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Parabolic approximation of damped wave equations via fractional powers: fast growing nonlinearities and continuity of the dynamics

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In this paper we consider a semilinear damped wave equation (with polynomially growing nonlinearity of power $\rho < \frac{N+2}{N-2}$) using parabolic approximations governed by the fractional powers $-\Lambda^\alpha$, $\alpha < 1$, of the wave operator $-\Lambda$. We give explicitly expressions for the fractional powers of the wave operator, compute their resolvent operators, their eigenvalues and exhibit a Lyapunov functional for the approximating equations (with the fractional powers). We obtain solutions of the semilinear damped wave equation as limit of solutions of the approximating equations, obtain the existence of global attractors and prove their continuity as $\alpha \to 1$. May, 2013 ICMC-USP

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, $N \geqslant 3$, $a > 0$ and consider the damped hyperbolic equation

$$\begin{cases} u_{tt} + au_t - \Delta u = f(u), \ t > 0, \ x \in \Omega, \\ u(0,x) = u_0(x), \ u_t(0,x) = v_0(x), \ x \in \Omega, \\ u(t,x) = 0, \ t \geqslant 0, \ x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in C^1(\mathbb{R})$ is such that

$$|f'(s)| \leqslant C(1 + |s|^{\rho-1}), \quad \text{for all } s \in \mathbb{R}, \tag{1.2}$$

for some $1 < \rho < \frac{N+2}{N-2}$ and $C > 0$ and

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \mu_1, \tag{1.3}$$

with $-\mu_1$ being the first eigenvalue of Δ_D (the Laplacian with Dirichlet boundary condition in Ω).

If $X = L^2(\Omega)$ and $A : D(A) \subset X \to X$ is defined by $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$ and $Au = -\Delta_D u$ for all $u \in D(A)$, then A is a positive self-adjoint operator and $-A$ generates a compact analytic semigroup on X. Denote by X^α the fractional power spaces associated to operator A; that is, $X^\alpha = D(A^\alpha)$ with the norm $\|A^\alpha \cdot\|_X : X^\alpha \to \mathbb{R}^+$. For $\alpha > 0$ we define $X^{-\alpha}$ as the completion of X with the norm $\|A^{-\alpha} \cdot\|_X$. With this notation $X^{\frac{1}{2}} = H^1_0(\Omega)$, $X^1 = H^2(\Omega) \cap H^1_0(\Omega)$ and $X^{-\alpha} = (X^\alpha)'$ (see [1] for the characterization of the negative scale).

The problem (1.1) can be written as an abstract Cauchy problem in the product space $X^{\frac{1}{2}} \times X$ as

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + \Lambda_{(a)}\begin{bmatrix} u \\ v \end{bmatrix} = F\Big(\begin{bmatrix} u \\ v \end{bmatrix}\Big), \quad t > 0, \begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \tag{1.4}$$

where the damped wave operator $\Lambda_{(a)} : D(\Lambda_{(a)}) \subset X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X$ is defined by

$$D(\Lambda_{(a)}) = X^1 \times X^{\frac{1}{2}}$$
$$\Lambda_{(a)}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -I \\ -\Delta_D & aI \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -v \\ -\Delta u + av \end{bmatrix},$$

$F : X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X^{-\frac{1}{2}}$ is defined by

$$F\begin{bmatrix} \phi \\ \varphi \end{bmatrix} := \begin{bmatrix} 0 \\ f^e(\phi) \end{bmatrix}, \ \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \in X^{\frac{1}{2}} \times X,$$

where $f^e : X^{\frac{1}{2}} \to X^{-\frac{1}{2}}$ denotes the operator $f^e(u)(x) := f(u(x))$, $u \in X^{\frac{1}{2}}$, $x \in \Omega$.

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It is easily seen that $0 \in \rho(\Lambda_{(a)})$ for all $a \in \mathbb{R}$ and writing the wave operator $\Lambda := \Lambda_{(0)}$ as

$$\Lambda = \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix},$$

we have

$$\Lambda^{-1} = \begin{bmatrix} 0 & A^{-1} \\ -I & 0 \end{bmatrix}.$$

Observe that, the adjoint of Λ, is given by

$$\Lambda^* = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix},$$

and $\Lambda = -\Lambda^*$; that is, the operator Λ is skew-adjoint. It follows that $i\Lambda$ is self-adjoint and, from Stone's theorem, Λ is the infinitesimal generator of a C_0-group of unitary operators on $X^{\frac{1}{2}} \times X$ (see Pazy [17, Theorem 10.8, pg. 41]).

Under the assumptions (1.2) with $\rho < \frac{N}{N-2}$ and (1.3), the existence of attractors $\mathcal{A}_1$ for problem 1.4 in $X^{\frac{1}{2}} \times X$ has been established in Babin and Vishik [4, 5] and in Arrieta, Carvalho and Hale [2] for $\rho = \frac{N}{N-2}$. In the later, the authors employed Alekseev's nonlinear variation of constants formula to prove the asymptotic compactness.

The semilinear (weakly) damped wave equation given in (1.1) have been considered before by many authors (see Pata and Zelik [15] and references therein).

In Carvalho, Cholewa and Dlotko [9] the authors considered (1.1) through a limit of a strongly damped wave equation, adding the term $2\eta(-\Delta_D)^{\frac{1}{2}}$ with $\eta > 0$ to the equation, so that the equation becomes 'parabolic' in nature (see Chen and Triggiani [11]), and passing to the limit as $\eta \to 0$. With the 'parabolic' structure ($\eta > 0$), they obtain local well posedness for the perturbed problem under the growth assumption (1.2) with $1 < \rho < \frac{N+2}{N-2}$ and with the usual semigroup approach. If the dissipativeness condition (1.3) also holds they obtain global well posedness, existence of global attractors and some uniform (with respect to η) bounds that allow a passage to the limit ($\eta = 0$). After this they obtain global solutions of (1.1) that satisfy the variation of constants formula and are able to establish the existence of global attractors.

The aim of this paper is study the asymptotic dynamics of the problem (1.1), with nonlinearity f satisfying (1.2) with $1 < \rho < \frac{N+2}{N-2}$ and (1.3), using an approximation by 'parabolic' type problems of 'lower' order which we begin to describe. If $-\Lambda$ denotes the wave operator (generator of a C_0-semigroup), we use the fractional power operators $-\Lambda^\alpha$, $\alpha \in (0, 1)$, (generator of an analytic semigroup) to approximate $-\Lambda$. This sort of approximation (though by a lower order operator) has the effect of a viscosity approximation and yields that the perturbed problem exhibits regularity properties whereas the limit does not.

We emphasize that, though it may appear cumbersome at the moment, we will be able to give explicit expressions to the fractional powers of Λ (in terms of the fractional powers of $-\Delta_D$) as well as to associate the following gradient differential equation to the perturbed problem

$$u_{tt} + 2\cos\frac{\pi\alpha}{2}\mathbb{A}^{\frac{1}{2}}u_t + \mathbb{A}u + au_t + a\cos\frac{\pi\alpha}{2}\mathbb{A}^{\frac{1}{2}}u = \sin\frac{\pi\alpha}{2}\mathbb{A}^{-\frac{1-\alpha}{2\alpha}}f(u) \tag{1.5}$$

where $\mathbb{A} = (-\Delta_D)^\alpha$. The Lyapunov function for this problem is given by

$$\mathcal{L}\Big(\begin{bmatrix} u \\ u_t \end{bmatrix}\Big) = \frac{1}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}} + \frac{1}{2}\|u\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{a}{2}\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac{1}{4}}} - \sin\frac{\pi\alpha}{2}\int_\Omega\int_0^u f(s)dsdx. \tag{1.6}$$

Under growth assumption (1.2) with $1 < \rho < \frac{N+2}{N-2}$ and exploiting the parabolic structure of (1.5), the local well posedness for (1.5) is obtained for α suitably close do 1. Using the gradient structure of (1.5), given by the Lyapunov function (1.6) and a suitable regularity analysis to accommodate the fact that the Lyapunov function is not defined in $H^1_0(\Omega) \times L^2(\Omega)$, we are able to establish, under the additional dissipativeness condition (1.3), the global well posedness, existence of global attractors and some uniform bounds for the solutions of (1.5), for α suitably close to 1. These striking features will make the passage to the limit as α tends to 1 possible, establishing the existence of global solutions, that satisfy the variation of constants formula, for (1.1) in $H^1_0(\Omega) \times L^2(\Omega)$ conditions (1.2) and (1.3). In fact, we prove continuity properties of the family $\{\mathcal{A}_\alpha : \alpha \in (0, 1)\}$ for (1.5) in $H^1_0(\Omega) \times L^2(\Omega)$ as α tends to 1.

This paper is organized as follows. In Section 2 we study the local well posedness of the problem (2.3) studying the the spectral properties of the operators Λ and Λ^α which are needed to characterize the fractional power spaces. In Section 3 we obtain the Lyapunov functional (1.6) for (1.5), discuss the global well posedness for (1.5) and prove that the semigroup associated to (1.5) has a global attractor $\mathcal{A}_\alpha$ in $H^1_0(\Omega) \times L^2(\Omega)$. Section 4 is devoted to study the convergence of the semigroup generated by $-\Lambda^\alpha$ to the semigroup generated by $-\Lambda$ as α tends to 1. We start with a result of convergence of the resolvents and conclude proving a Trotter-Kato type theorem. Finally, in Section 5 we discuss the convergence of the asymptotic dynamics of the problem (2.3) to that of (1.1) as α tends to 1. We conclude this section showing that (for $\rho < \frac{N}{N-2}$) the family of the global attractors behaves continuously at $\alpha = 1$.

2. LOCAL WELL POSEDNESS

We first recall that if $P : D(P) \subset Z \to Z$ is a sectorial operator with $\operatorname{Re}\sigma(P) > 0$, then

$$\begin{aligned} &\frac{d}{dt}z + Pz = G(z), \quad t > 0, \\ &z(0) = z_0, \end{aligned} \tag{2.1}$$

is locally well-posed under suitable conditions on the nonlinearity G which we make clear next.

Definition 2.1. Consider the problem (2.1). A *mild solution* of (2.1) in Z^1 is a continuous function $z(\cdot, z_0) : [0, \tau] \to Z^1$ such that $z(0, z_0) = z_0$, $G(z(\cdot, z_0)) : [0, \tau] \to Z^\alpha$

is continuous for some $\alpha > 0$ and that satisfies the integral equation

$$z(t, z_0) = e^{-Pt} z_0 + \int_0^t e^{-P(t-s)} G(z(s, z_0)) ds.$$

We say that (2.1) is locally well posed if, given $w_0 \in Z^1$, there exist $r > 0$ and $\tau > 0$ such that (2.1) has a mild solution $z(t, w) : [0, \tau] \to$ for every $w \in B_r^{Z^1}(w_0)$ and the map

$$B_r^{Z^1}(w_0) \ni w \mapsto z(\cdot, w) \in C([0, \tau], Z^1)$$

is continuous.

We remark that under suitable conditions on the map G a mild solution is also as strong solution; that is, it satisfies the equation in (2.1). The next result is an easy consequence of the results in [12, Theorem 3.3.3], and its proof is omitted.

Theorem 2.1. *Assume P as above and let $G : Z^1 \to Z^\alpha$ be a locally Lipschitz continuous in bounded subsets of Z^1, for some $\alpha \in (0, 1)$, then* (2.1) *is locally well posed in Z^1.*

In order to arrive at (1.5) and apply to it the above results, we will need to compute the fractional powers of Λ and to understand the fractional power spaces associated to it. This is what we do next. Note that

$$\lambda I + \Lambda = \begin{bmatrix} \lambda I & -I \\ A & \lambda I \end{bmatrix}, \ \lambda \in \mathbb{C},$$

and therefore, for all $\lambda \in \rho(-\Lambda)$, we have

$$(\lambda I + \Lambda)^{-1} = \begin{bmatrix} \lambda(\lambda^2 I + A)^{-1} & (\lambda^2 I + A)^{-1} \\ -A(\lambda^2 I + A)^{-1} & \lambda(\lambda^2 I + A)^{-1} \end{bmatrix}.$$

For $0 < \alpha < 1$, we can compute the fractional $\Lambda^{-\alpha}$ by the formula

$$\Lambda^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha} (\lambda I + \Lambda)^{-1} d\lambda$$

see [1, pg. 148] or [12, pg. 25]. With this, for any $0 < \alpha < 1$,

$$\Lambda^{-\alpha} = \begin{bmatrix} \cos \frac{\pi\alpha}{2} A^{-\frac{\alpha}{2}} & \sin \frac{\pi\alpha}{2} A^{\frac{-1-\alpha}{2}} \\ -\sin \frac{\pi\alpha}{2} A^{\frac{1-\alpha}{2}} & \cos \frac{\pi\alpha}{2} A^{-\frac{\alpha}{2}} \end{bmatrix} \in \mathcal{L}(X).$$

Also, it is not difficult to see that 0 is in the continuous spectrum of $\Lambda^{-\alpha}$ and that

$$\Lambda^{\alpha} = \begin{bmatrix} \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}} & -\sin\frac{\pi\alpha}{2}A^{\frac{-1+\alpha}{2}} \\ \sin\frac{\pi\alpha}{2}A^{\frac{1+\alpha}{2}} & \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}} \end{bmatrix} \tag{2.2}$$

for every $0 < \alpha < 1$.

Next we characterize the eigenvalues of Λ, in terms of the eigenvalues of A.

Proposition 2.1 (Spectral properties of Λ). *The eigenvalues $\{\lambda_n^{\pm}\}_{n\in\mathbb{N}}$ of the skew-adjoint operator Λ are given by*

$$\lambda_n^{\pm} = \pm i\sqrt{\mu_n},\ n \in \mathbb{N},$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ denote the eigenvalues of the operator $A = -\Delta_D$ with zero Dirichlet boundary conditions.

Proof: Since Λ has compact resolvent, all points in the spectrum $\sigma(\Lambda)$ of Λ are eigenvalues. The eigenvalue problem for Λ is

$$\begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix}\begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \lambda \begin{bmatrix} \varphi \\ \psi \end{bmatrix},\ \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D(\Lambda_{(a)}),$$

i.e.

$$A\varphi = -\lambda^2\varphi,\ \varphi \in D(A).$$

Recall that $A = -\Delta_D$ is a positive self-adjoint operator with compact resolvent. Denote by $\{\mu_n\}_{n\in\mathbb{N}}$ the eigenvalues of A ordered increasingly and repeated according to multiplicity. Hence, the eigenvalues of Λ are solutions of the equation $\lambda^2 = -\mu_n,\ n \in \mathbb{N}$, and therefore

$$\lambda = \lambda_n^{\pm} = \pm i\sqrt{\mu_n},\ n \in \mathbb{N}.$$

□

Proposition 2.2 (Spectral properties of $-\Lambda^{\alpha}$). *The spectrum of $-\Lambda^{\alpha}$ consists of eigenvalues only. They are given by*

$$\lambda_{\alpha,n}^{\pm} = e^{\pm i\frac{\pi(2-\alpha)}{2}}(\mu_n)^{\frac{\alpha}{2}},\ n \in \mathbb{N},$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ denotes the ordered sequence of eigenvalues of the operator A repeated according to multiplicity.

