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Existence of bounded solutions for delay linear inhomogenous equations

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We consider the linear non autonomous inhomogeneous system

$$\dot{x}(t) = L(t)x_t + f(t), \tag{1}$$

where f is a bounded continuous function. We show that a sufficient condition which ensure the existence of bounded solutions of delay system is the property of exponential dichotomy of the solution operator of the homogeneous system associated to (1). We use the sun-star framework and the perturbation theory for dual semigroups to study delay equation presented by Diekmann [8]. October, 2013 ICMC-USP

1. INTRODUCTION

We are concerned with the study of the existence of bounded solutions for non autonomous retarded functional differential equations (RFDE) given by

$$\dot{x}(t) = L(t)x_t + f(t), \quad t \geq s,\ x(t) \in \mathbb{C}^n \tag{2}$$

with initial condition

$$x_s = \varphi \in X$$

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where $X = C([-h, 0], \mathbb{C}^n)$ is the Banach space consisting by the continuous functions from $[-h, 0]$ into $\mathbb{C}^n$, equipped with the supremum norm and $L(t) : X \to \mathbb{C}^n$ is a linear operator given as the Riesz Representation Theorem,

$$L(t)\varphi = \int_0^h d_\theta[\zeta(t, \theta)]\varphi(-\theta).$$

with $\zeta(t, \cdot)$ a bounded variation function defined on $[0, h]$ with values in $\mathbb{C}^{n\times n}$. Moreover, f is a bounded continuous function.

In order to establish our main result we use the concept of exponential dichotomy, which is an important tool to describe qualitatively the asymptotic behavior of solutions of non autonomous differential equations and has been studied with much emphasis in the last fifty years by many authors ([1, 2, 3, 4, 5, 6, 10, 11, 14, 15, 16, 17]). This concept was introduced by Perron in his classical paper on stability in a finite-dimensional setting [13]. Although a condition of exponential dichotomy has not been established explicitly in that work, he gave a condition of existence of bounded solution for linear inhomogeneous equation, for bounded functions. The equivalence of this condition with exponential dichotomy condition was first established by Maĭzel [12]. In this direction Coppel developed a great number of results given in [1, 2, 3, 4, 5].

In the other hand an extension of O. Perron's problem to the more general infinite-dimensional Banach space was due to Daleckij and Krein [10], Massera and Schäffer [11]. Following this line of investigation, that characterizes the exponential dichotomy in term of Perron Type Theorem, we show that this condition is sufficient to ensure the existence of bounded solutions for the abstract integral equation (AIE)

$$u(t) = T_0(t-s)\varphi + \int_s^t T_0^{\odot *}(t-\tau)C(\tau)u(\tau)d\tau, \quad t \geq s \tag{3}$$

where $C(t) : X \to X^{\odot *}$, defined on the Banach space X, $\odot$-reflexive with respect to the $\mathcal{C}_0$-semigroup, is given by $C(t)\varphi = B(t)\varphi + (f(t), 0)$, where $B(t)$ is a family of bounded linear operators given by

$$B(t)\varphi = (L(t)\varphi, 0)$$

Notice that Equation (3) is the abstract integral equation of (2). Therefore, if we establish the correspondence between solutions of (3) and solutions of (2), we obtain the existence of bounded solutions for inhomogeneous RFDE.

This paper is organized in the following way: in Section 2 we present the general sun-star theory following the works of Diekmann [8] and Clement et al [7] and the book of Diekmann et al [9] and we present the result which establish the RFDE as bounded perturbation; in Section 3 we present the perturbation theory for evolutionary systems; in Section 4 we show a correspondence between solution of inhomogeneous delay system and his abstract integral equation which will be very useful for proving in Section 5 our main result. Finally we present an appendix of variants of variation-of-constant formula.

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2. LINEAR RFDE AS BOUNDED PERTURBATION

In this section we will give a summary of [7, 8, 9]. Consider the autonomous retarded functional differential equation (RFDE)

$$\dot{x} = Lx_t, \quad t \geq 0 \tag{4}$$

subject to initial condition

$$x_0 = \varphi,$$

with $\varphi \in X = C([-h,0],\mathbb{C})$ and $L : X \to \mathbb{C}$ the linear operator given as the Riesz Representation Theorem,

$$L\varphi = \int_0^h [d_\theta \zeta(\theta)]\varphi(-\theta).$$

The solution operator $T(t) : X \to X$ defined by the relation

$$T(t)\varphi = x_t(\cdot,\varphi)$$

is a $\mathcal{C}_0$-semigroup, with infinitesimal generator A defined by

$$\mathcal{D}(A) = \{\varphi \in X \mid \dot{\varphi} \in X,\ \dot{\varphi}(0) = L\varphi\}, \quad A\varphi = \dot{\varphi}$$

Notice that, the Equation (4) is embedded in the domain of the generator A. Therefore, to change the rule of the FDE, implies to change the domain of the infinitesimal generator. Furthermore, if we want to related solutions of several equations using the variation-of-constant formula, we have more technical complications.

In order to solve this problem, Dieckmann use duality theory of semigroups. The main idea is embedding the space X into the space $X^{\odot *}$, which is the adjoint of the space $X^{\odot}$ ($\odot$ is called *sun*) that is the maximal invariant subspace where the dual semigroup $T^*(t)$ is strongly continuous.

The aim of introduce the $\odot *$ theory is to make the semigroup $T(t)$ be a restriction of the semigroup $T^{\odot *}(t)$, and get the independence of the domain of A of the specific rule for the FDE. In fact, the particular equation is shifted from the domain into the action. Moreover, the space $X^{\odot *}$ is essential for the formulation of the variation-of-constant equation.

