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The Euler Obstruction and Torus Action

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In this paper we apply torus action in the study of surfaces with the property that their irreducible components are toric surfaces. In particular, we present a formula to compute the Euler obstruction of such surfaces. As an application of this formula we compute this invariant for some families of determinantal surfaces. In the last section, we make some remarks about Milnor number and toric surfaces in $\mathbb{C}^3$ and $\mathbb{C}^4$. May, 2013 ICMC-USP

1. INTRODUCTION

An action from a group G in a variety X is a powerful tool in mathematics. An important example is the torus action, from the torus $\mathbb{T}^n = (\mathbb{C}^*)^n$ in a algebraic normal variety X, that in the specific case that this action is almost transitive and effective the variety X is called toric variety.

The theory of toric varieties, as remarked by Fulton in his book [11], came from the work of several people, primarily in connection with the study of compactification problems. The reader may consult [9, 11, 19] for an overview about this theory. This theory can be seen as a cornerstone for the interaction between combinatorics and algebraic geometry, which relates the combinatorial study of convex polytopes with algebraic torus actions.

Another important class of surfaces is the class of determinantal surfaces, which has been largely investigated in algebraic geometry as in commutative algebra. In singularity theory this object has been deeply investigated by many authors, recently we can cite Frühbis-Krüger [10], Pereira, Ruas [21], among others.

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The toric varieties and determinantal varieties represent two interesting classes of surfaces, both have many special properties. Relating these two classes of surfaces Riemenschneider proved in [22, 23] that there exists a family $\mathcal{F}$ of affine toric surfaces whose each element is also determinantal surface. In this work we give an alternative proof of this fact, for a special subfamily $\mathcal{D}$ of $\mathcal{F}$, as we can see in the proof of Theorem 3.1.

In general, we can not say that all toric surface is related to a determinantal surface, but using the works [22, 23] of Riemenschneider, we present in Section 3 three classes of determinantal surfaces Y, with 1-dimensional singular set, and the property that their irreducible components are toric surfaces. These three classes of determinantal surfaces are actually examples of a special family of surfaces, called multitoric surfaces, that roughly speaking, are surfaces whose irreducible components are toric surfaces. We introduced the concept of multitoric surface in Section 2.

From [10, 21], we can see that not every determinantal surface contains a toric surface as one of its irreducible components.

One of the main invariants explored in this work is the Euler obstruction. The Euler obstruction was defined by MacPherson [15] as a tool to prove the conjecture about existence and unicity of the Chern classes in the singular case. Since that, the Euler obstruction has been deeply investigated by many authors such as Brasselet, Schwartz [5], Gonzalez-Sprinberg [12], Lê, Teissier [14], among others. The reader may consult [1, 2] for an overview about Euler obstruction and generalizations.

In the specific case of toric surfaces a beautiful formula for the Euler obstruction was proved by Gonzalez-Sprinberg [12], this formula was generalized by Matsui and Takeuchi in [16].

Using the work of Gonzalez-Sprinberg, in Section 2, we give a formula for the Euler obstruction of a multitoric surface, providing a very simple method to compute this invariant in this case. In the last section we give some remarks about Milnor number and toric surfaces.

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2. MULTITORIC SURFACES

In this section we introduce the concept of multitoric surfaces, that roughly speaking, are surfaces whose irreducible components are toric surfaces.

Let us first recall that a toric variety X is a normal complex algebraic variety with an action of the algebraic torus $\mathbb{T}^n = (\mathbb{C}^*)^n$, such that this action has an orbit $\mathcal{O}$ open and dense (in the Zariske topology) which is homeomorphic to the torus $\mathbb{T}^n$. In this paper, we consider the case of affine toric surfaces $X = X_\sigma$ associated with a cone $\sigma \subset \mathbb{R}^2$ (rational and strictly convex). For definitions and notations see [11].

Definition 2.1. We say that a surface $Y \subset \mathbb{C}^p$ is a multitoric surface if there exists an action $\varphi : \mathbb{T}^2 \times \mathbb{C}^p \to \mathbb{C}^p$ from $\mathbb{T}^2$ in $\mathbb{C}^p$ such that φ give to each irreducible component of Y a structure of toric surface and such that the intersection of each irreducible component of Y is the closure of an orbit of φ.

The definition of multitoric surface was motivated by the initial study of Example 3.1, that is an example of a surface that is toric and determinantal.

In this section, we present a formula for the Euler obstruction of a multitoric surface Y, that provides a very simple method to compute this invariant.

