] \right\|_{\infty} < \varepsilon \tag{22}$$

for every δ-fine partition $\{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, l\}$ of $[t_0, v]$. The existence of such a gauge follows from the definition of the Kurzweil integral. Choose a particular δ-fine partition $\{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, l\}$ of $[t_0, v]$. By (18) and (22), we have

$$\begin{aligned}
&\Big| y(v) - y(t_0) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s) \\
&\qquad - \int_{-r}^{0} \mathrm{d}_\theta \mu(v, \theta) y(v + \theta) + \int_{-r}^{0} \mathrm{d}_\theta \mu(t_0, \theta) y(t_0 + \theta) \Big| \\
&= \Big| \Big(\int_{t_0}^{v} \mathrm{D}G(x(\tau), t) \Big)(v) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s) \\
&\qquad - \int_{-r}^{0} \mathrm{d}_\theta \mu(v, \theta) y(v + \theta) + \int_{-r}^{0} \mathrm{d}_\theta \mu(t_0, \theta) y(t_0 + \theta) \Big| \\
&< \varepsilon + \Big| \sum_{i=1}^{l} \big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})\big](v) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s) \\
&\qquad - \int_{-r}^{0} \mathrm{d}_\theta \mu(s_i, \theta) y(s_i + \theta) + \int_{-r}^{0} \mathrm{d}_\theta \mu(s_{i-1}, \theta)(y(s_{i-1} + \theta) \Big| \\
&\leqslant \varepsilon + \sum_{i=1}^{l} \Big| \big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})\big](v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s) \\
&\qquad - \int_{-r}^{0} \mathrm{d}_\theta \mu(s_i, \theta) y(s_i + \theta) + \int_{-r}^{0} \mathrm{d}_\theta \mu(s_{i-1}, \theta) y(s_{i-1} + \theta) \Big|.
\end{aligned}$$

The definition of G yields

$$\begin{aligned}
\big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})\big](v) &= \int_{s_{i-1}}^{s_i} f(x(\tau_i)_s, s)\,\mathrm{d}g(s) \\
&\quad - \int_{-r}^{0} \mathrm{d}_\theta \mu(s_i, \theta) x(\tau_i)(s_i + \theta) + \int_{-r}^{0} \mathrm{d}_\theta \mu(s_{i-1}, \theta) x(\tau_i)(s_{i-1} + \theta),
\end{aligned}$$

which implies

$$\begin{aligned}
\Big| \big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})\big](v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s) \\
- \int_{-r}^{0} \mathrm{d}_\theta \mu(s_i, \theta) y(s_i + \theta) + \int_{-r}^{0} \mathrm{d}_\theta \mu(s_{i-1}, \theta) y(s_{i-1} + \theta) \Big| \\
\leqslant \left| \int_{s_{i-1}}^{s_i} f(x(\tau_i)_s, s)\,\mathrm{d}g(s) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s) \right| \\
+ \Big| \int_{-r}^{0} \mathrm{d}_\theta \mu(s_i, \theta)[x(\tau_i)(s_i + \theta) - y(s_i + \theta)] \\
- \int_{-r}^{0} \mathrm{d}_\theta \mu(s_{i-1}, \theta)[x(\tau_i)(s_{i-1} + \theta) - y(s_{i-1} + \theta)] \Big|.
\end{aligned}$$

By Lemma 4.2, for every $i \in \{1, \ldots, l\}$, we have $x(\tau_i)_s = x(s)_s = y_s$ for $s \in [s_{i-1}, \tau_i]$ and $y_s = x(s)_s = x(s_i)_s$ for $s \in [\tau_i, s_i]$. Thus

$$\begin{aligned}
\left| \int_{s_{i-1}}^{s_i} [f(x(\tau_i)_s, s) - f(y_s, s)]\,\mathrm{d}g(s) \right| = \left| \int_{\tau_i}^{s_i} [f(x(\tau_i)_s, s) - f(y_s, s)]\,\mathrm{d}g(s) \right| \\
= \left| \int_{\tau_i}^{s_i} [f(x(\tau_i)_s, s) - f(x(s_i)_s, s)]\,\mathrm{d}g(s) \right| \leqslant \int_{\tau_i}^{s_i} L(s) \|x(\tau_i)_s - x(s_i)_s\|_\infty\,\mathrm{d}g(s),
\end{aligned}$$

where the last inequality follows from condition (H3).

Using condition (H5),

$$\begin{aligned}
\Big| \int_{-r}^{0} \mathrm{d}_\theta \mu(s_i, \theta)[x(\tau_i)(s_i + \theta) - y(s_i + \theta)] \\
- \int_{-r}^{0} \mathrm{d}_\theta \mu(s_{i-1}, \theta)[x(\tau_i)(s_{i-1} + \theta) - y(s_{i-1} + \theta)] \Big| \\
\leqslant \int_{s_{i-1}}^{s_i} C(s) \int_{-r}^{0} \mathrm{d}_\theta \mu(s, \theta) |x(\tau_i)(s + \theta) - y(s + \theta)| \mathrm{d}s \\
\leqslant \int_{s_{i-1}}^{s_i} C(s) \operatorname{var}_{\theta \in [-r,0]} \mu(s, \theta) \|x(\tau_i)_s - y_s\|_\infty \mathrm{d}s \\
= \int_{\tau_i}^{s_i} C(s) \operatorname{var}_{\theta \in [-r,0]} \mu(s, \theta) \|x(\tau_i)_s - y_s\|_\infty \mathrm{d}s.
\end{aligned}$$

Again, we distinguish two cases:

(i) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \ldots, t_m\}$ contains a single point $t_k = \tau_i$.
(ii) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \ldots, t_m\}$ is empty.

