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The center problem for a 1:-4 resonant quadratic system

Brigita Ferčec, Jaume Giné, Matej Mencinger and Regilene Oliveira

Brigita Ferčec

Center for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova 2, SI-2000 Maribor, Slovenia
E-mail: brigita.fercec@gmail.com

Jaume Giné

Departament de Matemàtica, Universitat de Lleida, Av. Jaume II, 69, 25001 Lleida, Spain
E-mail: gine@matematica.udl.cat

Matej Mencinger

Faculty of Civil Engineering, University of Maribor, Smetanova 17, SI-2000 Maribor, Slovenia

Institute of Mathematics, Physics and Mechanics, Jadranska 19, SI-1000 Ljubljana, Slovenia
E-mail: matej.mencinger@um.si

Regilene Oliveira

Instituto de Ciências Matemáticas e de Computação. Universidade de São Paulo, Av. Trabalhador São-carlense, 400-Centro. Caixa Postal: 668, São Carlos, SP, Brazil
E-mail: regilene@icmc.usp.br

The main object of this paper is to find necessary and sufficient conditions for a $1 : -4$ resonant system of the form

$$\dot{x} = x - a_{10}x^2 - a_{01}xy - a_{12}y^2, \quad \dot{y} = -4y + b_{2,-1}x^2 + b_{10}xy + b_{01}y^2.$$

to have a center at the origin. Since applying a linear change of variables any system of this form can be transformed either to system with $a_{10} = 1$ or $a_{10} = 0$ only these two cases are considered. When $a_{10} = 1$ there appear 46 resonant center conditions and for $a_{10} = 0$ there are 9 center conditions. The necessary conditions are obtained using modular arithmetics. The sufficiency of each obtained condition is proven using a local analytic first integral - to find it or prove its existence distinct criteria are used. October, 2013 ICMC-USP

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1. INTRODUCTION

In this paper we consider differential autonomous systems in $\mathbb{C}^2$ of the form

$$\dot{x} = px + P(x,y), \quad \dot{y} = -qy + Q(x,y), \tag{1.1}$$

where $p, q \in \mathbb{N}$ and P and Q are complex polynomials. In particular, we consider the case $p = 1$, $q = 4$ and P, Q being quadratic polynomials. The main goal of the paper is to determine when the elementary singular point located at the origin is a resonant center where the definition of a resonant center comes from Dulac [10].

Definition 1.1. A $p : -q$ resonant elementary singular point of analytic system (1.1) is a resonant center if there exists a local analytic first integral of the form

$$\Phi(x,y) = x^q y^p + \sum_{j+k>p+q+1} \phi_{j-q,k-p} x^j y^k. \tag{1.2}$$

The $p : -q$ resonant center is a generalization of the concept of a real center to systems of differential equations in $\mathbb{C}^2$ of the form (1.1); see for instance [4, 8, 30]. The classical real center (which originates from the work of Poincaré and Lyapunov [23, 27]) was studied by several authors (see e.g. [3, 10, 16] and [12] for some other generalizations). Note that the $1 : -1$ resonant center in (see, for instance [30] for more details) $\mathbb{C}^2$ is actually the complexification of the classical real planar center [10]. The saddle-node and the node case were studied in [32, 34]. The study of $p : -q$ resonant centers is relatively new (see e.g. [2, 8, 17, 24, 25, 29, 34] and the references given there). Two general cases of the form $q \leq 0 < p$, $GCD(p,q) = 1$ which have been solved completely are the $1 : -1$ resonant cases with quadratic and homogeneous cubic nonlinearities [5, 10, 31]. The investigation of center cases for $1 : -2$ resonance of quadratic systems is practically sure that is completed in [14]. The same happens for the center cases for $1 : -3$ resonance, see [9]. The $1 : -4$ resonance presented in this paper is in the actual limit of the current computational facilities. The objective is to have the classification of lower resonances in order to use them for the study of particular systems with higher resonances via blow-up transformations.

The most studied resonant centers in (1.1) are the $1 : -\lambda$ quadratic Lotka-Volterra systems, i.e. systems of the form $\dot{x} = x(1 + ax + by)$, $\dot{y} = y(-\lambda + cx + dy)$, see [6]. In [21] certain sufficient conditions for integrability and linearizability for general $\lambda \in \mathbb{N}$ and necessary and sufficient conditions for $\lambda = \frac{p}{2}$ and for $\lambda = \frac{2}{p}$ (with $p \in \mathbb{N}$) are presented. Cases $3 : -4$ and $3 : -5$ are studied in [24]. Recently, the integrability of the Lotka-Volterra type systems (1.1) are studied in [15, 18, 25].

Inside the $1 : -1$ resonance centers, the case when P and Q are homogeneous quintic nonlinearities have been studied in [13, 19] but not finished. Concerning the generalized $p : -q$ resonant center many papers are considering $1 : -q$ and $2 : -q$ resonant centers (with mostly homogeneous quadratic, cubic or quartic nonlinearities added); see e.g. [2, 8]. In [34] for the general $p : -q$ resonant quadratic case there are exhibited fifteen independent sufficient conditions for existence of a center, along with the corresponding first integrals for each one.

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In [2] the $1:-q$ resonant center problem for certain cubic Lotka-Volterra system was studied for integers $q \leq 9$. For odd $q \leq 9$ the authors obtained necessary and sufficient conditions for existence of a center, whilst for even $q < 9$ only necessary conditions were obtained. This may indicate that the analysis of a $1:-q$ resonant center might be more difficult for q being even. The computational difficulties which occur when performing computations with Computer Algebra Systems (CAS) Singular and Mathematica for the present paper may definitely confirm this idea.

By the definition any nonconstant differentiable complex function which is constant on trajectories of (1.1) is a first integral of (1.1):

$$\dot{\Psi} := \frac{\partial \Psi}{\partial x}(px + P(x,y)) + \frac{\partial \Psi}{\partial y}(-qy + Q(x,y)) \equiv 0. \tag{1.3}$$

Throughout the work P and Q are assumed to be quadratic polynomials.

Thus, we can write system (1.1) in the form

$$\dot{x} = px - \sum_{\substack{j+k=1 \\ j \geq -1}}^{n-1} a_{jk} x^{j+1} y^k, \quad \dot{y} = -qy + \sum_{\substack{j+k=1 \\ j \geq -1}}^{n-1} b_{kj} x^k y^{j+1}. \tag{1.4}$$

According to Definition 1.1 to find conditions for existence of a resonant center of a $p:-q$ resonant elementary singular point of system (1.4) we look for a formal series in the form (1.2) satisfying (1.3). In other words, the approach is based on computing the saddle quantities [2, 5, 8, 14, 29, 34], $g_{kq,kp}$, which are polynomials in the coefficients $a_{i,j}$, $b_{i,j}$ of (1.4) with the property that a formal first integral (1.3) exists if and only if every saddle quantity $g_{kq,kp}$ vanish on the coefficients of (1.4), that is if and only if the coefficients $a_{i,j}$, $b_{i,j}$ of (1.4) lie in the variety $\mathbf{V}(\mathcal{B})$ in $\mathbb{C}^{(n+4)(n-1)}$ of the (Bautin) ideal

$$\mathcal{B} = \langle g_{kq,kp} : k \in \mathbb{N} \rangle$$

in the polynomial ring $\mathbb{C}[a_{10}, a_{01}, a_{-12}, \ldots, b_{2,-1}, b_{10}, b_{01}]$ which we abbreviate to $\mathbb{C}[a,b]$.

Therefore, to start the computational process we write down the initial string of (1.2) up to order $2N+1$

$$\Psi_{2N+1}(x,y) = x^q y^p + \sum_{j+k=p+q+1}^{2N+1} \phi_{j-q,k-p} x^j y^k. \tag{1.5}$$

Then for each $i = p+q+1, \ldots, 2N+1$ we equate coefficients of terms of order i in the expression

$$\frac{\partial \Psi_{2N+1}}{\partial x}(px + P(x,y)) + \frac{\partial \Psi_{2N+1}}{\partial y}(-qy + Q(x,y)). \tag{1.6}$$

Now let denote the coefficients of $x^{k_1+q}y^{k_2+p}$ in (1.6) by g_{k_1,k_2} and set them to be zero for $k_1+k_2 \leq 0$. For $k_1+k_2 \geq 1$ they are given [30, p. 117] recursively by:

$$\begin{aligned} g_{k_1,k_2} =& (pk_1 - qk_2)\phi_{k_1,k_2} - \\ & - \sum_{\substack{i_1+i_2=0 \\ i_1\geq -q, i_2 \geq -p}}^{k_1+k_2-1} [(i_1+q)a_{k_1-i_1,k_2-i_2} - (i_2+p)b_{k_1-i_1,k_2-i_2}]\phi_{i_1,i_2}. \end{aligned} \tag{1.7}$$

The coefficients g_{k_1,k_2} defined in (1.7) can obviously be set to zero, as long as k_1 and k_2 satisfy the conditions $k_1 \neq kq$ and $k_2 \neq kp$ (note that setting g_{k_1,k_2} to zero determines the coefficient ϕ_{k_1,k_2} by the previous ones, ϕ_{i_1,i_2}, as defined in (1.7) - using initial condition $\phi_{0,0} = 1$). However, the coefficients $g_{kq,kp}$ (i.e. g_{k_1,k_2} for $k_1 = kq$ and $k_2 = kp$) cannot be set to zero just by choosing a certain value for ϕ_{i_1,i_2}. Therefore, the corresponding polynomials for $g_{kq,kp}$ are defined to be the *saddle quantities*:

$$g_{kq,kp} = \sum_{\substack{i_1+i_2=0 \\ i_1\geq -q, i_2 \geq -p}}^{kq+kp-1} [(i_1+q)a_{kq-i_1,kp-i_2} - (i_2+p)b_{kq-i_1,kp-i_2}]\phi_{i_1,i_2}. \tag{1.8}$$

Obviously, if for a fixed system (1.4) with the coefficients (a^*, b^*), we have $g_{kq,kp}(a^*, b^*) = 0$ for all $k \in \mathbb{N}$, then we obtain a formal first integral Ψ defined in (1.5). Therefore to find necessary conditions for existence of formal first integral we need to find the set of all parameters (a, b) where all polynomials $g_{kq,kp}$ vanish, i.e. to find the variety of the Bautin ideal $\mathcal{B}$. By the Hilbert Basis Theorem (see e.g. [30, Th. 1.1.6]) the ideal $\mathcal{B}$ is finitely generated, i.e. there exists $K \in \mathbb{N}$ such that $\mathcal{B} = \mathcal{B}_K$, where $\mathcal{B}_K = \langle g_{kq,kp} : 1 \leq k \leq K\rangle$. Therefore, we have to find first few saddle quantities and then to compute the variety of the ideal they generate, where the variety of the ideal generated by polynomials $f_1, \dots f_s$ is the set of common solutions of polynomial system $f_1 = 0, \dots, f_s = 0$, i.e.

$$\mathbf{V}(\langle f_1, \dots f_s\rangle) = \{a = (a_1, \dots a_n) \in k^n : f_i(a) = 0, \quad \text{for every} \quad i = 1, \dots, s\}.$$

The variety of the ideal $\mathcal{B}$, $\mathbf{V}(\mathcal{B})$, is called a *center variety.* In the following two sections we consider necessary and sufficient conditions for the existence of a $1 : -4$ resonant center for system 1.4 with quadratic nonlinearities.

2. STATEMENT OF THE MAIN RESULTS

Let us consider system (1.4) for $p = 1$, $q = 4$ and P, Q being quadratic polynomials:

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 - a_{01}xy - a_{-12}y^2 \\ \dot{y} &= -4y + b_{2,-1}x^2 + b_{10}xy + b_{01}y^2, \end{aligned} \tag{2.1}$$

where $x, y, a_{ij}, b_{ji} \in \mathbb{C}$.