Let us first introduce some objects in order to define the Euler obstruction.

Let $(X,0) \subset (\mathbb{C}^N,0)$ be an equidimensional reduced complex analytic germ of dimension d in an open set $U \subset \mathbb{C}^N$. We consider a complex analytic Whitney stratification $\mathcal{V} = \{V_i\}$ of U adapted to X and we assume that $\{0\}$ is a stratum. We choose a small representative of $(X,0)$ such that 0 belongs to the closure of all the strata. We will denote it by X and we will write $X = \cup_{i=0}^{q} V_i$ where $V_0 = \{0\}$ and $V_q = X_{\text{reg}}$, the set of smooth points of X. We will assume that the strata $V_0, \dots, V_{q-1}$ are connected and that the analytic sets $\overline{V_0}, \dots, \overline{V_{q-1}}$ are reduced. We will set $d_i = \dim V_i$ for $i \in \{1, \dots, q\}$ (note that $d_q = d$).

Let $G(d,N)$ denote the Grassmanian of complex d-planes in $\mathbb{C}^N$. On the regular part X_{reg} of X the Gauss map $\phi : X_{\text{reg}} \to U \times G(d,N)$ is well defined by $\phi(x) = (x, T_x(X_{\text{reg}}))$.

Definition 2.2. The *Nash transformation* (or *Nash blow up*) $\widetilde{X}$ of X is the closure of the image $\text{Im}(\phi)$ in $U \times G(d,N)$. It is a (usually singular) complex analytic space endowed with an analytic projection map $\nu : \widetilde{X} \to X$ which is a biholomorphism away from $\nu^{-1}(\text{Sing}(X))$.

The fiber of the tautological bundle $\mathcal{T}$ over $G(d,N)$, at the point $P \in G(d,N)$, is the set of vectors v in the d-plane P. We still denote by $\mathcal{T}$ the corresponding trivial extension bundle over $U \times G(d,N)$. Let $\widetilde{T}$ be the restriction of $\mathcal{T}$ to $\widetilde{X}$, with projection map π. The bundle $\widetilde{T}$ on $\widetilde{X}$ is called *the Nash bundle* of X.

An element of $\widetilde{T}$ is written (x,P,v) where $x \in U$, P is a d-plane in $\mathbb{C}^N$ based at x and v is a vector in P. We have the following diagram:

$$\begin{array}{ccc} \widetilde{T} & \hookrightarrow & \mathcal{T} \\ \pi \downarrow & & \downarrow \\ \widetilde{X} & \hookrightarrow & U \times G(d,N) \\ \nu \downarrow & & \downarrow \\ X & \hookrightarrow & U. \end{array}$$

Adding the stratum $U \setminus X$ we obtain a Whitney stratification of U. Let us denote by $TU|_X$ the restriction to X of the tangent bundle of U. We know that a stratified vector field v on X means a continuous section of $TU|_X$ such that if $x \in V_\alpha \cap X$ then $v(x) \in T_x(V_\alpha)$. By Whitney condition one has the following lemma [5].

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Lemma 2.1. *Every stratified vector field v on a subset $A \subset X$ has a canonical lifting to a section $\tilde{v}$ of the Nash bundle $\tilde{T}$ over $\nu^{-1}(A) \subset \tilde{X}$.*

Now consider a stratified radial vector field $v(x)$ in a neighborhood of $\{0\}$ in X, *i.e.*, there is ϵ_0 such that for every $0 < \epsilon \leq \epsilon_0$, $v(x)$ is pointing outwards the ball $\mathbb{B}_\epsilon$ over the boundary $\mathbb{S}_\epsilon := \partial\mathbb{B}_\epsilon$.

The following interpretation of the local Euler obstruction has been given by Brasselet-Schwartz [5]. We refer to [20] for the original definition which uses 1-forms instead of vector fields.

Definition 2.3. Let v be a radial vector field on $X \cap \mathbb{S}_\epsilon$ and $\tilde{v}$ the lifting of v on $\nu^{-1}(X \cap \mathbb{S}_\epsilon)$ to a section of the Nash bundle. The local Euler obstruction (or simply the Euler obstruction) $Eu_X(0)$ is defined to be the obstruction to extending $\tilde{v}$ as a nowhere zero section of $\tilde{T}$ over $\nu^{-1}(X \cap \mathbb{B}_\epsilon)$.