Now, following [8], we can see that the linear autonomous RFDEs can be write as bounded perturbation of the follow trivial RFDE (called prototype problem):

$$\begin{aligned} \dot{x}(t) &= 0, \quad \text{for } t \geq 0, \\ x_0 &= \varphi \in X. \end{aligned} \tag{5}$$

Clearly the solution is

$$x(t) = \begin{cases} \varphi(t), & -h \leq t \leq 0, \\ \varphi(0), & t \geq 0 \end{cases} \tag{6}$$

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For each $t \geq 0$,

$$(T_0(t)\varphi)(\theta) = \begin{cases} \varphi(t+\theta), & \text{if} \quad -h \leq t+\theta \leq 0, \\ \varphi(0), & \text{if} \quad t+\theta \geq 0 \end{cases} \tag{7}$$

defines a bounded linear operator $T_0(t) : X \to X$. The operator $T_0(t)$ maps the initial state φ at time zero onto the state x_t at time t (translation). Furthermore T_0 is a $\mathcal{C}_0$-*semigroup* with infinitesimal generator given by

$$\mathcal{D}(A_0) = \{\varphi | \dot{\varphi} \in X, \dot{\varphi}(0) = 0\}, \quad A_0\varphi = \dot{\varphi} \tag{8}$$

Thus, in [8] the RFDE (4) it was be represented in an abstract framework, using the two component "shifting" and "extending". In fact, in symbols

$$\frac{d}{dt}x_t = (A_0^{\odot *} + B)x_t$$

where $B : X \to X^{\odot *}$ is defined by

$$B\varphi = (L\varphi, 0)$$

The operator B describe the rule for the extension, which is linear and bounded. This property is due to the range space $X^{\odot *}$ is large enough. Moreover, that even though x_t is conceived as an element of X, the differential equation is an identity for elements of $X^{\odot *}$.

The result given by Diekmann use a variation-of-constant formula, and it establish that:

THEOREM 2.1. *Let, with $\{T_0(t)\}$ and B as defined above, $\{T(t)\}$ be the semigroup defined by the abstract integral equation*

$$T(t)\varphi = T_0(t)\varphi + \int_0^t T_0^{\odot *}(t-\tau)BT(\tau)d\tau$$

Let $x(\cdot, \varphi)$ be the solution of the RFDE

$$\dot{x}(t) = L(t)x_t, \quad t \geq 0$$

with initial condition

$$x(\theta) = \varphi(\theta), \quad -h \leq \theta \leq 0.$$

Then

$$T(t)\varphi = x_t(\cdot; \varphi).$$

3. EVOLUTIONARY SYSTEMS AND BOUNDED PERTURBATION

In the study of non autonomous system, we have to take into account not only the time difference between the initial time and the present time matters but also the initial time itself. Hence we have to work with two-parameter families of operators $U(t,s)$, where s corresponds to the initial time and t to the current time. Consider the subset

$$\Delta = \{(t,s)|\alpha \leq s \leq t \leq \omega\} \subset \mathbb{R}^2 \tag{9}$$

where $\alpha, \omega \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $\alpha < \omega$ and where, here and in the following, one should read $\leq$ as $<$ whenever the left side equals $-\infty$ or the right hand side equals $+\infty$.

Definition 3.1. A two-parameter family $U = \{U(t,s)\}_{(t,s)\in\Delta}$ of bounded linear operators on Banach space X is called a *forward evolutionary system* on X whenever

i. $U(s,s) = I$ (the identity), $\quad \alpha \leq s \leq \omega$,
ii. $U(t,r)U(r,s) = U(t,s), \quad \alpha \leq s \leq r \leq t \leq \omega$.

Definition 3.2. A two-parameter family $V = \{V(s,t)\}_{(t,s)\in\Delta}$ of bounded linear operators on Banach space X is called a *backward evolutionary system* on X whenever

i. $V(s,s) = I$ (the identity), $\quad \alpha \leq s \leq \omega$,
ii. $V(s,r)V(r,t) = V(s,t), \quad \alpha \leq s \leq r \leq t \leq \omega$.

The following Lemma is important for our propose.

Lemma 3.1. *The adjoint of a forward evolutionary system is a backward evolutionary system. In notation, we have*

$$V(t,s) = U^*(t,s)$$

Definition 3.3. The forward evolutionary system U is said to be *strongly continuous* if for every $x \in X$ the mapping $(t,s) \mapsto U(t,s)x$ is continuous from Δ to X.

In the previous section, we see that the autonomous linear RFDE can be seen as linear perturbation of trivial RFDE. In this section working with non autonomous linear RFDE, we perturb the generator A_0 by a family $\{B(t)\}_{\alpha\leq t\leq\omega}$ of bounded linear operators from X into $X^{\odot *}$, and we assume that the family is strongly continuous, i.e. for every $\varphi \in X$, the mapping $t \mapsto B(t)\varphi$ is continuous from $[\alpha,\omega]$ to $X^{\odot *}$.

Therefore we have the variation-of-constant formula

$$U(t,s)\varphi = T_0(t-s)\varphi + \int_s^t T_0^{\odot *}(t-\tau)B(\tau)U(\tau,s)\varphi d\tau. \tag{10}$$

The integral has to be understood in the weak $*$ sense, i.e.

$$\langle \int_s^t T_0^{\odot *}(t-\tau)B(\tau)U(\tau,s)d\tau, x^{\odot}\rangle = \int_s^t \langle B(\tau)U(\tau,s)x, T_0^{\odot}(t-\tau)x^{\odot}\rangle$$

for arbitrary $x^{\odot} \in X^{\odot}$. At first, we notice that the integral takes values in $X^{\odot *}$ but one can show that in fact it takes values in the closed subspace $X^{\odot\odot} = X$. Within this setting the standard contraction argument apply and we can show that (10) admits a unique solution $U(t,s)$. By duality and restriction we obtain semigroups $\{U^*(t,s)\}$, $\{U^{\odot}(t,s)\}$ and $\{U^{\odot *}(t,s)\}$ on, respectively X^*, $X^{\odot}$ and $X^{\odot *}$ since it can be shown that the spaces of strong continuity do not depend on $B(t)$. Analogue the domain of the weak $*$ generators on the "big" spaces X^* and $X^{\odot *}$ are independent of $B(t)$.