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In case (i), it follows by (19) and (20) that

$$\int_{\tau_i}^{s_i} L(s)\|y_s - x(\tau_i)_s\|_\infty \,\mathrm{d}g(s) \leqslant \frac{\varepsilon}{4m+1}$$

and

$$\int_{\tau_i}^{s_i} C(s)\operatorname{var}_{\theta\in[-r,0]}\mu(s,\theta)\|x(\tau_i)_s - y_s\|_\infty \mathrm{d}s \leqslant \frac{\varepsilon}{4m+1}.$$

These conditions imply

$$\Big|\big[G(x(\tau_i),s_i) - G(x(\tau_i),s_{i-1})\big](v) - \int_{s_{i-1}}^{s_i} f(y_s,s)\,\mathrm{d}g(s) \\ - \int_{-r}^{0}\mathrm{d}_\theta\mu(s_i,\theta)y(s_i+\theta) + \int_{-r}^{0}\mathrm{d}_\theta\mu(s_{i-1},\theta)y(s_{i-1}+\theta)\Big| \leqslant \frac{2\varepsilon}{4m+1}.$$

In case (ii), we use Lemma 3.1 to obtain the estimate

$$\|x(s_i)_s - x(\tau_i)_s\|_\infty \leqslant \|x(s_i) - x(\tau_i)\|_\infty \leqslant h(s_i) - h(\tau_i) \leqslant \varepsilon,$$

for every $s \in [\tau_i, s_i]$, and thus

$$\Big|\big[G(x(\tau_i),s_i) - G(x(\tau_i),s_{i-1})\big](v) \\ - \int_{s_{i-1}}^{s_i} f(y_s,s)\,\mathrm{d}g(s) \int_{-r}^{0} d\mu(s_i,\theta)y(s_i+\theta) + \int_{-r}^{0}\mathrm{d}_\theta\mu(s_{i-1},\theta)y(s_{i-1}+\theta)\Big| \\ \leqslant \varepsilon\int_{\tau_i}^{s_i} L(s)\,\mathrm{d}g(s) + \varepsilon\int_{\tau_i}^{s_i} C(s)\operatorname{var}_{\theta\in[-r,0]}\mu(s,\theta)\mathrm{d}s.$$

Combining cases (i), (ii) and using the fact that case (i) occurs at most $2m$ times, we obtain

$$\sum_{i=1}^{l}\Big|\big[G(x(\tau_i),s_i) - G(x(\tau_i),s_{i-1})\big](v) \\ - \int_{s_{i-1}}^{s_i} f(y_s,s)\,\mathrm{d}g(s) - \int_{-r}^{0}\mathrm{d}_\theta\mu(s_i,\theta)y(s_i+\theta) + \int_{-r}^{0}\mathrm{d}_\theta\mu(s_{i-1},\theta)y(s_{i-1}+\theta)\Big| \\ \leqslant \varepsilon\int_{t_0}^{t_0+\sigma} L(s)\,\mathrm{d}g(s) + \varepsilon\int_{t_0}^{t_0+\sigma} C(s)\operatorname{var}_{\theta\in[-r,0]}\mu(s,\theta)\mathrm{d}s + \frac{4m\varepsilon}{4m+1}.$$

Consequently,

$$\Bigg\| y(v) - y(t_0) - \int_{t_0}^{v} f(y_s, s)\, \mathrm{d}g(s) - \int_{-r}^{0} \mathrm{d}_\theta \mu(v,\theta) y(v+\theta) + \int_{-r}^{0} \mathrm{d}_\theta \mu(t_0,\theta) y(t_0+\theta) \Bigg\| < \varepsilon \left[1 + \int_{t_0}^{t_0+\sigma} L(s)\, \mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} C(s) \operatorname{var}_{\theta\in[-r,0]} \mu(s,\theta) \mathrm{d}s \right]$$

which completes the proof. ▮

5. EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this section, our goal is to obtain a result on the existence and uniqueness of solutions of measure NFDEs via the correspondence between these equations and generalized ODEs.

We start by presenting a known result concerning existence-uniqueness of solutions for generalized ODEs. See [4, Theorem 2.15].

THEOREM 5.1. *Assume that X is a Banach space, $O \subset X$ open and $G : O \times [t_0, t_0+\sigma] \to X$ belongs to the class $\mathcal{F}(O \times [t_0, t_0+\sigma], h)$, where $h : [t_0, t_0+\sigma] \to \mathbb{R}$ is a left-continuous nondecreasing function. If $x_0 \in O$ is such that $x_0 + G(x_0, t_0+) - G(x_0, t_0) \in O$, then there exists $\delta > 0$ and a function $x : [t_0, t_0+\delta] \to X$ which is the unique solution of the initial value problem*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}G(x,t), \qquad x(t_0) = x_0.$$

Next, we present an existence-uniqueness theorem for measure NFDEs.