More precisely, let $\mathcal{O}(\tilde{v}) \in H^{2N}(\nu^{-1}(X \cap \mathbb{B}_\epsilon), \nu^{-1}(X \cap \mathbb{S}_\epsilon))$ be the obstruction cocycle to extending $\tilde{v}$ as a bowhere zero section of $\tilde{T}$ inside $\nu^{-1}(X \cap \mathbb{B}_\epsilon)$. The local Euler obstruction $Eu_X(0)$ is defined as the evaluation of the cocycle $\mathcal{O}(\tilde{v})$ on the fundamental class of the pair $((X \cap \mathbb{B}_\epsilon), \nu^{-1}(X \cap \mathbb{S}_\epsilon))$. The Euler obstruction is an integer.

The computation of local Euler obstruction is not so easy by using the definition. Various authors propose formulas which make the computation easier. Lê D.T. and B. Teissier provide a formula in terms of polar multiplicities [14].

In the paper [3], J.-P. Brasselet, Lê D. T. and J. Seade give a Lefschetz type formula for the local Euler obstruction. The formula shows that the local Euler obstruction, as a constructible function, satisfies the Euler condition relatively to generic linear forms.

In the following, using the work of Gonzalez-Sprinberg [12], we compute the Euler obstruction of a multitoric surface Y.

Theorem 2.1. *Let $Y \subset \mathbb{C}^p$ be a multitoric surface, and suppose that $Y = Y_1 \cup \cdots \cup Y_k \cup Y_{k+1} \cup \cdots \cup Y_{k+s}$, where $Y_{k+1}, \ldots, Y_{k+s}$ are the irreducible components of Y with isolated singularity at the origin. Then $\mathrm{Eu}_Y(0) = 3s + k - p_1 - \cdots - p_s$, where p_i is the smallest dimension of embedding of each component Y_{k+i} with singularity.*

Proof. In [12] Gonzalez-Sprinberg proved that

$$\mathrm{Eu}_{Y_{k+i}}(0) = 3 - p_i.$$

Therefore the result follows from the equality

$$\mathrm{Eu}_Y(0) = \mathrm{Eu}_{Y_1}(0) + \cdots + \mathrm{Eu}_{Y_k}(0) + \mathrm{Eu}_{Y_{k+1}}(0) + \cdots + \mathrm{Eu}_{Y_{k+s}}(0),$$

which was proved by MacPherson in [15]. ∎

In the case of higher dimensions we suggest the definition of multitoric varieties, as follow.

Definition 2.4. We say that a n-dimensional variety $Y \subset \mathbb{C}^p$ is a multitoric variety if there exists an action $\varphi : \mathbb{T}^n \times \mathbb{C}^p \to \mathbb{C}^p$ from $\mathbb{T}^n$ in $\mathbb{C}^p$ such that φ give to each irreducible component of Y a structure of a n-dimensional toric variety and such that the intersection of each irreducible component of Y is the closure of an orbit of φ.

3. APLICATIONS TO DETERMINANTAL SURFACES

In this section, using Theorem 2.1 we give a formula for the Euler obstruction of three classes of determinantal surfaces, providing a very simple method to compute this invariant in this case.

Let us first remember the definition of determinantal variety.

Now, let $Mat_{(n,p)}(\mathbb{C})$ be the set of all $n \times p$ matrices with complex entries, $\Delta_t \subset Mat_{(n,p)}(\mathbb{C})$ the subset formed by matrices that have rank less than t, with $1 \leq t \leq \min(n,p)$. It is possible to show that Δ_t is an irreducible singular algebraic variety of codimension $(n-t+1)(p-t+1)$ (see [7]). Moreover the singular set of Δ_t is exactly Δ_{t-1}. The set Δ_t is called *generic determinantal variety.*

Definition 3.1. Let $M = (m_{ij}(x))$ be a $n \times p$ matrix whose entries are complex analytic functions on $U \subset \mathbb{C}^r$, $0 \in U$ and f the function defined by the $t \times t$ minors of M. We say that X is a determinantal variety of codimension $(n-t+1)(p-t+1)$ if X is defined by the equation $f = 0$.

Remark 3. 1. We can look to a matrix $M = (m_{ij}(x))$ as a map $M : \mathbb{C}^r \to Mat_{(n,p)}(\mathbb{C})$, with $M(0) = 0$. Then, the determinantal variety in $\mathbb{C}^r$ is the set $X = M^{-1}(\Delta_t)$, with $1 \leq t \leq \min(n,p)$. The singular set of X is given by $M^{-1}(\Delta_{t-1})$.