Proof: The eigenvalue problem for $-\Lambda^{\alpha}$ is

$$-\begin{bmatrix} \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}} & -\sin\frac{\pi\alpha}{2}A^{\frac{-1+\alpha}{2}} \\ \sin\frac{\pi\alpha}{2}A^{\frac{1+\alpha}{2}} & \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}} \end{bmatrix}\begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \lambda \begin{bmatrix} \varphi \\ \psi \end{bmatrix},\ \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D(\Lambda^{\alpha}),$$

that is, $\lambda \in \mathbb{C}$ is an eigenvalue for $-\Lambda^\alpha$ if and only if there is a $0 \neq \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}}$ such that

$$\begin{cases} -\cos\dfrac{\pi\alpha}{2} A^{\frac{\alpha}{2}}\varphi + \sin\dfrac{\pi\alpha}{2} A^{\frac{-1+\alpha}{2}}\psi = \lambda\varphi \\ \\ -\sin\dfrac{\pi\alpha}{2} A^{\frac{1+\alpha}{2}}\varphi - \cos\dfrac{\pi\alpha}{2} A^{\frac{\alpha}{2}}\psi = \lambda\psi. \end{cases}$$

With this, we get $\lambda \in \mathbb{C}$ is an eigenvalue for $-\Lambda^\alpha$ if and only if

$$\lambda^2 + 2\lambda\cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} + A^\alpha = (\lambda - e^{i\frac{\pi(2-\alpha)}{2}} A^{\frac{\alpha}{2}})(\lambda - e^{-i\frac{\pi(2-\alpha)}{2}} A^{\frac{\alpha}{2}})$$

is not injective. Then, the eigenvalues λ of $-\Lambda^\alpha$ are solutions of equation

$$(\lambda - e^{i\frac{\pi(2-\alpha)}{2}} \mu_n^{\frac{\alpha}{2}})(\lambda - e^{-i\frac{\pi(2-\alpha)}{2}} \mu_n^{\frac{\alpha}{2}}) = 0;$$

that is, $\lambda_{\alpha,n}^{\pm} = e^{\pm i\frac{\pi(2-\alpha)}{2}} \mu_n^{\frac{\alpha}{2}},\ n \in \mathbb{N}$, and this concludes the proof. □

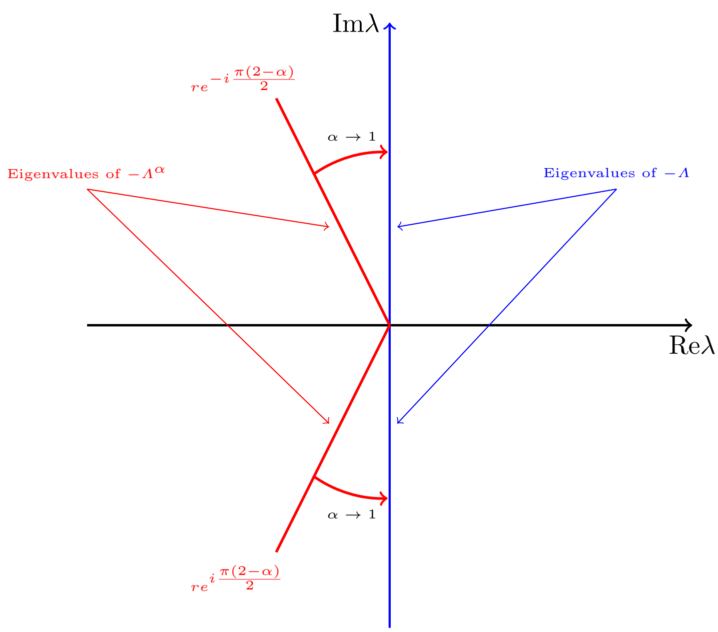

Figure 1: Spectrum of Λ^α.

Remark 2. 1. If we compare the Proposition 2.1 with the Proposition 2.2, we can conclude that the eigenvalues of the unbounded linear operator $-\Lambda^\alpha$, $\lambda_{\alpha,n}^{\pm}$, converge to the eigenvalues of the operator $-\Lambda$, $\lambda_n^{\pm}$, as α tends to 1. We can see that the eigenvalues $-\Lambda^\alpha$ lie in the semi-axes $\{re^{\pm i\frac{\pi(2-\alpha)}{2}} : r \geqslant 0\}$. These semi-axes form the edges of a sector of angle $\frac{\pi(2-\alpha)}{2}$ in the complex plane that, as α tends to 1 approaches the semi-plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geqslant 0\}$. This behavior reflects the loss of sectoriality that the operator $\varLambda^\alpha$ experiences as α tends to 1.

Thus, for each $\lambda \in \rho(-\varLambda^\alpha)$, $0 < \alpha < 1$, we have

$$\lambda I + \varLambda^\alpha = \begin{bmatrix} \lambda I + \cos\dfrac{\pi\alpha}{2} A^{\frac{\alpha}{2}} & -\sin\dfrac{\pi\alpha}{2} A^{\frac{-1+\alpha}{2}} \\ \sin\dfrac{\pi\alpha}{2} A^{\frac{1+\alpha}{2}} & \lambda I + \cos\dfrac{\pi\alpha}{2} A^{\frac{\alpha}{2}} \end{bmatrix}$$

and therefore, if $D_\alpha(\lambda) := \lambda^2 I + 2\lambda \cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} + A^\alpha = (\lambda - e^{i\frac{\pi(2-\alpha)}{2}} A^{\frac{\alpha}{2}})(\lambda - e^{-i\frac{\pi(2-\alpha)}{2}} A^{\frac{\alpha}{2}})$ is injective,

$$(\lambda I + \varLambda^\alpha)^{-1} = \begin{bmatrix} (\lambda I + \cos\dfrac{\pi\alpha}{2} A^{\frac{\alpha}{2}}) D_\alpha^{-1}(\lambda) & -\sin\dfrac{\pi\alpha}{2} A^{\frac{-1+\alpha}{2}} D_\alpha^{-1}(\lambda) \\ \sin\dfrac{\pi\alpha}{2} A^{\frac{1+\alpha}{2}} D_\alpha^{-1}(\lambda) & (\lambda I + \cos\dfrac{\pi\alpha}{2} A^{\frac{\alpha}{2}}) D_\alpha^{-1}(\lambda) \end{bmatrix}.$$

It is not difficult to prove that (see, for example, Theorem III.4.6.2 in [1]):

Proposition 2.3. *The family of operators $\{\varLambda^{-\alpha}; \alpha \geqslant 0\}$ is an analytic semigroup and therefore, $\varLambda^{-\alpha}$ converges in the uniform operator topology of $\mathcal{L}(X^{\frac{1}{2}} \times X)$ to $\varLambda^{-1}$, as $\alpha \to 1$.*

Lemma 2.1. *For $\alpha \in [0,1]$, the operator $\varLambda^{-\alpha} : X^{\frac{1-\alpha}{2}} \times X^{-\frac{\alpha}{2}} \to X^{\frac{1}{2}} \times X$ is an isometry. Consequently, if $Y^{1-\alpha}$ denotes the extrapolation of $Y^1 = X^{\frac{1}{2}} \times X$ generated by $\varLambda^\alpha$ and $Y^{1+\alpha}$ denotes $D(\Lambda^\alpha)$ with the graph norm, we have that*

$$Y^{1+\alpha} = X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}},$$
$$Y^{1-\alpha} = X^{\frac{1-\alpha}{2}} \times X^{-\frac{\alpha}{2}}.$$

The closure Λ_0 of $\Lambda : D(\Lambda) \subset Y^0 \to Y^0$ is a positive sectorial operator and $D(\Lambda_0^\beta) = Y^\beta$, $\beta \in [0,2]$ and $(\Lambda_0^\alpha)^\beta = \Lambda_0^{\beta\alpha}$.

Proof: The first equality is immediate for $\alpha = 0$ and $\alpha = 1$. For $\alpha \in (0,1)$ it follows from the fact that the complex interpolation and the fractional powers of accretive operators in

Hilbert spaces coincide. For the second equality, if $\begin{bmatrix} u \\ v \end{bmatrix}$ in $X^{\frac{1}{2}} \times X$ we have

$$\begin{aligned}
&\|\Lambda^{-\alpha} \begin{bmatrix} u \\ v \end{bmatrix}\|^2_{X^{\frac{1}{2}}\times X} = \langle \Lambda^{-\alpha}\begin{bmatrix} u \\ v \end{bmatrix}, \Lambda^{-\alpha}\begin{bmatrix} u \\ v \end{bmatrix}\rangle_{X^{\frac{1}{2}}\times X} = \\
&= \langle \cos\frac{\pi\alpha}{2} A^{-\frac{\alpha}{2}} u + \sin\frac{\pi\alpha}{2} A^{\frac{-1-\alpha}{2}} v, \cos\frac{\pi\alpha}{2} A^{-\frac{\alpha}{2}} u + \sin\frac{\pi\alpha}{2} A^{\frac{-1-\alpha}{2}} v\rangle_{X^{\frac{1}{2}}} \\
&+ \langle -\sin\frac{\pi\alpha}{2} A^{\frac{1-\alpha}{2}} u + \cos\frac{\pi\alpha}{2} A^{-\frac{\alpha}{2}} v, -\sin\frac{\pi\alpha}{2} A^{\frac{1-\alpha}{2}} u + \cos\frac{\pi\alpha}{2} A^{-\frac{\alpha}{2}} v\rangle_X \\
&= \langle \cos\frac{\pi\alpha}{2} A^{\frac{1-\alpha}{2}} u + \sin\frac{\pi\alpha}{2} A^{-\frac{\alpha}{2}} v, \cos\frac{\pi\alpha}{2} A^{\frac{1-\alpha}{2}} u + \sin\frac{\pi\alpha}{2} A^{-\frac{\alpha}{2}} v\rangle_X \\
&+ \langle -\sin\frac{\pi\alpha}{2} A^{\frac{1-\alpha}{2}} u + \cos\frac{\pi\alpha}{2} A^{-\frac{\alpha}{2}} v, -\sin\frac{\pi\alpha}{2} A^{\frac{1-\alpha}{2}} u + \cos\frac{\pi\alpha}{2} A^{-\frac{\alpha}{2}} v\rangle_X \\
&= \langle A^{\frac{1-\alpha}{2}} u, A^{\frac{1-\alpha}{2}} u\rangle_X + \langle A^{-\frac{\alpha}{2}} v, A^{-\frac{\alpha}{2}} v\rangle_X \\
&= \|\begin{bmatrix} u \\ v \end{bmatrix}\|^2_{X^{\frac{1-\alpha}{2}}\times X^{-\frac{\alpha}{2}}}.
\end{aligned}$$

For the remaining see, for example, [1, Section V.1.3]. □

Remark 2. 2. The fractional powers spaces associated to the operator

$$\Lambda = \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix},$$

coincide with those associated to $\Lambda_{(a)} = \Lambda + B_{(a)}$, where

$$B_{(a)} = \begin{bmatrix} 0 & 0 \\ 0 & aI \end{bmatrix}, \quad a > 0.$$

In fact, since $B_{(a)} \in \mathcal{L}(X)$, $D((\Lambda+B_{(a)})^\alpha) = D(\Lambda^\alpha)$ with equivalent norms. With this, we identify the fractional powers of $\Lambda+B_{(a)}$ using the fractional powers of Λ.

In what follows, let $\alpha \geqslant \frac{1}{3}$ and suppose that $f^e : X^{\frac{1+\alpha}{4}} \to X^{\frac{1-3\alpha}{4}}$ (where $X^{-s} = (X^s)'$, $s \geqslant 0$) is Lipschitz continuous in bounded sets and

$$f^e(u)(x) = f(u(x)), \quad u \in X^{\frac{1}{2}} \text{ and } x \in \Omega.$$

This implies that $F : X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}} \to X^{\frac{3-3\alpha}{4}} \times X^{\frac{1-3\alpha}{4}}$ given by

$$F\begin{bmatrix} \phi \\ \varphi \end{bmatrix} := \begin{bmatrix} 0 \\ f^e(\phi) \end{bmatrix}, \begin{bmatrix} \phi \\ \varphi \end{bmatrix} \in X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}},$$

with f^e as above and f satisfying (1.2), $\frac{1}{3} < \alpha < 1$ suitably close to 1 (see Figure 2).

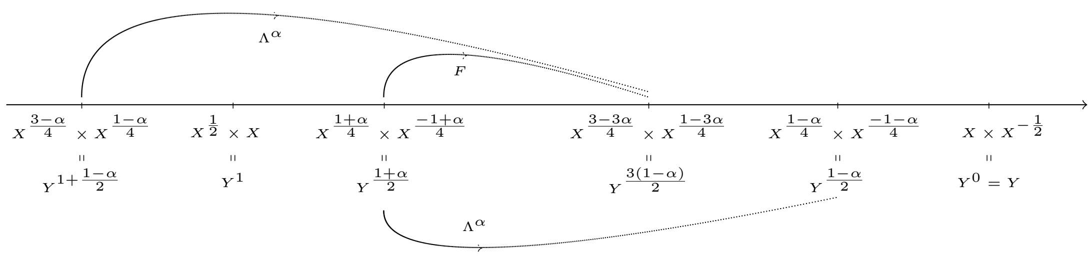

Figure 2: Partial description of the fractional power spaces scale for Λ.

Now, we consider the Cauchy problem

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + \Lambda^\alpha \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & aI \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = F(\begin{bmatrix} u \\ v \end{bmatrix}), \quad t > 0, \qquad \begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \tag{2.3}$$

in $Y^1 = X^{\frac{1}{2}} \times X$. Here, Λ^α is the operator in (2.2) and $F : X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}} \to X^{\frac{3-3\alpha}{4}} \times X^{\frac{1-3\alpha}{4}}$ is as above. The operator Λ^α is a positive sectorial operator (see [13]). With this, the system (2.3) can be seen as a parabolic type perturbation of the system (1.4) and we approximate solutions of (1.4) by solutions of the (2.3), $\frac{1}{3} < \alpha < 1$, with suitably chosen initial data.

Remark 2. 3. The exponent $\rho = \frac{N+(3\alpha-1)}{N-(\alpha+1)}$ is critical (in the sense of [7]) for the abstract parabolic problem in (2.3). In fact, for $\alpha \in (\frac{1}{3}, 1]$ and ρ as above, we have

$$X^{\frac{1+\alpha}{4}} \hookrightarrow H^{\frac{1+\alpha}{2}} \hookrightarrow L^{\frac{2N}{N-(\alpha+1)}} \xrightarrow{u^\rho} L^{\frac{2N}{\rho(N-(\alpha+1))}} \hookrightarrow H^{\frac{1-3\alpha}{2}} = X^{\frac{1-3\alpha}{4}},$$

$$(H^{\frac{-1+3\alpha}{2}} \hookrightarrow L^{\frac{2N}{2N-\rho(N-(\alpha+1))}})$$

with all inclusions being sharp.

Corollary 2.1. *For $\alpha_0 = \max\left\{\frac{1}{3}, \frac{(\rho-1)(N-1)}{\rho+3}\right\}$, assume that $\alpha \in (\alpha_0, 1)$ and that f satisfies* (1.2). *Then the problem* (2.3) *is locally well posed in $Y^{\frac{1+\beta}{2}}$ for all $\beta \in [\alpha, 1]$.*

Proof: Observe that $F : Y^{\frac{1+\alpha}{2}} \to Y^{\frac{3(1-\alpha)}{2}}$ is Lipschitz continuous in bounded sets and that $Y^{\frac{1+\beta}{2}} \hookrightarrow Y^{\frac{1+\alpha}{2}}$. In fact, let B be a bounded subset of Y^1 and consider $\begin{bmatrix} \varphi_1 \\ \psi_1 \end{bmatrix}, \begin{bmatrix} \varphi_2 \\ \psi_2 \end{bmatrix} \in B$.