First note that using a linear transformation every system (2.1) can be transformed to either a system (2.1) with $a_{10} = 1$ or $a_{10} = 0$. In both cases the (minimal primary) decomposition of the corresponding ideal $\mathcal{B}_K$ cannot be performed in polynomial rings of characteristic zero, i.e. in the ring $\mathbb{Q}[a, b]$ (where $[a, b]$ denotes the coefficients of system (2.1)). However, it becomes possible in the polynomial ring of a proper prime characteristic, i.e. in $\mathbb{Z}_p[a, b]$. See more details in the proof of Theorem 2.1, in Section 3.

The necessary conditions for $a_{10} = 1$ and $a_{10} = 0$ are considered in the following two theorems:

THEOREM 2.1. *System* (2.1) *with* $a_{10} = 1$ *has a* $1 : -4$ *resonant center at the origin if one of the following* 46 *conditions holds:*

1)$a_{01} = a_{-12} = 6b_{10}^2 - 18b_{10} + b_{01}b_{2,-1} + 12 = 0$;

2)$a_{01} = a_{-12} = 4b_{10}^2 - 12b_{10} + 9b_{01}b_{2,-1} = 0$;

3)$5b_{10} - 4 = 4a_{01} + 7b_{01} = a_{-12} = 25b_{01}b_{2,-1} + 24 = 0$;

4)$19b_{10} - 8 = 4a_{01} + 7b_{01} = a_{-12} = 361b_{01}b_{2,-1} + 168 = 0$;

5)$13b_{10} - 16 = 4a_{01} + 7b_{01} = a_{-12} = 169b_{01}b_{2,-1} + 72 = 0$;

6)$7b_{10} + 1 = 4a_{01} + 7b_{01} = a_{-12} = 49b_{01}b_{2,-1} + 12 = 0$;

7)$b_{2,-1} = a_{-12} = -4a_{01} - 3b_{01} + b_{01}b_{10} = 0$;

8)$b_{10} - 1 = a_{01} - b_{01} = a_{-12} = 49b_{01}b_{2,-1} - 24 = 0$;

9)$23b_{10} - 14 = a_{01} - b_{01} = a_{-12} = 529b_{01}b_{2,-1} - 168 = 0$;

10)$b_{10} = 4a_{01} + 3b_{01} = a_{-12} = 0$;

11)$19b_{10} - 16 = 4a_{01} + 11b_{01} = a_{-12} = 361b_{01}b_{2,-1} + 168 = 0$;

12)$23b_{10} - 22 = a_{01} - 2b_{01} = a_{-12} = 529b_{01}b_{2,-1} - 168 = 0$;

13)$b_{10} - 5 = 2a_{01} - b_{01} = a_{-12} = 0$;

14)$b_{2,-1} = b_{10} - 1 = 0$;

15)$b_{2,-1} = b_{10} = 0$;

16)$b_{2,-1} = b_{10} - 2 = 0$;

17)$13b_{10} - 16 = 338a_{01}b_{2,-1} + 169b_{01}b_{2,-1} - 180 = 108a_{01} + 9b_{01} + 65a_{-12}b_{2,-1} =$

$$= 1872a_{01}^2 + 1092b_{01}a_{01} + 78b_{01}^2 + 600a_{-12} = 0;$$

18)$-15b_{10} + 19a_{01}b_{2,-1} + 11b_{01}b_{2,-1} + 6 = 120 - 300b_{10} + 114b_{10}^2 + 49b_{01}b_{2,-1} =$

$$= 18a_{01}b_{10} - 48a_{01} - 18b_{01} + 12b_{01}b_{10} - 7a_{-12}b_{2,-1} = 72 - 2232b_{10} + 1329b_{01}b_{2,-1} +$$

$$+570b_{01}b_{10}b_{2,-1} - 2527a_{-12}b_{2,-1}^2 = 3192a_{01} + 1674b_{01} - 1221b_{01}b_{10} + 38a_{-12}b_{2,-1} +$$

$$+105b_{01}^2b_{2,-1} + 1083a_{-12}b_{10}b_{2,-1} = 504a_{01}^2 - 144a_{-12} + 42a_{01}b_{01} - 45b_{01}^2 + 306a_{-12}b_{10} +$$

$$= 30b_{01}^2b_{10} - 133a_{-12}b_{01}b_{2,-1} = 72a_{01}^3 + 96a_{01}a_{-12} + 54a_{01}^2b_{01} + 36a_{-12}b_{01} + 9a_{01}b_{01}^2 +$$

$$-15a_{-12}b_{01}b_{10} + 17a_{-12}^2b_{2,-1} = 0;$$

19)$b_{2,-1} = b_{10} - 8 = 4a_{01} - 5b_{01} = 0$;

20) $b_{2,-1} = b_{10} - 3 = a_{01} = 0$;

21) $b_{01} = a_{01} = a_{-12} = 0$;

22) $61b_{10} - 92 = 7a_{01} + b_{01} = 3721b_{01}b_{2,-1} - 4116 = 427a_{-12}b_{2,-1} - 27b_{01} = 549b_{01}^2 - 9604a_{-12} = 0$;

23) $131b_{10} - 112 = 36a_{01} - 97b_{01} = 17161b_{01}b_{2,-1} - 4536 = 131a_{-12}b_{2,-1} - 56b_{01} = 131b_{01}^2 - 81a_{-12} = 0$;

24) $8b_{10} + 9 = 19a_{01} + 12b_{01} = 32b_{01}b_{2,-1} - 57 = 152a_{-12}b_{2,-1} - 27b_{01} = 36b_{01}^2 - 361a_{-12} = 0$;

25) $22b_{10} - 9 = 53a_{01} + 54b_{01} = 121b_{01}b_{2,-1} + 159 = 583a_{-12}b_{2,-1} - 108b_{01} = 396b_{01}^2 + 2809a_{-12} = 0$;

26) $191b_{10} - 72 = 4a_{01} + 177b_{01} = 36481b_{01}b_{2,-1} + 504 = 1134b_{01} + 191a_{-12}b_{2,-1} = 1719b_{01}^2 - 4a_{-12} = 0$;

27) $b_{10} - 12 = 2a_{01} - 9b_{01} = 2b_{01}b_{2,-1} + 3 = 2a_{-12}b_{2,-1} - 27b_{01} = 9b_{01}^2 + a_{-12} = 0$;

28) $5b_{10} - 4 = 2a_{01} - 9b_{01} = 50b_{01}b_{2,-1} - 9 = 10a_{-12}b_{2,-1} - 9b_{01} = 5b_{01}^2 - a_{-12} = 0$;

29) $23b_{10} - 31 = 3a_{01} + 4b_{01} = 529b_{01}b_{2,-1} + 216 = 23a_{-12}b_{2,-1} - 48b_{01} = 46b_{01}^2 + 9a_{-12} = 0$;

30) $38b_{10} - 81 = 107a_{01} + 6b_{01} = 361b_{01}b_{2,-1} - 321 = 2033a_{-12}b_{2,-1} - 108b_{01} = 684b_{01}^2 - 11449a_{-12} = 0$;

31) $13b_{10} - 6 = 2a_{01} - 9b_{01} = 1352b_{01}b_{2,-1} - 147 = 52a_{-12}b_{2,-1} - 27b_{01} = 234b_{01}^2 - 49a_{-12} = 0$;

32) $19b_{10} - 28 = 31a_{01} + 23b_{01} = 361b_{01}b_{2,-1} + 2604 = 589a_{-12}b_{2,-1} - 189b_{01} = 171b_{01}^2 + 3844a_{-12} = 0$;

33) $b_{10} - 12 = 38a_{01} - 21b_{01} = b_{01}b_{2,-1} + 114 = 38a_{-12}b_{2,-1} - 63b_{01} = 21b_{01}^2 + 1444a_{-12} = 0$;

34) $4b_{10} - 3 = 148a_{01} - 81b_{01} = 256b_{01}b_{2,-1} - 111 = 27b_{01} + 148a_{-12}b_{2,-1} = 576b_{01}^2 + 1369a_{-12} = 0$;

35) $11b_{10} - 27 = a_{01} + 3b_{01} = 121b_{01}b_{2,-1} + 24 = 11a_{-12}b_{2,-1} - 36b_{01} = 33b_{01}^2 + 2a_{-12} = 0$;

36) $121b_{10} - 97 = a_{01} + 28b_{01} = 14641b_{01}b_{2,-1} + 504 = 504b_{01} + 121a_{-12}b_{2,-1} = 121b_{01}^2 - a_{-12} = 0$;

37) $4b_{10} - 3 = 3a_{01} - 41b_{01} = 392b_{01}b_{2,-1} - 27 = 14a_{-12}b_{2,-1} - 39b_{01} = 364b_{01}^2 - 9a_{-12} = 0$;

38) $71b_{10} + 8 = 22a_{01} + 31b_{01} = 5041b_{01}b_{2,-1} + 1584 = 781a_{-12}b_{2,-1} - 81b_{01} = 639b_{01}^2 + 1936a_{-12} = 0$;

39) $57b_{10} - 44 = a_{01} - 37b_{01} = 1083b_{01}b_{2,-1} - 28 = 19a_{-12}b_{2,-1} - 119b_{01} = 969b_{01}^2 - 4a_{-12} = 0$;

40) $89b_{10} - 248 = 58a_{01} - b_{01} = 7921b_{01}b_{2,-1} - 4176 = 2581a_{-12}b_{2,-1} - 81b_{01} = 801b_{01}^2 - 13456a_{-12} = 0$;

41) $11b_{10} + 8 = 7a_{01} + b_{01} = 121b_{01}b_{2,-1} + 84 = 77a_{-12}b_{2,-1} - 27b_{01} = 99b_{01}^2 + 196a_{-12} = 0$;

42) $b_{2,-1} = 2b_{10} + 1 = b_{01} = a_{01} = 0$;

43) $b_{2,-1} = 2b_{10} - 1 = b_{01} = a_{01} = 0$;

44) $b_{2,-1} = 2b_{10} - 5 = b_{01} = a_{01} = 0$;

45) $b_{2,-1} = 2b_{10} - 3 = b_{01} = a_{01} = 0$;

46) $-18346b_{10} + 17857b_{01}b_{2,-1} + 7100 = 10027b_{10} + 17857a_{01}b_{2,-1} - 14708 = -36738a_{01} - 25857b_{01} + 262753a_{-12}b_{2,-1} = 2551b_{10}^2 - 5224b_{10} + 1594 = -7966a_{01} - 440739b_{01} + 262753b_{01}b_{10} = 525506b_{10}a_{01} - 194666a_{01} + 60473b_{01} = 819b_{01}^2 + 3606a_{-12} - 9670a_{-12}b_{10} = 788b_{10}a_{-12} - 209a_{-12} + 819a_{01}b_{01} = 1638a_{01}^2 - 9355a_{-12} + 5469a_{-12}b_{10} = 0$.