Example 3.1. In $\mathbb{R}^2$, consider the cone σ generated by vectors $v_1 = 3e_1 - e_2$ and $v_2 = e_2$. By Gordon's Lemma [11], we have that the generators of the monoid associated to the cone σ are

$$a_1 = e_1^*, \quad a_2 = e_1^* + e_2^*, \quad a_3 = e_1^* + 2e_2^*, \quad a_4 = e_1^* + 3e_2^*.$$

Note that the relation between a_1, a_2, a_3 and a_4 are

$$\begin{aligned} a_1 + a_4 &= a_2 + a_3 \\ a_1 + a_3 &= 2a_2 \\ a_2 + a_4 &= 2a_3. \end{aligned} \tag{3.1}$$

Therefore we have the following multiplicative relations in $\mathbb{C}^4$

$$\begin{array}{l} z_1z_4 = z_2z_3 \\ z_1z_3 = z_2^2 \\ z_2z_4 = z_3^2. \end{array} \tag{3.2}$$

Then the toric surface X_σ is given by $V(I_\sigma)$, where

$$I_\sigma = \left\langle z_1z_4 - z_2z_3, z_1z_3 - z_2^2, z_2z_4 - z_3^2 \right\rangle.$$

But note that, taking the minors 2×2 of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix}$$

we have $f_1 = z_1z_3 - z_2^2$, $f_2 = z_1z_4 - z_2z_3$, $f_3 = z_2z_4 - z_3^2$ that are exactly the generators of ideal I_σ, so we have that the toric surface X_σ is also a determinantal surface of codimension $(2-2+1)(3-2+1) = 2$ in $\mathbb{C}^4$.

Remark 3. 2. In (3.1) note that adding the equations $a_1 + a_3 = 2a_2$ and $a_2 + a_4 = 2a_3$ we can obtain the first equation $a_1 + a_4 = a_2 + a_3$. This makes us think that we can ignore the first equation of the system (3.2), however this is not true, because the point $(1, 0, 0, 1) \in \mathbb{C}^4$ satisfies the last two equations of (3.2), but does not satisfy the equation $z_1z_4 = z_2z_3$.

What makes Example 3.1 so special, is the fact that we can use known tools either of toric varieties or of determinantal varieties for study some property that we want of X_σ, for example, in Section 4, using the work of Pereira and Ruas [21] for determinantal surface, we will compute the Milnor number of X_σ. And from Theorem 2.1 the Euler obstruction of the determinantal surface X_σ is 1.

Motivated by Example 3.1 in the following we define a special class $\mathcal{D}$ of toric surfaces.

Definition 3.2. Let us denote by $\mathcal{D}$ the class of toric surfaces defined by cones of $\mathbb{R}^2$ generated by vectors $v_1 = pe_1 - e_2$ and $v_2 = e_2$, where $p \in \mathbb{Z}_{>0}$ such that $p \geq 2$.

In [22, 23], Riemenschneider proved that all elements of the class $\mathcal{D}$ is a determinantal surface. For prove the next theorem we will utilize this fact, however the proof of this fact presented here was made in a totally independent way.

Theorem 3.1. *Consider $Y \subset \mathbb{C}^{p+1}$ the determinantal surface given by the 2×2 minors of the matrix*

$$A = \begin{pmatrix} z_1 & z_2 & \cdots & z_{p-1} & z_p \\ z_2 & z_3 & \cdots & z_p & z_{p+1} \end{pmatrix}$$

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where $p \geq 3$, then $Eu_Y(0) = 2 - p$.

Proof. Given $p \geq 3$ a positive integer, by Gordon's Lemma [11], we have that the generators of the monoid associated to the cone σ_p are

$$a_1 = e_1, \quad a_2 = e_1 + e_2, \quad a_3 = e_1 + 2e_2, \ldots, \quad a_{p+1} = e_1 + pe_2.$$

Moreover, as $q = 1$ the set $\{a_1, a_2, \ldots, \quad a_{p+1}\}$ form a minimal set of generators for this monoid.

We will prove that there are

$$C_2^p = \frac{p!}{2!(p-2)!} = \frac{p(p-1)}{2}$$

relations between the generators $\{a_1, a_2, \ldots, \quad a_{p+1}\}$ and we will show that the corresponding multiplicative relations can be obtained as the 2×2 minors of the matrix A. We will prove this by induction.

By Example 3.1 we have that the affirmation is true for $p = 3$, and suppose that the result is true for $p \in \mathbb{N}$ with $p > 3$.