In [8, Sec. XII.4] was proved that the RFDE

$$\dot{x}(t) = x(t-\tau(t))$$

where $\tau(t) \geq 0$ is a bounded function such that $\dot{\tau}(t) = 1$ for some interval of time. The dual semigroup $U(t,s)$ associated to this solution doesn't leave $X^{\odot} \subset X^*$ invariant.

Thus, we need the following assumption: The mapping $t \to B(t)$ is continuous from $[\alpha,\omega]$ to $\mathcal{L}(X, X^{\odot *})$.

This assumption guarantee that $X^{\odot}$ is invariant by the backward evolutionary system $V(t,s) = U^*(s,t)$. Taking adjoint and restriction, we have $V^{\odot}(s,t) = V(s,t)|_{X^{\odot}}$ and $U^{\odot *}(t,s) = (V^{\odot}(s,t))^*$, where $U^{\odot *}$ extend jUj^{-1}.

Example 3.1. The assumption above is been satisfying for the RFDE of the form

$$\dot{x}(t) = \sum_{j=1}^{n} a_j(t)x(t-t_j)$$

where the functions $a_j(t)$ are continuous.

4. INHOMOGENEOUS LINEAR RFDE

In this section, we consider the linear non autonomous inhomogeneous RFDE

$$\dot{x}(t) = L(t)x_t + f(t) \tag{11}$$

where $L(t)\varphi = \int_0^h [d_\theta \zeta(t,\theta)]\varphi(-\theta)$, with ζ is the matrix function $n \times n$ which entries are in the set of the NBV functions and $f : \mathbb{R} \to \mathbb{C}^n$ is a bounded continuous function.

Alternatively, we can use the notation

$$\langle \zeta(t,\cdot), x_t\rangle_n = \int_0^h d\zeta(t,\theta)x(t-\theta)$$

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and we can rewrite (11) as below

$$\dot{x}(t) = \langle \zeta(t,\cdot), x_t \rangle_n + f(t).$$

Moreover, we can suppose the initial condition

$$x(s+\theta) = \varphi(\theta), \quad -h \leq \theta \leq 0 \tag{12}$$

and consider the abstract integral equation

$$u(t) = T_0(t-s)\varphi + \int_s^t T_0^{\odot *}(t-\tau)C(\tau)u(\tau)d\tau, \;\; t \geq s \tag{13}$$

where $C(t) : X \to X^{\odot *}$ is defined by $C(t)\varphi = B(t)\varphi + (f(t), 0)$ where $B(t) : X \to X^{\odot *}$ is the family of linear bounded operator given by

$$B(t)\varphi = (\langle \zeta(t,\cdot), \varphi \rangle_n, 0). \tag{14}$$

Now, we state an auxiliary lemma, which will be used to prove that the AIE associated to the inhomogeneous RFDE, admit a unique solution and define a forward evolutionary system.

Lemma 4.1 ([9, Lema XII.2.8]). *Consider the set*

$$\Delta = \{(t,s) \mid -\infty \leq s \leq t \leq \infty\} \tag{15}$$

and $f : \Delta \to X^{\odot *}$ a continuous function. Define $v : \Delta \to X^{\odot *}$ by

$$v(t,s) = \int_s^t T_0^{\odot *}(t-\tau)f(\tau,s)d\tau.$$

Then v is continuous and take values in $j(X) = X$.

The following theorem is a generalization of the Theorem III.2.4 in [9] for inhomogeneous systems.

Theorem 4.1. *The variation-of-constant formula*

$$U(t,s)\varphi = T_0(t-s)\varphi + \int_s^t T_0^{\odot *}(t-\tau)C(\tau)U(\tau,s)\varphi d\tau \tag{16}$$

where $(t,s) \in \Delta$ given in (15) e $\varphi \in X$, define uniquely a forward evolutionary system U, strongly continuous. Moreover,

$$\|U(t,s)\| \leq Me^{(\omega_0 + MK(t,s))(t-s)}, \tag{17}$$

with M e ω_0 such that $\|T_0(t)\| \leq Me^{\omega_0 t}$ and

$$K(t,s) = \sup_{s\leq\tau\leq t} \|B(\tau)\| + \sup_{s\leq\tau\leq t} |f(\tau)|. \tag{18}$$

Proof. We star to prove the existence. For this, define inductively

$$\begin{cases} U_k(t,s)\varphi = \int_s^t T_0^{\odot *}(t-\tau)C(\tau)U_{k-1}(\tau,s)\varphi d\tau, & k \geq 1, \\ U_0(t,s) = T_0(t-s). \end{cases} \tag{19}$$

Then, by Lemma 4.1, we now that $U_k(t,s)$ apply X in X and $(t,s) \mapsto U_k(t,s)$ is continuous. Therefore, it follows by induction the next estimative

$$\|U_k(t,s)\| \leq Me^{w(t-s)} \frac{M^k[K(t,s)]^k(t-s)^k}{k!}$$

where $K(t,s)$ is as established in (18).