THEOREM 5.2. *Let $B_c = \{z \in G([t_0-r, t_0+\sigma], \mathbb{R}^n); \|z - \tilde{x}\| < c\}$, with $c \geqslant 1$, $P_c = \{x_t;\ x \in B_c,\ t \in [t_0, t_0+\sigma]\}$, $g : [t_0, t_0+\sigma] \to \mathbb{R}$ be a left-continuous and nondecreasing function and let $f : P_c \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ satisfy conditions* (H1), (H2), (H3). *Assume the normalized function $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ satisfies conditions* (H4) *and* (H5). *If $\phi \in P_c$ is such that the function*

$$z(t) = \begin{cases} \phi(t-t_0), & t \in [t_0-r, t_0], \\ \phi(0) + f(\phi, t_0)\Delta^+ g(t_0), & t \in (t_0, t_0+\sigma] \end{cases}$$

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belongs to B_c, then there exists $\delta > 0$ and a function $y : [t_0 - r, t_0 + \delta] \to \mathbb{R}^n$ which is the unique solution of the initial value problem

$$\begin{cases} y(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s) \\ \qquad\qquad + \displaystyle\int_{-r}^{0} \mathrm{d}_\theta \mu(t,\theta) y(t+\theta) - \int_{-r}^{0} \mathrm{d}_\theta \mu(t_0,\theta) y(t_0+\theta), \\ y_{t_0} = \phi. \end{cases} \tag{23}$$

Proof. Let G be a function defined by (8). According to Lemma 4.1, this function belongs to the class $\mathcal{F}(B_c \times [t_0, t_0 + \sigma], h)$, where

$$h(t) = \int_{t_0}^{t} [M(s) + L(s)]\mathrm{d}g(s) + \int_{t_0}^{t} C(s)\,\mathrm{var}_{\theta\in[-r,0]}\,\mu(s,\theta)\|x\|_\infty ds.$$

Define

$$x_0(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & \vartheta \in [t_0 - r, t_0], \\ \phi(0), & \vartheta \in [t_0, t_0 + \sigma]. \end{cases}$$

It is clear that $x_0 \in B_c$.

We also claim that $x_0 + G(x_0, t_0+) - G(x_0, t_0) \in B_c$. At first, note that $G(x_0, t_0) = 0$. The limit $G(x_0, t_0+)$ is taken with respect to the supremum norm and we know it must exist since G is regulated with respect to the second variable. This follows from the fact that $G \in \mathcal{F}(B_c \times [t_0, t_0 + \sigma], h)$ and also, note that $J(x_0, t_0^+) = 0$ by condition (H5). Thus it is sufficient to evaluate the pointwise limit $F(x_0, t_0+)(\vartheta)$ for every $\vartheta \in [t_0 - r, t_0 + \sigma]$. Using Theorem 3.1, we obtain

$$F(x_0, t_0+)(\vartheta) = \begin{cases} 0, & t \in [t_0 - r, t_0], \\ f(\phi, t_0)\Delta^+ g(t_0), & t \in (t_0, t_0 + \sigma]. \end{cases}$$

Hence $x_0 + G(x_0, t_0+) - G(x_0, t_0) = x_0 + F(x_0, t_0^+) = z \in B_c$.

Since all the assumptions of Theorem 5.1 are satisfied, there exists $\delta > 0$ and a unique solution $x : [t_0, t_0 + \delta] \to X$ of the initial problem value

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}G(x, t), \quad x(t_0) = x_0. \tag{24}$$

According to Theorem 4.2, the function $y : [t_0 - r, t_0 + \delta] \to \mathbb{R}^n$ given by

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \leqslant \vartheta \leqslant t_0, \\ x(\vartheta)(\vartheta), & t_0 \leqslant \vartheta \leqslant t_0 + \delta \end{cases}$$

is a solution of the measure neutral functional differential equation

$$\begin{cases} y(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s) \\ \qquad\qquad + \displaystyle\int_{-r}^{0} \mathrm{d}_\theta \mu(s,\theta) y(s+\theta) - \int_{-r}^{0} \mu(t_0,\theta) y(t_0+\theta), \\ y_{t_0} = \phi. \end{cases} \tag{25}$$

This solution must be unique, otherwise Theorem 4.1 would imply that x is not the only solution of the generalized ODE (24). Thus the result follows. ▮

6. CONTINUOUS DEPENDENCE ON PARAMETERS

In this section, we use a known result on continuous dependence of solution on parameters for generalized ODEs in order to obtain analogous results for measure NFDEs.

We need an auxiliary result which the following Arzelà-Ascoli-type result for regulated functions which can be found in [7, Theorem 2.18].

THEOREM 6.1. *The following conditions are equivalent:*

1.A set $\mathcal{A} \subset G([\alpha,\beta],\mathbb{R}^n)$ is relatively compact.

2.The set $\{x(\alpha); x \in \mathcal{A}\}$ is bounded and there is an increasing continuous function $\eta : [0,\infty) \to [0,\infty)$, $\eta(0) = 0$ and an increasing function $K : [\alpha,\beta] \to \mathbb{R}$ such that

$$\|x(t_2) - x(t_1)\| \leqslant \eta(K(t_2) - K(t_1))$$

for every $x \in \mathcal{A}$, $\alpha \leqslant t_1 \leqslant t_2 \leqslant \beta$.