Then, we have

$$\begin{aligned}
\|F\Big(\begin{bmatrix}\varphi_1\\ \psi_1\end{bmatrix}\Big) - F\Big(\begin{bmatrix}\varphi_2\\ \psi_2\end{bmatrix}\Big)\|_{Y^{\frac{3(1-\alpha)}{2}}} &\leqslant C\|f^e(\varphi_1) - f^e(\varphi_2)\|_{X^{\frac{1-3\alpha}{4}}}\\
&\leqslant C_1\|f^e(\varphi_1) - f^e(\varphi_2)\|_{L^{\frac{2N}{N+(3\alpha-1)}}(\Omega)}\\
&\leqslant C_2\|\varphi_1-\varphi_2\|_{L^{\frac{2N}{N-(\alpha+1)}}(\Omega)}\Big(1+\|\varphi_1\|^{\rho-1}_{L^{\frac{N(\rho-1)}{2\alpha}}(\Omega)}+\|\varphi_2\|^{\rho-1}_{L^{\frac{N(\rho-1)}{2\alpha}}(\Omega)}\Big)\\
&\leqslant C_3\|\begin{bmatrix}\varphi_1\\ \psi_1\end{bmatrix}-\begin{bmatrix}\varphi_2\\ \psi_2\end{bmatrix}\|_{Y^{\frac{1+\alpha}{2}}} \leqslant C_4\|\begin{bmatrix}\varphi_1\\ \psi_1\end{bmatrix}-\begin{bmatrix}\varphi_2\\ \psi_2\end{bmatrix}\|_{Y^{\frac{1+\beta}{2}}}.
\end{aligned}$$

Observe that, for $\alpha > \frac{(\rho-1)(N-1)}{\rho+3}$ we have that $\rho < \frac{N+(3\alpha-1)}{N-(\alpha+1)}$, then

$$X^{\frac{1+\alpha}{4}} \hookrightarrow L^{\frac{N(\rho-1)}{2\alpha}}(\Omega).$$

From Theorem 2.1 the result follows. □

3. GRADIENT STRUCTURE AND EXISTENCE OF GLOBAL ATTRACTOR

We start this section exhibiting a Lyapunov functional associated to (2.3). For each $\begin{bmatrix}u_0\\ v_0\end{bmatrix} \in X^{\frac{1+\alpha}{4}} \times X^{\frac{1-\alpha}{4}}$, note that the system in (2.3) is given by

$$\begin{cases} u_t + \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}}u - \sin\frac{\pi\alpha}{2}A^{\frac{-1+\alpha}{2}}v = 0,\\ v_t + \sin\frac{\pi\alpha}{2}A^{\frac{1+\alpha}{2}}u + \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}}v + av = f(u),\\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x). \end{cases} \tag{3.1}$$

Note that from first equation, we have that

$$\sin\frac{\pi\alpha}{2}v = A^{\frac{1-\alpha}{2}}\Big(u_t + \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}}u\Big).$$

Differentiating with respect to t, follows that

$$\sin\frac{\pi\alpha}{2}v_t = A^{\frac{1-\alpha}{2}}\Big(u_{tt} + \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}}u_t\Big). \tag{3.2}$$

Substituting (3.2) in the second equation in (3.1), after some calculations, we obtain that

$$A^{\frac{1-\alpha}{2}}u_{tt} + 2\cos\frac{\pi\alpha}{2}A^{\frac{1}{2}}u_t + A^{\frac{1+\alpha}{2}}u + aA^{\frac{1-\alpha}{2}}u_t + a\cos\frac{\pi\alpha}{2}A^{\frac{1}{2}}u = \sin\frac{\pi\alpha}{2}f(u). \tag{3.3}$$

Multiplying the above equation by u_t and integrating, we get

$$\frac{d}{dt}\Big(\frac{1}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}+\frac{1}{2}\|u\|^2_{X^{\frac{1+\alpha}{4}}}+\frac{a}{2}\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac{1}{4}}}-\sin\frac{\pi\alpha}{2}\int_\Omega\int_0^u f(s)dsdx\Big)$$
$$=-2\cos\frac{\pi\alpha}{2}\|u_t\|^2_{X^{\frac{1}{4}}}-a\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}. \tag{3.4}$$

Consider the energy functional $\mathcal{L}: X^{\frac{1+\alpha}{4}}\times X^{\frac{1-\alpha}{4}}\to\mathbb{R}$ defined by

$$\mathcal{L}\Big(\begin{bmatrix}u\\u_t\end{bmatrix}\Big)=\frac{1}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}+\frac{1}{2}\|u\|^2_{X^{\frac{1+\alpha}{4}}}+\frac{a}{2}\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac{1}{4}}}-\sin\frac{\pi\alpha}{2}\int_\Omega\int_0^u f(s)dsdx, \tag{3.5}$$

for all $\begin{bmatrix}u\\u_t\end{bmatrix}\in D(\mathcal{L})=X^{\frac{1+\alpha}{4}}\times X^{\frac{1-\alpha}{4}}$.

Remark 3. 1. Note that, if $\mathbb{A}=A^\alpha$, we can rewrite (3.3) in the form

$$u_{tt}+2\cos\frac{\pi\alpha}{2}\mathbb{A}^{\frac{1}{2}}u_t+\mathbb{A}u+au_t+a\cos\frac{\pi\alpha}{2}\mathbb{A}^{\frac{1}{2}}u=\sin\frac{\pi\alpha}{2}\mathbb{A}^{-\frac{1-\alpha}{2\alpha}}f(u) \tag{3.6}$$

and the equation (3.6) is a parabolic type perturbation (defined by the fractional powers of the associated to the wave operator) of the damped hyperbolic problem (1.1) (see [11]).

Remark 3. 2. Observe that, the domain of definition of our Lyapunov functional does not belong to fractional power spaces scale generated by the operator Λ and this is a very particular situation which does not appear elsewhere. In our analysis we will use the space $Y^{\frac{1+\alpha}{2}}=X^{\frac{1+\alpha}{4}}\times X^{\frac{-1+\alpha}{4}}$, knowing that

$$Y^{1+\frac{1-\alpha}{2}}=X^{\frac{3-\alpha}{4}}\times X^{\frac{1-\alpha}{4}}\hookrightarrow D(\mathcal{L})=X^{\frac{1+\alpha}{4}}\times X^{\frac{1-\alpha}{4}}\hookrightarrow X^{\frac{1+\alpha}{4}}\times X^{\frac{-1+\alpha}{4}}=Y^{\frac{1+\alpha}{2}},$$

(see Figure 2).

Now we show that, if (1.2) and (1.3) are satisfied and $\frac{(\rho-1)(N-1)}{\rho+3}<\alpha<1$ (that is, $\rho<\frac{N+(3\alpha-1)}{N-(\alpha+1)}$), the problem (2.3) is globally well posed and has a global attractor in $Y^{\frac{1+\alpha}{2}}$.

THEOREM 3.1. *Assume that f satisfies* (1.2) *with* $\rho\in(1,\frac{N+2}{N-2})$, $\alpha_0=\max\Big\{\frac{1}{3},\frac{(\rho-1)(N-1)}{\rho+3}\Big\}$ *and* (1.3) *holds. Then,* (2.3) *is globally well posed in* $Y^{\frac{1+\alpha}{2}}=X^{\frac{1+\alpha}{4}}\times X^{\frac{-1+\alpha}{4}}$ *whenever* $\alpha\in(\alpha_0,1)$. *Moreover,* (2.3) *has a global attractor* $\mathcal{A}_\alpha$ *in* $Y^{\frac{1+\alpha}{2}}$, *with*

$$\sup_{\alpha\in[0,1)}\sup_{w\in\mathcal{A}_\alpha}\|w\|_{Y^{\frac{1+\alpha}{2}}}<\infty.$$

Proof: Let B be a bounded subset of $Y^{\frac{1+\alpha}{2}}$. Since $F: Y^{\frac{1+\alpha}{2}} \to Y^{\frac{3(1-\beta)}{2}}$, for some $\beta < \alpha$, it follows from the results in [7] that there exists $\tau_B > 0$ so that all solutions starting in points of B must exist in $[0, \tau_B]$. If we denote by $\begin{bmatrix} u(t) \\ u_t(t) \end{bmatrix}$, the mild solution of (2.3) through $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B$ and $\tau \in (0, \tau_B)$ we have (see [7]) that $\sup_{\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B} \| \begin{bmatrix} u(\tau) \\ u_t(\tau) \end{bmatrix} \|_{Y^{1+\frac{1-\alpha}{2}}} < \infty$.

Now, since $Y^{1+\frac{1-\alpha}{2}} \subset D(\mathcal{L})$, we have that

$$\mathcal{L}\Big(\begin{bmatrix} u(t) \\ u_t(t) \end{bmatrix} \Big) \leqslant \mathcal{L}\Big(\begin{bmatrix} u(\tau) \\ u_t(\tau) \end{bmatrix} \Big),$$

for as long as the solution exists; that is,

$$\frac{1}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}} + \frac{1}{2}\|u\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{a}{2}\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac{1}{4}}} - \sin\frac{\pi\alpha}{2}\int_\Omega \int_0^u f(s)dsdx \leqslant \mathcal{L}\Big(\begin{bmatrix} u(\tau) \\ u_t(\tau) \end{bmatrix} \Big).$$

By Poincaré's inequality and (1.3),

$$\begin{aligned}
\frac{1}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}} + \frac{1}{2}\|u\|^2_{X^{\frac{1+\alpha}{4}}} &\leqslant \mathcal{L}\Big(\begin{bmatrix} u(\tau) \\ u_t(\tau) \end{bmatrix} \Big) - \frac{a}{2}\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac{1}{4}}} + \sin\frac{\pi\alpha}{2}\int_\Omega \int_0^u f(s)dsdx \\
&\leqslant \mathcal{L}\Big(\begin{bmatrix} u(\tau) \\ u_t(\tau) \end{bmatrix} \Big) + \sin\frac{\pi\alpha}{2}\int_\Omega \int_0^u f(s)dsdx \\
&\leqslant \mathcal{L}\Big(\begin{bmatrix} u(\tau) \\ u_t(\tau) \end{bmatrix} \Big) + c_\mu + \frac{\mu}{2}\|u\|^2_X, \ \ \mu > 0 \\
&\leqslant \mathcal{L}\Big(\begin{bmatrix} u(\tau) \\ u_t(\tau) \end{bmatrix} \Big) + c_\mu + \frac{1}{2}\lambda_1^{-\frac{1+\alpha}{2}}\mu\|u\|^2_{X^{\frac{1+\alpha}{4}}}, \ \ \mu > 0.
\end{aligned}$$

Thus

$$\frac{1}{2}\Big(1 - \lambda_1^{-\frac{1+\alpha}{2}}\mu\Big)\|u\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{1}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}} \leqslant \frac{1}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}} + \frac{1}{2}\|u\|^2_{X^{\frac{1+\alpha}{4}}} \leqslant C\Big(\begin{bmatrix} u(\tau) \\ u_t(\tau) \end{bmatrix} \Big), \ \mu > 0,$$

and

$$\| \begin{bmatrix} u(t) \\ u_t(t) \end{bmatrix} \|_{X^{\frac{1+\alpha}{4}} \times X^{\frac{1-\alpha}{4}}} \leqslant C\Big(\begin{bmatrix} u(\tau) \\ u_t(\tau) \end{bmatrix} \Big),$$

for as long as the solution exists.

This ensures that the mild solution of the problem (2.3) passing through $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y^{\frac{1+\alpha}{2}}$ at $t = 0$, does not blow up in the $X^{\frac{1+\alpha}{4}} \times X^{\frac{1-\alpha}{4}}$ norm. Consequently, it does not blow up in $Y^{\frac{1+\alpha}{2}}$ norm and it must exist for all $t \geqslant 0$. Hence, we can define in $Y^{\frac{1+\alpha}{2}}$ the nonlinear semigroup $\{T_\alpha(t);\ t \geqslant 0\}$ associated with (2.3). Also, the same reasoning leads to the conclusion that orbits of bounded subsets of $Y^{\frac{1+\alpha}{2}}$ are bounded subsets of $Y^{\frac{1+\alpha}{2}}$.

Furthermore, the existence of global attractors now follows easily from the fact that the set of equilibria of (2.3) is bounded and from the fact that the linear semigroup generated by $-\Lambda^\alpha$ is compact for $\alpha \in (0,1)$.

Now, any global solution $\phi : \mathbb{R} \to Y^{\frac{1+\alpha}{2}}$ in the attractor $\mathcal{A}_\alpha$ is easily seen to be bounded in $Y^{1+\frac{1-\alpha}{2}}$ (the proof of this fact can be reproduced from the proof of Lemma 3.1) and converges backwards and forwards to the set of equilibria (due to gradient structure). Hence $\sup_{a_\alpha \in \mathcal{A}_\alpha} \mathcal{L}(a_\alpha) \leqslant \sup_{E\in\mathcal{E}} \mathcal{L}(E) =: \mathcal{L}(\mathcal{E})$, where $\mathcal{E}$ is the set of equilibria for (3.1). Recall that;

$$\mathcal{E} = \left\{ E \in X^1 \times X^{\frac{1}{2}} : E = \begin{bmatrix} e \\ 0 \end{bmatrix} \text{ with } e \text{ satisfying } -\Delta e = f(e), \quad \text{in } \Omega \text{ and } e = 0 \text{ in } \partial\Omega \right\}.$$

Consequently, $\mathcal{L}(\mathcal{E})$ is independent of α and computations similar to those done above imply the uniform bounds. □

Lemma 3.1. *Assume that f satisfies* (1.2) *with $\rho \in (1, \frac{N+2}{N-2})$, $\alpha \in (\alpha_0, 1)$ and* (1.3). *Then, the solutions of the problem* (2.3) *originating at a bounded subset B of $Y^{\frac{1+\alpha}{2}}$, becomes bounded in $Y^{1+\frac{1-\alpha}{2}}$.*

Proof: Let B be a bounded subset of $Y^{\frac{1+\alpha}{2}}$ and $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B$. Since $F : Y^{\frac{1+\alpha}{2}} \to Y^{\frac{3(1-\beta)}{2}}$, for some $\frac{1}{2} < \beta < \alpha$, is a Lipschitz map on bounded sets, there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned}
&\|T_\alpha(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \|_{Y^{1+\frac{1-\alpha}{2}}} \leqslant \|e^{-\Lambda^\alpha t} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \|_{Y^{1+\frac{1-\alpha}{2}}} + \int_0^t \left\| e^{-\Lambda^\alpha (t-s)} F\left(T_\alpha(s) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right) \right\|_{Y^{1+\frac{1-\alpha}{2}}} ds \\
&\leqslant \|e^{-\Lambda^\alpha t} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \|_{Y^{1+\frac{1-\alpha}{2}}} + \int_0^t \|e^{-\Lambda^\alpha (t-s)}\|_{L(Y^{\frac{3(1-\beta)}{2}}, Y^{1+\frac{1-\alpha}{2}})} \left\| F\left(T_\alpha(s) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right) \right\|_{Y^{\frac{3(1-\beta)}{2}}} ds \\
&\leqslant \|e^{-\Lambda^\alpha t}\|_{L(Y^{\frac{1+\alpha}{2}}, Y^{\frac{3-\alpha}{2}})} \left\| \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right\|_{Y^{\frac{1+\alpha}{2}}} + \int_0^t \|e^{-\Lambda^\alpha (t-s)}\|_{L(Y^{\frac{3(1-\beta)}{2}}, Y^{1+\frac{1-\alpha}{2}})} \left\| F\left(T_\alpha(s) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right) \right\|_{Y^{\frac{3(1-\beta)}{2}}} ds \\
&\leqslant C_1 t^{-(1-\alpha)} e^{-\delta t} + C_2 \int_0^t (t-s)^{\frac{\alpha-3\beta}{2}} e^{-\delta(t-s)} \left(\left\| T_\alpha(s) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right\|_{Y^{\frac{1+\alpha}{2}}}^\rho + 1 \right) ds
\end{aligned}$$

and the boundedness in $Y^{\frac{1+\alpha}{2}}$ (Theorem 3.1) now implies the result. □

Corollary 3.1. *Assume that f satisfies* (1.2) *with $\rho \in (1, \frac{N+2}{N-2})$, $\alpha \in (\alpha_0, 1)$ and* (1.3). *Then,* (2.3) *is globally well posed in $X^{\frac{1}{2}} \times X$ and has a global attractor which coincide with $\mathcal{A}_\alpha$.*

4. CONVERGENCE OF THE LINEAR SEMIGROUPS

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The main purpose of this section is to prove *Trotter-Kato type theorem* for the convergence of the linear semigroups. We start with a result of convergence of the resolvents.

