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CADERNOS DE MATEMÁTICA **14**, 31–39 May (2013)
ARTIGO NÚMERO SMA#377

Equimultiplicity of topologically equisingular families of parametrized surfaces in $\mathbb{C}^3$

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We provide a positive answer to Zariski's conjecture for families of singular surfaces in $\mathbb{C}^3$, under the condition that the family has a smooth normalisation. As a corollary of the result, we obtain a surprising characterization of the Whitney equisingularity of one parameter families of $\mathcal{A}$ finitely determined map-germs $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, in terms of the constancy of only one invariant, the Milnor number of the double point locus. May, 2013 ICMC-USP

1. INTRODUCTION

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be germs of reduced holomorphic functions, $V_f = f^{-1}(0)$, $V_g = g^{-1}(0)$ the corresponding germs of hypersurfaces in $\mathbb{C}^n$. Zariski asked in [16] the following question: if the hypersurfaces V_f and V_g are topologically equivalent, is it true that the multiplicities $m_0(f)$ and $m_0(g)$ of f and g, respectively, are the same? The multiplicity of f (or of V_f) is the number of points of the intersection of V_f with a generic line passing close to the origin but not through the origin. The question asked by Zariski can also be formulated for families of hypersurfaces, that is, given a topologically equisingular family of hypersurfaces in $\mathbb{C}^n$, is it true that the family is equimultiple? These questions are still unsettled in general, but there are positive answers in a collection of special cases. (See [2] for a list of the main known results, [11] and [3] for some recent results on the subject.)

In this article we provide a positive answer to Zariski's conjecture for families of singular surfaces in $\mathbb{C}^3$, under the condition that the family has a smooth normalisation. As a corollary of the result, we obtain a surprising characterization of the Whitney equisingularity of one parameter families of $\mathcal{A}$ finitely determined map-germs $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, in terms of the constancy of only one invariant, the Milnor number of the double point locus. This question was posed and solved for a special class of map-germs in [12]. For corank one map-germs, Gaffney in ([5], Corollary 8.9) characterized the Whitney equisingularity of f_t in terms of the only invariant $e_D(f_t)$.

* Partially supported by CNPq grant # 303774/2008-8 and FAPESP grant # 08/54222-6.

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The inspiration for our results is the paper of J. Fernández de Bobadilla and M. Pe Pereira [4] in which they study topological equisingularity of a holomorphic family of reduced map germs by means of their normalisation. They introduce the notion of equisingularity at the normalisation for a family $\phi_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ and prove that in many cases, this implies the topological equisingularity of the family ϕ_t. We consider in this paper topologically equisingular families of singular surfaces in $\mathbb{C}^3$ whose normalisation is smooth. Building up on some of their results, and on formulas by Gaffney in [5], and by Marar, Nuño-Ballesteros and Peñaforte in [10], we prove the following results

Theorem 1.1. *Let $X = \Phi^{-1}(0)$ be a family of reduced hypersurfaces in $\mathbb{C}^3$, defined by $\Phi : (\mathbb{C}^3 \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$. We write $\phi_t(x) = \Phi(x, t)$ and $X_t = \phi_t^{-1}(0)$, which we also assume to be reduced for all sufficiently small t. Suppose that X has a smooth normalisation. Then, if X is topologically equisingular then $m_0(X_t)$ is constant.*

For a family of surfaces X in $\mathbb{C}^3$ whose normalisation is smooth, we can associate a family of parametrizations $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ whose images are X_t. In this situation, it follows from [4] that the $\mathcal{A}$ topological equisingularity of f_t implies that the Milnor numbers of the sets $f_t^{-1}(\Sigma_t)$ remain constant, where Σ_t is the singular set of X_t. The converse also holds with some additional hypothesis on X.

A particular class of parametrized singular surfaces consists on surfaces which are the image of an $\mathcal{A}$-finitely determined map-germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$, $F(x, y, t) = (f_t(x, y), t)$ be a one parameter unfolding of a finitely determined map-germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. In this case, the family of surfaces $X = F(\mathbb{C}^2 \times \mathbb{C})$ is topologically equisingular in a neighbourhood of 0 if and only $\mu(D^2(f_t))$ is constant, where $D^2(f_t)$ is the double point locus of f_t, for all sufficiently small $t \in \mathbb{C}$, and μ is the Milnor number of these sets.