Therefore, we define

$$U(t,s) = \sum_{k=0}^{\infty} U_k(t,s) \tag{20}$$

and notice that this sum converges uniformly in (t,s) on bounded intervals. Then, $(t,s) \mapsto U(t,s)\varphi$ is continuous for every $\varphi \in X$. Finally, using (19)–(20), we have that $U(t,s)$ satisfies (16). In fact,

$$\begin{aligned} U(t,s)\varphi &= \sum_{k=0}^{\infty} U_k(t,s) = U_0(t,s) + \sum_{k=1}^{\infty} U_k(t,s) \\ &= T_0(t-s)\varphi + \sum_{k=1}^{\infty} \int_s^t T_0^{\odot *}(t-\tau)C(\tau)U_{k-1}(\tau,s)\varphi d\tau \\ &= T_0(t-s)\varphi + \int_s^t T_0^{\odot *}(t-\tau)C(\tau)U(\tau,s)\varphi d\tau. \end{aligned}$$

It remains to prove the uniqueness, for this we consider the evolutionary family $\{W(t,s)\}_{t\geq s}$ of bounded linear operators in X, such that $(t,s) \mapsto W(t,s)\varphi$ is continuous for every $\varphi \in X$ and the equation

$$W(t,s)\varphi = T_0(t-s)\varphi + \int_s^t T_0^{\odot *}(t-\tau)C(\tau)W(\tau,s)\varphi d\tau \tag{21}$$

is hold. Subtracting (21) of (16), we obtain

$$U(t,s)\varphi - W(t,s)\varphi = \int_s^t T_0^{\odot *}(t-\tau)C(\tau)[U(\tau,s) - W(\tau,s)]\varphi d\tau.$$

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Hence,

$$e^{-wt}\|U(t,s)\varphi - W(t,s)\varphi\| \leq MK(t,s)\int_s^t e^{-w\tau}\|U(\tau,s)\varphi - W(\tau,s)\varphi\| d\tau.$$

Now, using the Gronwall's inequality, we get $U(t,s)\varphi = W(t,s)\varphi$ for every $\varphi \in X$. Then $U(t,s) = W(t,s)$. Finally in order to prove that $U(t,s)$ is a strongly continuous evolutionary family, firstly we note that $U(s,s)\varphi = T_0(0)\varphi = \varphi$, therefore $U(s,s) = I$. The second property of semigroups follows using (16) in (t,r) and replacing φ by $U(r,s)\varphi$. Then, notice that (21) is hold for $W(t,s) = U(t,r)U(r,s)$. In consequence $U(t,s) = U(t,r)U(r,s)$ for $s \leq r \leq t$. Finally, as $T_0(t)$ is a $\mathcal{C}_0$-semigroup, by (16) and Lemma 4.1, we have that $(t,s) \mapsto U(t,s)\varphi$ is continuous for every $\varphi \in X$ and the proof of the theorem is complete. ∎

The next theorem show that the solutions of the AIE (16) are in corresponding to solutions of the initial problem value for the inhomogeneous RFDEs (11).

THEOREM 4.2. *Let X, $T_0(t)$ and $C(t)$ as described above, $U(t,s)$ an evolutionary system defined by the abstract integral equation* (16)*. Then $x(t)$ defined by*

$$x(s+\theta) = \varphi(\theta), \quad -h \leq \theta \leq 0, \tag{22}$$

$$x(t) = (U(t,s)\varphi)(0), \quad t \geq s \tag{23}$$

satisfies (11)*. Conversely, if x is the solution of* (11) *satisfy the initial condition* (22)*, then, for $t \geq s$ and $\theta \in [-h,0]$,*

$$(U(t,s)\varphi)(\theta) = \begin{cases} \varphi(t-s+\theta), & t+\theta \leq s, \\ x(t+\theta), & t+\theta \geq s. \end{cases} \tag{24}$$

Now we enunciate a variant of the Lemma III.4 in [9], which is useful to prove the theorem.

LEMMA 4.2. *Let e_i the i-th unit vector in $\mathbb{C}^n$ and let $r_i^{\odot *} = (e_i, 0)$. For some $\eta \in L^1_{\text{loc}}$,*

$$\int_s^t T_0^{\odot *}(t-\tau)\eta(\tau)r_i^{\odot *} d\tau = e_i \int_s^{\max(s,t+\cdot)} \eta(\sigma)d\sigma. \tag{25}$$

Proof (Proof of the Theorem 4.2). Fix $s \in \mathbb{R}$, $\varphi \in X$ and define the continuous functions y and x

$$y(t) = \langle \zeta(t,\cdot), U(t,s)\varphi\rangle_n + f(t), \quad t \geq s, \tag{26}$$
$$x(t) = (U(t,s)\varphi)(0), \quad t \geq s.$$

By the definition of $C(t)$, we have

$$C(t)U(t,s)\varphi = B(t)U(t,s)\varphi + (f(t),0) = (\langle \zeta(t,\cdot), U(t,s)\varphi\rangle_n, 0) + (f(t),0)$$
$$= (y(t),0). \tag{27}$$

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Remember the definition of the semigroup $T_0(t)$,

$$(T_0(t-s)\varphi)(\theta) = \begin{cases} \varphi(t+\theta-s), & -h \le t+\theta-s \le 0, \\ \varphi(0), & t+\theta-s \ge 0, \end{cases} \tag{28}$$

and consider $\theta = 0$ in equation (16) and using (23), (27), (28), then we have

$$x(t) = \varphi(0) + \sum_{i=1}^{n} \int_s^t T_0^{\odot *}(\tau - s) y_i(\tau) r_i^{\odot *}. \tag{29}$$

Now, using the Lemma 4.2, we obtain

$$x(t) = \varphi(0) + \sum_{i=1}^{n} e_i \int_s^{\max(s,t)} y_i(\tau) d(\tau) = \varphi(0) + \int_s^t y(\tau) d(\tau)$$

and we conclude that x is continuously differentiable for $t \ge s$ and

$$\dot{x}(t) = y(t), \quad t \ge s. \tag{30}$$

In the other hand, the AIE (16) for $t \ge s$ and $\theta \in [-h, 0]$

$$(U(t,s)\varphi)(\theta) = (T_0(t-s)\varphi)(\theta) + \left(\int_s^t T_0^{\odot *}(t-\tau) C(\tau) U(\tau, s)\varphi d\tau \right)(\theta)$$

together with the definition of $T_0(t-s)$ provide:

