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The center problem for a $2:-3$ resonant cubic Lotka–Volterra system

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In this paper we obtain conditions on the coefficients of a cubic Lotka-Volterra system of the form

$$\begin{aligned} \dot{x} &= x(2 - a_{20}x^2 - a_{11}xy - a_{02}y^2), \\ \dot{y} &= y(-3 + b_{20}x^2 + b_{11}xy + b_{02}y^2), \end{aligned} \tag{0.1}$$

which fulfillment yields the existence in a neighborhood of the origin of a first integral of the form $\phi(x, y) = x^3y^2 + h.o.t.$, in which case the origin is termed a $2 : -3$ resonant center. This system was studied in [13], where, due to computational constrains, the consideration was limited to the cases where either one or both coefficients a_{11}, b_{11} in system (0.1) were equal to zero, or both coefficients were equal to 1. Here we are studying the case where the coefficient a_{11} is equal to 1 and b_{11} is arbitrary. The obtained results represent the study of the center problem for general system (0.1), since by a linear substitution any system of the form (0.1) can be transformed either to system (0.1) with $a_{11} = 1$ or to one of systems studied in [13]. Computation of the resonant saddle quantities (focus quantities) and the decomposition of the variety of the ideal generated by an initial string of them were used to obtain necessary conditions of integrability and the theory of Darboux first integrals and some other methods to show the sufficiency. Since the decompositions of the variety mentioned above was performed using modular computations the obtained 19 conditions of integrability represent the complete list of the integrability conditions only with very high probability and there remains an open problem to verify this statement. May, 2013 ICMC-USP

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1. INTRODUCTION

In this paper we consider polynomial vector fields in $\mathbb{C}^2$ with a $p:-q$ resonant elementary singular point, i.e.,

$$\dot{x} = p\,x + P(x,y), \quad \dot{y} = -q\,y + Q(x,y), \tag{1.1}$$

where $p, q \in \mathbb{Z}$ with $p, q > 0$ and P and Q are polynomials. For such type of systems there arises the problem to determine if the elementary singular point located at the origin is a resonant center. A resonant center is a generalization of the concept of a real center to systems of ordinary differential equations in $\mathbb{C}^2$ of the form (1.1), see for instance [4, 22]. The classical real center problem goes back to Poincaré and Lyapunov, see [17, 20], and has been studied extensively in hundreds of works, see for instance [3, 11, 12, 22] and references therein. We have the following definition of a resonant center coming from Dulac [7].

Definition 1.1. A $p:-q$ resonant elementary singular point of analytic system (1.1) is a resonant center if there exists a local meromorphic first integral of the form

$$\Phi = x^q y^p + \sum_{k+s>p+q} \phi_{ks} x^k y^s. \tag{1.2}$$

The most simple case is when P and Q in (1.1) are quadratic polynomials. The integrability problem for the $1:-2$ resonance case is solved in [9] where 20 cases of the integrability are given. Also in this particular case of $1:-2$ resonance the linearizability problem is solved in [4] where necessary and sufficient conditions (15 cases) are given. Some sufficient center conditions for the $p:-q$ resonant singular point of the quadratic system are known in the literature, while other resonances in the general quadratic case are open. The most studied case is the quadratic Lotka-Volterra system, i.e.,

$$\dot{x} = x(1 + ax + by), \quad \dot{y} = y(-\lambda + cx + dy), \qquad (\lambda > 0). \tag{1.3}$$

Necessary and sufficient conditions for integrability and linearizability for $1:-\lambda$ resonance with $\lambda \in \mathbb{N}$ of system (1.3) are known, see [4] and references therein. In [14] the methods introduced in [4] and [9] for $\lambda \in \mathbb{N}$ allow to produce certain sufficient conditions for general λ. Moreover in [14] some methods are given which allow to find the complete set of necessary and sufficient conditions for integrability and linearizability of system (1.3) in the particular case $\lambda = p/2$ or $\lambda = 2/p$ with $p \in \mathbb{N}^+$. In [18] sufficient conditions for integrable Lotka-Volterra systems (1.3) with $3:-q$ resonance are given. In the particular cases of $3:-5$ and $3:-4$ resonances, necessary and sufficient conditions for integrability of system (1.3) are obtained.

The next studied case is the one when P and Q are homogeneous cubic polynomials. In such case the $1:-3$ resonant centers on $\mathbb{C}^2$ were studied in [16]. Moreover in [2] the necessary conditions and distinct sufficient conditions are derived for $1:-q$ Lotka-Volterra resonant systems i.e., systems of the form $\dot{x} = x(1 - a_{20}x^2 - a_{11}xy - a_{02}y^2)$, $\dot{y} = y(-q + b_{20}x^2 + b_{11}xy + b_{02}y^2)$. Recently, integrability of the Lotka-Volterra type systems of degree 4 was studied in [19].