PROPOSITION 4.1. *Let $\lambda \in \mathbb{C}$, with $Re\lambda \geqslant 0$. The family $\{(\lambda I + \Lambda^{\alpha})^{-1} : \alpha \in [0,1)\}$ converges in the uniform operator topology of $\mathcal{L}(X^{\frac{1}{2}} \times X)$ to $(\lambda I + \Lambda)^{-1}$, as α tends to* 1.

Proof: Note that to prove $(\lambda I + \Lambda^{\alpha})^{-1} \to (\lambda I + \Lambda)^{-1}$ in the uniform operator topology of $\mathcal{L}(X^{\frac{1}{2}} \times X)$ is equivalent to prove that

$$\| \sin\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} D_{\alpha}^{-1}(\lambda) - A^{\frac{1}{2}}(\lambda^2 I + A)^{-1}\|_{\mathcal{L}(X)} \to 0,$$

and

$$\|(\lambda I + \cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}}) D_{\alpha}^{-1}(\lambda) - \lambda(\lambda^2 I + A)^{-1}\|_{\mathcal{L}(X)} \to 0,$$

as $\alpha \to 1$, where $A = -\Delta_D$ with zero Dirichlet boundary conditions in $L^2(\Omega)$.

Proof of first part: Observe that, for $\lambda \in \rho(A) \cap \rho(B)$, we can write

$$(\lambda I - A)^{-1} - (\lambda I - B)^{-1} = A(\lambda I - A)^{-1}(B^{-1} - A^{-1})B(\lambda I - B)^{-1}. \tag{4.1}$$

Furthermore, recall that

$$D_{\alpha}(\lambda) = \lambda^2 I + 2\lambda \cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} + A^{\alpha} = (\lambda - e^{i\frac{\pi(2-\alpha)}{2}} A^{\frac{\alpha}{2}})(\lambda - e^{-i\frac{\pi(2-\alpha)}{2}} A^{\frac{\alpha}{2}}),$$

for $\alpha \in (0,1)$. Note that,

$$e^{\pm i\frac{\pi(2-\alpha)}{2}} \to \pm i, \quad \text{as} \quad \alpha \to 1.$$

Additionally,

$$(\lambda^2 + A) = (\lambda - iA^{\frac{1}{2}})(\lambda + iA^{\frac{1}{2}}).$$

For $\alpha \in (0,1]$, we have $\lambda e^{\pm i\frac{\pi(2-\alpha)}{2}} \in \rho(A^{\frac{\alpha}{2}})$ whenever $\mathrm{Re}\lambda \geqslant 0$ (since $\sigma(A^{\frac{\alpha}{2}})$ consists of positive eigenvalues bounded away from zero). Remember that, since that A is positive definite and self-adjoint, then so is $A^{\frac{\alpha}{2}}$, for all $\alpha > 0$ (see [12, pg. 27]).

Thus, for each $\mathrm{Re}\lambda \geqslant 0$ we get

$$D_{\alpha}^{-1}(\lambda) = (\lambda - e^{i\frac{\pi(2-\alpha)}{2}} A^{\frac{\alpha}{2}})^{-1}(\lambda - e^{-i\frac{\pi(2-\alpha)}{2}} A^{\frac{\alpha}{2}})^{-1},$$

$$(\lambda^2 + A)^{-1} = (\lambda - iA^{\frac{1}{2}})^{-1}(\lambda + iA^{\frac{1}{2}})^{-1},$$

and $\frac{\pi}{2} \leqslant \frac{\pi(2-\alpha)}{2} = |\arg(\lambda e^{\pm i\pi\frac{(2-\alpha)}{2}})| \leqslant \pi$.

From this, it follows that

$$\begin{aligned}
&\sin\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} D_{\alpha}^{-1}(\lambda) - A^{\frac{1}{2}}(\lambda^2 + A)^{-1} \\
&\quad = \sin\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}}(\lambda - e^{i\frac{\pi(2-\alpha)}{2}} A^{\frac{\alpha}{2}})^{-1}(\lambda - e^{-i\frac{\pi(2-\alpha)}{2}} A^{\frac{\alpha}{2}})^{-1} - A^{\frac{1}{2}}(\lambda - iA^{\frac{1}{2}})^{-1}(\lambda + iA^{\frac{1}{2}})^{-1}
\end{aligned}$$

To conclude, we only need to show that

$$(\lambda - e^{-i\frac{\pi(2-\alpha)}{2}}A^{\frac{\alpha}{2}})^{-1} - (\lambda + iA^{\frac{1}{2}})^{-1} \to 0, \tag{4.2}$$

and

$$(\lambda - e^{i\frac{\pi(2-\alpha)}{2}}A^{\frac{\alpha}{2}})^{-1} - A^{\frac{1}{2}}(\lambda - iA^{\frac{1}{2}})^{-1} \to 0, \tag{4.3}$$

as $\alpha \to 1$.

In fact, from (4.1) we have

$$\begin{aligned}&(\lambda - e^{-i\frac{\pi(2-\alpha)}{2}}A^{\frac{\alpha}{2}})^{-1} - (\lambda + iA^{\frac{1}{2}})^{-1}\\ &\quad = A^{\frac{\alpha}{2}}(\lambda - e^{-i\frac{\pi(2-\alpha)}{2}}A^{\frac{\alpha}{2}})^{-1}[-e^{-i\frac{\pi(2-\alpha)}{2}}A^{-\frac{1}{2}} - iA^{-\frac{\alpha}{2}}]A^{\frac{1}{2}}(\lambda + iA^{\frac{1}{2}})^{-1}\end{aligned}$$

and

$$\begin{aligned}&(\lambda - e^{i\frac{\pi(2-\alpha)}{2}}A^{\frac{\alpha}{2}})^{-1} - (\lambda - iA^{\frac{1}{2}})^{-1}\\ &\quad = A^{\frac{\alpha}{2}}(\lambda - e^{-i\frac{\pi(2-\alpha)}{2}}A^{\frac{\alpha}{2}})^{-1}[e^{i\frac{\pi(2-\alpha)}{2}}A^{-\frac{1}{2}} - iA^{-\frac{\alpha}{2}}]A^{\frac{1}{2}}(\lambda - iA^{\frac{1}{2}})^{-1}.\end{aligned}$$

Observe that we can write

$$-e^{-i\frac{\pi(2-\alpha)}{2}}A^{-\frac{1}{2}} - iA^{-\frac{\alpha}{2}} = e^{-i\frac{\pi(2-\alpha)}{2}}[A^{-\frac{\alpha}{2}} - A^{-\frac{1}{2}}] + [-e^{-i\frac{\pi(2-\alpha)}{2}} - i]A^{-\frac{\alpha}{2}}$$

and

$$[e^{i\frac{\pi(2-\alpha)}{2}}A^{-\frac{1}{2}} - iA^{-\frac{\alpha}{2}}] = e^{i\frac{\pi(2-\alpha)}{2}}[A^{-\frac{1}{2}} - A^{-\frac{\alpha}{2}}] + [e^{i\frac{\pi(2-\alpha)}{2}} - i]A^{-\frac{\alpha}{2}}.$$

Then, (4.2)-(4.3) follows from above and from the convergence of $A^{-\frac{\alpha}{2}}$ to $A^{-\frac{1}{2}}$ in $\mathcal{L}(X)$ (see, for example, Theorem III.4.6.2 in [1]).

Proof of second part: Note that we can write

$$\begin{aligned}&(\lambda I + \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}})D_\alpha^{-1}(\lambda) - \lambda(\lambda^2 + A)^{-1}\\ &\quad = (\lambda I + \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}})(\lambda - e^{i\frac{\pi(2-\alpha)}{2}}A^{\frac{\alpha}{2}})(\lambda - e^{-i\frac{\pi(2-\alpha)}{2}}A^{\frac{\alpha}{2}}) - \lambda(\lambda - iA^{\frac{1}{2}})^{-1}(\lambda + iA^{\frac{1}{2}})^{-1}\end{aligned}$$

Using the same idea of the first part, is not difficult to show that

$$(\lambda I + \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}})D_\alpha^{-1}(\lambda) - \lambda(\lambda^2 + A)^{-1} \to 0, \quad \alpha \to 1.$$

□

Remark 4. 1. For each $\alpha > 0$, Λ^α is closed and

$$\begin{aligned}\langle -\Lambda^\alpha \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \rangle_{X^{\frac{1}{2}} \times X} &= \langle \begin{bmatrix} -\cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} u + \sin\frac{\pi\alpha}{2} A^{\frac{-1+\alpha}{2}} v \\ -\sin\frac{\pi\alpha}{2} A^{\frac{1+\alpha}{2}} u - \cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \rangle_{X^{\frac{1}{2}} \times X} \\ &= -\cos\frac{\pi\alpha}{2} \langle A^{\frac{1+\alpha}{2}} u, A^{\frac{1}{2}} u \rangle_X + \sin\frac{\pi\alpha}{2} \langle A^{\frac{\alpha}{2}} v, A^{\frac{1}{2}} u \rangle_X \\ &\quad - \sin\frac{\pi\alpha}{2} \langle A^{\frac{1+\alpha}{2}} u, v \rangle_X - \cos\frac{\pi\alpha}{2} \langle A^{\frac{\alpha}{2}} v, v \rangle_X \\ &= -\cos\frac{\pi\alpha}{2} \langle A^{\frac{2+\alpha}{4}} u, A^{\frac{2+\alpha}{4}} u \rangle_X + \sin\frac{\pi\alpha}{2} \langle A^{\frac{\alpha}{2}} v, A^{\frac{1}{2}} u \rangle_X \\ &\quad - \sin\frac{\pi\alpha}{2} \overline{\langle A^{\frac{\alpha}{2}} v, A^{\frac{1}{2}} u \rangle}_X - \cos\frac{\pi\alpha}{2} \langle A^{\frac{\alpha}{4}} v, A^{\frac{\alpha}{4}} v \rangle_X \end{aligned}$$

Hence, for $\begin{bmatrix} u \\ v \end{bmatrix} \in D(\Lambda^\alpha)$ and $\alpha \in [0,1]$,

$$\mathrm{Re} \langle -\Lambda^\alpha \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \rangle_{X^{\frac{1}{2}} \times X} \leqslant -\cos\frac{\pi\alpha}{2} \langle A^{\frac{2+\alpha}{4}} u, A^{\frac{2+\alpha}{4}} u \rangle_X - \cos\frac{\pi\alpha}{2} \langle A^{\frac{\alpha}{4}} v, A^{\frac{\alpha}{4}} v \rangle_X \leqslant 0,$$

showing that $-\Lambda^\alpha$ is dissipative. Noting that the equation

$$(I + \Lambda^\alpha) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix}$$

has, for each $\begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} \in Y^0$, a unique solution

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (1 + \cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}}) D_\alpha^{-1}(1) \bar{u} - \sin\frac{\pi\alpha}{2} A^{\frac{-1+\alpha}{2}} D_\alpha^{-1}(1) \bar{v} \\ \sin\frac{\pi\alpha}{2} A^{\frac{1+\alpha}{2}} D_\alpha^{-1}(1) \bar{u} + (1 + \cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}}) D_\alpha^{-1}(1) \bar{v} \end{bmatrix} \in D(\Lambda^\alpha),$$

we have, from Lumer-Phillips theorem that Λ^α is the infinitesimal generator of C_0-semigroups of contractions in Y^0.

If we denote by $\{e^{-\Lambda^\alpha t};\ t \geqslant 0\}$ the linear semigroup generated by $-\Lambda^\alpha$, $0 \leqslant \alpha \leqslant 1$, we have the following result ensuring the convergence (uniform in compact subsets of Y^1 and in bounded intervals of $\mathbb{R}^+$) of $(t, U_0) \mapsto e^{-\Lambda^\alpha t} U_0$ to $(t, U_0) \mapsto e^{-\Lambda t} U_0$ (see also, [6]).

Theorem 4.1. *Let J be a compact subset in Y^1. Then, we have*

$$\sup_{w \in J} \sup_{t \in [0,T]} \| e^{-\Lambda^\alpha t} w - e^{-\Lambda t} w \|_{Y^1} \to 0, \text{ as } \alpha \to 1,$$

where $T > 0$.

Proof: Given $\epsilon > 0$ and $T > 0$, choosing $\delta = \epsilon/4$, $n \in \mathbb{N}$ and $\{w_1, w_2, \dots, w_n\}$ in J, such that $J \subset \cup_{i=1}^n B_\delta^{Y^1}(w_i)$, where $B_\delta^{Y^1}(w_i)$ denote the ball of radius δ centered in w_i in Y^1, $i = 1, \dots, n$. From Trotter-Kato theorem (see Pazy [17]), there exists $\alpha_\epsilon = \alpha(\epsilon) \in (0,1)$ such that

$$\|e^{-\Lambda^\alpha t} w_i - e^{-\Lambda t} w_i\|_{Y^1} < \frac{\epsilon}{2}, \text{ for all } t \in [0,T],\ \alpha \in (\alpha_\epsilon, 1),\ i = 1, \dots, n.$$

Now, if $w \in J$ and $i \in \{1, 2, \dots, n\}$ is such that $w \in B_\delta^{Y^1}(w_i)$, we get

$$\begin{aligned}\sup_{t\in[0,T]} \|e^{-\Lambda^\alpha t} w - e^{-\Lambda t} w\|_{Y^1} &\leqslant \sup_{t\in[0,T]} \|e^{-\Lambda^\alpha t}(w - w_i)\|_{Y^1} + \sup_{t\in[0,T]} \|(e^{-\Lambda^\alpha t} - e^{-\Lambda t}) w_i\|_{Y^1} \\ &\quad + \sup_{t\in[0,T]} \|e^{-\Lambda t}(w - w_i)\|_{Y^1} \\ &< 2\delta + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

□

5. ASYMPTOTIC DYNAMICS

In this section, we will show that, for f satisfying (1.2) with $\rho \in (1, \frac{N+2}{N-2})$, the solutions of (2.3) converge (as α tends to 1) to global weak solutions of (1.4) (along subsequences and in a sense to be specified). This procedure is similar to that of [9, Theorem 1.2].

Before we prove a result that we guarantee a Hölder-type condition on the function f (see [9, Lemma 2.2] for an analogous result).