THEOREM 2.2. *System (2.1) with $a_{10} = 0$ has a $1 : -4$ resonant center at the origin if and only if one of the following 9 conditions holds:*

1)$b_{2,-1} = b_{10} = 0$*;*
2)$b_{2,-1} = b_{01} = a_{-12} = 0$*;*
3)$a_{01} = a_{-12} = 6b_{10}^2 + b_{01}b_{2,-1} = 0$*;*
4)$a_{01} = a_{-12} = 4b_{10}^2 + 9b_{01}b_{2,-1} = 0$*;*
5)$b_{01} = a_{01} = a_{-12} = 0$*;*
6)$b_{10} = 2a_{01} - b_{01} = a_{-12} = 0$*;*
7)$b_{10} = 4a_{01} + 3b_{01} = a_{-12} = 0$*;*
8)$b_{10} = 2a_{01} + b_{01} = a_{-12} = 0$*;*
9)$19a_{01}+11b_{01} = 114b_{10}^2+49b_{01}b_{2,-1} = 30b_{01}b_{10}-133a_{-12}b_{2,-1} = 35b_{01}^2+361a_{-12}b_{10} = 0$*;*

3. PROOFS OF THE MAIN RESULTS

In this section we prove the main results of this paper: Theorems 2.1 and 2.2. Since the Darboux method of integration is the main tool for proving the sufficiency of the conditions in Theorems 2.1 and 2.2 let first recall some definitions concerning this method. Note also that it is commonly used as a tool for investigating the center problem [8, 12, 30]. In this section we use it to prove the existence of first integrals in most cases of polynomial systems (2.1) listed in Theorems 2.1 and 2.2. The method is in details described in [30]. A good survey of recent applications of Darboux method to polynomial systems in $\mathbb{R}^n$ and $\mathbb{C}^n$ is also [26]. We apply the method to system (2.1) and set $\widetilde{P}$ ($\widetilde{Q}$) for $x - a_{10}x^2 - a_{01}xy - a_{-12}y^2$ ($-4y + b_{2,-1}x^2 + b_{10}xy + b_{01}y^2$), respectively. Thus, system (2.1) is now written as

$$\dot{x} = \widetilde{P}(x, y), \quad \dot{y} = \widetilde{Q}(x, y), \tag{3.1}$$

where $x, y \in \mathbb{C}$, $\widetilde{P}$ and $\widetilde{Q}$ are polynomials without constant terms that have no nonconstant common factor, and $m = max(deg(\widetilde{P}), deg(\widetilde{Q}))$.

We define the *algebraic partial integral* or *Darboux factor* of system (3.1) to be a polynomial $f(x, y)$ such that

$$\frac{\partial f}{\partial x}\widetilde{P} + \frac{\partial f}{\partial y}\widetilde{Q} = Kf,$$

where $K(x, y)$ is called a *cofactor*. It turns out that $K(x, y)$ is a polynomial of degree at most $m - 1$.

An *integrating factor* on an open set Ω for system (3.1) is a differentiable function $\mu(x, y)$ on Ω such that

$$\frac{\partial \mu}{\partial x}\widetilde{P} + \frac{\partial \mu}{\partial y}\widetilde{Q} = -\mu(\widetilde{P}_x + \widetilde{Q}_y)$$

holds throughout on Ω, where $\widetilde{P}_x + \widetilde{Q}_y$ stands for the divergence of $(\widetilde{P}, \widetilde{Q})$.

It is easily seen that if there are Darboux factors $f_1, f_2, \dots, f_k$ with the cofactors $K_1, K_2, \dots, K_k$ satisfying

$$\sum_{i=1}^{k} \alpha_i K_i = 0, \tag{3.2}$$

then $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$ is a first integral of (3.1), and if

$$\sum_{i=1}^{k} \alpha_i K_i + \widetilde{P}_x + \widetilde{Q}_y = 0 \tag{3.3}$$

then the equation admits the (Darboux) integrating factor

$$\mu = f_1^{\alpha_1} \cdots f_k^{\alpha_k}. \tag{3.4}$$

3.1. Proof of Theorem 2.1

Following the approach described in [17, 30] and using a straightforward modification of the computer code in [30, p. 128] we compute first 6 saddle quantities $g_{4,1}, \dots, g_{24,6}$. First saddle quantity is

$$\begin{aligned} g_{4,1} = &- 1512 b_{01} b_{10}^4 + 6048 a_{01} a_{10} b_{10}^3 + 9072 a_{10} b_{01} b_{10}^3 + 684 a_{-12} b_{2,-1} b_{10}^3 \\ &- 18144 a_{01} a_{10}^2 b_{10}^2 - 16632 a_{10}^2 b_{01} b_{10}^2 + 9072 a_{01}^2 b_{2,-1} b_{10}^2 - 3654 b_{01}^2 b_{2,-1} b_{10}^2 \\ &+ 7452 a_{10} a_{-12} b_{2,-1} b_{10}^2 - 2916 a_{01} b_{01} b_{2,-1} b_{10}^2 + 12096 a_{01} a_{10}^3 b_{10} \\ &+ 10476 a_{01} a_{-12} b_{2,-1}^2 b_{10} - 1086 a_{-12} b_{01} b_{2,-1}^2 b_{10} + 9072 a_{10}^3 b_{01} b_{10} \\ &+ 10962 a_{10} b_{01}^2 b_{2,-1} b_{10} - 6480 a_{01}^2 a_{10} b_{2,-1} b_{10} - 19800 a_{10}^2 a_{-12} b_{2,-1} b_{10} \\ &+ 15516 a_{01} a_{10} b_{01} b_{2,-1} b_{10} + 1120 a_{-12}^2 b_{2,-1}^3 + 3024 a_{01}^3 b_{2,-1}^2 - 567 b_{01}^3 b_{2,-1}^2 \\ &- 756 a_{01} b_{01}^2 b_{2,-1}^2 - 5088 a_{01} a_{10} a_{-12} b_{2,-1}^2 + 2268 a_{01}^2 b_{01} b_{2,-1}^2 \\ &+ 4464 a_{10} a_{-12} b_{01} b_{2,-1}^2 - 4896 a_{01}^2 a_{10}^2 b_{2,-1} - 6804 a_{10}^2 b_{01}^2 b_{2,-1} \\ &+ 8640 a_{10}^3 a_{-12} b_{2,-1} - 12744 a_{01} a_{10}^2 b_{01} b_{2,-1}. \end{aligned}$$

The other saddle quantities are too large to be presented here. Then, we set in the obtained polynomials $a_{10} = 1$. To obtain the necessary conditions for integrability we find the minimal decomposition of the variety of the ideal $\mathcal{B}_6 = \langle g_{4,1}, \dots, g_{24,6} \rangle$. This is a very difficult computational problem and the computational tool which we use is the routine `minAssGTZ` [7] of the computer algebra system SINGULAR [22] which is based on the Gianni-Trager-Zacharias algorithm [20].

Since computations are too laborious they can not be completed in the field of rational numbers. Therefore, we choose the approach based on making use of modular computations [1, 28, 33]. We choose some primes $p = 32003$, 104729, 4256233, 7368787, 15485863, 179595127, 433494437 and 479001599 and compute the decomposition over the field $\mathbb{Z}_p$ for each p listed above.

First note that during the computations over the fields of finite characteristic p listed above we arrived at two different problems (both probably caused by computing the decomposition of the center variety over the field of a finite characteristic p). The first problem was the phenomenon that in some cases (of the decomposition) we obtained different polynomials (i.e. conditions) for different characteristic of the field $\mathbb{Z}_p$. Secondly we observed that when computing the decomposition over the field $\mathbb{Z}_p$ we obtained 46 components for some values of p while for some other values of p we obtained 47 components of the center variety.

For each decomposition we performed rational reconstruction algorithm to obtain ideals in $\mathbb{Q}[a, b]$. Then we checked by a direct computation using CAS Mathematica, if the 6 saddle quantities $g_{4,1}, \ldots, g_{24,6}$ are equal to zero under the obtained conditions. However, in cases $17-18$, $22-41$ and 46 it turned out that the (after the decomposition and rational reconstruction) obtained conditions were actually not the center variety conditions (since they didn't yield the saddle quantities to be zero). We noted that in all this "problematic" cases the saddle quantities were not equal to zero and the reason for that was the presence of a particular polynomial of the form $f_p = b_{01}b_{2,-1} + K$. Actually it was the value of K in f_p which was problematic (i.e. the essence of the problem).

Such kind of errors probably appear because of using computations over the fields of finite characteristic. Usually such problems are solved by choosing another prime p in order to compute the decomposition again and improve the result. But in this cases we tried eight different primes p listed above but the problem remaind the same.

Therefore, we tried to "correct" the value for constant K in the polynomial f_p in all this problematic cases. To this end in the problematic cases we used all the other polynomials (i.e. conditions) to compute the "right" value for K. For b_{01} and $b_{2,-1}$ satisfying the other (i.e. "non-problematic") conditions we solved the equation $f_p = 0$ for K. However, note that the problematic polynomials f_p in the cases $17-18$, $22-41$ and 46 cannot be omitted (they imply also that $b_{01}b_{2,-1} \neq 0$).

The second problem was the fact that the decomposition over some fields of finite characteristic contained 46 cases for some values of p and 47 cases for some other values of p. We noted that for those values of p for which 46 components were obtained there was always one large component which seemed to be decomposed into two smaller ones similar to those two from the decompositions consisting of 47 components (which was obtained for some other value of p). Since the large component under no chosen prime p was lying in the center variety after rational reconstruction we compared its decomposition for different primes and pick out from them those polynomials which were common. Next we computed in $\mathbb{Q}[a, b]$ the decomposition of the variety of the ideal generated by these common polynomials. Then, after checking that the saddle quantities $g_{4,1}, \ldots, g_{24,6}$ are equal to zero we obtained the "critical" $46-$th component listed in Theorem 2.1. Furthermore, we noted that in the field $\mathbb{C}[a, b]$ this "critical" $46-$th component can be decomposed into two components whose polynomials contain some square roots, which possibly explains why in the decompositions with 47 components we always obtain (the last) two components containing quotients of big numbers in numerators and denominators (i.e. this quotients were the approximations for this square roots).

Checking if some conditions in computations of the decomposition over the field of finite characteristic were lost is a very difficult computational problem which can again not be performed (completely) over rational numbers. We first compute intersection $P = \cap_{i=1}^{46} P_i$ over the field of characteristic zero, where P_i denotes the component i from Theorem 2.1. We obtain 112 polynomials $p_1, \dots, p_{112}$. We would like to check if $\sqrt{\mathcal{B}_6} = \sqrt{P}$. Computing over the field of characteristic 0 Gröbner basis of each ideal $\langle 1 - wg_{4k,k}, P : k = 1, \dots, 6\rangle$ we find that they are all $\{1\}$. This implies that $\sqrt{\mathcal{B}_6} \subset \sqrt{P}$. To check the opposite inclusion, $\sqrt{P} \subset \sqrt{\mathcal{B}_6}$ we must use computations with modular arithmetics. We choose prime $p = 179595127$ and after computing Gröbner basis of each ideal $\langle 1 - wp_k, \mathcal{B}_6 : k = 1, \dots, 112\rangle$ over the field $\mathbb{Z}_p$ we find that they are all $\{1\}$. Then we repeat computations over prime 32003 and find that the Gröbner basis of ideal $\langle 1 - wp_k, \mathcal{B}_6 : k = 1, \dots, 112\rangle$ is $\{1\}$ for each $k = 1, \dots, 112$. We can conclude that equality $\sqrt{P} = \sqrt{\mathcal{B}_6}$ holds with high probability.

3.2. Necessity of cases of Theorem 2.2

We use the same saddle quantities $g_{4,1}, \dots, g_{24,6}$ as in the proof of Theorem 2.1 and set $a_{10} = 0$. Using again the routine `MinAssGTZ` of CAS system SINGULAR we compute the decomposition of the variety $\mathbf{V}(\langle g_{4,1}, \dots, g_{24,6}\rangle)$. Computations cannot be completed in the field of rational numbers, therefore we choose prime $p = 32003$ and compute the decomposition in the finite field $\mathbb{Z}_p[a_{10}, a_{01}, a_{-12}, b_{2,-1}, b_{10}, b_{01}]$. We obtain nine components and after applying the rational reconstruction algorithm we obtain nine components listed in Theorem 2.2. Following the decomposition algorithm [28] we first check that all saddle quantities are zero under each condition. Then we check that no condition is lost. We denote by P_i, $i = 1, \dots, 9$ components from Theorem 2.2 and we compute the intersection

$$P = \cap_{i=1}^{9} P_i.$$

We obtain nine polynomials $q_1, \dots, q_9$ in P. Now we compute over the field of characteristic zero Gröbner bases of ideals $\langle 1 - wq_i, B_6 : i = 1, \dots, 9\rangle$ over the field of characteristic 0 and Gröbner bases of ideals $\langle 1 - wg_{4i,i}, P : i = 1, \dots, 6\rangle$ (over the field of characteristic 0) and find that they are all $\{1\}$. Therefore, no condition is lost.