The unfoldings F for which $\mu(D^2(f_t))$ are constant are called μ-constant unfoldings. They are excellent unfoldings, as defined by T. Gaffney in [5]. The topological triviality of F was obtained in [1] by integrating controlled vector fields tangent to the strata of the stratification of F given by the stable types in source and target. The topological triviality of μ- constant unfoldings also follows as a corollary of the main theorem of Fernández de Bobadilla and Pereira in [4].

A natural question is whether μ-constant in a one-parameter family F also implies the Whitney equisingularity of the family. This is indeed the case as we show in the following theorem which completely describes the equisingularity of F in terms of the equisingularity of $D^2(F)$, the double point locus in source, that happens to be a family of reduced plane curves. For families of reduced plane curves, it is well known that topological triviality and Whitney equisingularity are equivalent notions. It is very surprising that this is also true for families of singular surfaces in $\mathbb{C}^3$ parametrized by $\mathcal{A}$-finitely determined map germs:

Theorem 1.2. *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be a one-parameter unfolding of an $\mathcal{A}$-finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. The following statements are equivalent:*

i)$\mu(D^2(f_t))$ is constant for $t \in T$, where T is a small domain at 0 in $\mathbb{C}$;

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ii)F is Whitney equisingular.

This result improves Theorem 5.3 in [10].

2. NOTATIONS AND PREVIOUS RESULTS

Let $\Phi : (\mathbb{C}^3 \times \mathbb{C}, 0) \to (\mathbb{C}, 0), \Phi(0,t) = 0$ be the germ of a holomorphic function, and $X = \Phi^{-1}(0)$. For each $t \in (\mathbb{C}, 0)$ we write $\phi_t(x) = \Phi(x,t)$, and $X_t = {\phi_t}^{-1}(0)$. The function germ Φ defines a family of surfaces in $\mathbb{C}^3$. We assume that Φ and ϕ_t are reduced germs at the origin.

Definition 2.1. The family ϕ_t is *topologically equisingular* if there is a homeomorphism germ $H : (\mathbb{C}^3 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$, $H(x,t) = (h_t(x), t)$, $h_t(0) = 0$ such that $H(X) = X_0 \times \mathbb{C}$ at the origin, and $X_0 = {\phi_0}^{-1}(0)$.

The family Φ is $\mathcal{R}$-topologically trivial if, in addition, h_t preserves the fibers of ϕ_t.

As in [4] we relate these equisingularity notions with the *equisingularity at the normalisation.* Let

$$n : \hat{X} \to X$$

be the normalization mapping. We denote by $\hat{\Sigma}(X) = n^{-1}(\Sigma(X))$, where $\Sigma(X)$ is the singular set of X. If the context is clear, to simplify notation, we write Σ and $\hat{\Sigma}$ for the singular set of X and its inverse image by n, respectively.

For any subspace A of X, we denote the space $n^{-1}(A)$ by $\hat{A}$, and for any function $g : X \to Z$, the composition $\hat{g} = g \circ n : \hat{X} \to Z$ is denoted by $\hat{g}$.

Definition 2.2. The family ϕ_t is equisingular at the normalisation if there is a homeomorphism

$$\alpha : (\hat{X}, \hat{\Sigma}, \hat{T}) \to (\hat{X}_0, \hat{\Sigma}_0, n_0^{-1}(0)) \times T,$$

such that $\hat{\tau} = \tau \circ \alpha$, where $\tau : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ is the projection in the second factor.

We recall the following properties from [4].

Lemma 2.1. *([4] Lemma 3)*
If X is equisingular at the normalisation, then the following hold.

(i)The number of irreducible components of the germ $(X_t, 0)$ is independent of t.

(ii)The mapping $n_t : \hat{X}_t \to X_t$ is the topological normalisation, that is, topologically right-left equivalent to the normalisation mapping of X_t.