Theorem 5.1. *Assume that f satisfies* (1.2) *with $\rho \in (1, \frac{N+2}{N-2})$, $\alpha \in (\alpha_0, 1)$ (α_0 defined in Theorem 3.1). Then, there is a constant $C > 0$ such that, for $\theta = \frac{(\rho-1)(N-1-\alpha)}{4\alpha}$ and for all u_1, $u_2 \in X^{\frac{1+\alpha}{4}}$, we have*

$$\|f(u_1) - f(u_2)\|_{X^{\frac{1-3\alpha}{4}}} \leqslant C \|u_1 - u_2\|^{1-\theta}_{X^{\frac{1-3\alpha}{4}}} \|u_1 - u_2\|^{\theta}_{X^{\frac{1+\alpha}{4}}} \left(1 + \|u_1\|^{\rho-1}_{X^{\frac{1+\alpha}{4}}} + \|u_2\|^{\rho-1}_{X^{\frac{1+\alpha}{4}}}\right)$$

Proof: First, observe that

$$X^{\frac{1+\alpha}{4}} \hookrightarrow L^{\frac{2N}{N-1-\alpha}}(\Omega),$$

and then

$$L^{\frac{2N}{N-1+3\alpha}}(\Omega) \hookrightarrow X^{\frac{1-3\alpha}{4}}.$$

From this, using (1.2) and Hölder inequality, we get

$$\begin{aligned}\|f(u_1)-f(u_2)\|_{X^{\frac{1-3\alpha}{4}}} &\leqslant C\|f(u_1)-f(u_2)\|_{L^{\frac{2N}{N-1+3\alpha}}(\Omega)}\\ &\leqslant C\|u_1-u_2\|_{L^{\frac{2Nr}{N-1+3\alpha}}(\Omega)}\Big(1+\|u_1\|^{\rho-1}_{L^{\frac{2Nr^*(\rho-1)}{N-1+3\alpha}}(\Omega)}+\|u_2\|^{\rho-1}_{L^{\frac{2Nr^*(\rho-1)}{N-1+3\alpha}}(\Omega)}\Big)\\ &\leqslant C\|u_1-u_2\|_{X^{\frac{s}{2}}}\Big(1+\|u_1\|^{\rho-1}_{X^{\frac{1+\alpha}{4}}}+\|u_2\|^{\rho-1}_{X^{\frac{1+\alpha}{4}}}\Big),\end{aligned}$$

where $r=\frac{N-1+3\alpha}{(N-1+3\alpha)-(\rho-1)(N-1-\alpha)}$, $\frac{1}{r}+\frac{1}{r^*}=1$ and $s=\frac{(\rho-1)(N-1-\alpha)-(3\alpha-1)}{2}$. Now, since $\rho\in(1,\frac{N+(3\alpha-1)}{N-(\alpha+1)})$, it follows that

$$\frac{1-3\alpha}{2}<\frac{(\rho-1)(N-\alpha-1)-(3\alpha-1)}{2}<\frac{1+\alpha}{2}.$$

Thus, from the moment inequality we obtain that

$$\|u_1-u_2\|_{X^{\frac{s}{2}}}\leqslant C(\alpha)\|u_1-u_2\|^{\frac{2s-1+3\alpha}{4\alpha}}_{X^{\frac{1+\alpha}{4}}}\|u_1-u_2\|^{\frac{1+\alpha-2s}{4\alpha}}_{X^{\frac{1-3\alpha}{4}}}.$$

This completes the proof. □

Denote $\tilde{X}^{\frac{1}{2}}$ and $\tilde{X}$ respectively by $\bigcap_{\alpha\in[\frac{1}{3},1)}X^{\frac{1+\alpha}{4}}$ and $\overline{\bigcup_{\alpha\in[\frac{1}{3},1)}X^{\frac{1-\alpha}{4}}}^{X}$. Now define the space

$$\tilde{Y}^1:=\Big\{\begin{bmatrix}u\\v\end{bmatrix}\in\tilde{X}^{\frac{1}{2}}\times\tilde{X}:\sup_{\alpha\in[\frac{1}{3},1)}\|u\|_{X^{\frac{1+\alpha}{4}}}<\infty\Big\}\tag{5.1}$$

endowed with the norm

$$\|\begin{bmatrix}u\\v\end{bmatrix}\|_{\tilde{Y}^1}=\sup_{\alpha\in[\frac{1}{3},1)}\|u\|_{X^{\frac{1+\alpha}{4}}}+\lim_{\alpha\to1}\|v\|_{X^{\frac{1-\alpha}{4}}}.\tag{5.2}$$

Lemma 5.1. *Let $\alpha\in(\frac{1}{3},1)$. The norms $\lim_{\alpha\to1}\|u\|_{X^{\frac{1+\alpha}{4}}}$ and $\sup_{\alpha\in[\frac{1}{3},1)}\|u\|_{X^{\frac{1+\alpha}{4}}}$ in $\tilde{X}^{\frac{1}{2}}$ are equivalents.*

Proof: First observe that if $u\in X^{\frac{1+\alpha}{4}}$ then

$$\|u\|_{X^{\frac{1+\alpha}{4}}}\leqslant\sup_{\alpha\in[\frac{1}{3},1)}\|u\|_{X^{\frac{1+\alpha}{4}}}.$$

Then

$$\lim_{\alpha\to1}\|u\|_{X^{\frac{1+\alpha}{4}}}\leqslant\sup_{\alpha\in[\frac{1}{3},1)}\|u\|_{X^{\frac{1+\alpha}{4}}}.$$

Furthermore, fix β such that $\frac{1}{3} \leqslant \beta < 1$. Then for all $\beta < \alpha < 1$ we have that $X^{\frac{1+\alpha}{4}} \hookrightarrow X^{\frac{1+\beta}{4}}$. If $u \in X^{\frac{1+\alpha}{4}}$ there exists a constant $c > 0$ (independent of α) such that

$$c\|u\|_{X^{\frac{1+\beta}{4}}} \leqslant \|u\|_{X^{\frac{1+\alpha}{4}}}.$$

This we obtain that

$$c\|u\|_{X^{\frac{1+\beta}{4}}} \leqslant \lim_{\alpha\to 1} \|u\|_{X^{\frac{1+\alpha}{4}}}$$

Now taking the supremum in β follows that

$$c \sup_{\beta\in[\frac{1}{3},1)} \|u\|_{X^{\frac{1+\beta}{4}}} \leqslant \lim_{\alpha\to 1} \|u\|_{X^{\frac{1+\alpha}{4}}}.$$

∎

THEOREM 5.2. *Let $\alpha \in [\frac{1}{3}, 1)$. Then the space $\tilde{X}^{\frac{1}{2}}$ endowed with the norm $\lim_{\alpha\to 1} \|u\|_{X^{\frac{1+\alpha}{4}}}$ coincide with $X^{\frac{1}{2}}$.*

Proof: Let $u \in X^{\frac{1}{2}}$ then $u \in \tilde{X}^{\frac{1}{2}}$ and

$$\|u\|_{\tilde{X}^{\frac{1}{2}}} = \lim_{\alpha\to 1} \|u\|_{X^{\frac{1+\alpha}{4}}} = \lim_{\alpha\to 1} \|A^{\frac{1+\alpha}{4}}u\|_X = \lim_{\alpha\to 1} \|A^{\frac{\alpha-1}{4}}A^{\frac{1}{2}}u\|_X = \|A^{\frac{1}{2}}u\|_X = \|u\|_{X^{\frac{1}{2}}}.$$

By the other side, if $u \in \tilde{X}^{\frac{1}{2}}$ then $u \in X^{\frac{1+\alpha}{4}}$ and $A^{\frac{-1+\alpha}{4}}u \in X^{\frac{1}{2}}$ for all $\alpha \in [\frac{1}{3}, 1)$. Writing $u_\alpha := A^{\frac{-1+\alpha}{4}}u$ for all $\alpha \in [\frac{1}{3}, 1)$ and using the fact that $\lim_{\alpha\to 1} \|A^{\frac{1+\alpha}{4}}u\|_X = \|u\|_X$ we have

$$\|u_\alpha - u_{\alpha'}\|_{X^{\frac{1}{2}}} = \|A^{\frac{-1+\alpha}{4}}u - A^{\frac{-1+\alpha'}{4}}u\|_{X^{\frac{1}{2}}} = \|A^{\frac{1+\alpha}{4}}u - A^{\frac{1+\alpha'}{4}}u\|_X \overset{\alpha,\alpha'\to 1^-}{\longrightarrow} 0,$$

and by reflexivity of the space $\tilde{X}^{\frac{1}{2}}$ we get

$$\|u_\alpha - u\|_{X^{\frac{1}{2}}} \overset{\alpha\to 1^-}{\longrightarrow} 0$$

and this means that $u \in X^{\frac{1}{2}}$. ∎

THEOREM 5.3. *Let $\alpha \in [\frac{1}{3}, 1)$. Then the space $\tilde{X}$ endowed with the norm $\lim_{\alpha\to 1} \|u\|_{X^{\frac{1-\alpha}{4}}}$ coincide with X.*

Proof: Observe that if $u \in \cup_{\alpha\in(\frac{1}{3},1)} X^{\frac{1-\alpha}{4}}$ then $u \in X^{\frac{1-\alpha}{4}}$ for some $\alpha \in (\frac{1}{3}, 1)$. This implies that $u \in X$. In particular, $u \in X^{\frac{1-\beta}{4}}$ for some $\beta > \alpha$. From this, we have

$$\|u\|_{\tilde{X}} = \lim_{\alpha\to 1} \|u\|_{X^{\frac{1-\alpha}{4}}} = \lim_{\alpha\to 1} \|A^{\frac{\beta-\alpha}{4}}A^{\frac{1-\beta}{4}}u\|_X = \|u\|_X.$$

Whereas if $u = \lim_{n\to+\infty} v_{\alpha_n}$ in X where $v_{\alpha_n} \in X^{\frac{1-\alpha}{4}}$ for some $\alpha \in (\frac{1}{3}, 1)$. Then

$$\|u\|_{\tilde{X}} = \lim_{n\to+\infty} \|v_{\alpha_n}\|_{\tilde{X}} = \lim_{n\to+\infty} \lim_{\alpha\to 1} \|v_{\alpha_n}\|_{X^{\frac{1-\alpha}{4}}} = \lim_{n\to+\infty} \|v_{\alpha_n}\|_X = \|u\|_X.$$

Furthermore, if $x \in X$ observe that

$$\|u\|_X = \lim_{\alpha \to 1^-} \|A^{\frac{\alpha-1}{4}} u\|_X.$$

Note that $v_\alpha := A^{\frac{\alpha-1}{4}} u$ belongs to $X^{\frac{1-\alpha}{4}}$ and

$$\|u - v_\alpha\|_X \stackrel{\alpha\to 1^-}{\longrightarrow} 0,$$

we obtain that $u \in \tilde{X}$. ∎

Remark 5. 1. From Theorems 5.2 and 5.3 we conclude that the space $\tilde{Y}^1$, defined in (5.1) endowed with the norm defined in (5.2) coincides with the space $X^{\frac{1}{2}} \times X$ endowed with the usual norm. Thus we obtain in Theorem 5.5 a result of existence of global solutions for the problem (1.4) in the 'natural' space of waves $X^{\frac{1}{2}} \times X$. But we continue using the notation $\tilde{Y}^1$ endowed with the norm defined in (5.2).

Definition 5.1. A bounded function $\begin{bmatrix} u \\ v \end{bmatrix}\left(\cdot, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right) : [0,\infty) \to \tilde{Y}^1$ is a global weak solution of the problem (1.4) if $\begin{bmatrix} u \\ v \end{bmatrix}\left(\cdot, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right) : [0,\infty) \to Y$ and $F\left(\begin{bmatrix} u \\ v \end{bmatrix}\left(\cdot, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right)\right) :$ $[0,\infty) \to Y$ is continuous, for all $\alpha \in (\frac{1}{3}, 1)$, and

$$\begin{bmatrix} u \\ v \end{bmatrix}\left(t, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right) = e^{-\Lambda t}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + \int_0^t e^{-\Lambda(t-s)} F\left(\begin{bmatrix} u \\ v \end{bmatrix}\left(s, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right)\right) ds, \quad \text{for all } t \geqslant 0. \quad (5.3)$$

Theorem 5.4 (Arzelá-Ascoli). *Suppose that S is a separable metric space and that M is a metric space. Let Ξ be an equicontinuous family in $C(S;M)$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in Ξ such that for each $s \in S$ the closure of $\{\varphi_n(s);\ n \in \mathbb{N}\}$ is a compact subset of M. Then there is a subsequence $\{\varphi_{n_k}\}_{k\in\mathbb{N}}$ and $\varphi \in C(S;M)$ such that $\varphi_{n_k} \stackrel{k\to\infty}{\longrightarrow} \varphi$ uniformly in compact subsets of S.*

See [16, Section 7.10] for a proof.

To describe a class of global weak solutions of the problem (2.3) with $\alpha = 1$ which are obtained as limits of the solutions of (2.3) with $\alpha = \alpha_n \to 1$ we will prove the following theorem.

Theorem 5.5. *Assume that f satisfies (1.2) with $\rho \in (1, \frac{N+2}{N-2})$ and (1.3). Hence, if $\alpha_n \to 1^-$, $\left\{\begin{bmatrix} u_0^n \\ v_0^n \end{bmatrix}\right\}_{n\in\mathbb{N}}$ is a sequence in $\tilde{Y}^1$, $\left\{\begin{bmatrix} u_0^n \\ v_0^n \end{bmatrix}\right\}_{n\in\mathbb{N}}$ is weakly (in $\tilde{Y}^1$) convergent to $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \tilde{Y}^1$ and $\begin{bmatrix} u \\ v \end{bmatrix}\left(\cdot, \begin{bmatrix} u_0^n \\ v_0^n \end{bmatrix}, \alpha_n\right)$ denotes the mild solution of (2.3) with $\alpha = \alpha_n$,* *then there is a subsequence* $\left\{\begin{bmatrix} u \\ v \end{bmatrix}\left(\cdot, \begin{bmatrix} u_0^{n_k} \\ v_0^{n_k} \end{bmatrix}, \alpha_{n_k}\right)\right\}_{k\in\mathbb{N}}$ *of* $\left\{\begin{bmatrix} u \\ v \end{bmatrix}\left(\cdot, \begin{bmatrix} u_0^{n} \\ v_0^{n} \end{bmatrix}, \alpha_{n}\right)\right\}_{n\in\mathbb{N}}$ *and a bounded function* $\begin{bmatrix} \varphi \\ \psi \end{bmatrix} : [0,\infty) \to \tilde{Y}^1$ *which is a global weak solution of the problem* (1.4)*, such that for all* $T > 0$*,*

$$\sup_{t\in[0,T]} \left\| \begin{bmatrix} u \\ v \end{bmatrix}\left(t, \begin{bmatrix} u_0^{n_k} \\ v_0^{n_k} \end{bmatrix}, \alpha_{n_k}\right) - \begin{bmatrix} \varphi \\ \psi \end{bmatrix}(t) \right\|_Y \to 0,$$

and

$$\begin{bmatrix} u \\ v \end{bmatrix}\left(t, \begin{bmatrix} u_0^{n_k} \\ v_0^{n_k} \end{bmatrix}, \alpha_{n_k}\right) \overset{w-\tilde{Y}^1}{\longrightarrow} \begin{bmatrix} \varphi \\ \psi \end{bmatrix}(t), \text{ for each } t \in [0,T].$$