3.3. Sufficiency of cases of Theorem 2.2

There are nine cases here. In all nine cases we applied the Darboux method to prove the existence of the first integral. Below is the case-by-case analysis:

Case 1. In this case system (2.1) is written as

$$\dot{x} = x - a_{-12}y^2 - a_{01}xy, \qquad \dot{y} = -4y + b_{01}y^2,$$

and we find two invariant lines $l_1 = 1 - b_{01}y/4$ and $l_2 = y$ yielding the Darboux integrating factor of the form $\mu = l_1^{(a_{01}-5b_{01})/(4b_{01})} l_2^{-3/4}$. By Theorem 4.13 of [4] there exists a first integral of the form (5).

Case 2. Here the corresponding system (2.1) is

$$\dot{x} = x - a_{01}xy, \qquad \dot{y} = -4y + b_{01}y^2,$$

and it admits the Darboux first integral $\Psi = x^4 y \left(1 - \frac{b_{01} y}{4}\right)^{-(b_{01}-4a_{01})/(b_{01})}$.

Case 3. In this case system (2.1) takes the form

$$\dot{x} = x, \qquad \dot{y} = -4y - \frac{6b_{10}^2 x^2}{b_{01}} + b_{10} yx + b_{01} y^2,$$

and it has two invariant curves

$$l_1 = 1 + \frac{5b_{10}^2 x^2}{4} - 2b_{10} x + \frac{5}{12} b_{01} b_{10} yx - \frac{b_{01} y}{4},$$
$$l_2 = 1 + \frac{25 b_{10}^3 x^3}{12} + \frac{15 b_{10}^2 x^2}{4} - \frac{25}{24} b_{01} b_{10}^2 yx^2 + 3b_{10} x - \frac{5}{6} b_{01} b_{10} yx - \frac{b_{01} y}{4},$$

and one invariant line $l_3 = x$. We are able to compute the Darboux integrating factor of the form $\mu = (l_1 l_2)^{-1} l_3^3$ and after integration we obtain a first integral of the form (5).

Case 4. Here the system is of the form

$$\dot{x} = x, \quad \dot{y} = -4y - \frac{4b_{10}^2 x^2}{9b_{01}} + b_{10} yx + b_{01} y^2.$$

We compute two invariant curves $l_1 = 1 - \frac{b_{10} x}{3} - \frac{b_{01} y}{4}$,

$$l_2 = 1 + \frac{125 b_{10}^4 x^4}{1944} + \frac{25 b_{10}^3 x^3}{81} - \frac{125}{648} b_{01} b_{10}^3 yx^3 + \frac{5 b_{10}^2 x^2}{6} - \frac{25}{72} b_{01} b_{10}^2 yx^2$$
$$+ \frac{4 b_{10} x}{3} - \frac{5}{12} b_{01} b_{10} yx - \frac{b_{01} y}{4},$$

and one invariant line $l_3 = x$ yielding the Darboux integrating factor $\mu = (l_1 l_2)^{-1} l_3^3$ and a first integral of the form (5).

Case 5. In this case we compute the Darboux integrating factor

$$\mu = x^3 \left(1 + \frac{x^4 y b_{10}^6}{120 b_{2,-1}} + \frac{x^5 b_{10}^5}{120} + \frac{x^4 b_{10}^4}{24} + \frac{x^3 b_{10}^3}{6} + \frac{x^2 b_{10}^2}{2} + x b_{10}\right)^{-1}$$

and after integration we obtain a first integral of the required form.

Case 6. System under conditions of this case admits the first integral

$$\Psi = -\frac{6yx^4}{b_{2,-1}} + x^6 + \frac{3 b_{01} y^2 x^4}{2 b_{2,-1}}.$$

Case 7. In this case system (2.1) is written as

$$\dot{x} = x + \frac{3 b_{01} yx}{4}, \quad \dot{y} = -4y + b_{2,-1} x^2 + b_{01} y^2.$$

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We have found two invariant curves

$$l_1 = 1 + \frac{1}{4}b_{01}b_{2,-1}x^2 + \frac{b_{01}^2y^2}{16} - \frac{b_{01}y}{2}, \quad \text{and} \quad l_2 = 1 + \frac{3}{128}b_{01}^2b_{2,-1}^2x^4 + \frac{3}{8}b_{01}b_{2,-1}x^2 - \frac{3}{32}b_{01}^2b_{2,-1}yx^2 - \frac{b_{01}^3y^3}{64} + \frac{3b_{01}^2y^2}{16} - \frac{3b_{01}y}{4},$$

and compute the Darboux first integral $\Psi = l_1^{-3/2}l_2$.

Case 8. Here the corresponding system is of the form

$$\dot{x} = x + \frac{b_{01}yx}{2}, \qquad \dot{y} = -4y + b_{2,-1}x^2 + b_{01}y \cdot 2$$

and it has two invariant curves $l_1 = 1 + \frac{1}{8}b_{01}b_{2,-1}x^2 - \frac{b_{01}y}{4}$, and

$$l_2 = \frac{3}{64}b_{01}^2b_{2,-1}^2x^4 + \frac{3}{128}b_{01}^3b_{2,-1}y^2x^2 + \frac{3}{8}b_{01}b_{2,-1}x^2 - \frac{3}{16}b_{01}^2b_{2,-1}yx^2 - \frac{b_{01}^3y^3}{64} + \frac{3b_{01}^2y^2}{16} - \frac{3b_{01}y}{4},$$

yielding the Darboux first integral $\Psi = l_1^{-3}l_2$.

Case 9. In this case system (2.1) is written as

$$\dot{x} = x + \frac{35b_{01}^2y^2}{361b_{10}} + \frac{11b_{01}xy}{19}, \quad \dot{y} = -4y - -\frac{114b_{10}^2x^2}{49b_{01}} + b_{10}yx + b_{01}y^2.$$

In this case we found three invariant curves

$$l_1 = 1 - \frac{6b_{10}x}{7} - \frac{6b_{01}y}{19}, \quad l_2 = -\frac{19b_{10}x^2}{7b_{01}} + yx - \frac{7b_{01}y^2}{76b_{10}} - \frac{7y}{b_{10}},$$

$$\begin{aligned} l_3 = & \frac{6021872768b_{10}^6x^6}{16807b_{01}^6} + \frac{24463858120b_{10}^5x^5}{7203b_{01}^6} - \frac{950822016b_{10}^5yx^5}{2401b_{01}^5} \\ & + \frac{645234257915b_{10}^4x^4}{49392b_{01}^6} + \frac{62554080b_{10}^4y^2x^4}{343b_{01}^4} - \frac{3070362760b_{10}^4yx^4}{1029b_{01}^5} \\ & + \frac{317418559107b_{10}^3x^3}{12544b_{01}^6} - \frac{2194880b_{10}^3y^3x^3}{49b_{01}^3} + \frac{153778780b_{10}^3y^2x^3}{147b_{01}^4} \\ & - \frac{60676805995b_{10}^3yx^3}{7056b_{01}^5} + \frac{43320b_{10}^2y^4x^2}{7b_{01}^2} - \frac{548720b_{10}^2y^3x^2}{3b_{01}^3} \\ & + \frac{5504368077b_{10}^2x^2}{224b_{01}^6} + \frac{2779095325b_{10}^2y^2x^2}{1344b_{01}^4} - \frac{179390896451b_{10}^2yx^2}{16128b_{01}^5} \\ & - \frac{456b_{10}y^5x}{b_{01}} + \frac{95665b_{10}y^4x}{6b_{01}^2} - \frac{61353755b_{10}y^3x}{288b_{01}^3} + \frac{3322273253b_{10}y^2x}{2304b_{01}^4} \end{aligned}$$

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$$+\frac{611596453b_{10}x}{64b_{01}^6}-\frac{1384139341b_{10}yx}{256b_{01}^5}+14y^6-\frac{3325y^5}{6b_{01}}+\frac{4485425y^4}{576b_{01}^2}$$
$$-\frac{108269315y^3}{2304b_{01}^3}+\frac{59296055y^2}{576b_{01}^4},$$

yielding the following integrating factor $\mu = l_1^{-2/3} l_2^{-5/6} l_3^{-1/3}$. By Theorem 4.13 of [4] there exists first integral of the required form.

3.4. Sufficiency of cases of Theorem 2.1

In this subsection we prove the sufficiency of all 46 conditions of Theorem 2.1 by doing a case-by-case analysis of all 46 cases. Note that for proving the existence of a first integrals in different cases we use either the Darboux method, the sum method (i.e. for the (formal) first integral we set a trial solution of the form $\Psi(x,y)=\sum_{k=1}^{\infty} f_k(x)y^k$, where functions f_k are determined recursively by some first order differential equations and finally use mathematical induction to prove that Ψ actually takes the desired form yielding the existence of the first integral; see case 14-16 on page 232 for details) or the monodromy arguments (as in case 17 on page 233) described in [6]. Below is the case-by-case analysis of all 46 cases.

Case 1. In this case system (2.1) takes the form

$$\dot{x}=x-x^2, \quad \dot{y}=-4y-\frac{6b_{10}^2-18b_{10}+12}{b_{01}}x^2+b_{10}xy+b_{01}y^2.$$

This system has three algebraic curves: $l_1 = x$, $l_2 = 1-x$, and

$$l_3 = 1+\tfrac{5b_{10}^2x^2}{4}-\tfrac{11b_{10}x^2}{4}+\tfrac{3x^2}{2}-2b_{10}x-\tfrac{b_{01}yx}{2}+\tfrac{5}{12}b_{01}b_{10}yx+2x-\tfrac{b_{01}y}{4},$$

which allow us to construct a Darboux integrating factor of the form

$$\mu = l_1^3 l_2^{-9+5b_{10}} l_3^{-2}.$$

Integration yields a first integral, whose series expansion is of the form

$$\Psi = x^4y + h.o.t. \tag{5}$$

Case 2. The corresponding system for this case is

$$\dot{x}=x-x^2, \quad \dot{y}=-4y-\frac{4b_{10}^2-12b_{10}}{9b_{01}}x^2+b_{10}xy+b_{01}y^2.$$

It has three algebraic curves: $l_1 = x$, $l_2 = 1-x$ and

$$l_3 = 1-\frac{b_{10}x}{3}-\frac{b_{01}y}{4}$$

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which allow us to construct a Darboux integrating factor of the form $\mu = l_1^3 l_2^{5(-3+b_{10})/3} l_3^{-2}$ and first integral of the form (5).

Case 3. Here the corresponding system is

$$\dot{x} = x - x^2 + \frac{7b_{01}xy}{4}, \quad \dot{y} = -4y - \frac{24x^2}{25b_{01}} + \frac{4xy}{5} + +b_{01}y^2.$$

This system admits three algebraic curves: $l_1 = x$, $l_2 = 1 + \frac{x}{5} - \frac{b_{01}y}{4}$, and

$$l_3 = 1 + \frac{16x^4}{625} - \frac{8}{125}b_{01}yx^3 + \frac{16x^3}{125} + \frac{3}{50}b_{01}^2y^2x^2 - \frac{2}{25}b_{01}yx^2 - \frac{1}{40}b_{01}^3y^3x + \frac{1}{10}b_{01}^2y^2x + \frac{2b_{01}yx}{5} - \frac{8x}{5} + \frac{b_{01}^4y^4}{256} - \frac{b_{01}^3y^3}{16} + \frac{3b_{01}^2y^2}{8} - b_{01}y,$$

yielding the Darboux integrating factor $\mu = l_1^3 l_2^1 l_3^{-5/2}$ and a first integral of the form (5).

Case 4. In this case we have three algebraic curves: $l_1 = \frac{2x}{19} - \frac{b_{01}y}{4} + 1$, $l_2 = \frac{49x^2}{361} - \frac{7b_{01}yx}{38} - \frac{26x}{19} + \frac{b_{01}^2y^2}{16} - \frac{b_{01}y}{2} + 1$ and $l_3 = x$ yielding integrating factor $\mu = l_1^{-2} l_2^{-7/2} l_3^3$. After integration we obtain a first integral of the form (5).