1. for $t + \theta \le s$

$$(U(t,s)\varphi)(\theta) = \varphi(t-s+\theta) + \int_s^{\max(s,t+\theta)} y(\tau) d\tau = \varphi(t-s+\theta), \tag{31}$$

2. for $t + \theta \ge s$

$$\begin{aligned} (U(t,s)\varphi)(\theta) &= \varphi(0) + \int_s^{\max(s,t+\theta)} y(\tau) d\tau \\ &= \varphi(0) + \int_s^{t+\theta} y(\tau) d\tau = x(t+\theta). \end{aligned} \tag{32}$$

Therefore, joining the two cases, we obtain

$$U(t,s)\varphi(\theta) = \begin{cases} \varphi(t-s+\theta), & t+\theta \le s, \\ x(t+\theta), & t+\theta \ge s. \end{cases} \tag{33}$$

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Since we can extent x to the interval $[s-h,s]$ by the initial condition (22), we can rewrite (33) such

$$U(t,s)\varphi(\theta) = x_t(\theta) \tag{34}$$

Finally, using (30) and the definition of $y(t)$ in (26), we have

$$\dot{x}(t) = \langle \zeta(t,\cdot), U(t,s)\varphi\rangle_n + f(t)$$

and (34) implies that

$$\dot{x}(t) = \langle \zeta(t,\cdot), x_t\rangle_n + f(t),$$

this shows the first statement of the theorem.

Reciprocally, if x is the solution of the RFDE (11) with initial condition (22), the integration of (11) provides

$$x(t+\theta) - x(s) = \int_s^{t+\theta} (\langle \zeta(\tau,\cdot), x_\tau\rangle_n + f(\tau))d\tau. \tag{35}$$

Using the initial condition and the definition of $T_0(t-s)$, we can rewrite (35) in the following form

$$x_t(\theta) = T_0(t-s)\varphi(\theta) + \int_s^{\max(s,t+\theta)} (\langle \zeta(\tau,\cdot), x_\tau\rangle_n + f(\tau))d\tau.$$

Now, using the Lemma 4.2 in the other direction and of the definition of $C(t)$,

$$\begin{aligned} x_t &= T_0(t-s)\varphi + \sum_{i=1}^n \int_s^t T_0^{\odot *}(t-\tau)\Big[(\langle \zeta(\tau,\cdot), x_\tau\rangle_n)_i + f_i(\tau)\Big] r_i^{\odot *} d\tau \\ &= T_0(t-s)\varphi + \int_s^t T_0^{\odot *}(t-\tau)C(\tau)x_\tau d\tau. \end{aligned}$$

The uniqueness of the solutions of (16) implies that

$$U(t,s)\varphi = x_t,$$

which ends the proof of the theorem. ▮

5. RESULT

Consider the initial value problem for the linear non autonomous RFDE

$$\begin{cases} \dot{x}(t) = L(t)x_t, \qquad t \geq s, \quad x(t) \in \mathbb{C}^n \\ x_s = \varphi, \end{cases} \tag{36}$$

remember that $\varphi \in X = C([-h,0],\mathbb{C}^n)$ and $L(t)\varphi = \int_0^h [d_\theta \zeta(t,\theta)]\varphi(-\theta)$. Moreover, the variant-of-constant formula is given by

$$U(t,s)\varphi = T_0(t-s)\varphi + \int_s^t T_0^{\odot *}(t-\tau)B(\tau)U(\tau,s)\varphi d\tau \tag{37}$$

where $B(t)\varphi = (L(t)\varphi, 0)$.

In the sequence, we give a definition of exponential dichotomy for a general evolutionary family $\{U(t,s)\}_{t\geq s}$.

Definition 5.1. An evolutionary family $\{U(t,s)\}_{t\geq s}$ has the property of *exponential dichotomy on* $\mathbb{R}$ (with constant $\alpha > 0$) is there ir a constant $M = M(\alpha) > 0$ and projection operators $P(s) : X^{\odot *} \to X^{\odot *}$, with $s \mapsto P(s)\varphi$ continuous and bounded for every $\varphi \in X^{\odot *}$, such that $Q(s) = I - P(s)$ and the following conditions hold:

1. $P(t)U^{\odot *}(t,s) = U^{\odot *}(t,s)P(s)$;
2. The restriction $U^{\odot *}(t,s)|_{\operatorname{Im} Q(s)}$ is invertible as an operator of $\operatorname{Im} Q(s)$ in $\operatorname{Im} Q(t)$ and we define $U(s,t)$ as the inverse operator;
3. $\|U^{\odot *}(t,s)P(s)\| \leq Me^{-\alpha(t-s)}$, for $s \leq t$;
4. $\|U^{\odot *}(t,s)Q(s)\| \leq Me^{-\alpha(s-t)}$, for $t \leq s$.

Let V_1 the subspace of $X^{\odot *}$ consisted by the initial values of all bounded solutions of the homogeneous equation (36), and let V_2 any fixed subspace of X complementary of V_1. We have that $\operatorname{Im} P(s) = V_1$ and $\ker P(s) = V_2$.

Let $BC(\mathbb{R},\mathbb{C}^n)$ the set of bounded continuous functions. Our aim of this section is to show the following theorem.