Proof: Let $\begin{bmatrix} u_0^{n} \\ v_0^{n} \end{bmatrix} \in B \subset \tilde{Y}^1$, where B is a bounded set. Observe that, the sequence

$$\begin{bmatrix} u \\ v \end{bmatrix}\left(\cdot, \begin{bmatrix} u_0^{n} \\ v_0^{n} \end{bmatrix}, \alpha_{n}\right), \ n \in \mathbb{N},$$

is relatively compact in $C([0,T],Y)$, for all $T > 0$. In fact, since

$$\mathcal{L}\Big(\begin{bmatrix} u \\ v \end{bmatrix}\left(t, \begin{bmatrix} u_0^{n} \\ v_0^{n} \end{bmatrix}, \alpha_{n}\right) \Big) \leqslant \mathcal{L}\Big(\begin{bmatrix} u_0^{n} \\ v_0^{n} \end{bmatrix} \Big),$$

and, proceeding as in the proof of Theorem 3.1, given $\alpha_n \in (\alpha_0, 1)$ and $t \geqslant 0$ there is a constant $C_B = C(B) > 0$ such that

$$\sup_{n\in\mathbb{N}} \sup_{t\geqslant 0} \left\| \begin{bmatrix} u \\ v \end{bmatrix}\left(t, \begin{bmatrix} u_0^{n} \\ v_0^{n} \end{bmatrix}, \alpha_{n}\right) \right\|_{X^{\frac{1+\alpha_n}{4}} \times X^{\frac{1-\alpha_n}{4}}} \leqslant C_B. \tag{5.4}$$

In addition, from this and from (3.1) there exists $\bar{C}_B > 0$ such that, for $t \geqslant 0$, $\alpha_n \in (\alpha_0, 1)$ and $\begin{bmatrix} u_0^{n} \\ v_0^{n} \end{bmatrix} \in B$,

$$\sup_{n\in\mathbb{N}} \sup_{t\geqslant 0} \left\| \begin{bmatrix} u_t \\ v_t \end{bmatrix}\left(t, \begin{bmatrix} u_0^{n} \\ v_0^{n} \end{bmatrix}, \alpha_{n}\right) \right\|_{Y} \leqslant \bar{C}_B. \tag{5.5}$$

In fact, for $\begin{bmatrix} u_0^{n} \\ v_0^{n} \end{bmatrix} \in B \subset \tilde{Y}^1$ and let $\begin{bmatrix} u \\ v \end{bmatrix}(\cdot, \begin{bmatrix} u_0^{n} \\ v_0^{n} \end{bmatrix}, \alpha)$ be the mild solution of the problem with initial condition $\begin{bmatrix} u_0^{n} \\ v_0^{n} \end{bmatrix}$. Then

$$\begin{cases} u_t = -\cos\frac{\pi\alpha_n}{2} A^{\frac{\alpha_n}{2}} u + \sin\frac{\pi\alpha_n}{2} A^{\frac{-1+\alpha_n}{2}} v, \\ v_t = -\sin\frac{\pi\alpha_n}{2} A^{\frac{1+\alpha_n}{2}} u - \cos\frac{\pi\alpha_n}{2} A^{\frac{\alpha_n}{2}} v - av + f(u). \end{cases}$$

Observe that

$$\| \begin{bmatrix} u_t \\ v_t \end{bmatrix} (t, \begin{bmatrix} u_0^n \\ v_0^n \end{bmatrix}, \eta)\|_Y = \|u_t\|_X + \|v_t\|_{X^{-\frac{1}{2}}},$$

and from (1.2), there exists a positive constant c such that

$$|f(s)| \leqslant c(1+|s|^\rho), \quad \forall s \in \mathbb{R}.$$

Consequently, we have that

$$\begin{aligned} \|u_t\|_X &\leqslant \| \cos \frac{\pi\alpha_n}{2} A^{\frac{\alpha_n}{2}} u\|_X + \| \sin \frac{\pi\alpha_n}{2} A^{\frac{-1+\alpha_n}{2}} v\|_X \\ &\leqslant \|u\|_{X^{\frac{\alpha_n}{2}}} + \|v\|_{X^{\frac{-1+\alpha_n}{2}}} \leqslant c_1 \|u\|_{X^{\frac{1+\alpha_n}{4}}} + c_2 \|v\|_{X^{\frac{-1+\alpha_n}{4}}}, \end{aligned}$$

and

$$\begin{aligned} \|v_t\|_{X^{-\frac{1}{2}}} &\leqslant \| \sin \frac{\pi\alpha_n}{2} A^{\frac{1+\alpha_n}{2}} u\|_{X^{-\frac{1}{2}}} + \| \cos \frac{\pi\alpha_n}{2} A^{\frac{\alpha_n}{2}} v\|_{X^{-\frac{1}{2}}} + \|av\|_{X^{-\frac{1}{2}}} + \|f(u)\|_{X^{-\frac{1}{2}}} \\ &\leqslant \|u\|_{X^{\frac{\alpha_n}{2}}} + \|v\|_{X^{\frac{-1+\alpha_n}{2}}} + a\|v\|_{X^{-\frac{1}{2}}} + c(1+\|u\|^\rho_{X^{\frac{1+\alpha_n}{4}}}) \\ &\leqslant c_2(\|u\|_{X^{\frac{1+\alpha_n}{4}}} + \|u\|^\rho_{X^{\frac{1+\alpha_n}{4}}}) + c_3\|v\|_{X^{\frac{-1+\alpha_n}{4}}}. \end{aligned}$$

Completing the proof of (5.5).

From (5.4) and (5.5), we obtain that family $\left\{ \begin{bmatrix} u \\ v \end{bmatrix} \left(\cdot, \begin{bmatrix} u_0^n \\ v_0^n \end{bmatrix}, \alpha_n \right) \right\}$ is equicontinuous on $C([0,\infty), Y)$. It follows from Arzelá-Ascoli Theorem (Theorem 5.4) that, for each sequence $\{\alpha_n\}$ convergent to 1 and $\left\{ \begin{bmatrix} u_0^n \\ v_0^n \end{bmatrix} \right\}_{n\in\mathbb{N}} \subset B$ convergent in Y to $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$, there is a subsequence $\{\alpha_{n_k}\}$ and a function $\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in C([0,\infty), Y)$ such that

$$\sup_{t\in[0,T]} \left\| \begin{bmatrix} u \\ v \end{bmatrix} \left(t, \begin{bmatrix} u_0^{n_k} \\ v_0^{n_k} \end{bmatrix} \right) - \begin{bmatrix} \varphi \\ \psi \end{bmatrix} (t) \right\|_Y \to 0, \text{ for each } T > 0. \tag{5.6}$$

Furthermore, as a consequence of (5.4), for each $\beta \in (\alpha_0, 1)$,

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,\infty)} \left\| \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix} \right\|_{Y^{\frac{1+\beta}{2}}} \leqslant C. \tag{5.7}$$

Given $\beta \in (0,1)$, (5.4), (5.6), (5.7) and the momentum inequality imply that

$$\sup_{t\in[0,T]} \left\| \begin{bmatrix} u \\ v \end{bmatrix} \left(t, \begin{bmatrix} u_0^{n_k} \\ v_0^{n_k} \end{bmatrix} \right) - \begin{bmatrix} \varphi \\ \psi \end{bmatrix} (t) \right\|_{Y^\beta} \stackrel{k\to\infty}{\longrightarrow} 0, \text{ for each } T > 0.$$

It remains to prove that $\begin{bmatrix}\varphi\\ \psi\end{bmatrix}$ satisfies

$$\begin{bmatrix}\varphi\\ \psi\end{bmatrix}(t)=e^{-\Lambda t}\begin{bmatrix}\varphi(0)\\ \psi(0)\end{bmatrix}+\int_0^t e^{-\Lambda(t-s)}F\left(\begin{bmatrix}\varphi\\ \psi\end{bmatrix}(s)\right)ds,\qquad \text{for all}\ \ t\geqslant 0. \tag{5.8}$$

Remember that

$$\begin{bmatrix}u\\ v\end{bmatrix}\left(t,\begin{bmatrix}u_0^n\\ v_0^n\end{bmatrix},\alpha_n\right)=e^{-\Lambda^{\alpha_n}t}\begin{bmatrix}u_0^n\\ v_0^n\end{bmatrix}+\int_0^t e^{-\Lambda^{\alpha_n}(t-s)}F\left(\begin{bmatrix}u\\ v\end{bmatrix}\left(s,\begin{bmatrix}u_0^n\\ v_0^n\end{bmatrix},\alpha_n\right)\right)ds.$$

From Theorem 4.1 we only have to prove that

$$\int_0^t e^{-\Lambda^{\alpha_{n_k}}(t-s)}F\left(\begin{bmatrix}u\\ v\end{bmatrix}\left(s,\begin{bmatrix}u_0^{n_k}\\ v_0^{n_k}\end{bmatrix},\alpha_{n_k}\right)\right)ds\overset{k\to\infty}{\longrightarrow}\int_0^t e^{-\Lambda(t-s)}F\left(\begin{bmatrix}\varphi(s)\\ \psi(s)\end{bmatrix}\right)ds,$$

in Y, uniformly for $t\in[0,T]$. Using again Theorem 4.1, it is sufficient that

$$F\left(\begin{bmatrix}u\\ v\end{bmatrix}\left(s,\begin{bmatrix}u_0^{n_k}\\ v_0^{n_k}\end{bmatrix},\alpha_{n_k}\right)\right)\overset{k\to\infty}{\longrightarrow}F\left(\begin{bmatrix}\varphi(s)\\ \psi(s)\end{bmatrix}\right),$$

in Y, uniformly for $s\in[0,T]$. This is obtained in the following manner. If $r>0$ is such that

$$\sup_{n\in\mathbb{N}}\sup_{s\in[0,T]}\max\left\{\left\|\begin{bmatrix}u\\ v\end{bmatrix}(s,\begin{bmatrix}u_0^n\\ v_0^n\end{bmatrix},\alpha_n)\right\|_{Y^{\frac{1+\beta}{2}}},\left\|\begin{bmatrix}\varphi(s)\\ \psi(s)\end{bmatrix}\right\|_{Y^{\frac{1+\beta}{2}}}\right\}\leqslant r,\ \ \beta\in(\alpha_0,1)$$

then proceeding as in the proof of Corollary 2.1 and using Theorem 5.1, we get

$$\begin{aligned}
&\left\|F\left(\begin{bmatrix}u\\ v\end{bmatrix}\left(s,\begin{bmatrix}u_0^{n_k}\\ v_0^{n_k}\end{bmatrix},\alpha_{n_k}\right)\right)-F\left(\begin{bmatrix}\varphi(s)\\ \psi(s)\end{bmatrix}\right)\right\|_{X^{\frac{3-3\beta}{4}}\times X^{\frac{1-3\beta}{4}}}\\
&=\left\|f\left(u\left(s,\begin{bmatrix}u_0^{n_k}\\ v_0^{n_k}\end{bmatrix},\alpha_{n_k}\right)\right)-f(\varphi(s))\right\|_{X^{\frac{1-3\beta}{4}}}\\
&\leqslant C\left\|u\left(s,\begin{bmatrix}u_0^{n_k}\\ v_0^{n_k}\end{bmatrix},\alpha_{n_k}\right)-\varphi(s)\right\|^{1-\theta}_{X^{\frac{1-3\beta}{4}}}\left\|u\left(s,\begin{bmatrix}u_0^{n_k}\\ v_0^{n_k}\end{bmatrix},\alpha_{n_k}\right)-\varphi(s)\right\|^{\theta}_{X^{\frac{1+\beta}{4}}}\cdot\\
&\qquad\cdot\left(1+\left\|u\left(s,\begin{bmatrix}u_0^{n_k}\\ v_0^{n_k}\end{bmatrix},\alpha_{n_k}\right)\right\|^{\rho-1}_{X^{\frac{1+\beta}{4}}}+\|\varphi(s)\|^{\rho-1}_{X^{\frac{1+\beta}{4}}}\right)\\
&\leqslant C\left\|u\left(s,\begin{bmatrix}u_0^{n_k}\\ v_0^{n_k}\end{bmatrix},\alpha_{n_k}\right)-\varphi(s)\right\|^{1-\theta}_{X^{\frac{1-3\beta}{4}}}(2r)^{\theta}(1+2r^{\rho-1}),
\end{aligned}$$

where $\theta\in(0,1)$ is given in Lemma 5.1. This completes the proof. □

Theorem 5.6. *Suppose that f satisfies (1.2) with $\rho\in(1,\frac{N+2}{N-2})$ and (1.3). Choose $\alpha_0\geqslant\frac{1}{3}$ such that $\rho\in(1,\frac{N+(3\alpha-1)}{N-(\alpha+1)})$ for all $\alpha\in(\alpha_0,1)$. Then, there is an $r_0>0$ such that* *for any B bounded in $\tilde{Y}^1$ there is a time $t_B > 0$ such that*

$$\begin{bmatrix} u \\ u_t \end{bmatrix} \left(t, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \alpha \right) \in B_{r_0}^{\tilde{Y}^1}(0) \quad \text{for all} \quad t \geqslant t_B, \quad \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B, \quad \alpha \in (\alpha_0, 1).$$

Proof: For $\delta \in (0, \frac{1}{2}]$, we define the functional $\mathcal{L}_\delta : X^{\frac{1+\alpha}{4}} \times X^{\frac{1-\alpha}{4}} \to \mathbb{R}$, by

$$\begin{bmatrix} u \\ u_t \end{bmatrix} \mapsto \mathcal{L}_\delta \left(\begin{bmatrix} u \\ u_t \end{bmatrix} \right) = \mathcal{L} \left(\begin{bmatrix} u \\ u_t \end{bmatrix} \right) + \delta \int_\Omega A^{\frac{1-\alpha}{2}} u \, u_t dx, \quad \begin{bmatrix} u \\ u_t \end{bmatrix} \in X^{\frac{1+\alpha}{4}} \times X^{\frac{1-\alpha}{4}},$$

where $\mathcal{L} : X^{\frac{1+\alpha}{4}} \times X^{\frac{1-\alpha}{4}} \to \mathbb{R}$ was given in (3.5).

From (1.3), it follows that for all $\mu > 0$

$$-\sin \frac{\pi\alpha}{2} \int_\Omega \int_0^u f(s) ds dx \geqslant -\sin \frac{\pi\alpha}{2} (c_\mu + \frac{\mu}{2} \|u\|_X^2),$$

for all $\alpha \in (\alpha_0, 1)$. Let $c_1, c_2, c_3 > 0$ be constants of embeddings for $X^{\frac{1+\alpha}{4}} \subset X$, $X^{\frac{1-\alpha}{4}} \subset X$ and $X^{\frac{1+\alpha}{4}} \subset X^{\frac{1-\alpha}{2}}$ (where we have used that $\alpha \geqslant \frac{1}{3}$), respectively.