Case 5. Here we find following invariant curves: $l_1 = \frac{81x^2}{169} - \frac{9b_{01}yx}{26} - \frac{22x}{13} + \frac{b_{01}^2y^2}{16} - \frac{b_{01}y}{2} + 1$, $l_2 = \frac{6x^2}{169} - \frac{29b_{01}yx}{156} + \frac{3x}{13} + \frac{b_{01}^2y^2}{16} - \frac{b_{01}y}{2} + 1$ and $l_3 = x$. We obtain Darboux integrating factor $\mu = l_1^{-\frac{5}{2}} l_2^{-2} l_3^3$, which yields first integral of the form (5).

Case 6. Under these conditions system (2.1) becomes

$$\dot{x} = x - x^2 + \frac{7b_{01}yx}{4}, \quad \dot{y} = -4y - \frac{12x^2}{49b_{01}} - \frac{yx}{7} + b_{01}y^2.$$

It admits two invariant curves: $l_1 = 1 + \frac{x^2}{49} - \frac{b_{01}yx}{14} - \frac{8x}{7} + \frac{b_{01}^2y^2}{16} - \frac{b_{01}y}{2}$ and $l_2 = x$ yielding Darboux integrating factor $\mu = l_1^{-9/2} l_2^3$ and a first integral of the form (5).

Case 7. Here we have three invariant curves: $l_1 = 1 - x - \frac{b_{01}y}{4}$, $l_2 = x$ and $l_3 = y$. We can construct the Darboux first integral $H = l_1^{(-4+b10)} l_2^4 l_3$ which is of the form (5).

Case 8. The corresponding system here is

$$\dot{x} = x - x^2 - b_{01}yx, \quad \dot{y} = -4y + \frac{24x^2}{49b_{01}} + yx + b_{01}y^2.$$

This system admits three algebraic invariant curves: $l_1 = 1 - \frac{x}{7} - \frac{b_{01}y}{4}$, $l_2 = 1 + \frac{9x^4}{2401} + \frac{6}{343}b_{01}yx^3 - \frac{12x^3}{343} + \frac{1}{49}b_{01}^2y^2x^2 - \frac{6}{49}b_{01}yx^2 + \frac{30x^2}{49} + \frac{4b_{01}yx}{7} - \frac{12x}{7}$ and $l_3 = x$ bringing Darboux integrating factor $\mu = l_1^{-5/2} l_2^2 l_3^3$ and a first integral of the form (5).

Case 9. In this case we have three algebraic curves: $l_1 = 1 + \frac{105x^2}{529} + \frac{10b_{01}yx}{23} - \frac{30x}{23}$, $l_2 = -\frac{2x}{23} - \frac{b_{01}y}{4}$ and $l_3 = x$, which allow us to construct the Darboux integrating factor of the form $\mu = l_1^{-7/2} l_2^2 l_3^3$. Integration brings a first integral of the form (5).

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Case 10. In this case system (2.1) takes the form

$$\dot{x} = x - x^2 + \frac{3b_{01}yx}{4}, \quad \dot{y} = -4y + b_{2,-1}x^2 + b_{01}y^2$$

and it has three algebraic curves: $l_{1,2} = 1 + \frac{1}{2}\left(\pm\sqrt{1 - b_{01}b_{2,-1}} - 1\right)x - \frac{b_{01}y}{4}$ and $l_3 = x$ yielding the Darboux integrating factor $\mu = l_1^{a_1}l_2^{a_2}l_3^{b}$, where $a_{1,2} = -\frac{5\left(\sqrt{1-b_{01}b_{2,-1}}\pm 1\right)}{2\sqrt{1-b_{01}b_{2,-1}}}$ and $b = 3$. By [4, Th. 4.13] there exists first integral of the form (5).

Case 11. The corresponding system is

$$\dot{x} = x - x^2 + \frac{11b_{01}yx}{4}, \quad \dot{y} = -4y - \frac{168x^2}{361b_{01}} + \frac{16yx}{19} + b_{01}y^2.$$

It has three algebraic curves $l_1 = 1 + \frac{2x}{19} - \frac{b_{01}y}{4}$,

$$\begin{aligned} l_2 =& 1 + \frac{729x^6}{47045881} - \frac{729b_{01}yx^5}{4952198} + \frac{2754x^5}{2476099} + \frac{1215b_{01}^2y^2x^4}{2085136} - \frac{2079b_{01}yx^4}{260642} \\ &+ \frac{2655x^4}{130321} - \frac{135b_{01}^3y^3x^3}{109744} + \frac{621b_{01}^2y^2x^3}{27436} - \frac{645b_{01}yx^3}{6859} + \frac{636x^3}{6859} \\ &+ \frac{135b_{01}^4y^4x^2}{92416} - \frac{183b_{01}^3y^3x^2}{5776} + \frac{485b_{01}^2y^2x^2}{2888} - \frac{39}{361}b_{01}yx^2 + \frac{39x^2}{361} \\ &- \frac{9b_{01}^5y^5x}{9728} + \frac{53b_{01}^4y^4x}{2432} - \frac{45}{304}b_{01}^3y^3x + \frac{21}{76}b_{01}^2y^2x + \frac{b_{01}yx}{2} - \frac{30x}{19} \\ &+ \frac{b_{01}^6y^6}{4096} - \frac{3b_{01}^5y^5}{512} + \frac{15b_{01}^4y^4}{256} - \frac{5b_{01}^3y^3}{16} + \frac{15b_{01}^2y^2}{16} - \frac{3b_{01}y}{2}, \end{aligned}$$

and $l_3 = x$ yielding the Darboux integrating factor $\mu = l_1^2l_2^{-5/2}l_3^3$ and integration brings a first integral of the form (5).

Case 12. For this case system (2.1) takes the form

$$\dot{x} = x - x^2 - 2b_{01}yx, \quad \dot{y} = -4y + \frac{168x^2}{529b_{01}} + \frac{22yx}{23} + b_{01}y^2,$$

and it has the following algebraic curves $l_1 = 1 - \frac{2x}{23} - \frac{b_{01}y}{4}$,

$$\begin{aligned} l_2 =& 1 + \frac{2401x^6}{148035889} + \frac{1372b_{01}yx^5}{6436343} - \frac{4074x^5}{6436343} + \frac{294b_{01}^2y^2x^4}{279841} - \frac{1856b_{01}yx^4}{279841} \\ &+ \frac{105x^4}{12167} + \frac{28b_{01}^3y^3x^3}{12167} - \frac{280b_{01}^2y^2x^3}{12167} + \frac{840b_{01}yx^3}{12167} - \frac{1036x^3}{12167} \\ &+ \frac{1}{529}b_{01}^4y^4x^2 - \frac{14}{529}b_{01}^3y^3x^2 + \frac{70}{529}b_{01}^2y^2x^2 - \frac{224}{529}b_{01}yx^2 + \frac{399x^2}{529} \\ &- \frac{4}{23}b_{01}^2y^2x + \frac{28b_{01}yx}{23} - \frac{42x}{23}, \end{aligned}$$

and $l_3 = x$. We can construct the Darboux integrating factor of the form $\mu = l_1^6 l_2^{-5/2} l_3^3$ and integration brings a first integral of the form (5).

Case 13. Under these conditions system (2.1) is written as

$$\dot{x} = x - x^2 - \frac{b_{01}yx}{2}, \quad \dot{y} = -4y + b_{2,-1}x^2 + 5yx + b_{01}y^2,$$

and it has the Darboux integrating factor $\mu = x^3$ which yields first integral

$$\Psi = x^4 y - \frac{b_{2,-1}x^6}{6} - yx^5 - \frac{1}{4}b_{01}y^2x^4.$$

Case 14 - Case 16. In case 14 and case 16 we find only $f_1 = y$ and in case 15 we find $f_1 = y$ and $f_2 = 1 - \frac{b_{01}y}{4}$. However, in all three cases we can not construct Darboux first integral or Darboux integrating factor. Noting that conditions in these cases are $b_{2,-1} = 0$ and $b_{10} = A$, where $A = 1, 0$ and 2, respectively, system (2.1) is written as

$$\dot{x} = x - x^2 - a_{01}xy - a_{-12}y^2, \quad \dot{y} = -4y + Axy + b_{01}y^2.$$

We look for a formal first integral in the form $\Psi(x,y) = \sum_{k=1}^{\infty} f_k(x)y^k$. The functions f_k are determined recursively by the differential equation

$$-a_{-12}f'_{k-2}(x) - a_{01}xf'_{k-1}(x) + (k-1)b_{01}f_{k-1} + (Ax-4)kf_k(x) + x(1-x)f'_k(x) = 0$$

If $k = 1, 2, 3, 4$ and setting the integration constant equal to 1 we observe that

$$f_1(x) = \frac{-x^4}{(x-1)^{4-A}}, \qquad f_2(x) = \frac{x^8 + \cdots}{(x-1)^{2(4-A)}},$$
$$f_3(x) = \frac{x^{12} + \cdots}{(x-1)^{3(4-A)}}, \qquad f_4(x) = \frac{x^{16} + \cdots}{(x-1)^{4(4-A)}}.$$

So, suppose by induction that $f_k(x) = \frac{p_{4k}(x)}{(x-1)^{k(4-A)}}$, where $p_i(x)$ denotes a polynomial of degree at most i and $k = 1, \ldots, n-1$. Let check the form of $f_k(x)$ for $k = n$. In order to complete this task we solve the differential equation

$$f'_n(x) = \frac{(4-Ax)n}{x(1-x)}f_n(x) + \frac{a_{01}xf'_{n-1} + a_{-12}f'_{n-2} - b_{01}(n-1)f_{n-1}}{x(1-x)}, \tag{6}$$

using the induction assumption about f_{n-1} and f_{n-2}.

As the general solution of linear differential equation of the form $f'(x) = g(x)f(x) + h(x)$ is

$$f(x) = Ce^{\int g(x)dx} + e^{\int g(x)dx} \int e^{-\int g(x)dx} h(x)dx. \tag{7}$$

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and, in our case we have $g(x) = \frac{(4-Ax)n}{x(1-x)}$ and

$$h(x) = \frac{p_{4n-3}(x)}{x(x-1)^{(n-1)(4-A)+2}},$$

it follows that $e^{\int g(x)dx} = \frac{x^{4n}}{(x-1)^{n(4-A)}}$ and

$$\begin{aligned} e^{-\int g(x)dx}h(x) =& \frac{(x-1)^{n(4-A)}}{x^{4n}} \cdot \frac{p_{4n-3}(x)}{x(x-1)^{(n-1)(4-A)+2}} \\ =& \frac{p_{4n-A-1}(x)}{x^{4n+1}}. \end{aligned}$$

Rewriting $e^{-\int g(x)dx}h(x)$ as

$$\frac{p_{4n-A-1}(x)}{x^{4n+1}} = \frac{a_0 + a_1 x + \cdots a_{4n-A-1}x^{4n-A-1}}{x^{4n+1}} = \frac{a_0}{x^{4n+1}} + \frac{a_1}{x^{4n}} + \cdots + \frac{a_{4n-A-1}}{x^{A+2}},$$

and integrating, yields

$$\int e^{-\int g(x)dx}h(x)dx = \frac{\overline{a}_0}{x^{4n}} + \frac{\overline{a}_1}{x^{4n-1}} + \cdots + \frac{\overline{a}_{4n-A-1}}{x^{A+1}}$$

for some $\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_{4n-A-1}$. Therefore, using (7) choosing integration constant $C = 1$ at each step i of computing functions $f_i(x)$ for $i = 1, 2, \ldots$ we obtain the solution of (6)

$$\begin{aligned} f_n(x) =& \frac{x^{4n}}{(x-1)^{n(4-n)}} + \frac{x^{4n}}{(x-1)^{n(4-n)}} \Big[\frac{\overline{a}_0}{x^{4n}} + \frac{\overline{a}_1}{x^{4n-1}} + \cdots + \frac{\overline{a}_{4n-A-1}}{x^{A+1}} \Big] \\ =& \frac{\left[1 \cdot x^{4n} + \overline{a}_{4n-A-1}x^{4n-A-1} + \cdots + \overline{a}_0\right]}{(x-1)^{n(4-A)}} = \frac{\overline{p}_{4n}}{(x-1)^{n(4-A)}}, \end{aligned}$$

where $\overline{p}_{4n}$ denotes a polynomial of degree at most $4n$. For choices $A = 1, 0$ and 2 satisfying cases 14, 15 and 16, respectively it follows that there exist analytic first integral of the form $\Psi(x, y) = \sum_{k=1}^{\infty} f_k(x) y^k$ whose power series expansion is of the form (5).