(iii)Each connected component of $\hat{X}_t$ contains a unique point of $n_t^{-1}(0)$.

(iv)For any $t \in T$, the space $(\hat{X}_t)_{red}$ is smooth outside $n_t^{-1}(0)$.

(v)For any $t \in T$, the curve $(\hat{\Sigma}_t)_{red}$ is smooth outside $n_t^{-1}(0)$.

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(vi)If $\hat{\Sigma}_t$ is a flat, reduced family of curves, then $(\hat{\Sigma}_t, n_t^{-1}(0))$ is a μ- constant family of multi-germs.

(vii)If $\hat{X}_t$ is equal to $\mathbb{C}^2$ for any t, then $\hat{\Sigma}_t$ is a μ-constant family.

Remark 2. 1. Let X be equisingular at the normalisation, $\hat{\Sigma}^1, \dots, \hat{\Sigma}^k$ the irreducible components of $\hat{\Sigma}$. Then the components of $\hat{\Sigma}_t$ can also be indexed by $\hat{\Sigma}_t^1, \dots, \hat{\Sigma}_t^k$, in the sense that $\hat{\Sigma}_t^i$ is contained in $\hat{\Sigma}^i$ for all $t \in T$.

3. THE RESULT

In this section we prove our main result, the conjecture of Zariski for topologically equisingular families of surfaces in $\mathbb{C}^3$, under the hypothesis that the family has smooth normalisation.

Theorem 3.1. *Let $X = \Phi^{-1}(0)$ be a family of reduced hypersurfaces in $\mathbb{C}^3$, defined by $\Phi : (\mathbb{C}^3 \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$. We write $\phi_t(x) = \Phi(x,t)$ and $X_t = \phi_t^{-1}(0)$, which we also assume to be reduced for all sufficiently small t. Suppose that X has a smooth normalisation. Then, if X is topologically equisingular then $m_0(X_t)$ is constant.*

The family X has a smooth normalisation. Then, after a change of coordinates, we can write the normalisation of X as $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$, so that X is the germ at the origin of the image $F(\mathbb{C}^2 \times \mathbb{C})$. We also write $F(x,y,t) = (f_t(x,y), t)$, $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, $f_0 = f$.

Let Σ be the singular set of X and $\hat{\Sigma} = F^{-1}(\Sigma)$ its inverse image in $\mathbb{C}^2 \times \mathbb{C}$. Let Σ_t denote the singular set of X_t and $\hat{\Sigma}_t = f_t^{-1}(\Sigma_t)$.

If X is topologically equisingular, then it follows from Lemma 5 in [4] that X is topologically equisingular at the normalisation. Notice that this implies that Theorem 3.1 holds if we replace *topologically equisingular* by *topologically equisingular at the normalisation.*

The following properties follow immediately from Lemma 2.1 (Lemma 3 in [4]).

Lemma 3.1.
If X is topologically equisingular, then the following hold.

(i)The mapping $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is topologically right-left equivalent to the normalisation mapping of X_t.

(ii)The family $\hat{\Sigma}_t$ is a μ-constant family of reduced curves.

We take a small representative $F : W \times T \to \mathbb{C}^3 \times T$, such that W is a neighborhood of 0 in $\mathbb{C}^2$, T is a domain at 0 in $\mathbb{C}$.

Let $l : \mathbb{C}^3 \to \mathbb{C}$ be a generic linear projection defining the hyperplane H_0, that is $H_0 = l^{-1}(0)$. We assume l is generic, so that $Y_0 = f(\mathbb{C}^2) \cap H_0$ is a generic reduced plane

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curve (so with isolated singularity at 0). In particular, f is transverse to H_0 away from zero. Let $Y_t = f_t(\mathbb{C}^2) \cap H_0$ be the family of germs of plane curves in X and $\tilde{Y}_t = f_t^{-1}(H_0)$ the corresponding family of germs of curves in $(\mathbb{C}^2, 0)$.