For $p + 1$ we have that the generators of the monoid associated to the cone σ_{p+1} are

$$a_1 = e_1, \quad a_2 = e_1 + e_2, \quad a_3 = e_1 + 2e_2, \ldots, \quad a_{p+1} = e_1 + pe_2, \quad a_{p+2} = e_1 + (p+1)e_2$$

and by the hypothesis of induction we have that there are C_2^p relations between the generators

$$a_1 = e_1, \quad a_2 = e_1 + e_2, \quad a_3 = e_1 + 2e_2, \ldots, \quad a_{p+1} = e_1 + pe_2,$$

and we have also that the corresponding multiplicative relations between this generators are

$$\begin{array}{l} z_1 z_3 = z_2^2 \\ z_1 z_4 = z_2 z_3 \\ z_2 z_4 = z_3^2 \\ \vdots \\ z_{p-1} z_{p+1} = z_p^2. \end{array} \tag{3.3}$$

Now note that the last generator $a_{p+2} = e_1 + (p+1)e_2$ relates to the generators

$$a_1 = e_1, \quad a_2 = e_1 + e_2, \quad a_3 = e_1 + 2e_2, \ldots, \quad a_p = e_1 + (p-1)e_2$$

generating p news additive relations

$$\begin{array}{l} a_1 + a_{p+2} = a_2 + a_{p+1} \\ a_2 + a_{p+2} = a_3 + a_{p+1} \\ \vdots \\ a_p + a_{p+2} = 2a_{p+1} \end{array} \tag{3.4}$$

whose the corresponding multiplicative relations are

$$\begin{aligned} z_1 z_{p+2} &= z_2 z_{p+1} \\ z_2 z_{p+2} &= z_3 z_{p+1} \\ &\vdots \\ z_p z_{p+2} &= z_{p+1}^2. \end{aligned} \tag{3.5}$$

Then from (3.3) and (3.5) we have that $X_{\sigma_{p+1}}$ is given by $V(I_{\sigma_{p+1}})$, where $I_{\sigma_{p+1}}$ is the ideal generated by the polynomials

$$\begin{array}{ccc} f_1 = z_1 z_3 - z_2^2 & \dots & f_{\frac{p(p-1)}{2}} = z_{p-1} z_{p+1} - z_p^2 \\ \vdots & & \\ f_{\frac{p(p-1)}{2}+1} = z_1 z_{p+2} - z_2 z_{p+1} & \dots & f_{\frac{(p+1)p}{2}} = z_p z_{p+2} - z_{p+1}^2 \end{array}$$

what are exactly the 2×2 minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & \cdots & z_p & z_{p+1} \\ z_2 & z_3 & \cdots & z_{p+1} & z_{p+2} \end{pmatrix}.$$

Therefore, we have that $X_{\sigma_{p+1}}$ is a determinantal surface of codimension

$$(2-2+1)((p+1)-2+1) = p,$$

and the result follows from Theorem 2.1. ∎

Remark 3. 3. In the case $p = 2$ we have that X_{σ_2} is the cone in $\mathbb{C}^3$, that is a determinantal surface but that is also a complete intersection with isolated singularity, what we can not expect in general, as we can see in Example 3.1.

In general, we can not say that all toric surface is related to a determinantal surface, like the elements of $\mathcal{D}$, but we can prove that there exist classes of determinantal surfaces Y that are multitoric surfaces. Using this fact in the following we find formulas for the Euler obstruction of Y.

THEOREM 3.2. *Let $Y \subset \mathbb{C}^p$ be the determinantal surface given by the 2×2 minors of the matrix*

$$\begin{pmatrix} z_1 & z_2 & \cdots & z_{p-3} & z_{p-2}^b & z_{p-2}^{b-1} z_{p-1}^c \\ z_2^a & z_3 & \cdots & z_{p-2} & z_{p-1} & z_p \end{pmatrix}$$

where a, b, c are positive integers with $b \geq 2$. Then, $Eu_Y(0) = 4 - p$.