Case 17. Under conditions of this case system (2.1) is written as

$$\begin{aligned} \dot{x} &= x - x^2 - a_{01}yx + \frac{78a_{01}^2 y^2}{25} + \frac{13b_{01}^2 y^2}{100} + \frac{91}{50}a_{01}b_{01}y^2, \\ \dot{y} &= -4y + \frac{180x^2}{169(2a_{01} + b_{01})} + \frac{16yx}{13} + b_{01}y^2. \end{aligned} \tag{8}$$

In this case we are able to find one invariant curve

$$f_1 = x^2 + \frac{13}{5}(2a_{01} + b_{01})yx + \frac{169}{100}\left(4a_{01}^2 + 4b_{01}a_{01} + b_{01}^2\right) y^2 - \frac{169}{30}(2a_{01} + b_{01})y,$$

which is not enough to construct Darboux first integral or Darboux integrating factor. Therefore, we use the monodromy arguments (see e.g. [6] for details). In order to avoid some difficulties which appear in the computations of the singular points in this case we reparameterize coefficients a_{01} and b_{01} by $a_{01} = (-7s_1^2 + s_2^2)/30$ and $b_{01} = (22s_1^2 - s_2^2)/15$, which yields the system

$$
\begin{aligned}
\dot{x} &= x - x^2 - \frac{13}{300} s_1^2(4s_1^2 - s_2^2)y^2 + \frac{1}{30} xy(7s_1^2 - s_2^2), \\
\dot{y} &= -4y + \frac{180x^2}{169s_1^2} + \frac{16xy}{13} + \frac{(22s_1^2 - s_2^2)y^2}{15},
\end{aligned} \tag{9}
$$

and the invariant curve has the form $f_1 = x^2 + 13s_1^2xy/5 + 169s_1^2y(3s_1^2y - 10)/300$. We compute eigenvalues of all singular points of system (9) on the curve $f_1 = 0$. The ratios of eigenvalues are 3 and 3 for finite points and 2 for the point at the infinity. We recall that here the sum of the ratio of eigenvalues on an invariant conic must be 4. Thus, all singular points on $f_1 = 0$ except the saddle point at the origin are linearizable nodes whose monodromy is the identity and therefore, the monodromy of the saddle at the origin is also the identity. Hence the saddle at the origin is integrable.

Case 18. In order to avoid some difficulties which appear in computations with square roots we reparameterize coefficients a_{01} and b_{01} by $a_{01} = \frac{1}{16}(9s_1^2 - s_2^2)$ and $b_{01} = \frac{1}{4}(-5s_1^2 + s_2^2)$. Thus, in this case system (2.1) becomes

$$
\begin{aligned}
\dot{x} =& x - x^2 + \frac{s_2^2 - 9s_1^2}{16} xy + \frac{21s_1^4 + 43s_1^3s_2 + 27s_1^2s_2^2 + 5s_1s_2^3}{64} y^2, \\
\dot{y} =& \frac{-y\left(5ys_1^2 - s_2^2y + 16\right)(7s_1 + 5s_2)^2 + 16s_2(7s_1 + 5s_2)xy + 384x^2}{4(7s_1 + 5s_2)^2}.
\end{aligned} \tag{10}
$$

We can find two invariant curves $f_1 = 1 + \frac{2(s_1 - s_2)x}{7s_1 + 5s_2} + \frac{1}{16}(s_1 - s_2)(3s_1 + s_2)y$ and $f_2 = 49y^2s_1^4 + 168s_2y^2s_1^3 + 214s_2^2y^2s_1^2 + 224xys_1^2 - 784ys_1^2 + 120s_2^3y^2s_1 - 1120s_2ys_1 + 384s_2xys_1 + 256x^2 + 25s_2^4y^2 - 400s_2^2y + 160s_2^2xy$ which it turns out are not enough to construct the Darboux integrating factor or the Darboux first integral. Therefore, in this case we also use the monodromy arguments (see e.g. [6]). We compute eigenvalues of all singular points of system (2.1) on the curve $f_2 = 0$. The ratios of eigenvalues are 2 and 4 for finite points and 2 for the point at the infinity. Hence, as in the previous case, all singular points on $f_2 = 0$ except the saddle point at the origin are linearizable nodes whose monodromy is the identity and therefore, the monodromy of the saddle at the origin is also the identity. Hence the saddle at the origin is integrable.

Case 19. In this case the corresponding system (2.1) is of the form

$$
\dot{x} = x - x^2 - \frac{5b_{01}yx}{4} - a_{-12}y^2, \quad \dot{y} = -4y + b_{01}y^2 + 8xy.
$$

It admits two invariant curves $l_1 = y$ and $l_2 = \frac{4x^2}{b_{01}} + yx - \frac{4x}{b_{01}} + \frac{4a_{-12}y^2}{9b_{01}}$ which yields Darboux first integral of the form $\Psi = l_1^{\frac{b}{4}} l_2^b = x^4y + h.o.t.$

Case 20. Here the corresponding system is

$$\dot{x} = x - x^2 - a_{-12}y^2, \quad \dot{y} = -4y + b_{01}y^2 + 3xy$$

and it has three invariant lines $l_{1,2} = 1 - x + \frac{1}{8}\left(-b_{01} \pm \sqrt{b_{01}^2 - 16a_{-12}}\right) y$ and $l_3 = y$ yielding the Darboux integrating factor $l_1^{a_1} l_2^{a_2} l_3^{b}$, where

$$a_{1,2} = -\frac{5\left(\pm b_{01} + \sqrt{b_{01}^2 - 16a_{-12}}\right)}{8\sqrt{b_{01}^2 - 16a_{-12}}} \quad \text{and} \quad b = -\frac{3}{4}.$$

By [4, Th. 4.13] there exists a first integral of the form (5).

Case 21. Under conditions of these case system (2.1) becomes

$$\dot{x} = x - x^2, \quad \dot{y} = -4y + b_{2,-1}x^2 + b_{10}yx.$$

It has two invariant curves $l_1 = 1 - x$ and $l_2 = x$. Using constructed Darboux integrating factor of the form $\mu = l_1^{-5+b_{10}} l_2^3$ we obtain a first integral of the form (5).

In cases 22–41 all conditions are of the form: $b_{10} = A$, $a_{01} = Bb_{01}$, $a_{-12} = Cb_{01}^2$ and $b_{2,-1} = \frac{D}{b_{01}}$. Therefore, the system (2.1) with coefficients A, B, C, and D corresponding to one of cases 22–41 is of the form

$$\dot{x} = x - x^2 - Bb_{01}xy - Cb_{01}^2y^2, \quad \dot{y} = -4y + \frac{D}{b_{01}}x^2 + Axy + b_{01}y^2.$$

Case 22. In this case we obtain three invariant curves

$$\begin{aligned} l_1 =& 1 - \frac{40x}{61} - \frac{10b_{01}y}{49}, \quad l_2 = x^2 + \frac{61b_{01}yx}{343} - \frac{61x}{49} + \frac{3721b_{01}^2y^2}{470596}, \\ l_3 =& 1 + \frac{6125x^2}{7442} + \frac{125b_{01}yx}{854} - \frac{110x}{61} + \frac{125b_{01}^2y^2}{19208} - \frac{5b_{01}y}{28}, \end{aligned}$$

which bring the Darboux integrating factor $\mu = l_1^{-7/2} l_2^3 l_3^{-2}$ and using it we obtain a first integral of the required form.

Case 23. In this case we find three invariant curves

$$\begin{aligned} l_1 =& 1 - \frac{5x}{131} - \frac{5b_{01}y}{36}, \quad l_2 = 14x^2 + \frac{1703b_{01}yx}{36} - 131x + \frac{17161b_{01}^2y^2}{729}, \\ l_3 =& \frac{279936x^3}{17161b_{01}^3} + \frac{16848yx^2}{131b_{01}^2} - \frac{979776x^2}{81875b_{01}^3} + \frac{288y^2x}{b_{01}} - \frac{300672yx}{625b_{01}^2} + 131y^3 \\ &+ \frac{66024y^2}{625b_{01}} + \frac{169776y}{625b_{01}^2}, \end{aligned}$$

and we construct the integrating factor of the form $\mu = l_1^{-1/3} l_2^{-1/3} l_3^{-5/6}$. By [4, Th. 4.13] there exists a first integral of the form (5).

Case 24. In this case we construct the Darboux first integral of the form $\Psi = l_1^{-10} l_2^4 l_3$, where

$$\begin{aligned} l_1 =& 1 - \frac{5x}{8} - \frac{5b_{01}y}{38}, \; l_2 = x^2 + \frac{8b_{01}yx}{19} - 4x + \frac{16b_{01}^2y^2}{361}, \\ l_3 =& x^2 + \frac{8b_{01}yx}{19} + \frac{16b_{01}^2y^2}{361} - \frac{64b_{01}y}{19}. \end{aligned}$$

Case 25. In this case system has three invariant curves

$$\begin{aligned} l_1 =& 1 - \frac{15x}{11} - \frac{20b_{01}y}{53}, \quad l_2 = 1 + \frac{5x}{22} - \frac{45b_{01}y}{212}, \\ l_3 =& x^2 - \frac{44b_{01}yx}{53} + 11x + \frac{484b_{01}^2y^2}{2809}, \end{aligned}$$

yielding the Darboux integrating factor $\mu = l_1^{-7/2} l_2^{-2} l_3^3$ and a first integral of the form (5).

Case 26. In this case we find two invariant curves

$$\begin{aligned} l_1 =& x^2 - \frac{2483b_{01}yx}{36} - \frac{191x}{9} + \frac{36481b_{01}^2y^2}{36}, \\ l_2 =& \frac{625x^4}{1330863361} - \frac{625b_{01}yx^3}{6967871} + \frac{14500x^3}{6967871} + \frac{1875b_{01}^2y^2x^2}{291848} - \frac{9000b_{01}yx^2}{36481} \\ &+ \frac{6450x^2}{36481} - \frac{625b_{01}^3y^3x}{3056} + \frac{7125}{764}b_{01}^2y^2x - \frac{1475b_{01}yx}{191} - \frac{260x}{191} \\ &+ \frac{625b_{01}^4y^4}{256} - \frac{875b_{01}^3y^3}{8} + \frac{975b_{01}^2y^2}{8} - 10b_{01}y, \end{aligned}$$

and we construct the Darboux integrating factor $\mu = l_1^3 l_2^{-7/2}$. After integration we obtain a first integral (5).

Case 27. Here we have two invariant curves

$$l_1 = x^2 + \frac{b_{01}yx}{3} - \frac{2x}{3} - \frac{2b_{01}^2y^2}{3}, \quad l_2 = x^2 - 8b_{01}yx - 9b_{01}^2y^2 + 4b_{01}y,$$

yielding the Darboux first integral of the form

$$\Psi = l_1^4 l_2 = x^4 y + P_6(x,y) + P_7(x,y) + P_8(x,y) + P_9(x,y) + P_{10}(x,y),$$

where $P_k(x,y)$ denotes homogeneous polynomial of degree k.