Taking a generic line in H_0, close to but not through the origin, we can easily see that

$$m_0(Y_t, 0) = m_0(f_t(\mathbb{C}^2), 0).$$

We claim that $\mu(Y_t, 0)$ is constant. Since Y_t is a family of plane curves, this implies that $m_0(Y_t, 0)$ is also constant.

We prove the claim in the following two propositions, which are direct applications respectively of the proof of Lemma 4, in [4], and Theorem A in [7].

Let $H_z = l^{-1}(z)$, $z \neq 0$, sufficiently close to 0. For a fixed $z \neq 0$, we consider the family of multigerms $(f_t(\mathbb{C}^2) \cap H_z, \Sigma_t \cap H_z)$, where $\Sigma_t \cap H_z$ are the source points of the multigerm. The inverse images $f_t^{-1}(H_z)$ define a family of multigerms of curves in W, at the points $\hat{\Sigma}_t \cap f_t^{-1}(H_z)$, denoted $(f_t^{-1}(H_z), (\hat{\Sigma}_t \cap f_t^{-1}(H_z))$. We may also use the notations $(F(W \times T) \cap (H_z \times T), \Sigma \cap (H_z \times T))$ and $(F^{-1}(H_z \times T), \hat{\Sigma} \cap F^{-1}(H_z \times T))$.

Proposition 3.1. *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be the normalization of $X = F(\mathbb{C}^2 \times \mathbb{C})$ in a neighbourhood of the origin . For sufficiently small $z \neq 0$ in $\mathbb{C}$, and $T' \subset T$, the family of multigerms of curves $(f_t(\mathbb{C}^2) \cap H_z, \Sigma_t \cap H_z)$ is μ-constant.*

Proof. The proof is just a particular case of the argument given in [4], Claim in Lemma 4, pages 883 and 884. We include the proof for completeness.

It is easy to see that for sufficiently small z, $f : \mathbb{C}^2 \to \mathbb{C}^3$ is transverse to H_z at any point in $\hat{\Sigma}_0 \setminus \{0\}$. Hence all components of the multigerm $(f^{-1}(H_z), (\hat{\Sigma}_0 \cap f^{-1}(H_z))$ are smooth.

Now, the family X, being topologically equisingular, is equisingular at the normalisation. Then, by Lemma 3.2., the family $\hat{\tau} : \hat{\Sigma} \to T$ is a topologically equisingular family of reduced curves. Hence,

$$\hat{\tau} : \hat{\Sigma} \cap F^{-1}(H_z \times T) \to T$$

is a local diffeomorphism near each point of $\hat{\Sigma}_0 \cap f^{-1}(H_z)$.

Now, the result follows from the following facts holding for the normalisation F:

- The mapping $F|F^{-1}(H_z \times T) : (F^{-1}(H_z \times T), F^{-1}(H_z \times T) \cap \hat{\Sigma}) \to T$ is a submersion in a neighbourhood U of $\hat{\Sigma}_0 \cap f^{-1}(H_z)$ in $F^{-1}(H_z \times T)$. Taking local charts for the submersion, we can choose a sufficiently small neighbourhood U of $\hat{\Sigma}_0 \cap f^{-1}(H_z)$ in $F^{-1}(H_z \times T)$ and a neighbourhood T' of 0 in T such that the projection

$$(U, \hat{\Sigma} \cap U) \to T'$$

is a locally trivial fibration whose fiber is diffeomorphic to the disjoint union of punctured discs $(D, *)$.

- Write $F(U, \hat{\Sigma} \cap U) = (V, \Sigma \cap (H_z \times T))$. The mapping

$$F : (U, \hat{\Sigma} \cap U) \to (V, \Sigma \cap (H_z \times T'))$$

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is a (non-embedded) simultaneous resolution of singularities.

Then it follows that the multigerm of curves $(V_t \cap H_z, \Sigma_t \cap H_z)$ has constant Milnor number, where for each t, $\mu(f_t(\mathbb{C}^2) \cap H_z)$ is the sum of the Milnor numbers at each point in $\Sigma_t \cap H_z$ (see [14], [4] for details).