Next, we present a result on the continuous dependence of solutions on the initial data for generalized ODEs. Such result is a Banach space-valued version of Theorem 2.4 from [8]. The proof for the case $X = \mathbb{R}^n$ from [8] remains true in this more general setting.

THEOREM 6.2. *Let X be a Banach space, $O \subset X$ be an open set and $h_k : [a,b] \to \mathbb{R}$, $k = 0,1,2,\dots$, be a sequence of nondecreasing left-continuous functions such that $h_k(b) - h_k(a) \leqslant c$ for some $c > 0$ and every $k = 0,1,2,\dots$. Assume that, for every $k = 0,1,2,\dots$, $G_k : O \times [a,b] \to X$ belongs to the class $\mathcal{F}(O \times [a,b], h_k)$ and moreover*

$$\lim_{k\to\infty} G_k(x,t) = G_0(x,t), \quad x \in O,\ t \in [a,b],$$
$$\lim_{k\to\infty} G_k(x,t+) = G_0(x,t+) \quad x \in O,\ t \in [a,b).$$

For every $k = 1,2,\dots$, let $x_k : [a,b] \to O$ be a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}G_k(x,t).$$

If there exists a function $x_0 : [a,b] \to O$ *such that* $\lim_{k\to\infty} x_k(t) = x_0(t)$ *uniformly for* $t \in [a,b]$*, then* x_0 *is a solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}G_0(x,t), \quad t \in [a,b].$$

We point out that Theorem 2.4 in [8] assumes that the functions G_k are defined on $O \times (-T,T)$, where $[a,b] \subset (-T,T)$, and similarly the functions h_k are defined in the open interval $(-T,T)$. However, it is easy to extend the functions defined on $[a,b]$ to $(-T,T)$ by letting $G_k(x,t) = G_k(x,a)$ for $t \in (-T,a)$, $G_k(x,t) = G_k(x,b)$ for $t \in (b,T)$, and similarly for h_k. Note that the extended functions G_k now belong to the class $\mathcal{F}(O \times (-T,T), h_k)$, as assumed in [8].

Now, we are able to prove a theorem on the continuous dependence on parameters of solutions of measure NFDEs.

THEOREM 6.3. *Let* $B_c = \{z \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n); \|z - \tilde{x}\| < c\}$*, with* $c \geqslant 1$*,* $P_c = \{x_t;\ x \in B_c,\ t \in [t_0, t_0+\sigma]\}$*,* $g : [t_0, t_0+\sigma] \to \mathbb{R}$ *be a nondecreasing left-continuous function and* $f_k : P_c \times [t_0, t_0+\sigma] \to \mathbb{R}^n$*,* $k = 0,1,2,\ldots$*, be a sequence of functions which satisfy conditions* (H1)-(H3) *for the same functions* $L, M : [t_0, t_0+\sigma] \to \mathbb{R}$ *for every* $k = 0,1,2,\ldots$*. Suppose the normalized function* $\mu_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ *satisfies conditions* (H4) and (H5) *for the same function* $C : [t_0, t_0+\sigma] \to \mathbb{R}$ *for every* $k = 0,1,2,\ldots$*. Moreover, suppose*

1.For every $y \in B_c$*,*

$$\lim_{k\to\infty} \int_{t_0}^{t} f_k(y_s, s)\,\mathrm{d}g(s) = \int_{t_0}^{t} f_0(y_s, s)\,\mathrm{d}g(s)$$

uniformly with respect to $t \in [t_0, t_0+\sigma]$*.*

2.For every $y \in B_c$*,*

$$\lim_{k\to\infty} \int_{-r}^{0} \mathrm{d}_\theta \mu_k(t,\theta) y(t+\theta) = \int_{-r}^{0} \mathrm{d}_\theta \mu_0(t,\theta) y(t+\theta)$$

uniformly with respect to $t \in [t_0, t_0+\sigma]$*.*

Consider a sequence of functions $\phi_k \in P_c$*,* $k = 0,1,2,\ldots$*, such that*

$$\lim_{k\to\infty} \phi_k = \phi_0$$

uniformly in $[-r,0]$*. Let* $y_k \in B_c$*,* $k = 1,2,\ldots$*, be solutions of*

$$\begin{cases} y_k(t) = y_k(t_0) + \displaystyle\int_{t_0}^{t} f_k((y_k)_s, s)\,\mathrm{d}g(s) + \int_{-r}^{0} \mathrm{d}_\theta \mu_k(t,\theta) y_k(t+\theta) \\ \qquad\qquad - \displaystyle\int_{-r}^{0} \mathrm{d}_\theta \mu_k(t_0,\theta) y_k(t_0+\theta), \\ (y_k)_{t_0} = \phi_k \end{cases} \tag{26}$$

in $[t_0-r,t_0+\sigma]$. *If there exists a function* $y_0 \in B_c$ *such that* $\lim_{k\to\infty} y_k = y_0$ *in* $[t_0,t_0+\sigma]$, *then* $y_0 : [t_0-r,t_0+\sigma] \to \mathbb{R}^n$ *is a solution of*

$$\begin{cases} y_0(t) = y_0(t_0) + \displaystyle\int_{t_0}^{t} f_0((y_0)_s, s)\,\mathrm{d}g(s) + \int_{-r}^{0} \mathrm{d}_\theta \mu_0(t,\theta) y_0(t+\theta) \\ \qquad\qquad - \displaystyle\int_{-r}^{0} \mathrm{d}_\theta \mu_0(t_0,\theta) y_0(t_0+\theta), \\ (y_0)_{t_0} = \phi_0. \end{cases} \tag{27}$$