Proof. We will prove that Y is a bitoric surface, then the result follows from Theorem 2.1. As a motivation for the general case, suppose that $p = 5$. In [22, 23] Riemenschneider

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proved that given a, b, c positive integers, there exist m, q positive integers satisfying $0 < q < m$ and $(m, q) = 1$, such that the toric surface X_σ obtained by the cone $\sigma\mathbb{R}^2$ generated by vectors $v_1 = e_2$ and $v_2 = me_1 - qe_2$ is the surface $X_\sigma = V(I_\sigma)$, where I_σ is the ideal generated by the binomials

$$\begin{array}{lll} f_1 = z_1 z_3 - z_2^{a+1} & f_2 = z_1 z_4 - z_2^a z_3^b & f_3 = z_1 z_5 - z_2^a z_3^{b-1} z_4^c \\ f_4 = z_2 z_4 - z_3^{b+1} & f_5 = z_2 z_5 - z_3^b z_4^c & f_6 = z_3 z_5 - z_4^{c+1}. \end{array}$$

Now, if $b \geq 2$, note that the 2×2 minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3^b & z_3^{b-1} z_4^c \\ z_2^a & z_3 & z_4 & z_5 \end{pmatrix}$$

are f_1, f_2, f_3, f_4, f_5 and $g = z_3^{b-1} f_6$
Then, between the generators of I_σ and J_Y, where $Y = V(J_Y)$, there is difference only between the binomials

$$z_3 z_5 - z_4^{c+1} \quad \text{and} \quad z_3^b z_5 - z_3^{b-1} z_4^{c+1},$$

so is only the equation $z_3 = 0$ that makes difference between the surfaces X_σ and Y. Then we have $X_\sigma \subset Y$ and

$$Z_Y := Y \setminus X_\sigma = \{0\} \times \{0\} \times \{0\} \times \mathbb{C}^* \times \mathbb{C}.$$

Now observe that the closure of the set Z_Y in Y is

$$\overline{Z_Y} = \{0\} \times \{0\} \times \{0\} \times \mathbb{C} \times \mathbb{C},$$

i.e., the irreducible components of Y are X_σ and Z_Y that are toric surfaces.

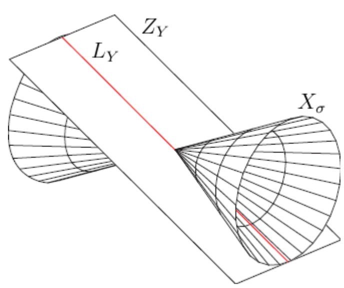

Figure 1: An illustration for the determinantal surface Y.

In [22, 23] Riemenschneider proved that

$$u_1 = (1, 0), \quad u_2 = (1, 1), \quad u_3 = (a+1)u_2 - u_1, \quad u_4 = (b+1)u_3 - u_2, \quad u_5 = (c+1)u_4 - u_3$$

form a basis for the cone associated with the monoid σ, then the action φ from $\mathbb{T}^2$ in $\mathbb{C}^5$ given by

$$\varphi(t=(t_1,t_2),(z_1,z_2,z_3,z_4,z_5)) = (t^{u_1}z_1, t^{u_2}z_2, t^{u_3}z_3, t^{u_4}z_4, t^{u_5}z_5),$$

where $t^{u_i} = t_1^{u_i^1}t_2^{u_i^2}$ and $u_i = (u_i^1, u_i^2)$, $1 \leq i \leq 5$ gives to X_σ a structure of toric surface. Furthermore note that $\overline{Z_Y}$ is the union of the orbits

$$\mathcal{O}_{(0,\ldots,0)}, \quad \mathcal{O}_{(0,\ldots,0,1)}, \quad \mathcal{O}_{(0,\ldots,0,1,0)}, \quad \mathcal{O}_{(0,\ldots,0,1,1)},$$

of φ, where $\mathcal{O}_{(0,\ldots,0,1,1)}$ is homeomorphic to the torus $\mathbb{T}^2$ and $\overline{\mathcal{O}}_{(0,\ldots,0,1,1)} = \overline{Z_Y}$. Note also, that $L_Y := X_\sigma \cap \overline{Z_Y} = \overline{\mathcal{O}}_{(0,0,0,0,1)}$. Then, Y is a bitoric surface with a singular irreducible component.

The proof that Y is a bitoric surface in the general case is analogous.

Therefore, from Theorem 2.1 we have that $Eu_Y(0) = 3 - p + 1 = 4 - p$. ∎

Let us observe that Theorem 3.2 assures that the Euler obstruction of Y depends only of the dimension of embedding of Y, giving us a very simple method to compute the Euler obstruction in this case. Then, motivated by Theorem 3.2 we studied others class of determinantal surface and we have the following theorems.

THEOREM 3.3. *Let $Y \subset \mathbb{C}^p$ be a determinantal surface given by the 2×2 minors of the matrix*

$$B = \begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_{p-3} & z_{p-2}^c & z_{p-2}^{c-1}z_{p-1}^d \\ z_2^a z_3^{b-1} & z_3^b & z_4 & \cdots & z_{p-2} & z_{p-1} & z_p \end{pmatrix}$$

where a,b,c,d are positive integers with $b,c \geq 2$. Then, $Eu_Y(0) = 5 - p$.