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Case 28. Here we find two invariant curves passing through the origin

$$\begin{aligned} l_1 =& x^2 + 5b_{01}yx - 10x + \frac{50b_{01}^2y^2}{9}, \\ l_2 =& \frac{3x^4}{125b_{01}^4} + \frac{2yx^3}{5b_{01}^3} - \frac{84x^3}{25b_{01}^4} + \frac{12y^2x^2}{5b_{01}^2} - \frac{192yx^2}{5b_{01}^3} + \frac{12x^2}{5b_{01}^4} + \frac{6y^3x}{b_{01}} \\ & - \frac{132y^2x}{b_{01}^2} + \frac{144yx}{b_{01}^3} + 5y^4 - \frac{120y^3}{b_{01}} - \frac{60y^2}{b_{01}^2} - \frac{80y}{b_{01}^3}, \end{aligned}$$

consequently, we can construct the Darboux integrating factor of the form $\mu = l_1^{-1/3} l_2^{-5/6}$. By Theorem 4.13 of [4] there exists a first integral of required form.

Case 29. In this case we find two invariant curves

$$l_1 = 1 - \frac{5x}{23} + \frac{5b_{01}y}{12}, \quad \text{and} \quad l_2 = 18x^2 - \frac{253b_{01}yx}{2} + \frac{529b_{01}^2y^2}{3} + \frac{529b_{01}y}{2}.$$

They are not enough to construct the Darboux first integral or the Darboux integrating factor. Therefore, we attempt to use the monodromy argument (see e.g. [6]). We compute eigenvalues of all singular points of system (2.1) corresponding to case 29 on the curve $l_2 = 0$. The ratios of eigenvalues are: -4 (for $(0,0)$) and 3 for finite singular point $(\frac{23}{25}, -\frac{3}{25b_{01}})$ and 3 and 2 for singular points at the infinity. Thus, all singular points on $l_2 = 0$ except the saddle point at the origin are linearizable nodes whose monodromy is the identity and therefore, the monodromy of the saddle at the origin is also the identity. Hence the saddle at the origin is integrable.

Case 30. Here we find the Darboux integrating factor of the form $\mu = l_1^{-9/2} l_2^{-2} l_3^3$, where l_1, l_2 and l_3 are algebraic invariant curves of the form

$$\begin{aligned} l_1 =& 1 - \frac{15x}{19} - \frac{20b_{01}y}{107}, \quad l_2 = 1 - \frac{35x}{38} - \frac{35b_{01}y}{428}, \\ l_3 =& x^2 + \frac{19b_{01}yx}{107} - \frac{19x}{16} + \frac{361b_{01}^2y^2}{45796}. \end{aligned}$$

After integration using μ we obtain a first integral of the form (5).

Case 31. In this case system has two invariant curves

$$\begin{aligned} l_1 =& 1 + \frac{525x^2}{2704} + \frac{425b_{01}yx}{364} - \frac{35x}{26} + \frac{225b_{01}^2y^2}{196} + \frac{5b_{01}y}{7}, \\ l_2 =& x^2 + \frac{26b_{01}yx}{3} - \frac{52x}{3} + \frac{1352b_{01}^2y^2}{147}, \end{aligned}$$

which yields the Darboux integrating factor $\mu = l_1^{-7/2} l_2^3$ and a first integral of the required form.

Case 32. Here we find three invariant curves of the form

$$\begin{aligned} l_1 =& 1 - \frac{40x}{19} - \frac{10b_{01}y}{31}, \quad l_2 = x^2 - \frac{19b_{01}yx}{93} + \frac{19x}{9} + \frac{361b_{01}^2y^2}{34596}, \\ l_3 =& \frac{5984416080x^4}{130321b_{01}^4} - \frac{128697120yx^3}{6859b_{01}^3} + \frac{1595844288x^3}{6859b_{01}^4} + \frac{1037880y^2x^2}{361b_{01}^2} \\ & - \frac{4289904yx^2}{361b_{01}^3} - \frac{206868704x^2}{1805b_{01}^4} - \frac{3720y^3x}{19b_{01}} - \frac{92256y^2x}{19b_{01}^2} + \frac{22879488yx}{95b_{01}^3} \\ & + 5y^4 + \frac{372y^3}{b_{01}} + \frac{149916y^2}{5b_{01}^2} - \frac{476656y}{5b_{01}^3}. \end{aligned}$$

We construct the Darboux integrating factor of the form $\mu = l_1^{1/2}(l_2l_3)^{-1}$ and by Theorem 4.13 of [4] there exists a first integral of the form (5).

Case 33. Using two invariant curves of the form

$$\begin{aligned} l_1 =& 1 + \frac{75x^2}{2} + \frac{50b_{01}yx}{19} - 10x - \frac{75b_{01}^2y^2}{2888} - \frac{5b_{01}y}{38} \quad \text{and} \\ l_2 =& x^2 - \frac{2b_{01}yx}{19} - \frac{21b_{01}^2y^2}{1444} + \frac{b_{01}y}{19} \end{aligned}$$

we can construct the Darboux integrating factor $\mu = l_1^{\frac{1}{4}} l_2^{-\frac{3}{4}}$ and a first integral of the form (5).

Case 34. Using the Darboux integrating factor of the form $\mu = l_1^{-\frac{9}{2}} l_2^{-\frac{7}{2}} l_3^3$, where

$$l_1 = 1 - \frac{5x}{32} - \frac{45b_{01}y}{148}, \quad l_2 = 1 - \frac{35x}{32} + \frac{35b_{01}y}{148} \quad \text{and}$$

$$l_3 = x^2 + \frac{232b_{01}yx}{111} - \frac{32x}{3} - \frac{2048b_{01}^2y^2}{4107}$$

we compute a first integral of required form.

Case 35. Here we obtain three invariant curves $l_1 = 1 - \frac{5x}{11} + \frac{5b_{01}y}{4}$,

$$l_2 = 1 - \frac{15x}{11} - \frac{5b_{01}y}{2}, \quad \text{and} \quad l_3 = x^2 - \frac{77b_{01}yx}{4} + \frac{363b_{01}^2y^2}{8} + \frac{121b_{01}y}{4}.$$

We compute the Darboux integrating factor $\mu = l_1^{1/4}(l_2l_3)^{-3/4}$ yielding the first integral of the form (5).

Case 36. Using three invariant curves of the form $l_1 = 1 + \frac{5x}{121} - \frac{5b_{01}y}{4}$,

$$\begin{aligned} l_2 =& x^2 - \frac{242b_{01}yx}{9} - \frac{121x}{9} + \frac{14641b_{01}^2y^2}{81}, \quad \text{and} \quad l_3 = -\frac{14580x^3}{1771561b_{01}^3} \\ & + \frac{6885yx^2}{14641b_{01}^2} + \frac{84x^2}{14641b_{01}^3} - \frac{90y^2x}{11b_{01}} - \frac{222yx}{121b_{01}^2} + 45y^3 - \frac{6y^2}{b_{01}} + \frac{y}{b_{01}^2}, \end{aligned}$$

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we construct the Darboux integrating factor $\mu = l_1^{-1/2} l_2^{-1/3} l_3^{-5/6}$. By Theorem 4.13 of [4] there exists a first integral of the form (5).

Case 37. In this case system admits two invariant curves

$$l_1 = \frac{27x^3}{3136b_{01}^3} + \frac{27yx^2}{112b_{01}^2} + \frac{27x^2}{70b_{01}^3} + \frac{9y^2x}{4b_{01}} + \frac{99yx}{20b_{01}^2} - \frac{27x}{5b_{01}^3} + 7y^3 + \frac{364y^2}{15b_{01}},$$

and

$$l_2 = \frac{27x^3}{3136b_{01}^3} + \frac{27yx^2}{112b_{01}^2} - \frac{27x^2}{4480b_{01}^3} + \frac{9y^2x}{4b_{01}} - \frac{153yx}{160b_{01}^2} + 7y^3 + \frac{231y^2}{160b_{01}} + \frac{21y}{40b_{01}^2}.$$

We compute the Darboux integrating factor $\mu = l_1^{-1/3} l_2^{-5/6}$ and the existence of first integral of the form (5) is ensured by Theorem 4.13 of [4].

Case 38. Here we find first integral of the form (5) using the Darboux integrating factor $\mu = l_1^{-\frac{9}{2}} l_2^3$, where

$$l_1 = 1 - \frac{80x}{71} - \frac{5b_{01}y}{11} \quad \text{and} \quad l_2 = x^2 - \frac{71b_{01}yx}{22} + 71x + \frac{5041b_{01}^2y^2}{1936}.$$

Case 39. Here we find three invariant lines of the form

$$\begin{aligned} l_1 =& 1 + \frac{43750x^5}{1805076171} + \frac{578125b_{01}yx^4}{190008018} + \frac{896875x^4}{126672012} + \frac{250000b_{01}^2y^2x^3}{1666737} \\ &+ \frac{678125b_{01}yx^3}{1111158} + \frac{8750x^3}{555579} + \frac{15625b_{01}^3y^3x^2}{4332} + \frac{4446875b_{01}^2y^2x^2}{233928} \\ &- \frac{4375b_{01}yx^2}{12996} + \frac{1625x^2}{3249} + \frac{171875b_{01}^4y^4x}{4104} + \frac{1034375b_{01}^3y^3x}{4104} \\ &+ \frac{23125}{684}b_{01}^2y^2x + \frac{500b_{01}yx}{171} - \frac{100x}{57} + \frac{53125b_{01}^5y^5}{288} + \frac{690625b_{01}^4y^4}{576} \\ &+ \frac{74375b_{01}^3y^3}{144} + \frac{2125b_{01}^2y^2}{16} + \frac{25b_{01}y}{2}, \\ l_2 =& \frac{16x^4}{3518667b_{01}^4} + \frac{32yx^3}{61731b_{01}^3} + \frac{64x^3}{48735b_{01}^4} + \frac{8y^2x^2}{361b_{01}^2} + \frac{176yx^2}{1805b_{01}^3} - \frac{224x^2}{243675b_{01}^4} \\ &+ \frac{8y^3x}{19b_{01}} + \frac{224y^2x}{95b_{01}^2} - \frac{1664yx}{4275b_{01}^3} + 3y^4 + \frac{92y^3}{5b_{01}} + \frac{124y^2}{75b_{01}^2} + \frac{16y}{75b_{01}^3}, \end{aligned}$$

and

$$\begin{aligned} l_3 =& \frac{16x^4}{3518667b_{01}^4} + \frac{32yx^3}{61731b_{01}^3} + \frac{48x^3}{34295b_{01}^4} + \frac{8y^2x^2}{361b_{01}^2} + \frac{568yx^2}{5415b_{01}^3} + \frac{288x^2}{9025b_{01}^4} \\ &+ \frac{8y^3x}{19b_{01}} + \frac{244y^2x}{95b_{01}^2} + \frac{504yx}{475b_{01}^3} - \frac{288x}{475b_{01}^4} + 3y^4 + \frac{102y^3}{5b_{01}} + \frac{408y^2}{25b_{01}^2}. \end{aligned}$$

We compute the Darboux first integral $\Psi = l_1^{-3} l_2 l_3^4$ whose series expansion is of the form (5).

Case 40. In this case we find two invariant curves

$$l_1 = 1 - \frac{80x}{89} - \frac{5b_{01}y}{29}, \quad \text{and} \quad l_2 = 27x^2 + \frac{267b_{01}yx}{58} - \frac{89x}{3} + \frac{7921b_{01}^2y^2}{40368},$$

and we construct the Darboux integrating factor $\mu = l_1^{-\frac{11}{2}} l_2^3$ and then first integral of the form (5).

Case 41. In this case we compute the Darboux first integral $\Psi = l_1^{-4} l_2^4 l_3$, where

$$\begin{aligned} l_1 =& -\frac{25x^2}{242} + \frac{25b_{01}yx}{154} - \frac{10x}{11} - \frac{25b_{01}^2y^2}{392} - \frac{5b_{01}y}{28}, \\ l_2 =& x^2 - \frac{11b_{01}yx}{7} + 11x + \frac{121b_{01}^2y^2}{196}, \\ l_3 =& x^2 - \frac{11b_{01}yx}{7} + \frac{121b_{01}^2y^2}{196} + \frac{121b_{01}y}{14}. \end{aligned}$$

Case 42. The corresponding system (2.1) in this case is of the form

$$\dot{x} = x - x^2 - a_{-12}y^2, \qquad \dot{y} = -4y - xy/2\,.$$

We find two invariant curves for these system $l_1 = 1 - x + a_{-12}y^2/8$ and $l_2 = y$ which allow us to construct the Darboux integrating factor of the form $\mu = l_1^{-17/8} l_2^{-3/4}$ and the existence of first integral of the form (5) is assured by Theorem 4.13 of [4].