Proof. The assumptions imply that, for every $x \in B_c$, $\lim_{k\to\infty} G_k(x,t) = G_0(x,t)$ uniformly with respect to $t \in [t_0,t_0+\sigma]$. By the Moore-Osgood theorem, $\lim_{k\to\infty} G_k(x,t+) = G_0(x,t+)$ for every $x \in B_c$ and $t \in [t_0,t_0+\sigma)$. Besides, $G_0(x,t) \in G([t_0-r,t_0+\sigma],\mathbb{R}^n)$.

According to Lemma 4.1, $G_k \in \mathcal{F}(B_c \times [t_0,t_0+\sigma],h)$ for every $k = 1,2,\ldots$, where

$$h(t) = \int_{t_0}^{t} (M(s)+L(s))\mathrm{d}g(s) + \int_{t_0}^{t} C(s)\,\mathrm{var}_{\theta\in[-r,0]}\,\mu(s,\theta)\|x\|_\infty ds,$$

for every $t \in [t_0,t_0+\sigma]$. Then since $\lim_{k\to\infty} G_k(x,t) = G_0(x,t)$, we also have $G_0 \in \mathcal{F}(B_c \times [t_0,t_0+\sigma],h)$.

Given $k = 0,1,2,\ldots$ and $t \in [t_0,t_0+\sigma]$, let

$$x_k(t)(\vartheta) = \begin{cases} y_k(\vartheta), & \vartheta \in [t_0-r,t], \\ y_k(t), & \vartheta \in [t,t_0+\sigma]. \end{cases}$$

According to Theorem 4.1, x_k is a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}G_k(x,t).$$

Thus for $k = 1,2,\ldots$ and $t_0 \leqslant t_1 \leqslant t_2 \leqslant t_0+\sigma$, we have

$$\begin{aligned} |y_k(t_2)-y_k(t_1)| &= \Big|\int_{t_1}^{t_2} f_k((y_k)_s,s)\,\mathrm{d}g(s) + \int_{-r}^{0} \mathrm{d}_\theta\mu_k(t_2,\theta)y_k(t_2+\theta) \\ &\quad - \int_{-r}^{0} \mathrm{d}_\theta\mu_k(t_1,\theta)y_k(t_1+\theta)\Big| \leqslant h(t_2)-h(t_1) < h(t_2)-h(t_1)+(t_2-t_1), \end{aligned}$$

where $\eta(t) = t$ for every $t \in [0,\infty)$ and $K(t) = h(t)+t$ for every $t \in [t_0,t_0+\sigma]$.

Note that K is an increasing function. Moreover the sequence $\{y_k(t_0)\}_{k=1}^\infty$ is bounded. Thus condition 2 from Theorem 6.1 is fulfilled and hence the sequence $\{y_k\}_{k=1}^\infty$ contains a subsequence which is uniformly convergent in $[t_0,t_0+\sigma]$. Without loss of generality, we can denote this subsequence again by $\{y_k\}_{k=1}^\infty$. Since $(y_k)_{t_0} = \phi_k$, it follows that $\{y_k\}_{k=1}^\infty$ is, in fact, uniformly convergent in $[t_0-r,t_0+\sigma]$.

By the definition of x_k, we have

$$\lim_{k\to\infty} x_k(t) = x_0(t)$$

uniformly with respect to $t \in [t_0, t_0+\sigma]$. Then Theorem 6.2 implies that x_0 is a solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}G_0(x,t)$$

in $[t_0, t_0+\sigma]$. The proof is finished by applying Theorem 4.2, which guarantees that y_0 satisfies

$$\begin{cases} y_0(t) = y_0(t_0) + \int_{t_0}^{t} f_0((y_0)_s, s)\,\mathrm{d}g(s) + \int_{-r}^{0} \mathrm{d}_\theta[\mu_0(t,\theta)]y_0(t+\theta) \\ \qquad - \int_{-r}^{0} \mathrm{d}_\theta[\mu_0(t_0,\theta)]y_0(t_0+\theta), \\ (y_0)_{t_0} = \phi_0, \end{cases} \tag{28}$$

in $[t_0 - r, t_0+\sigma]$. ∎

7. AN EXAMPLE

We now present an example which illustrates how to get a solution of a generalized ODE, given a measure NFDE and its solution. This is done by means of the correspondence between the equations, as provided by Section 4, computing explicitly the generalized ODE and its solution.