We remark that the irreducible components of the multigerm $(V_t, \Sigma_t \cap H_z)$ are in 1 to 1 correspondence with the discs in U_t and then its cardinality is independent of t. We can give indices to these components, in a way that for any index i, and any $t, t' \in T'$ the components V_t^i and $V_{t'}^i$ are images of discs lying in the same component of U. ∎

Proposition 3.2.
With the above notation, the family of germs of plane curves at the origin $\tilde{Y}_t = f_t^{-1}(H_0)$ *has constant Milnor number.*

Proof. Let $g_t : W \to \mathbb{C}$ be the representative of the family of germs of analytic functions $g_t = l \circ f_t$, then $\tilde{Y}_t = g_t^{-1}(0)$. The non-zero level sets of g_t are the curves $f_t^{-1}(H_z)$, $z \neq 0$, in $\mathbb{C}^2$. It follows from the above proposition that for sufficiently small $z \neq 0$, the level sets of g_t are smooth for $z \neq 0$. In other words, all singularities of g_t are in $g_t^{-1}(0)$. Then we can apply Theorem A in [7] to get that 0 is the unique singularity of g_t in W for sufficiently small t. Then $\mu(g_t)$ is constant (see for instance [6]). ∎

To conclude the proof of Theorem 3.1, we now want to prove that the family of generic sections $Y_t = f_t(\mathbb{C}^2) \cap H_0$ at the origin in X_t is a μ constant family of curves. This will imply $m_0(Y_t)$ is constant, hence $m_0(f_t(\mathbb{C}^2, 0))$ is also constant.

Since the family X is equisingular at the normalisation, it follows that for each t sufficiently small, in the diagram

$$(\mathbb{C}^2, 0) \xrightarrow{f_t} (\mathbb{C}^3, 0) \xrightarrow{l} (\mathbb{C}, 0)$$

f_t is the normalisation of the surface $X_t = f_t(\mathbb{C}^2)$ and $l|_{X_t} : X_t \to \mathbb{C}$ is a flat family of plane curves.

As above, $g_t = l \circ f_t$, $\tilde{Y}_t = g_t^{-1}(0)$, and to simplify notation we denote $(Y_t)_z = f_t(\mathbb{C}^2) \cap H_z$. Then we can apply the following formula of Lejeune and Teissier (see [8])

$$\delta(Y_t) = \delta(\tilde{Y}_t) + \delta(Y_t)_z.$$

The invariant δ is upper semicontinuous, hence the constancy of $\delta(Y_t)$ follows from the previous propositions. In fact, Y_t and $\tilde{Y}_t$ are reduced germs of curves at the origin hence, $\mu(Y_t) = 2\delta(Y_t) - r + 1$ and $\mu(\tilde{Y}_t) = 2\delta(\tilde{Y}_t) - r + 1$ as a consequence of Milnor's formula. Notice that from Proposition 3.2 we get that $\mu(\tilde{Y}_t)$ is constant, hence the number of branches r of the family does not depend on t. Moreover $f_t|_{\tilde{Y}_t} : \tilde{Y}_t \to Y_t$ is a 1-1 map-germ at the origin, for each t sufficiently small, hence both families have the same number of branches. The singular points of the generic fiber $(Y_t)_z$ are the points $p_t \in \Sigma_t \cap H_z$, and the Milnor formula $\mu((Y_t)_z, p_t) = 2\delta((Y_t)_z, p_t) + r_p - 1$ holds for each point $p_t \in \Sigma_t \cap H_z$, where r_p

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is the number of irreducible branches through p_t, which clearly does not depend on t. We now use Proposition 3.1 to get that $\delta((Y_t)_z)$ remains constant.

4. WHITNEY EQUISINGULARITY OF FAMILIES $F_T : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$

By Mather-Gaffney's criterion [15], a map-germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is $\mathcal{A}$-finitely determined if and only if we can find a sufficiently small neighbourhood W of 0, such that $f^{-1}(0) \cap W = \{0\}$ and the only singularities of $f(W) \setminus \{0\}$ are transverse double points.

Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$, $F(x, y, t) = (f_t(x, y), t)$, $f_t(0, 0) = 0$ be a one parameter unfolding of f.