Proof. Developing a process analogous to what we did in the proof of Theorem 3.2, we prove that Y has 3 irreducible components, *i.e.*, using [22, 23] we prove that given a,b,c,d positive integers, there exist a toric surface $X_\sigma = V(I_\sigma)$ contained in Y whose smaller dimension of embedding is p and such that the ideal I_σ is generated by

$$C_2^p = \frac{p!}{2!(p-2)!} = \frac{p(p-1)}{2}$$

binomials. Comparing the binomials given by the minors of the matrix B and the generators of I_σ is easy to see that there exist difference only between two these binomials, this difference generates two planes in $\mathbb{C}^p$ which are the irreducible components smooth of Y. Furthermore, we use the action from $\mathbb{T}^2$ in $\mathbb{C}^p$ given by combinatory from the cone σ for prove that Y is a tritoric surface. Then, the result follows from Theorem 2.1. ∎

THEOREM 3.4. *Let $Y \subset \mathbb{C}^{p+1}$ be a determinantal surface given by the 2×2 minors of the matrix*

$$\begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_{p-1} & z_p^b z_{p+1} \\ z_2^a & z_3 & z_4 & \cdots & z_p & z_{p+1}^2 \end{pmatrix}$$

where a, b are positive integers. Then, $Eu_Y(0) = 5 - 2p$.

Proof. Note that Y have two irreducible components, that are $V(I_1)$ and $V(I_2)$, with $I_1 = \langle I_{\sigma_1}, z_{p+1}\rangle$ and $I_2 = I_{\sigma_2}$, where the ideal I_{σ_1} is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_{p-1} \\ z_2^a & z_3 & z_4 & \cdots & z_p \end{pmatrix}$$

and the ideal I_{σ_2} is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_{p-1} & z_p^b \\ z_2^a & z_3 & z_4 & \cdots & z_p & z_{p+1} \end{pmatrix},$$

i.e., $V(I_1)$ is the toric surface X_{σ_1} immersed in $\mathbb{C}^{p+1}$ and $V(I_2)$ is the toric surface X_{σ_2}, where $\sigma_1 \subset \mathbb{R}^2$ is the cone generated by vectors $v_1^1 = (0,1)$, $v_2^1 = w_p e_1 - u_p e_2$ and $\sigma_2 \subset \mathbb{R}^2$ is the cone generated by vectors $v_1^2 = (0,1)$, $v_2^2 = w_{p+1}e_1 - u_{p+1}e_2$ with

$$\begin{array}{rcl} (u_1,w_1) & = & (1,0), \\ (u_2,w_2) & = & (1,1), \\ (u_3,w_3) & = & (a+1)(u_2,w_2) - (u_1,w_1), \\ (u_4,w_4) & = & 2(u_3,w_3) - (u_2,w_2), \\ (u_5,w_5) & = & 2(u_4,w_4) - (u_3,w_3), \\ & \vdots & \\ (u_p,w_p) & = & 2(u_{p-1},w_{p-1}) - (u_{p-2},w_{p-2}), \\ (u_{p+1},w_{p+1}) & = & (b+1)(u_p,w_p) - (u_{p-1},w_{p-1}) \end{array}.$$

Furthermore, we note that the action $\varphi : \mathbb{T}^2 \times \mathbb{C}^{p+1} \longrightarrow \mathbb{C}^{p+1}$ given by

$$\varphi((t_1,t_2),(z_1,\ldots,z_{p+1})) = (t_1z_1, t_1^{u_2}t_2^{w_2}z_2, \ldots, t_1^{u_p}t_2^{w_p}z_p, t_1^{u_{p+1}}t_2^{w_{p+1}}z_{p+1})$$

gives to each irreducible component of Y the structure of a toric surface. Therefore, Y is a bitoric surface and the result follows from Theorem 2.1. ∎

As we saw in this section, we have some classes of surfaces that are multitoric and also determinantal, but not all multitoric surface is determinantal.

EXAMPLE 3.2. Let $\sigma \subset \mathbb{R}^2$ be the cone generated by vectors $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where p, q are positive integers satisfying $0 < q < p$ and $(p,q) = 1$ such that the continued fraction of p by $p - q$ is

$$\frac{p}{p-q} = a - \frac{1}{2 - \frac{1}{b}}$$

with a, b positive integers such that $a, b \geq 2$. The toric surface $X_\sigma \subset \mathbb{C}^5$, associated with σ, is determinantal. However, the bitoric surface $Y = X_\sigma \cup V(z_1z_3 - z_2^a, z_4, z_5)$ is not determinantal.