Consider the Cauchy problem for the following measure neutral functional differential equation

$$\begin{gathered} \mathrm{D}[y(t) - ay(t-1)] = by(t-1)\mathrm{D}u, \quad t \geqslant 0 \\ y_0 = \varphi, \end{gathered} \tag{29}$$

where

$$u(s) = s + H_1(s), \tag{30}$$

with H_1 being the Heaviside function concentrated at 1, i.e. H_1 is given by

$$H_1(s) = \begin{cases} 0, & \text{if } s \leqslant 1, \\ 1, & \text{otherwise.} \end{cases} \tag{31}$$

The solution of (29) satisfies the integral form

$$y(t) = y(0) + ay(t-1) - ay(-1) + \int_0^t by(s-1)\mathrm{d}u(s). \tag{32}$$

Fix $T > 0$. We want to write the measure NFDE (29) as a generalized ODE of the form

$$\begin{aligned} \frac{dx}{d\tau} &= \mathrm{D}G(x,t), \\ x(0)(\vartheta) &= \varphi([-1,\vartheta,0]), \end{aligned} \tag{33}$$

where, for each $t \in [0,T]$, $x(t)$ is a function defined in the interval $[-1,T]$.

Let $G(y,t) = F(y,t) + J(y,t)$, with F and J defined by (9) and (10) respectively. Then F and J are described as follows

$$F(y,t)(\vartheta) = \int_0^{[0,\vartheta,t]} by(s-1)\mathrm{d}u(s) \tag{34}$$

$$J(y,t)(\vartheta) = ay([0,\vartheta,t]-1) - ay(-1), \tag{35}$$

where $y(t)$ is defined for $t \in [-1,T]$ and $\vartheta \in [-1,T]$.

Note that, for all $t \in [0,T]$ and all $\vartheta \in [-1,0]$, we have $[0,\vartheta,t] = 0$. Therefore, for any $y \in G([-1,T],\mathbb{R}^n)$, $t \in [0,T]$ and $\vartheta \in [-1,0]$, we have $F(y,t)(\vartheta) = 0 = J(y,t)(\vartheta)$ and, hence,

$$\int_0^t \mathrm{D}G(x(\tau),s)(\vartheta) = 0.$$

Therefore, since the integral form of (33) is

$$x(t) = x(0) + \int_0^t \mathrm{D}G(x(\tau),s),$$

we have

$$x(t)(\vartheta) = x(0)(\vartheta) = \varphi(\vartheta), \quad t \in [0,T],\ \vartheta \in [-1,0]. \tag{36}$$

Since the function u is given by a Heaviside function, we have to consider three cases which we discuss in the sequel.

Case 1. Let $0 < t \leqslant 1$. Suppose x is a solution of (33). We want to prove that the corresponding y given by Theorem 4.2 satisfies the integral equation (32). In order to compute $\int_0^t \mathrm{D}F(x(\tau),s)$ and $\int_0^t \mathrm{D}J(x(\tau),s)$, we consider a partition $0 = s_0 < s_1 < \cdots < s_n = t$ of the interval $[0,t]$. For an arbitrary choice of tags $\tau_i \in [s_{i-1},s_i]$, we have

$$\begin{aligned} F(x(\tau_i),s_i)(\vartheta) &= \int_0^{[0,\vartheta,s_i]} bx(\tau_i)(s-1)ds = \int_0^{[0,\vartheta,s_i]} b\varphi(s-1)ds \\ &= \varphi([0,\vartheta,s_i]) - \varphi(0) \end{aligned} \tag{37}$$

where we used (36), since $s-1 \in [-1,0]$. We also have

$$J(x(\tau_i),s_i)(\vartheta) = ax(\tau_i)([0,\vartheta,t]-1) - ax(\tau_i)(-1). \tag{38}$$

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Now, we analyze two possible cases, when $\vartheta < t$ and $\vartheta \geqslant t$.

Suppose $\vartheta < t$. There exists some integer $0 < k \leqslant n$ such that $\vartheta \in [s_{k-1}, s_k]$. Therefore the Riemann sum for the Kurzweil integral of $F(x(\tau), t)$ becomes

$$\begin{aligned}
&\sum_{i=1}^{n}\big[F(x(\tau_i), s_i)(\vartheta) - F(x(\tau_i), s_{i-1})(\vartheta)\big] \\
&\quad = \sum_{i=1}^{n}[\varphi([0,\vartheta,s_i]) - \varphi([0,\vartheta,s_{i-1}])] \\
&\quad = \sum_{i=1}^{I-1}[\varphi([0,\vartheta,s_i]) - \varphi([0,\vartheta,s_{i-1}])] + [\varphi([0,\vartheta,s_I]) - \varphi([0,\vartheta,s_{I-1}])] \\
&\qquad + \sum_{i=I+1}^{n}[\varphi([0,\vartheta,s_i]) - \varphi([0,\vartheta,s_{i-1}])]
\end{aligned}$$