We take a small representative $F : W \times T \to \mathbb{C}^3 \times T$, such that W is a neighborhood of 0 in $\mathbb{C}^2$, T is a domain at 0 in $\mathbb{C}$. Notice that F is the normalisation map of $F(\mathbb{C}^2 \times \mathbb{C})$ at the origin.

Let $D^2(F) \subset \mathbb{C}^2 \times \mathbb{C}$, and $F(D^2(F)) \subset \mathbb{C}^3 \times \mathbb{C}$ be the double point set of F in the source and its image by F, respectively. We also write $D^2(f_t) \subset \mathbb{C}^2$ and $f_t(D^2(f_t)) \subset \mathbb{C}^3$ for the double point sets of f_t. See [9] for definitions. According to the notations introduced in the previous section, $D^2(F) = \hat{\Sigma}$ and $F(D^2(F)) = \Sigma$.

Gaffney defined in [5] the *excellent unfoldings*. An excellent unfolding has a natural stratification whose strata in the complement of the parameter space T are the stable types in source and target. For families F as above, the strata in the source are the following

$$\left\{\mathbb{C}^2 \setminus D^2(F), D^2(F) \setminus T, T\right\}.$$

In the target, the strata are:

$$\left\{\mathbb{C}^3 \setminus F(\mathbb{C}^2 \times \mathbb{C}), F(\mathbb{C}^2 \times \mathbb{C}) \setminus \overline{F(D^2(F))}, F(D^2(F)) \setminus T, T\right\}.$$

Notice that F preserves the stratification, that is, F sends a stratum into a stratum.

Definition 4.1. An unfolding F as above is Whitney equisingular if the above stratifications in source and target are Whitney equisingular along T.

By Thom' s second isotopy lemma for complex analytic maps, it follows that any Whitney equisingular unfolding is topologically trivial (see [5] for details).

Recall that the unfolding F is μ-constant if $\mu(D^2(f_t))$ does not depend on t.

The following results are known:

Theorem 4.1 (Theorem 8.7, [5], Theorem 5.2, [1]). *Let F be a one parameter unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. If F is μ- constant, then F is an excellent unfolding.*

Theorem 4.2 (Theorem 5.3, [10]). *Let F be an unfolding of a finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. Then F is Whitney equisingular if and only if $\mu(D^2(f_t))$ and $\mu_1(f_t(\mathbb{C}^2), 0)$ are constant.*

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The invariant $\mu_1(f(\mathbb{C}^2),0)$ is defined in [10] as follows. Let H_0 be a generic hyperplane in $\mathbb{C}^3$, $Y_0 = f(\mathbb{C}^2) \cap H_0$ and $\tilde{Y}_0 = f^{-1}(H_0)$. For a generic H_0, $(Y_0, 0)$ and $(\tilde{Y}_0, 0)$ are germs of reduced plane curves. Then, $\mu_1(f(\mathbb{C}^2),0) = \mu(Y_0,0)$.

We now state and prove the following

Theorem 4.3. *Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be a one-parameter unfolding of an $\mathcal{A}$-finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. The following statements are equivalent:*

(a)$\mu(D^2(f_t))$ is constant for $t \in T$, where T is a small domain at 0 in $\mathbb{C}$;
(b)F is Whitney equisingular.

Proof. The condition $(b) \Rightarrow (a)$ is clear. So, we prove here that $(a) \Rightarrow (b)$. The condition $\mu(D^2(f_t))$ constant implies equisingularity at the normalisation, and the topological equisingularity of F ([4]). From the results of the previous section, it follows that $Y_t = f_t(\mathbb{C}^2) \cap H_0$ at the origin is a family of reduced plane curves at the origin, and $\delta(Y_t)$ is constant. Then $\mu(Y_t)$ is also constant and we saw above that $\mu(Y_t) = \mu_1(f_t(\mathbb{C}^2))$. The result now follows from Theorem 4.2. ∎

Acknowlegment: I am grateful to J.J. Nuño-Ballesteros for many helpful conversations.

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