For any $0 < \delta \leqslant \frac{1}{4c_2c_3}$, we obtain

$$-\frac{1}{8} \Big[\|u\|^2_{X^{\frac{1+\alpha}{4}}} + \|v\|^2_{X^{\frac{1-\alpha}{4}}} \Big] \leqslant \delta \int_\Omega A^{\frac{1-\alpha}{2}} uv dx \leqslant \frac{1}{8} \Big[\|u\|^2_{X^{\frac{1+\alpha}{4}}} + \|v\|^2_{X^{\frac{1-\alpha}{4}}} \Big],$$

which leads to

$$\begin{aligned} \frac{3}{8} \Big[\|u\|^2_{X^{\frac{1+\alpha}{4}}} + \|v\|^2_{X^{\frac{1-\alpha}{4}}} \Big] &\leqslant \frac{1}{2} \|u\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{1}{2} \|v\|^2_{X^{\frac{1-\alpha}{4}}} + 2\delta \int_\Omega A^{\frac{1-\alpha}{2}} uv dx \\ &\leqslant \frac{5}{8} \Big[\|u\|^2_{X^{\frac{1+\alpha}{4}}} + \|v\|^2_{X^{\frac{1-\alpha}{4}}} \Big]. \end{aligned} \tag{5.9}$$

For $\mu \in (\mu_1, \frac{1}{c_1})$ and $\delta \leqslant \min\{\frac{1-\mu c_1}{2c_3}, \frac{1}{2c_2}\}$ we obtain that, for $\alpha \in (\alpha_0, 1)$,

$$\begin{aligned} \mathcal{L}_\delta \left(\begin{bmatrix} u \\ u_t \end{bmatrix} \right) &= \frac{1}{2} \|u_t\|^2_{X^{\frac{1-\alpha}{4}}} + \frac{1}{2} \|u\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{a}{2} \cos \frac{\pi\alpha}{2} \|u\|^2_{X^{\frac{1}{4}}} - \sin \frac{\pi\alpha}{2} \int_\Omega \int_0^u f(s) ds dx \\ &\quad + \delta \int_\Omega A^{\frac{1-\alpha}{2}} u u_t dx \\ &\geqslant \frac{1}{2} \|u_t\|^2_{X^{\frac{1-\alpha}{4}}} + \frac{1}{2} \|u\|^2_{X^{\frac{1+\alpha}{4}}} - \sin \frac{\pi\alpha}{2} (c_\mu + \frac{\mu}{2} \|u\|^2_X) - \frac{\delta}{2} \|A^{\frac{1-\alpha}{2}} u\|^2_X - \frac{\delta}{2} \|u_t\|^2_X \\ &\geqslant \frac{1}{2} \|u_t\|^2_{X^{\frac{1-\alpha}{4}}} + \frac{1}{2} \|u\|^2_{X^{\frac{1+\alpha}{4}}} - (c_\mu + \frac{\mu c_1}{2} \|u\|^2_{X^{\frac{1+\alpha}{4}}}) - \frac{\delta c_3}{2} \|u\|^2_{X^{\frac{1+\alpha}{4}}} - \frac{\delta c_2}{2} \|u_t\|^2_{X^{\frac{1-\alpha}{4}}}; \end{aligned}$$

that is, for $\eta_1 = \frac{1-\mu c_1}{4}$,

$$\mathcal{L}_\delta \left(\begin{bmatrix} u \\ u_t \end{bmatrix} \right) \geqslant \eta_1 \left(\|u\|^2_{X^{\frac{1+\alpha}{4}}} + \|u_t\|^2_{X^{\frac{1-\alpha}{4}}} \right) - c_\mu, \tag{5.10}$$

for all $\begin{bmatrix} u \\ u_t \end{bmatrix} \in X^{\frac{1+\alpha}{4}} \times X^{\frac{1-\alpha}{4}}$.

Writing $f(s) = f_0(s) + f(0)$ (in a similar way as [9]) with

$$|f_0(s)| \leqslant c(|s|^\rho + |s|), \quad s \in \mathbb{R},$$

for some $c > 0$ we can find a constant $\bar{c} > 1$ such that

$$-\int_\Omega \int_0^u f_0(s) ds dx \leqslant \bar{c}\|u\|^2_{X^{\frac{1+\alpha}{4}}} (1 + \|u\|^{\rho-1}_{X^{\frac{1+\alpha}{4}}})$$

and, we have seen in Theorem 3.1 that, given $r > 0$ there is a constant $c(r)$ such that, if

$$\sup \left\{ \left\| \begin{bmatrix} u \\ u_t \end{bmatrix} \left(t, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \alpha\right) \right\|_{Y^{\frac{1+\alpha}{2}}} : \left\| \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right\|_{Y^{\frac{1+\alpha}{2}}} \leqslant r, t \geqslant 0 \right\} \leqslant c(r).$$

Hence

$$-\bar{d} \int_\Omega \int_0^u f_0(s) ds dx \leqslant \|u\|^2_{X^{\frac{1+\alpha}{4}}} \tag{5.11}$$

for $\|u\|_{X^{\frac{1+\alpha}{4}}} \leqslant c(r)$ and $\bar{d} = \frac{1}{\bar{c}(1+c(r)^{\rho-1})}$. So $\bar{d} \leqslant 1$.

These properties of $\mathcal{L}_\delta$ and f lead to the following: let $\begin{bmatrix} u \\ u_t \end{bmatrix} \left(t, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \alpha\right)$ ($\begin{bmatrix} u \\ u_t \end{bmatrix}$ for short) be the solution of (2.3). Then, from (3.3) follows that

$$\begin{aligned}
\frac{d}{dt} \int_\Omega A^{\frac{1-\alpha}{2}} u u_t dx &= \langle A^{\frac{1-\alpha}{2}} u_t, u_t \rangle_{L^2(\Omega)} + \langle u, A^{\frac{1-\alpha}{2}} u_{tt} \rangle_{L^2(\Omega)} \\
&= \|u_t\|^2_{X^{\frac{1-\alpha}{4}}} - \langle u, 2\cos\frac{\pi\alpha}{2} A^{\frac{1}{2}} u_t \rangle_{L^2(\Omega)} - \langle u, A^{\frac{1+\alpha}{2}} u \rangle_{L^2(\Omega)} \\
&- \langle u, aA^{\frac{1-\alpha}{2}} u_t \rangle_{L^2(\Omega)} - \langle u, a\cos\frac{\pi\alpha}{2} A^{\frac{1}{2}} u \rangle_{L^2(\Omega)} + \langle u, \sin\frac{\pi\alpha}{2} f(u) \rangle_{L^2(\Omega)} \\
&= \|u_t\|^2_{X^{\frac{1-\alpha}{4}}} - 2\cos\frac{\pi\alpha}{2} \int_\Omega u A^{\frac{1}{2}} u_t dx - \|u\|^2_{X^{\frac{1+\alpha}{4}}} \\
&- a \int_\Omega u A^{\frac{1-\alpha}{2}} u_t dx - a\cos\frac{\pi\alpha}{2} \|u\|_{X^{\frac{1}{4}}} + \sin\frac{\pi\alpha}{2} \int_\Omega f(u) u dx.
\end{aligned}$$

Fix $\mu \in (\mu_1, \frac{1}{c_1})$. Note that

$$\sin\frac{\pi\alpha}{2} \int_\Omega u f(u) dx \leqslant \mu c_1 \|u\|^2_{X^{\frac{1+\alpha}{4}}} + \bar{c}_\mu,$$

and, for any $\varepsilon > 0$,

$$2\left|\cos\frac{\pi\alpha}{2} \int_\Omega u A^{\frac{1}{2}} u_t dx\right| \leqslant \varepsilon \|u\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{1}{\varepsilon} \|u_t\|^2_{X^{\frac{1-\alpha}{4}}},$$

and

$$2\left|\int_\Omega A^{\frac{1-\alpha}{2}}uu_tdx\right| \leqslant \varepsilon c_1\|u\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{c_3}{\varepsilon}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}.$$

It follows that

$$\begin{aligned}\frac{d}{dt}\int_\Omega A^{\frac{1-\alpha}{2}}uu_tdx \leqslant -\Big(1-\varepsilon-\frac{a\varepsilon c_3}{2}-\mu c_1\Big)\|u\|^2_{X^{\frac{1+\alpha}{4}}} - a\cos\frac{\pi\alpha}{2}\|u\|_{X^{\frac14}} \\ + \Big(1+\frac{1}{\varepsilon}+\frac{ac_2}{2\varepsilon}\Big)\|u_t\|^2_{X^{\frac{1-\alpha}{4}}} + \bar{c}_\mu.\end{aligned} \tag{5.12}$$

From (3.4) and (5.12), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{L}_\delta\left(\begin{bmatrix}u\\u_t\end{bmatrix}\right) \leqslant -\delta\Big(1-\mu c_1-\varepsilon-\frac{a\varepsilon c_1}{2}\Big)\|u\|^2_{X^{\frac{1+\alpha}{4}}} - \delta a\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac14}} \\ -\Big(a-\frac{\delta}{\varepsilon}-\delta-\frac{\delta ac_2}{2\varepsilon}\Big)\|u_t\|^2_{X^{\frac{1-\alpha}{4}}} + \delta\bar{c}_\mu.\end{aligned}$$

Fix $\alpha\in(\alpha_0,1)$, and $\varepsilon>0$ so that $\frac{1-\mu c_1}{2}\geqslant\varepsilon+\frac{a\varepsilon c_1}{2}$ and then $\delta>0$ so that $\frac{a}{2}\geqslant\frac{\delta}{\varepsilon}+\delta+\frac{\delta ac_2}{2\varepsilon}$. Thus

$$\frac{d}{dt}\mathcal{L}_\delta\left(\begin{bmatrix}u\\u_t\end{bmatrix}\right) \leqslant -\delta\frac{1-\mu c_1}{2}\|u\|^2_{X^{\frac{1+\alpha}{4}}} - \frac{a}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}} - \delta a\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac14}} + \delta\bar{c}_\mu.$$

Now, for $\eta_2=\min\{\frac{a}{2},\delta\frac{(1-\mu c_1)}{2},2\delta\}$,

$$\frac{d}{dt}\mathcal{L}_\delta\left(\begin{bmatrix}u\\u_t\end{bmatrix}\right) \leqslant -\eta_2\Big(\|u\|^2_{X^{\frac{1+\alpha}{4}}} + \|u_t\|^2_{X^{\frac{1-\alpha}{4}}} + \frac{a}{2}\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac14}}\Big) + \delta\bar{c}_\mu. \tag{5.13}$$

Let $\|u_0\|_{X^{\frac{1+\alpha}{4}}}+\|v_0\|_{X^{\frac{1-\alpha}{4}}}\leqslant r$. Then, from (5.11),

$$-\frac{\eta_2}{2}\Big(\|u\|^2_{X^{\frac{1+\alpha}{4}}}+\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}\Big) \leqslant \frac{\eta_2\bar{d}}{2}\int_\Omega\int_0^u f(s)dsdx + C(\eta,\bar{d},r).$$

Using that $\alpha\in(\alpha_0,1)$, we have

$$\begin{aligned}-\frac{\eta_2}{2}\Big(\|u\|^2_{X^{\frac{1+\alpha}{4}}}+\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}\Big) &\leqslant -\frac{\eta_2}{2}\sin\frac{\pi\alpha}{2}\Big(\|u\|^2_{X^{\frac{1+\alpha}{4}}}+\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}\Big) \\ &\leqslant \frac{\eta_2\bar{d}}{2}\sin\frac{\pi\alpha}{2}\int_\Omega\int_0^u f(s)dsdx + C(\eta,\bar{d},r).\end{aligned}$$

Then, it follows from (5.9) and (5.13) that

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}_\delta\left(\begin{bmatrix} u\\ u_t\end{bmatrix}\right) &\leqslant -\eta_2\Big(\frac{1}{2}\|u\|^2_{X^{\frac{1+\alpha}{4}}}+\frac{1}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}+\frac{a}{2}\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac{1}{4}}}\Big)+\delta\bar{c}_\mu\\
&+\frac{\eta_2\bar{d}}{2}\sin\frac{\pi\alpha}{2}\int_\Omega\int_0^u f(s)dsdx+C(\eta,\bar{d},r)\\
&\leqslant -\frac{\eta_2\bar{d}}{2}\Big(\frac{1}{2}\|u\|^2_{X^{\frac{1+\alpha}{4}}}+\frac{1}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}+\frac{a}{2}\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac{1}{4}}}+\delta\int_\Omega A^{\frac{1-\alpha}{2}}uu_tdx\Big)\\
&+\frac{\eta_2\bar{d}}{2}\sin\frac{\pi\alpha}{2}\int_\Omega\int_0^u f(s)dsdx+C_\mu\\
&\leqslant -\bar{\eta}_2\mathcal{L}_\delta\left(\begin{bmatrix} u\\ u_t\end{bmatrix}\right)+C_\mu.
\end{aligned}$$

From this we obtain that

$$\|u\|^2_{X^{\frac{1+\alpha}{4}}}+\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}\leqslant Me^{-\eta_2 t}+N,$$

Then,

$$\|\begin{bmatrix} u\\ u_t\end{bmatrix}\|_{\tilde{Y}^1}\leqslant \bar{C}.$$

□

From Theorem 3.11 in [6] we obtain the continuity of nonlinear semigroups, as α goes to 1, using the continuity of linear semigroups and variation of constants formula. Furthermore, we have that the attractors are upper semicontinuous at $\alpha = 1$ by Proposition 4.1 in [6].

Now, we define the set $\mathcal{A}$ consisting of the (weak in $\tilde{Y}^1$) limit points of the solutions of (2.3) with $\alpha \to 1$

$$\begin{aligned}\mathcal{A} := \Big\{\begin{bmatrix} w\\ z\end{bmatrix} : \text{ there are sequences } \ t_n\to\infty,\ \Big\{\begin{bmatrix} u_n\\ v_n\end{bmatrix}\Big\} \text{ bounded in } \tilde{Y}^1,\\ \text{and } \alpha_n\to 1 \text{ such that } \begin{bmatrix} u\\ v\end{bmatrix}\Big(t_n,\begin{bmatrix} u_n\\ v_n\end{bmatrix},\alpha_n\Big)\overset{w-\tilde{Y}^1}{\longrightarrow}\begin{bmatrix} w\\ z\end{bmatrix}\Big\}.\end{aligned}\tag{5.14}$$

In the next we will denote the ball $B^{\tilde{Y}^1}_{r_0}(0)$ given in the Theorem 5.6 by B_0.

Definition 5.2. $\begin{bmatrix} \phi\\ \varphi\end{bmatrix}\in\mathcal{LS}$ if and only if one of the following conditions holds.

(i) $\begin{bmatrix} \phi\\ \varphi\end{bmatrix}(0)\in\mathcal{A}$ and $\begin{bmatrix} \phi\\ \varphi\end{bmatrix}$ is a global weak solution of (2.3) with $\alpha = 1$ being (uniform for t in compact subsets of $[0,\infty)$) limit in Y of a sequence of solutions of (2.3) of the form $\begin{bmatrix} u\\ v\end{bmatrix}(t,\begin{bmatrix} u\\ v\end{bmatrix}(t_n,\begin{bmatrix} u_n\\ v_n\end{bmatrix},\alpha_n),\alpha_n)$, where $t_n\to\infty$, $\alpha_n\to 1$ and $\Big\{\begin{bmatrix} u_n\\ v_n\end{bmatrix}\Big\}\subset B_0$;

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(ii) $\begin{bmatrix}\phi\\ \varphi\end{bmatrix}(0) \in \tilde{Y}^1\backslash\mathcal{A}$ and $\begin{bmatrix}\phi\\ \varphi\end{bmatrix}$ is a global weak solution of (2.3) with $\alpha = 1$ being (uniform for t in compact subsets of $[0,\infty)$) limit in Y of a sequence of solutions of (2.3) of the form $\begin{bmatrix}u\\ v\end{bmatrix}(\cdot, \begin{bmatrix}\phi(0)\\ \psi(0)\end{bmatrix}, \alpha_n)$, where $t_n \to \infty$, where $\alpha_n \to 1$.