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All studied cases involve rather laborious calculations related to decompositions of affine varieties defined by the resonant saddle quantities (which are also called focus quantities [22]). The complexity of calculations for different pairs p and q is difficult to estimate in advance and it depends on the structure of the resonant saddle quantities as described in Section 3 of [22]. Often decompositions of varieties of ideals generated by resonant saddle quantities cannot be performed in polynomial rings of characteristic 0, but it becomes possible replacing the ring $\mathbb{Q}[A]$ (where A denotes the coefficients of system (1.1)) by the ring $\mathbb{Z}_p[A]$ where p is a prime number. Although the modular approach (which for the first time was successfully applied to study such kind of problems in [8, 9]) does not provide a rigorous proof of the completeness of the decomposition it is practically sure that the obtained decomposition is complete. The probability of the opposite event is almost zero (the reader can consult [1] for more details on the modular approach).

In this paper we focus our attention to the homogeneous cubic $2 : -3$ resonant Lotka-Volterra systems, i.e., systems of the form (0.1)

$$\begin{aligned}\dot{x} &= x(2 - a_{20}x^2 - a_{11}xy - a_{02}y^2),\\ \dot{y} &= y(-3 + b_{20}x^2 + b_{11}xy + b_{02}y^2).\end{aligned}$$

This system was studying in [13], where due to computational constrains, the consideration was limited to the cases where either one or both coefficients a_{11}, b_{11} in system (0.1) were equal to zero, or both coefficients were equal to 1.

The aim of our present paper is to complete the study of the integrability problem for system (0.1). Here we are considering the case where the coefficient a_{11} is equal to 1 and b_{11} is arbitrary. In [13] the authors were not able to find the decomposition of the variety of the ideal generated by focus quantities. Now using more powerful computational facilities we believe that we are able to perform this task. By a linear transformation any system (0.1) can be transformed either to a system with $a_{11} = 1$ or to one of systems considered in [13]. Moreover, the conditions presented in Theorem 1.1 for the case when $a_{11} = 1$ and b_{11} is arbitrary give us the sufficient conditions for integrability of system (0.1) and the necessary conditions with high probability, as it is discussed in the proof of the theorem and at the end of this paper.

Theorem 1.1. *System (0.1) with $a_{11} = 1$ and b_{11} arbitrary has a center if one of the following conditions holds:*

1. $b_{11} = 0, -6 + a_{20}b_{02} = 0, b_{20} = 0, 3a_{02} + 4b_{02} = 0$;
2. $b_{11} = 0, 6 + a_{20}b_{02} = 0, b_{20} = 0, a_{02} - b_{02} = 0$;
3. $-7 + 2b_{11} = 0, -15 + 4b_{02}b_{20} = 0, a_{02} - b_{02} = 0, 5a_{20} + 2b_{20} = 0$;
4. $-23 + 4b_{11}, -105 + 16b_{02}b_{20} = 0, a_{02} - b_{02} = 0, 5a_{20} + 2b_{20} = 0$;
5. $13 + b_{11} = 0, -30 + b_{02}b_{20} = 0, a_{02} - b_{02} = 0, 5a_{20} + 2b_{20} = 0$;
6. $-21 + 8b_{11} = 0, -45 + 64b_{02}b_{20} = 0, a_{02} - b_{02} = 0, 5a_{20} + 2b_{20} = 0$;
7. $-10 + b_{02}b_{20}, -11 + b_{11}, a_{02} - 5b_{02} = 0, 5a_{20} + 2b_{20} = 0$;
8. $360 - 516b_{11} + 240b_{11}^2 - 36b_{11}^3 + 396a_{02}b_{20} - 52b_{02}b_{20} - 306a_{02}b_{11}b_{20} + 24b_{02}b_{11}b_{20} + 54a_{02}b_{11}^2b_{20} + 27a_{02}^2b_{20}^2 - 9a_{02}b_{02}b_{20}^2 = 0, a_{20} - b_{20} = 0$;

9. $-7+2b_{11}=0, -3+4b_{02}b_{20}, a_{02}-b_{02}, a_{20}-2b_{20}=0$;

10. $-72+49b_{02}b_{20}=0, -3+b_{11}=0, a_{02}-b_{02}, 3a_{20}-b_{20}=0$;

11. $-18+7b_{02}b_{20}=0, -30+7b_{11}=0, a_{02}-b_{02}=0, 3a_{20}-b_{20}=0$;