Theorem 5.1. *Fixed* $s \in \mathbb{R}$. *Suppose that the solution operator* $\{U(t,s)\}_{t\geq s}$ *of the homogeneous RFDE* (36) *has the property of exponential dichotomy. Then, there is* $r \geq 0$ *such that for all* $f \in BC(\mathbb{R},\mathbb{C}^n)$, *the inhomogeneous RFDE*

$$\dot{x}(t) = L(t)x_t + f(t), t \geq s$$

admits a unique solution x_t *with* $x_s \in \ker P(s)$ *such that*

$$\|x_t\| \leq r\|f\|.$$

Proof. Fix $f \in BC(\mathbb{R},\mathbb{C}^n)$. Let $P(t)$ a projection function which satisfies the following condition 5.1. Consider

$$G(t,s) = \begin{cases} P(t)U^{\odot *}(t,s)P(s), & t > s, \\ -Q(t)U^{\odot *}(t,s)Q(s), & t < s. \end{cases} \tag{38}$$

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Consider $F(t) = (f(t), 0) \in X^{\odot *}$ and define the operator $\hat{G}$ by

$$(\hat{G}F)(t) = \int_{-\infty}^{\infty} G(t,\tau)F(\tau)d\tau. \tag{39}$$

Notice that the integral above converges. In fact, using the properties 3. e 4. of the Definition 5.1 we have that

$$\begin{aligned}\int_{-\infty}^{\infty} \|G(t,s)\| ds &= \int_{-\infty}^{t} \|P(t)U^{\odot *}(t,s)P(s)\| ds + \int_{t}^{\infty} \|Q(t)U^{\odot *}(t,s)Q(s)\| ds \\ &\leq \int_{-\infty}^{t} Me^{-\alpha(t-s)} ds + \int_{t}^{\infty} Me^{-\alpha(s-t)} ds \leq \frac{2M}{\alpha}\end{aligned}$$

Therefore $\hat{G}$ is bounded in $X^{\odot *}$ with $\|\hat{G}F(t,f)\| \leq \frac{2M}{\alpha}\|f\|_{\infty}$. Now, we consider $x_t = (\hat{G}F)(t)$. Then, for $t \geq s$

$$\begin{aligned}
&x_t - U(t,s)x_s = (\hat{G}F)(t) - U(t,s)(\hat{G}F)(s) \\
&= \int_{-\infty}^{t} P(t)U^{\odot *}(t,\tau)P(\tau)F(\tau)d\tau - \int_{t}^{\infty} Q(t)U^{\odot *}(t,\tau)Q(\tau)F(\tau)d\tau \\
&\quad - U(t,s)\left[\int_{-\infty}^{s} P(s)U(s,\tau)^{\odot *}P(\tau)F(\tau)d\tau - \int_{s}^{\infty} Q(s)U(s,\tau)^{\odot *}Q(\tau)F(\tau)d\tau\right] \\
&= \int_{-\infty}^{s} P(t)U^{\odot *}(t,\tau)P(\tau)F(\tau)d\tau + \int_{s}^{t} P(t)U^{\odot *}(t,\tau)P(\tau)F(\tau)d\tau \\
&\quad - \int_{t}^{\infty} Q(t)U^{\odot *}(t,\tau)Q(\tau)F(\tau)d\tau - \int_{-\infty}^{s} U^{\odot *}(t,s)P(s)U(s,\tau)^{\odot *}P(\tau)F(\tau)d\tau \\
&\quad + \int_{s}^{t} U^{\odot *}(t,s)Q(s)U(s,\tau)^{\odot *}Q(\tau)F(\tau)d\tau + \int_{t}^{\infty} Q(t)U^{\odot *}(t,\tau)Q(\tau)F(\tau)d\tau \\
&= \int_{s}^{t} U^{\odot *}(t,\tau)P(\tau)F(\tau)d\tau + \int_{s}^{t} U^{\odot *}(t,\tau)Q(\tau)F(\tau)d\tau \\
&= \int_{s}^{t} U^{\odot *}(t,\tau)F(\tau)d\tau.
\end{aligned}$$

Therefore,

$$x_t = U(t,s)x_s + \int_{s}^{t} U^{\odot *}(t,\tau)F(\tau)d\tau. \tag{40}$$

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Finally, in order to show that x_t satisfies (16), we see by the Corollary 5.3 in the appendix we have the following equality

$$\int_s^t U^{\odot*}(t,\tau)F(\tau)d\tau = \int_s^t T_0^{\odot*}(t-\tau)F(\tau)d\tau + \int_s^t T_0^{\odot*}(t-\tau)B(\tau)\int_s^\tau U^{\odot*}(\tau,\sigma)F(\sigma)d\sigma d\tau \tag{41}$$

Using the expression for $U(t,s)$ given in (37) and (41) we obtain

$$\begin{aligned} x_t =& T_0(t-s)\varphi + \int_s^t T_0^{\odot*}(\tau)B(\tau)U(\tau,s)\varphi d\tau + \int_s^t T_0^{\odot*}(t-\tau)F(\tau)d\tau \\ &+ \int_s^t T_0^{\odot*}(t-\tau)B(\tau)\int_s^\tau U^{\odot*}(\tau,\sigma)F(\sigma)d\sigma d\tau \\ =& T_0(t-s)\varphi + \int_s^t T_0^{\odot*}(\tau)B(\tau)\overbrace{\left[U(\tau,s)\varphi d\tau + \int_s^\tau U^{\odot*}(\tau,\sigma)F(\sigma)d\sigma d\tau\right]}^{x_\tau} \\ &+ \int_s^t T_0^{\odot*}(t-\tau)F(\tau)d\tau \\ =& T_0(t-s)\varphi + \int_s^t T_0^{\odot*}(\tau)B(\tau)x_\tau d\tau + \int_s^t T_0^{\odot*}(t-\tau)F(\tau)d\tau \end{aligned}$$

This means, x_t is the solution of the integral equation

$$x_t = T_0(t-s)\varphi + \int_s^t T_0^{\odot*}(t-\tau)C(\tau)x_\tau d\tau.$$

It follows of the Theorem 4.2 that x_t is the solution of the inhomogeneous RFDE (11).