In Theorems 3.2, 3.3 and 3.4 we found a formula for the Euler obstruction of determinantal surfaces using known results for toric surfaces. In the next section, will use known tools of determinantal surfaces for obtain information about the Milnor number of toric surfaces with isolated singularity at the origin such that its codimension is 2 in $\mathbb{C}^4$.

4. SOME REMARKS ON THE MILNOR NUMBER OF TORIC SURFACES

In [17] J. Milnor obtained importants results about the topology of hypersurfaces with isolated singularities. For example he proved that the Milnor fiber of the germ of complex analytic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, with isolated singular point, has the homotopy type of a bouquet of spheres of real dimension n. The number of spheres of this bouquet is called *Milnor number* of f and denoted by $\mu(f)$.

One of actual problems in singularity theory is try to extend the definition of Milnor number. In [21] Pereira and Ruas, using topological methods, have introduced the Milnor number of determinantal varieties of codimension 2 in $\mathbb{C}^4$ and in $\mathbb{C}^5$ with isolated singularity at the origin, as follow.

Definition 4.1. Let $(X, 0) \subset (\mathbb{C}^r, 0)$ be the germ of a codimension 2 determinantal variety with isolated singularity at the origin, $dim(X) = 2, 3$. The Milnor number of X, denoted $\mu(X)$, is defined by $\mu(X) = b_d(X_t)$, where X_t is the generic fiber of X and $b_d(X_t)$ is the d-th Betti number of X_t, $d = dim(X)$.

Using [22, 23] we compute the Milnor number of all toric surface X_σ that is a complete intersection and also of all toric surface $X_\sigma \subset \mathbb{C}^4$.

Proposition 4.1. *Let $\sigma \subset \mathbb{R}^2$ be the cone generated by vectors $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where $0 < q < p$ and p, q are coprimes, then:*
1)If X_σ is a complete intersection, then $\mu(X_\sigma) = p - 1$;
2)If $X_\sigma \subset \mathbb{C}^4$, then $\mu(X_\sigma) = a_2 + a_3 - 3$, where a_2 and a_3 are the integers coming from the continued fraction of $\frac{p}{p-q}$, as described in [22, 23].

Proof. In [22] Riemenschneider proved that X_σ is a complete intersection with isolated singularity if and only if $e = 3$, that is if and only if σ is generated by $v_1 = e_2$ and $v_2 = pe_1 - (p-1)e_2$, where $p \in \mathbb{Z}$ e $p \geq 2$, and in this case, the ideal I_σ is generated by $z_1 z_3 - z_2^p$. Then we have 1).

Now, if $X_\sigma = V(I_\sigma) \subset \mathbb{C}^4$ Riemenschneider proved that the generators of I_σ are given by the 2×2 minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3^{a_3-1} \\ z_2^{a_2-1} & z_3 & z_4 \end{pmatrix}$$

Now, note that X_σ is a surface and we have $(2-2+1)(3-2+1) = 2 = codim(X_\sigma)$, therefore X_σ is determinantal and in [21] Pereira and Ruas show that the Milnor number of determinantal surfaces that are analytically equivalent to M is equal to $a_2 + a_3 - 3$. ∎

The above proposition allows us to compute the Milnor number of a toric surface X_σ of codimension 2 in $\mathbb{C}^4$ of a very simple way, as we can see in the following example.

EXAMPLE 4.1. Let $X_\sigma \subset \mathbb{C}^4$ be the toric surface associated to the cone $\sigma \subset \mathbb{R}^2$ generated by vectors $v_1 = e_2$ and $v_2 = 14e_1 - 11e_2$. From the continued fraction process we have

$$\frac{14}{3} = 5 - \frac{1}{3},$$

then $X_\sigma = V(I_\sigma)$ where I_σ is the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3^2 \\ z_2^4 & z_3 & z_4 \end{pmatrix}$$

therefore, using the Proposition 4.1 we have that $\mu(X_\sigma) = 5 + 3 - 3 = 5$.

As a continuation of this work, we intend to study invariants of functions defined on multitoric varieties. More specifically, we want to consider the Euler obstruction of a function [4] and the notion of Milnor number that arises from the work of Lê [13]. Other references to these invariants are [6, 8, 18].

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