12. $-21+8b_{11}=0, 64b_{20}b_{02}-45=0, a_{02}-b_{02}=0, 5a_{20}-6b_{20}=0$;

13. $b_{11}=0, 18+b_{02}b_{20}=0, 296964\,a_{02}+395952\,b_{02}+5345352\,b_{02}b_{11}+419004\,b_{02}b_{11}^2+394200\,b_{02}b_{11}^3-354707\,b_{02}b_{11}^4=0, a_{20}+3b_{20}=0$;

14. $-34+11b_{11}=0, 121a_{20}b_{02}-42=0, a_{02}-b_{02}=0, 5a_{20}-b_{20}=0$;

15. $64b_{20}b_{02}-189=0, a_{02}-b_{02}=0, -27+8b_{11}=0, 9a_{20}+2b_{20}=0$;

16. $b_{02}=0, -2+b_{11}, a_{02}=0$;

17. $b_{02}=0, -3+b_{11}=0, a_{02}=0$;

18. $a_{20}=b_{20}=0$;

19. $-3a_{02}-4b_{02}+3a_{02}b_{11}+2b_{02}b_{11}=0, -3a_{20}+a_{20}b_{11}-2b_{20}+2b_{11}b_{20}=0, 3a_{02}a_{20}+a_{20}b_{02}-2b_{02}b_{20}=0$;

Proof. *Computation of the conditions.* Following the approach described in [13, 22] and using a straightforward modification of the computer code in [22, Figure 6.1], we compute 12 resonant saddle quantities $g_{3,2}, \ldots, g_{36,24}$, where $g_{q(2k+1),p(2k+1)}=0$ for $k=0,\ldots,5$, $g_{6,4}=(1512a_{11}^4a_{20}+216a_{02}a_{11}^2a_{20}^2+36a_{11}^2a_{20}^2b_{02}-1764a_{11}^3a_{20}b_{11}-288a_{02}a_{11}a_{20}^2b_{11}-72a_{11}a_{20}^2b_{02}b_{11}+672a_{11}^2a_{20}b_{11}^2+72a_{02}a_{20}^2b_{11}^2+20a_{20}^2b_{02}b_{11}^2-84a_{11}a_{20}b_{11}^3+1008a_{11}^4b_{20}+2196a_{02}a_{11}^2a_{20}b_{20}+63\,a_{02}^2\,a_{20}^2\,b_{20}+576\,a_{11}^2\,a_{20}\,b_{02}\,b_{20}+21\,a_{02}\,a_{20}^2\,b_{02}\,b_{20}-1848\,a_{11}^3\,b_{11}\,b_{20}-1386a_{02}a_{11}a_{20}b_{11}b_{20}-272a_{11}a_{20}b_{02}b_{11}b_{20}+1008a_{11}^2b_{11}^2b_{20}+198a_{02}a_{20}b_{11}^2b_{20}+20a_{20}b_{02}b_{11}^2b_{20}-168a_{11}b_{11}^3b_{20}+360a_{02}a_{11}^2b_{20}^2+126a_{02}^2a_{20}b_{20}^2-976a_{11}^2b_{02}b_{20}^2-468a_{02}a_{11}b_{11}b_{20}^2++512a_{11}b_{02}b_{11}b_{20}^2+108a_{02}b_{11}^2b_{20}^2-40b_{02}b_{11}^2b_{20}^2-84a_{02}b_{02}b_{20}^3)/504$. The other polynomials are too large to be presented here, however they can be easily computed using any available computer algebra systems. Then, we set in the obtained polynomials $a_{11}=1$.

To obtain the necessary conditions for integrability we need to find the set of all parameters a_{ks}, b_{sk} where all polynomials $g_{6,4}, g_{12,8}, \ldots, g_{36,24}$ vanish, that is to find the decomposition of the variety of the ideal $\mathcal{B}_{12}=\langle g_{6,4}, g_{12,8}, \ldots, g_{36,24}\rangle$ (the ideal is defined by 6 nonzero polynomials).

This is a difficult computational problem requiring an efficient software and powerful computers. The computational tool we used is the routine *minAssGTZ* [6] of the computer algebra system SINGULAR [15] which computes the decomposition using the method described in [10].

As it is well known such computations are usually too laborious and cannot be completed working in the field of rational numbers. However the approach based on making use of modular computations as described in [21] turned out to be very efficient.