Case 43 - Case 45. For all three cases we have $b_{01} = 0$, $b_{2,-1} = 0$, $a_{01} = 0$ and $b_{10} = A$, where A is $1/2$, $5/2$ and $3/2$ for cases 43–45, respectively. Therefore, the corresponding system is of the form

$$\dot{x} = x - x^2 - a_{-12}y^2, \quad \dot{y} = \quad -4y + Axy.$$

In each of these three cases we are able to find only one invariant line $l_1 = y$ which is not enough to construct the Darboux first integral or the Darboux integrating factor. Therefore we look for a formal first integral expressed in the form $\Psi(x,y) = \sum_{k=1}^{\infty} f_k(x) y^k$. The functions f_k are determined recursively by the first order differential equation

$$-a_{-12} f'_{k-2}(x) - 4k f_k(x) + kAx f_k(x) + x(1-x) f'_k(x) = 0.$$

Let $p_i(x)$ denotes a polynomial of degree at most i. Using induction on k we show that $f_k(x) = \frac{p_{4k}(x)}{(x-1)^{(4-A)k}}$. For the basis of induction, by setting the integration constant to one, we have $f_1(x) = \frac{x^4}{(x-1)^{(4-A)}}$. Next, assuming that the induction hypothesis holds for $j < k$, we have $f_j(x) = \frac{p_{4j}(x)}{(x-1)^{(4-A)j}}$. Then we have the first order linear differential equation

$$f_k'(x) = \frac{(4-Ax)k}{x(1-x)} f_k(x) + \frac{a_{-12} f_{k-2}'(x)}{x(1-x)}. \tag{11}$$

Recall that the solution of the linear differential equation $f'(x) = g(x)f(x) + h(x)$ is (7). Noting that $f_{k-2} = \frac{p_{4(k-2)}}{(x-1)^{(4-A)(k-2)}}$ and applying similar computational procedure as in cases 14–16 we find that the solution for (11) is

$$\begin{aligned}
f_k(x) =& \frac{x^{4k}}{(x-1)^{(4-A)k}} + \frac{x^{4k}}{(x-1)^{(4-A)k}} \int \frac{p_{4k-2A-2}(x)}{x^{4k+1}}\,dx \\
& \frac{x^{4k}}{(x-1)^{(4-A)k}} \left(1 + \int \left(\frac{a_0}{x^{4k+1}} + \frac{a_1}{x^{4k}} + \frac{a_2}{x^{4k-1}} + \cdots + \frac{a_{4k-2A-2}}{x^{2A+3}}\right) dx\right) \\
& \frac{x^{4k}}{(x-1)^{(4-A)k}} \left(1 + \frac{\overline{a}_0}{x^{4k}} + \frac{\overline{a}_1}{x^{4k-1}} + \frac{\overline{a}_2}{x^{4k-2}} + \cdots + \frac{\overline{a}_{4k-2A-2}}{x^{2A+2}}\right) \\
=& \frac{\overline{p}_{4k}(x)}{(x-1)^{(4-A)k}}
\end{aligned}$$

where $\overline{p}_{4k}(x)$ denotes polynomials of degree at most $4k$. This proves the induction and the proof of cases 43–45 is completed for $A = \frac{1}{2}, \frac{5}{2}, \frac{3}{2}$.

Case 46. After computing the coefficients from the conditions of this case we find that this case can be decomposed into two subcases:

a) $b_{2,-1} = -\frac{3\left(-539504+250675\sqrt{10}\right)}{6507601 a_{01}}$, $a_{-12} = \dfrac{234\left(410175\sqrt{10}a_{01}^2 + 1368511 a_{01}^2\right)}{74632321}$,

$b_{01} = -\frac{2\left(7725\sqrt{10}a_{01}+24529a_{01}\right)}{8639}$, $b_{10} = \dfrac{2612+525\sqrt{10}}{2551}$,

b) $b_{2,-1} = \frac{3\left(539504+250675\sqrt{10}\right)}{6507601 a_{01}}$, $a_{-12} = -\dfrac{234\left(410175\sqrt{10}a_{01}^2 - 1368511 a_{01}^2\right)}{74632321}$,

$b_{01} = \frac{2\left(7725\sqrt{10}a_{01}-24529a_{01}\right)}{8639}$, $b_{10} = \dfrac{2612-525\sqrt{10}}{2551}$.

In both cases we find two invariant curves. Let $l_{a,b}$ and $f_{a,b}$ denote invariant curves for both subcases. Index a corresponds to invariant curve for system (2.1) filled with conditions from a) and index b corresponds to invariant curve for system (2.1) filled with conditions from b). Using this notation we have

$$\begin{aligned}
l_{a,b} =& \tfrac{\left(44132822\mp706860\sqrt{10}\right)x^3}{78091212a_{01}^2} + \tfrac{\left(238\mp2970\sqrt{10}\right)yx^2}{5102a_{01}} \pm \tfrac{7\left(\mp51980720+16414566\sqrt{10}\right)x^2}{382650a_{01}^2} \\
&+2y^2x \pm \tfrac{41\left(\mp10780+3414\sqrt{10}\right)yx}{300a_{01}} + \tfrac{\left(5972420\mp1888566\sqrt{10}\right)x}{300a_{01}^2} \mp \tfrac{2\left(2970\sqrt{10}a_{01}\pm238a_{01}\right)y^3}{25917} \\
&\pm\tfrac{13}{150}\left(\mp5720+1806\sqrt{10}\right)y^2,
\end{aligned}$$

$$
\begin{aligned}
f_{a,b} = {} & \frac{\left(3861805637\mp 1162343360\sqrt{10}\right)x^6}{94002296445a_{01}^2} + \frac{\left(6625166\mp 3041960\sqrt{10}\right)yx^5}{36849195a_{01}} \\
& \pm \frac{2\left(\mp 853872612205+270026378041\sqrt{10}\right)x^5}{921229875a_{01}^2} + y^2x^4 \pm \frac{2\left(\mp 259106075+81917729\sqrt{10}\right)yx^4}{72225a_{01}} \\
& + \frac{3\left(440609747\mp 139334678\sqrt{10}\right)x^4}{66875a_{01}^2} \mp \frac{8\left(\pm 3599117+2318725\sqrt{10}\right)a_{01}y^3x^3}{24958071} \\
& \pm \frac{32\left(\mp 4893737+1549508\sqrt{10}\right)y^2x^3}{14445} + \frac{8\left(16131771463\mp 5100608812\sqrt{10}\right)yx^3}{1805625a_{01}} \\
& \pm \frac{64\left(\mp 6451604305+2040198883\sqrt{10}\right)x^3}{15046875a_{01}^2} \pm \frac{8\left(\pm 82712334587+19372813330\sqrt{10}\right)a_{01}^2y^4x^2}{215612775369} \\
& \pm \frac{4\left(\mp 1253404102870+394018564231\sqrt{10}\right)a_{01}y^3x^2}{623951775} \mp \frac{4\left(\mp 9118925659+2886803191\sqrt{10}\right)y^2x^2}{601875} \\
& \pm \frac{4\left(\mp 3095596649260+978840105391\sqrt{10}\right)yx^2}{45140625a_{01}} + \frac{4\left(194210144462\mp 61415401673\sqrt{10}\right)x^2}{75234375a_{01}^2} \\
& \mp \frac{32\left(\pm 169898542620406+62274561197195\sqrt{10}\right)a_{01}^3y^5x}{3104464610687985} \\
& \pm \frac{16\left(\mp 107163377491130+34634486590247\sqrt{10}\right)a_{01}^2y^4x}{598924376025} \\
& \mp \frac{8\left(\mp 736346688059+229612484741\sqrt{10}\right)a_{01}y^3x}{577733125} \mp \frac{8\left(\mp 106721196235+33582210361\sqrt{10}\right)y^2x}{45140625} \\
& + \frac{8\left(8528758152668\mp 2696860731977\sqrt{10}\right)yx}{225703125a_{01}} \\
& \pm \frac{16\left(\pm 29687040274008l997+86796109377846580\sqrt{10}\right)a_{01}^4y^6}{8939823257244500805} \\
& \pm \frac{16\left(\mp 68557718200692598O+20249465757488211l\sqrt{10}\right)a_{01}^3y^5}{25870538422399875} \\
& \pm \frac{4\left(\mp 1019171138095172 3+3116773245583502\sqrt{10}\right)a_{01}^2y^4}{14973109400625} \\
& \pm \frac{16\left(\mp 419435752079855+131894867253953\sqrt{10}\right)a_{01}y^3}{389969859375} \\
& \pm \frac{4\left(\mp 2381664790141+752392059784\sqrt{10}\right)y^2}{225703125} \pm \frac{8\left(\mp 2846157446809+899975292211\sqrt{10}\right)y}{225703125a_{01}}
\end{aligned}
$$

In both cases we can construct the Darboux integrating factor of the form $\mu_{a,b} = l_{a,b}^{1/5} f_{a,b}^{-7/10}$. By [4, Th. 4.13] there exists first integral of the form (5) for system of cases a) and b), respectively.

4. CONCLUSIONS

In order to obtain the necessary conditions for the existence of a $1 : -4$ resonant center for (2.1) the decomposition of the center variety was found. However, we arrived at huge computational difficulties when computing the decomposition. The decomposition gives 46 cases (of necessary conditions) for Theorem 2.1 and 9 cases for Theorem 2.2. The computations for the decomposition in Theorem 2.1 were still extremely laborious. The corresponding computational problem was solved by using the routine `minAssGTZ` [7] of CAS Singular which is based on the Gianni-Trager-Zacharias algorithm [20]. Like e.g. in [8] also in case of (2.1) which was the main problem of the present paper the decomposition of varieties of ideals generated by saddle quantities (1.8) could not be performed in polynomial ring of characteristic zero, but it becomes possible in the rings of finite characteristics $\mathbb{Z}_p[a, b]$. The modular approach was for the first time successfully applied in $p : -q$ resonant center problems in [11, 14]. Preparing this paper the authors have used modular approach for the following prime numbers: $p = 32003, 104729, 4256233, 7368787, 15485863, 179595127, 433494437$ and 479001599. The computational problems are further described in the proof of Theorem 2.1.

To perform the rational reconstruction we used the following Mathematica code [28, 33]:

```
RATCONVERT[c,m]:=Block[u={1,0,m},v={0,1,c},r],
While[Sqrt[m/2] ≤ v[[3]],r=u-Quotient[u[[3]],v[[3]]] v; u=v; v=r];
If[Abs[v[[2]]] ≥ Sqrt[m/2], err, v[[3]]/v[[2]]]
```

Given an integer number c and a natural number m the function produces a couple of integer numbers v_2 and v_3 such that $v_3/v_2 \equiv c \, mod \, m$ and $|v_2|$, $|v_3| \leq \sqrt{m/2}$.

Finally for the existence of the first integral in the 55 cases of Theorems 2.1 and 2.2 we have used the Darboux method, the construction of an integrating factor, the so-called sum method [2, 8, 28], as well as the monodromy arguments [6]. From the results presented in this work we have the following conjecture:

Conjecture 4.1. The conditions $1-46$ of Theorem 2.1 are the necessary and sufficient conditions for integrability of system (2.1).

In order to verify this conjecture we have to check that over the field of characteristic 0 all Gröbner basis of $\langle 1 - wp_k, \mathcal{B}_6 \rangle$ for $k = 1, \ldots, 112$ are equal to $\{1\}$. We were not able to complete computations over the field of rational numbers.

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