In order to prove the second part of the theorem, since the superposition principle, for all $f \in BC(\mathbb{R},\mathbb{C}^n)$, the inhomogeneous equation (11) has a unique bounded solution $x(\cdot)$ with $x_s = \varphi_f \in V_2$. Consider, now, the linear operator $S : BC(\mathbb{R},\mathbb{C}^n) \to V_2 \subset X$ which associate f with φ_f as described above and for $t \geq s$ let $V(t)$ the family of linear operators defined by

$$\begin{aligned} V(t) : BC(\mathbb{R},\mathbb{C}^n) &\to X \\ f \quad &\mapsto x_t(Sf,f), \end{aligned}$$

where x is the solution of (11). The first part of the theorem, the limitation of x_t for every $f \in BC(\mathbb{R},\mathbb{C}^n)$ give us

$$\sup_t |V(t)f| < \infty.$$

Therefore, by uniformly boundedness principle there is a constant $r \geq 0$ such that $\|V(t)\| \leq r$, which finish the proof of the theorem. ∎

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APPENDIX: VARIANTS OF VARIATION-OF-CONSTANT FORMULA

When we work with non autonomous system, we consider families of two-parameters $\{U(t,s)\}$ and we have the variation-of-constant formula given by

$$U(t,s)\varphi = T_0(t-s)\varphi + \int_s^t T_0^{\odot *}(t-\tau)B(\tau)U(\tau,s)\varphi d\tau.$$

Consider

$$W(t,s)x^{\odot *} = \int_s^t U^{\odot *}(\tau,s)x^{\odot *}d\tau, \tag{42}$$

$$W_0(t,s)x^{\odot *} = \int_s^t T_0^{\odot *}(\tau-s)x^{\odot *}d\tau. \tag{43}$$

Lemma 5.1. *Assume that the map $t \mapsto B(t)$ is continuous from $[\alpha,\omega]$ into $\mathcal{L}(X, X^{\odot *})$. For all $x^* \in X^*$, $x^{\odot *} \in X^{\odot *}$, $t \geq s$,*

$$\langle x^*, \int_s^t U^{\odot *}(\tau,s)d\tau\rangle = \langle x^{\odot *}, \int_s^t U^*(\tau,s)x^*d\tau\rangle.$$

Proof. Fix $x^* = x^{\odot} \in X^{\odot}$. By the definition of the weak-$*$ integral, it follows that

$$\begin{aligned}\langle x^*, \int_s^t U^{\odot *}(\tau,s)d\tau\rangle &= \int_s^t \langle x^*, U^{\odot *}(\tau,s)x^{\odot *}\rangle d\tau = \int_s^t \langle U^{\odot}(\tau,s)x^*, x^{\odot *}\rangle d\tau \\ &= \langle \int_s^t U^{\odot}(\tau,s)d\tau x^*, x^{\odot *}\rangle = \langle \int_s^t U^*(\tau,s)d\tau x^*, x^{\odot *}\rangle. \end{aligned} \tag{44}$$

Now we notice that for $x^* \in X^{\odot}$, $\frac{1}{h}\int_0^h U^*(s,\sigma)x^*d\sigma \in X^{\odot}$. In fact, by hypothesis about of the family $B(t)$, the Theorem 4.5 in [9] ensures that $X^{\odot}$ is invariant by $U^*(t,s)$, then $U^*(s,\sigma) \in X^{\odot}$. Therefore

$$\begin{aligned}&\left\| U^*(t,s)\frac{1}{h}\int_0^h U^*(s,\sigma)x^*d\sigma - \frac{1}{h}\int_0^h U^*(s,\sigma)x^*d\sigma\right\| \\ &\quad = \frac{1}{h}\left\|\int_0^h U^*(t,s)U^*(s,\sigma)x^* - U^*(s,\sigma)x^*d\sigma\right\| \\ &\quad \leq \frac{1}{h}\int_0^h \|U^*(t,s)U^*(s,\sigma)x^* - U^*(s,\sigma)x^*\| \to 0, \text{ when } t \to s.\end{aligned}$$

Thus, we can approximate x^* por $\frac{1}{h}\int_0^h U^*(s,\sigma)x^* d\sigma \in X^{\odot}$ and the left side of (44) became

$$\langle \frac{1}{h}\int_0^h U^*(\sigma,s)x^* d\sigma, \int_s^t U^{\odot *}(\tau,s)x^* d\tau\rangle = \frac{1}{h}\int_0^h \langle U^*(\sigma,s)x^*, \int_s^t U^{\odot *}(\tau,s)x^* d\tau\rangle$$
$$= \frac{1}{h}\int_0^h \langle x^*, U(\sigma,s)\int_s^t U^{\odot *}(\tau,s)x^* d\tau\rangle.$$

Since $\int_s^t U^{\odot *}(\tau,s)x^* d\tau \in X$ and provided $U(\sigma,s)$ is strongly continuous, we obtain

$$\langle \frac{1}{h}\int_0^h U^*(\sigma,s)x^* d\sigma, \int_s^t U^{\odot *}(\tau,s)x^* d\tau\rangle \to \langle x^*, \int_s^t U^{\odot *}(\tau,s)x^{\odot *} d\tau\rangle,\ h\downarrow 0.$$

Of the last equality of the right side of (44), for x^* approximated by $\frac{1}{h}\int_0^h U^*(s,\sigma)x^* d\sigma \in X^{\odot}$, we obtain

$$\langle x^{\odot *}, \int_s^t U^*(\tau,s)(\frac{1}{h}\int_0^h U^*(s,\sigma)x^* d\sigma)d\tau\rangle \to \langle x^{\odot *}, \int_s^t U^*(\theta,s)x^* d\theta\rangle,\ h\downarrow 0.$$