For the first summand on the right-hand side of the last equality, we have $\vartheta \geqslant s_i \geqslant s_{i-1}$. Hence $[0,\vartheta,s_i] = s_i$ and $[0,\vartheta,s_{i-1}] = s_{i-1}$. For the second summand, since $\vartheta \in [s_{I-1}, s_I]$, we have $[0,\vartheta,s_I] = \vartheta$ and $[0,\vartheta,s_{I-1}] = s_{I-1}$. For the third summand, since $\vartheta \leqslant s_{i-1} \leqslant s_i$, we have $[0,\vartheta,s_i] = \vartheta$ and $[0,\vartheta,s_{i-1}] = \vartheta$. Therefore the Riemann sum for the Kurzweil integral of $F(x(\tau), t)$ becomes

$$\begin{aligned}
&\sum_{i=1}^{n}\big[F(x(\tau_i), s_i)(\vartheta) - F(x(\tau_i), s_{i-1})(\vartheta)\big] \\
&\quad = \sum_{i=1}^{I-1}[\varphi(s_i) - \varphi(s_{i-1})] + [\varphi(\vartheta) - \varphi(s_{I-1})] + \sum_{i=I+1}^{n}[\varphi(\vartheta) - \varphi(\vartheta)] \\
&\quad = \varphi(s_{I-1}) - \varphi(s_0) + \varphi(\vartheta) - \varphi(s_{I-1}) = \varphi(\vartheta) - \varphi(0) \\
&\quad = \int_0^{\vartheta} b\varphi(s-1)\mathrm{d}s,
\end{aligned}$$

Similarly, the Riemann sum for Kurzweil integral of $J(x(\tau), t)$ is

$$\begin{aligned}
&\sum_{i=1}^{n}\big[J(x(\tau_i),s_i)(\vartheta)-J(x(\tau_i),s_{i-1})(\vartheta)\big]\\
&\qquad =a\sum_{i=1}^{n}[x(\tau_i)([0,\vartheta,s_i]-1)-x(\tau_i)([0,\vartheta,s_{i-1}]-1)]\\
&\qquad =a\sum_{i=1}^{n}[x(\tau_i)([s_{i-1},\vartheta,s_i]-1)-x(\tau_i)(s_{i-1}-1)]\\
&\qquad =a\sum_{i=1}^{I-1}[x(\tau_i)([s_{i-1},\vartheta,s_i]-1)-x(\tau_i)(s_{i-1}-1)]\\
&\qquad\quad +a[x(\tau_I)([s_{I-1},\vartheta,s_I]-1)-x(\tau_I)(s_{I-1}-1)]\\
&\qquad\quad +\sum_{i=I+1}^{n}[x(\tau_i)([s_{i-1},\vartheta,s_i]-1)-x(\tau_i)(s_{i-1}-1)]\\
&\qquad =a\sum_{i=1}^{I-1}[x(\tau_i)(s_i-1)-x(\tau_i)(s_{i-1}-1)]\\
&\qquad\quad +a[x(\tau_I)([s_{I-1},\vartheta,s_I]-1)-x(\tau_I)(s_{I-1}-1)]\\
&\qquad\quad +\sum_{i=I+1}^{n}[x(\tau_i)(s_{i-1}-1)-x(\tau_i)(s_{i-1}-1)]\\
&\qquad =ax(\tau_i)(\vartheta-1)-ax(\tau_i)(-1)\\
&\qquad =a\varphi(\vartheta-1)-a\varphi(-1)
\end{aligned}$$

For $\vartheta \geqslant t$, we have $\vartheta \geqslant s_i \geqslant s_{i-1}$. Therefore $[0,\vartheta,s_i]=s_i$ and $[0,\vartheta,s_{i-1}]=s_{i-1}$ and, hence,

$$\begin{aligned}
&\sum_{i=1}^{n}\big[F(x(\tau_i),s_i)(\vartheta)-F(x(\tau_i),s_{i-1})(\vartheta)\big]\\
&\qquad =\sum_{i=1}^{n}[\varphi([0,\vartheta,s_i])-\varphi([0,\vartheta,s_{i-1}])]=\sum_{i=1}^{n}[\varphi(s_i)-\varphi(s_{i-1})]\\
&\qquad =\varphi(s_N)-\varphi(s_0)=\varphi(t)-\varphi(0)=\int_0^t b\varphi(s-1)\mathrm{d}s.
\end{aligned}$$

Similarly, the Riemann sum for the Kurzweil integral of $J(x(\tau),t)$ is given by

$$\begin{aligned}
&\sum_{i=1}^{n}\big[J(x(\tau_i),s_i)(\vartheta)-J(x(\tau_i),s_{i-1})(\vartheta)\big]\\
&\quad=a\sum_{i=1}^{n}[x(\tau_i)([0,\vartheta,s_i]-1)-x(\tau_i)([0,\vartheta,s_{i-1}]-1)]\\
&\quad=a\sum_{i=1}^{n}[x(\tau_i)(s_i-1)-x(\tau_i)(s_{i-1}-1)]=ax(\tau_i)(t-1)-ax(\tau_i)(-1)\\
&\quad=a\varphi(t-1)-a\varphi(-1)
\end{aligned}$$