Next we show that the class $\mathcal{LS}$ has bounded dissipativeness property in $\tilde{Y}^1$ and that the set $\mathcal{A}$, defined in (5.14) is an 'attractor' for the limit problem (1.4) restricted class $\mathcal{LS}$; that is, we show that it is a closed bounded invariant subset of $\tilde{Y}^1$, which is weakly compact in $\tilde{Y}^1$ and compact in $H^s_{\{I\}}(\Omega) \times H^{s-1}(\Omega)$, $s \in [0,1)$. Moreover $\mathcal{A}$ attracts bounded subset of $\tilde{Y}^1$ in $H^s_{\{I\}}(\Omega)\times H^{s-1}(\Omega)$, for each $s\in[0,1)$. The proof of the next result follows using the same argument of the proof of Theorem 1.6 in [9].

Theorem 5.7. *Under the assumptions of Theorem 5.5, $\mathcal{LS}$ has the following properties:*

(i)(Existence) given $\begin{bmatrix}u_0\\ v_0\end{bmatrix} \in \tilde{Y}^1$ there exists $\begin{bmatrix}\phi\\ \varphi\end{bmatrix} \in \mathcal{LS}$ with $\begin{bmatrix}\phi\\ \varphi\end{bmatrix}(0) = \begin{bmatrix}u_0\\ v_0\end{bmatrix}$;

(ii)(Bounded dissipative) there is a bounded subset B_0 of $\tilde{Y}^1$ such that for any B bounded in $\tilde{Y}^1$, each $\begin{bmatrix}\phi\\ \varphi\end{bmatrix}$ from the class $\mathcal{LS}$ with $\begin{bmatrix}\phi\\ \varphi\end{bmatrix}(0) \in B$ enters B_0 in a certain time $\tau_B \geqslant 0$ and stays in B_0 for all $t \geqslant \tau_B$;

(iii)(Attractor) the set $\mathcal{A}$ is a closed bounded subset in $\tilde{Y}^1$ which satisfies

(a)(Compactness) $\mathcal{A}$ is weakly compact in $\tilde{Y}^1$ and strongly compact in $H^s_{\{I\}}(\Omega) \times H^{s-1}(\Omega)$ for any $s \in [0,1]$,

(b)(Invariance)

$$\left\{\begin{bmatrix}\phi\\ \varphi\end{bmatrix}(t);\ \begin{bmatrix}\phi\\ \varphi\end{bmatrix} \in \mathcal{LS}, \begin{bmatrix}\phi\\ \varphi\end{bmatrix}(0) \in \mathcal{A},\ t \geqslant 0\right\} \subset \mathcal{A}$$

and for each $\begin{bmatrix}w\\ z\end{bmatrix} \in \mathcal{A}$ and any $t \geqslant 0$ there is a certain $\begin{bmatrix}\bar\phi\\ \bar\varphi\end{bmatrix}(0) \in \mathcal{A}$ such that $\begin{bmatrix}\bar\phi\\ \bar\varphi\end{bmatrix}(t) = \begin{bmatrix}w\\ z\end{bmatrix}$,

(c)(Attracting property)

$$\sup_{\begin{bmatrix}u\\ v\end{bmatrix}\in\mathcal{LS},\begin{bmatrix}u\\ v\end{bmatrix}(0)\in B}\ \inf_{\begin{bmatrix}w\\ z\end{bmatrix}\in\mathcal{A}} \left\| \begin{bmatrix}u\\ v\end{bmatrix}(t) - \begin{bmatrix}w\\ z\end{bmatrix}\right\|_{H^s_{\{I\}}(\Omega)\times H^{s-1}(\Omega)} \overset{t\to\infty}{\longrightarrow}\ 0, \tag{5.15}$$

for any set B bounded in $\tilde{Y}^1$ and each $s\in[0,1]$ and $\alpha \in [\frac{1}{3},1)$.

Proof: Part (i) follows from Theorem 5.5 and part (ii) follows from Theorem 5.6. For part (iii), let us first prove compactness properties in part (iii)-(a). If B_0 is the uniformly absorbing set, then from Theorem 5.6 $\mathcal{A}$ is contained in B_0, B_0 is contained in $\tilde{Y}^1$ and $\tilde{Y}^1$ is separable. Therefore, it suffices to prove that $\mathcal{A}$ is sequentially weakly compact. Hence, given a sequence $\{\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]\} \subset \mathcal{A}$, taking subsequences if necessary, we may assume that there is a $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in \tilde{Y}^1$ such that $\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]$, as $n \to \infty$, weakly in $\tilde{Y}^1$ and strongly in Y. Since each $\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]$ is in $\mathcal{A}$, there is a $t_n > n$, $1 - \alpha_n < \frac{1}{n}$ and $\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right] \in B_0$ such that $\| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \alpha_n) - \left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right] \|_Y < \frac{1}{n}$. Thus, there are sequences $t_n \to \infty$, $\alpha_n \to \infty$ and $\{\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]\}$ in B_0 such that

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \alpha_n) - \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \quad \text{as } n \to \infty$$

strongly in Y. Since only finitely many elements of the sequence $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \alpha_n)\}$ may be outside B_1 we also conclude that it contains a subsequence weakly convergent in Y^1. This proves that $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in \mathcal{A}$ and that $\mathcal{A}$ is weakly compact in $\tilde{Y}^1$ and strongly compact in Y^s for $s \in [0, 1)$. Closedness of $\mathcal{A}$ in $\tilde{Y}^1$ is straightforward.

For the proof of 'positive invariance' part in (iii)-(b) choose $\begin{bmatrix} \phi \\ \psi \end{bmatrix} \in \mathcal{LS}$ with $\begin{bmatrix} \phi \\ \psi \end{bmatrix}(0) \in \mathcal{A}$. From the first case in Definition 5.2 we infer that

$$\begin{cases} \text{there is a sequence of the solutions of (2.3)} \\ \text{of the form } \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \alpha_n), \alpha_n), \text{ where} \\ t_n \nearrow \infty,\ \alpha_n \nearrow 1 \text{ and } \{\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]\} \subset B_0, \text{ such that} \\ \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t + t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \alpha_n) \overset{L^2(\Omega)\times H^{-1}(\Omega)}{\longrightarrow} \begin{bmatrix} \phi \\ \psi \end{bmatrix}(t) \\ \text{uniformly for } t \text{ in compact subsets of } [0, \infty). \end{cases} \tag{5.16}$$

By Theorem 5.6, for each fixed $t \geqslant 0$, all but finitely many elements of $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t + t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \alpha_n)\}$ are in B_0 and the elements $\begin{bmatrix} \phi \\ \psi \end{bmatrix}(t)$ in (5.16) can be thus viewed as weak limit in $\tilde{Y}^1$ of a certain subsequence of $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t + t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \alpha_n)\}$. Recalling (5.14) we thus conclude that $\begin{bmatrix} \phi \\ \psi \end{bmatrix}(t) \in \mathcal{A}$.

As for the 'negative invariance' part in (iii)-(b) choose $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in \mathcal{A}$ and apply (5.14) to infer that

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \alpha_n) \overset{w-H^1_0(\Omega)\times L^2(\Omega)}{\longrightarrow} \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]$$

for certain $t_n \nearrow \infty$, $\{\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]\} \subset B_0$ and $\alpha_n \nearrow 1$. By Theorem 5.6, for each fixed $s \geqslant 0$, all but finitely many elements of $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t_n - s, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \alpha_n)\}$ are in B_0. We can argue as in first part of the proof to conclude that all but finitely many elements of $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (t_n - s, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \alpha_n)\}$ are in B_1, where B_1 is a bounded set in Y^1. Hence, whenever $s > 0$ is fixed, a suitable subsequence $\{n_k\}$ can be chosen for which

$$\begin{aligned} &\text{all elements of } \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \big(t_{n_k} - s, \left[\begin{smallmatrix} u_{n_k} \\ v_{n_k} \end{smallmatrix}\right], \alpha_{n_k}\big)\} \text{ are in } B_1, \\ &\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \big(t_{n_k}, \left[\begin{smallmatrix} u_{n_k} \\ v_{n_k} \end{smallmatrix}\right], \alpha_{n_k}\big) \longrightarrow \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \text{ strongly in } Y \text{ and weakly in } Y^1, \\ &\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \big(t_{n_k} - s, \left[\begin{smallmatrix} u_{n_k} \\ v_{n_k} \end{smallmatrix}\right], \alpha_{n_k}\big) \longrightarrow \left[\begin{smallmatrix} \tilde{w} \\ \tilde{z} \end{smallmatrix}\right] \text{ strongly in } Y \text{ and weakly in } Y^1. \end{aligned} \tag{5.17}$$

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Therefore, recalling (5.14), we have

$$\left[\begin{smallmatrix}\tilde w\\ \tilde z\end{smallmatrix}\right] \in \mathcal{A}. \tag{5.18}$$

We also infer from Theorem 5.5 that there is a certain subsequence $\{n_{k_l}\}$ and a bounded function $\left[\begin{smallmatrix}\tilde\phi\\ \tilde\psi\end{smallmatrix}\right] : [0,\infty) \to H^1_0(\Omega)\times L^2(\Omega)$, being a global weak solution of (2.3) with $\alpha = 1$, such that

$$\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\Big(t, \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t_{n_{k_l}} - s, \left[\begin{smallmatrix}u_{n_{k_l}}\\ v_{n_{k_l}}\end{smallmatrix}\right], \alpha_{n_{k_l}}), \alpha_{n_{k_l}}\Big) \xrightarrow{Y} \left[\begin{smallmatrix}\tilde\phi\\ \tilde\psi\end{smallmatrix}\right](t) \tag{5.19}$$

uniformly on compact subintervals of $[0,\infty)$; in particular

$$\left[\begin{smallmatrix}\tilde\phi\\ \tilde\psi\end{smallmatrix}\right](0) = \left[\begin{smallmatrix}\tilde w\\ \tilde z\end{smallmatrix}\right]. \tag{5.20}$$

Recalling Definition 5.2 we infer from the properties (5.17)-(5.20) that $\left[\begin{smallmatrix}\tilde\phi\\ \tilde\psi\end{smallmatrix}\right] \in \mathcal{LS}$. Since

$$\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\Big(s, \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t_{n_{k_l}} - s, \left[\begin{smallmatrix}u_{n_{k_l}}\\ v_{n_{k_l}}\end{smallmatrix}\right], \alpha_{n_{k_l}}), \alpha_{n_{k_l}}\Big) = \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\Big(t_{n_{k_l}}, \left[\begin{smallmatrix}u_{n_{k_l}}\\ v_{n_{k_l}}\end{smallmatrix}\right], \alpha_{n_{k_l}}\Big) \xrightarrow{Y} \left[\begin{smallmatrix}\tilde\phi\\ \tilde\psi\end{smallmatrix}\right](s)$$

and (5.17) holds, we obtain that actually $\left[\begin{smallmatrix}\tilde\phi\\ \tilde\psi\end{smallmatrix}\right](s) = \left[\begin{smallmatrix}w\\ z\end{smallmatrix}\right]$.

To prove part (iii)-(c), suppose now that (5.15) fails. Hence, there exists $\epsilon > 0$, bounded subset B of Y^1, $s \in [0,1)$, and sequences $t_n \overset{n\to\infty}{\longrightarrow} \infty$, $\{\left[\begin{smallmatrix}\phi_n\\ \psi_n\end{smallmatrix}\right]\} \subset \mathcal{LS}$ with $\{\left[\begin{smallmatrix}\phi_n\\ \psi_n\end{smallmatrix}\right](0)\} \subset B$ such that for each $n \in \mathbb{N}$

$$\inf_{\left[\begin{smallmatrix}w\\ z\end{smallmatrix}\right]\in\mathcal{A}} \|\left[\begin{smallmatrix}\phi_n\\ \psi_n\end{smallmatrix}\right](t_n) - \left[\begin{smallmatrix}w\\ z\end{smallmatrix}\right]\|_{Y^s} > \epsilon. \tag{5.21}$$

From part (ii) we can assume, without loss of generality, that $\{\left[\begin{smallmatrix}\phi_n\\ \psi_n\end{smallmatrix}\right](t_n)\} \subset B_0$ whereas from (5.21) and 'positive invariance' part in (iii)-(b) we have that $\{\left[\begin{smallmatrix}\phi_n\\ \psi_n\end{smallmatrix}\right](0)\} \subset Y^1\backslash\mathcal{A}$ which allows us to restrict our attention to the second case in Definition 5.2. By Definition 5.2, for each fixed $n \in \mathbb{N}$, there exists $\left[\begin{smallmatrix}u_n\\ v_n\end{smallmatrix}\right] = \left[\begin{smallmatrix}\phi_n\\ \psi_n\end{smallmatrix}\right](0) \in B$ and $0 < 1 - \alpha_n \leqslant \frac{1}{n}$ such that

$$\sup_{t\in[0,t_n]} \|\left[\begin{smallmatrix}\phi_n\\ \psi_n\end{smallmatrix}\right](t) - \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t, \left[\begin{smallmatrix}u_n\\ v_n\end{smallmatrix}\right], \alpha_n)\|_Y \leqslant \frac{1}{n}. \tag{5.22}$$

From Theorem 5.6 we know that, for some $t_B > 0$, $\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t, \left[\begin{smallmatrix}u_n\\ v_n\end{smallmatrix}\right], \alpha_n) \in B_0$ for $t \geqslant t_B$. Therefore, we can choose a subsequence $\{\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t_{n_k}, \left[\begin{smallmatrix}u_{n_k}\\ v_{n_k}\end{smallmatrix}\right], \alpha_{n_k})\}$ convergent weakly in Y^1 to a certain $\left[\begin{smallmatrix}\hat w\\ \hat z\end{smallmatrix}\right] \in Y^1$.

Since $\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t_{n_k}, \left[\begin{smallmatrix}u_{n_k}\\ v_{n_k}\end{smallmatrix}\right], \alpha_{n_k}) = \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]\big(t_{n_k} - t_B, \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t_B, \left[\begin{smallmatrix}u_{n_k}\\ v_{n_k}\end{smallmatrix}\right], \alpha_{n_k}), \alpha_{n_k}\big)$ we have $\left[\begin{smallmatrix}\hat w\\ \hat z\end{smallmatrix}\right] \in \mathcal{A}$. Therefore, via (5.22), after choosing subsequences $\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t_{n_{k_l}}, \left[\begin{smallmatrix}u_{n_{k_l}}\\ v_{n_{k_l}}\end{smallmatrix}\right], \alpha_{n_{k_l}})$ convergent in Y^s and $\left[\begin{smallmatrix}\phi_{n_{k_l}}\\ \psi_{n_{k_l}}\end{smallmatrix}\right](t_{n_{k_l}})$ convergent in Y^s we obtain that $\left[\begin{smallmatrix}\phi_{n_{k_l}}\\ \psi_{n_{k_l}}\end{smallmatrix}\right](t_{n_{k_l}}) \xrightarrow{Y^s} \left[\begin{smallmatrix}\hat w\\ \hat z\end{smallmatrix}\right]$, which contradicts (5.21). ∎

Now, we will show that the global attractors behaves continuously at $\alpha = 1$. For this, it remains only to show lower semicontinuity of the global attractors. We assume that nonlinearities f satisfies (1.2) with $\rho < \frac{N}{N-2}$ and (1.3) and that the equilibrium of the problem (1.4) are all hyperbolic (as in [10]) and, consequently, there are finitely many of them. The continuity of the equilibria as α varies is immediate and the continuity of the local unstable manifolds follows from Theorem 5.13 in [6]. The lower semicontinuity of the global attractors at $\alpha = 1$ now follows from Theorem 6.1 in [6].

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