This completes the proof of the lemma. ▮

Corollary 5.1. *$W(t)^*$ applies X^* in $X^{\odot}$ and $W(t)^* = W^{\odot}(t)$, where $W^{\odot}(t) : X^* \to X^{\odot}$ is defined by*

$$W^{\odot}(t,s)x^* = \int_s^t U^*(\tau,s)x^* d\tau.$$

Corollary 5.2. *Let $\{x_n^{\odot *}\}$ a sequence in $X^{\odot *}$ converges weak-$*$ for $x_\infty^{\odot *}$. Then $\{W(t)x_n^{\odot *}\}$ converge weakly in X for $W(t)x_\infty^{\odot *}$.*

Proof. For $x^* \in X^{\odot}$ and using the before corollary, notice that

$$\langle W(t,s)x_n^{\odot *}, x^*\rangle = \langle \int_s^t U^{\odot *}(\tau,s)x_n^{\odot *} d\tau, x^*\rangle = \int_s^t \langle U^{\odot *}(\tau,s)x_n^{\odot *}, x^*\rangle d\tau$$
$$= \int_s^t \langle x_n^{\odot *}, U^{\odot}(\tau,s)x^*\rangle d\tau = \langle x_n^{\odot *}, \int_s^t U^{\odot}(\tau,s)x^*\rangle d\tau$$
$$\to \langle x_\infty^{\odot *}, \int_s^t U^{\odot}(\tau,s)x^* d\tau\rangle = \langle x_\infty^{\odot *}, \int_s^t U^*(\tau,s)x^* d\tau\rangle$$
$$= \langle x_\infty^{\odot *}, W(t)^* x^*\rangle = \langle W(t)x_\infty^{\odot *}, x^*\rangle,$$

and the proof is complete. ▮

Now we are going to prove the variation-of-constant formula for the semigroup $W(t,s)$.

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THEOREM 5.2. *For all $x^{\odot *} \in X^{\odot *}$, $t \geq s$,*

$$W(t,s)x^{\odot *} = W_0(t,s)x^{\odot *} + \int_s^t T_0^{\odot *}(t-\tau)B(\tau)W(\tau,s)x^{\odot *}d\tau. \tag{45}$$

Proof. Let $\varphi \in X$ and $x^{\odot} \in X^{\odot}$, it follows of the variant-of-constant formula that

$$\begin{aligned}\langle x^{\odot}, \int_s^t U(\tau,s)\varphi d\tau\rangle =& \langle x^{\odot}, \int_s^t T_0(\tau-s)\varphi d\tau\rangle \\ &+ \langle x^{\odot}, \int_s^t \int_s^\tau T_0^{\odot *}(\tau-\sigma)B(\sigma)U(\sigma,s)\varphi d\sigma d\tau\rangle \\ =& \langle x^{\odot}, \int_s^t T_0(\tau-s)\varphi d\tau\rangle \\ &+ \int_s^t \int_s^\tau \langle B(\sigma)^* T_0^{\odot}(\tau-\sigma)x^{\odot}, U(\sigma,s)\varphi\rangle d\sigma d\tau.\end{aligned}$$

Changing the integration order in the last term, we obtain

$$\langle x^{\odot}, W(t,s)\varphi\rangle = \langle x^{\odot}, W_0(t,s)\varphi\rangle + \int_s^t \langle B^*(t)T_0^{\odot}(\sigma)x^{\odot}, W(t-\sigma,s)\varphi\rangle d\sigma.$$

Let $x^{\odot *} \in X^{\odot *}$ and set $\{x_n\}$ a sequence in X converges to $x^{\odot *}$ in the sense weak-$*$ (for example, take $x_n = n(nI - A^{\odot *})^{-1}x^{\odot *}$). In the last identity consider $\varphi = x^n$. It follows by Corollary 5.2 and by dominated convergence theorem that

$$\langle x^{\odot}, W(t,s)x^{\odot *}\rangle = \langle x^{\odot}, W_0(t,s)x^{\odot *}\rangle + \int_s^t \langle B^*(t)T_0^{\odot}(\sigma)x^{\odot}, W(t-\sigma,s)x^{\odot *}\rangle d\sigma,$$

thus (45) follow immediately. ∎

COROLLARY 5.3. *For $x^{\odot *} \in X^{\odot *}$,*

$$\begin{aligned}\int_s^t U^{\odot *}(t,\tau)x^{\odot *}d\tau =& \int_s^t T_0^{\odot *}(t-\tau)x^{\odot *}d\tau \\ &+ \int_s^t T_0^{\odot *}(t-\tau)B(\tau)\int_s^\tau U^{\odot *}(\tau,\sigma)x^{\odot *}d\sigma d\tau.\end{aligned}$$

Proof. For $t = s$ the equality is hold. Let $x^{\odot} \in X^{\odot}$, integrating both side from s to t and making duality with $x^{\odot}$, it follows from the Theorem 5.2, that

$$\begin{aligned}
&\int_s^t \langle \int_s^\sigma U^{\odot*}(\sigma,\tau)x^{\odot*}d\tau - \int_s^\sigma T_0^{\odot*}(\sigma-\tau)x^{\odot*}d\tau, x^{\odot}\rangle d\sigma \\
&= \int_s^t \langle (W(t,\sigma) - W_0(t,\sigma))x^{\odot*}, x^{\odot}\rangle d\sigma \\
&= \langle \int_s^t (W(t,\sigma) - W_0(t,\sigma))x^{\odot*}d\sigma, x^{\odot}\rangle \\
&= \langle \int_s^t \int_\sigma^t T_0^{\odot*}(t-\tau)B(\tau)W(\tau,\sigma)x^{\odot*}d\tau d\sigma, x^{\odot}\rangle \\
&= \langle \int_s^t \int_s^\gamma T_0^{\odot*}(\gamma-\tau)B(\tau)\int_s^\tau U^{\odot*}(\tau,\sigma)x^{\odot*}d\sigma d\tau d\gamma, x^{\odot}\rangle,
\end{aligned}$$

this complete the proof. ▮

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