From the previous calculations, it is easy to see that the Riemann sums are independent of the particular partition and, hence,

$$\int_0^t \mathrm{D}F(x(\tau),s)=\int_0^{[0,\vartheta,t]} b\varphi(s-1)\mathrm{d}s \tag{39}$$

and

$$\int_0^t \mathrm{D}J(x(\tau),s)=a\varphi([0,\vartheta,t]-1)-a\varphi(-1).$$

Therefore,

$$\begin{aligned}
x(t)(\vartheta)&=x(0)(\vartheta)+\int_0^t \mathrm{D}F(x(\tau),s)+\int_0^t \mathrm{D}J(x(\tau),s)\\
&=\varphi([-1,\vartheta,0])+\int_0^{[0,\vartheta,t]} b\varphi(s-1)\mathrm{d}s+\varphi([0,\vartheta,t]-1)-a\varphi(-1)\\
&=\varphi([-1,\vartheta,t]),\quad 0\leqslant t\leqslant 1,\quad -1\leqslant\vartheta\leqslant 1,
\end{aligned}$$

where we use again φ for the solution of the NFDE

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t)-ay(t-1)]=by(t-1). \tag{40}$$

subject to the initial condition $y_0=\varphi$, whose integral form is given by

$$\varphi(t)-a\varphi(t-1)-\varphi(0)+a\varphi(-1)=\int_0^t b\varphi(s-1)\mathrm{d}s,\quad t>0. \tag{41}$$

and can be solved by the method of steps, for instance. Note that u restricted to the interval $[0,1]$ is the identity function. Thus $\mathrm{d}u(s)=\mathrm{d}s$ for $s\in[0,1]$ and (32) can be replaced by (41).

Case 2. Consider $1 < t \leqslant 2$. Then,

$$\begin{aligned} F(x(\tau_i), s_j) &= \int_0^{[0,\vartheta,s_j]} bx(\tau_i)(s-1)\mathrm{d}s + \int_0^{[0,\vartheta,s_j]} bx(\tau_i)(s-1)\mathrm{d}H_1(s) \\ &= \int_0^{[0,\vartheta,s_j]} bx(\tau_i)(s-1)\mathrm{d}s + b\varphi(0)H_1(s_j) \end{aligned} \tag{42}$$

Computing the Riemann sum as the previous case, but taking into account the last term in (42), we get

$$\int_1^t \mathrm{D}F(x(\tau), s) = \int_1^{[1,\vartheta,t]} b\varphi(s-1)\mathrm{d}s + b\varphi(0).$$

Using the same calculations for the case $0 < t \leqslant 1$, we obtain

$$\begin{aligned} \int_1^t \mathrm{D}J(x(\tau), s) &= \int_0^t \mathrm{D}J(x(\tau), s) - \int_0^1 \mathrm{D}J(x(\tau), s) \\ &= a\varphi([0,\vartheta,t]-1) - a\varphi([0,\vartheta,1]-1) \\ &= a\varphi([1,\vartheta,t]-1) - a\varphi(0). \end{aligned} \tag{43}$$

From equations (39) and (43) and by recalling that φ is the solution of the NFDE (40), we get, for $t \in (1, 2]$,

$$\begin{aligned} x(t)(\vartheta) &= \varphi([-1,\vartheta,1]) + \int_1^{[1,\vartheta,t]} b\varphi(s-1)\mathrm{d}s + a\varphi([1,\vartheta,t]-1) - a\varphi(0) + b\varphi(0) \\ &= \varphi([-1,\vartheta,t]) + b\varphi(0), \quad 1 < t \leqslant 2, \quad -1 \leqslant \vartheta \leqslant 2. \end{aligned}$$

Note that by cases 1 and 2, we can write the solution $x(t)$ of the GODE (33), for $t \in [0, 2]$, in the following form

$$x(t) = x_1(t) + x_2(t)$$

where $x_1(t)(\vartheta) = \varphi([-1,\vartheta,t])$ and $x_2(t) = a\varphi(0)H_1([0,\vartheta,t])$.

By (30), we observe that, for $s > 2$, $\mathrm{d}u(s) = \mathrm{d}s$. Therefore no jumps occur in the solution $x(t)$ for $t > 2$.

We now describe the last case.

Case 3. Consider $t > 2$. Using similar computations as before, we have

$$\begin{aligned} x_1(t)(\vartheta) &= x_1(2)(\vartheta) + \Big(\int_2^t \mathrm{D}G(x_1(\tau), s)\Big)(\vartheta) \\ &= \varphi([-1,\vartheta,2]) + \int_2^{[2,\vartheta,t]} b\varphi(s-1)\mathrm{d}s + a\varphi([2,\vartheta,t]-1) - a\varphi(1) \\ &= \varphi([-1,\vartheta,2]) + \varphi([2,\vartheta,t]) - \varphi(2) \\ &= \varphi([-1,\vartheta,t]), \quad t > 2. \end{aligned}$$

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Define the function $w(s) = a\varphi(0)H_1(s)$, for $-1 \leqslant s \leqslant 2$. Also, let w be the solution of (40) subject to the initial condition $y_2 = w_2$. Then $x_2(t)(\theta) = w([-1, \vartheta, t])$ for $t \in [0, 2]$. Replacing y by w in the previous computations, we get

$$\begin{aligned} x_2(t)(\vartheta) &= x_2(2)(\vartheta) + \Big(\int_2^t \mathrm{D}G(x_2(\tau), s) \Big)(\vartheta) \\ &= w([-1, \vartheta, t]), \quad t > 2. \end{aligned}$$

Finally, by cases 1, 2 and 3, Now, from the three cases, for $t > 0$, we obtain

$$x(t)(\vartheta) = \varphi([-1, \vartheta, t]) + w([-1, \vartheta, t])$$

and a substitution of the function $y(t) = \varphi(t) + w(t)$ into the integral form (41) shows that it is the solution of the measure NFDE (29).

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