Computing the decomposition in the ring $\mathbb{Q}_{32003}[a_{20}, a_{02}, b_{20}, b_{11}, b_{02}]$ we obtain 19 components. Applying now the rational reconstruction algorithm of [23] (an implementation in Mathematica is presented in [13]) we obtain the components 1)–11), 13), 17)–19) listed in the theorem and 3 “fake” components, which in $\mathbb{Z}_{32003}[a_{20}, a_{02}, b_{20}, b_{11}, b_{02}]$ are the following:

12’) $b_{11}-4003, b_{20}b_{02}+7500, -14434\,b_{11}^4b_{02}-7677\,b_{11}^3b_{02}-13988\,b_{11}^2b_{02}+12573\,b_{11}b_{02}+a_{02}+3136\,b_{02}, a_{20}+12800\,b_{20}$;

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14') $b_{11} + 8725, b_{20}b_{02} - 5027, 5872\, b_{11}^4 b_{02} + 13284\, b_{11}^3 b_{02} + 1265\, b_{11}^2 b_{02} + 10327\, b_{11} b_{02} + a_{02} - 1917\, b_{02}, a_{20} + 12801\, b_{20}$;
15') $b_{11} + 3997, b_{20}b_{02} - 503, 3691\, b_{11}^4 b_{02} + 3091\, b_{11}^3 b_{02} - 15426\, b_{11}^2 b_{02} + 13175\, b_{11} b_{02} + a_{02} + 15576\, b_{02}, a_{20} + 7112\, b_{20}$.

The lifting to the field of rational numbers using the algorithm of [23] gives:

12") $b_{11} - \frac{21}{8}, b_{02}b_{20} - \frac{45}{64}, -\frac{65 b_{02} b11^4}{51} + \frac{93 b_{02} b_{11}^3}{25} + \frac{10 b_{02} b_{11}^2}{151} + \frac{11 b_{02} b_{11}}{28} + a_{02} - \frac{79 b_{02}}{51}, a_{20} - \frac{6 b_{20}}{5}$;
14") $b_{11} - \frac{34}{11}, b_{02}b_{20} + \frac{143}{70}, -\frac{12 b_{02} b_{11}^4}{109} - \frac{14 b_{02} b_{11}^3}{53} + \frac{131 b_{02} b_{11}^2}{76} + \frac{107 b_{02} b_{11}}{31} + a_{02} + \frac{159 b_{02}}{50}, a_{20} - \frac{b_{20}}{5}$;
15") $b_{11} - \frac{27}{8}, b_{02}b_{20} + \frac{125}{127}, -\frac{43 b_{02} b_{11}^4}{26} + \frac{153 b_{02} b_{11}^3}{145} - \frac{13 b_{02} b_{11}^2}{139} - \frac{46 b_{02} b_{11}}{17} + a_{02} - \frac{77 b_{02}}{113}, a_{20} + \frac{2 b_{20}}{9}$.

However the check with the radical membership test shows that under each of conditions 12"), 14"), 15") not all generators of $\mathcal{B}_{12}$ vanish, so these conditions are not correct. Usual approach in decomposition of varieties using modular computations in such situation is to recompute the decomposition with a bigger prime number. However since such recomputing is time-consuming we use our empirical observation that usually "simple" polynomials of "fake" components are correct. So we add to the ideal $\mathcal{B}_{12}$ two polynomials of 12") – $b_{11} - \frac{21}{8}$ and $a_{20} - \frac{6b_{20}}{5}$ – and compute the decomposition with *minAssGTZ* in the ring $\mathbb{Q}[a_{20}, a_{02}, b_{20}, b_{11}, b_{02}]$ obtaining condition 12) of the theorem. In similar way conditions 14) and 15) are derived.

Since modular computation and the ad hoc choice of polynomials (as described above) were involved we have to check the correctness of the obtained decomposition. Following the procedure described in details in [21] using the function *intersect* of SINGULAR we compute

$$J = \cap_{k=1}^{19} J_k, \tag{1.4}$$

where J_k are ideals defined by conditions 1)-19) of Theorem 1.1. Then, using the radical membership test, we verify that each polynomial of $\mathcal{B}_{12}$ vanishes on $\mathbf{V}(J)$. That means,

$$\mathbf{V}(J) \subset \mathbf{V}(\mathcal{B}_{12}) \tag{1.5}$$

and, therefore, the condition of 1)–19) of the theorem are necessary conditions for integrability of system (0.1) with $a_{11} = 1$ and b_{11} arbitrary.

Proof of sufficiency. In the cases 1, 2 and 13 we have $b_{11} = 0$ so they satisfy the condition β (4, 3 and 5 respectively) of Theorem 3 in [13]. The case 8 and 18 (where $a_{20} = b_{20}$) were considered in the proof of Theorem 2 in [13] where the authors applied a blow up of the singularity to complete the proof.

Case 3. In this case system (0.1) takes the form

$$\dot{x} = x(2 - a_{20}x^2 - a_{11}xy - a_{02}y^2), \qquad \dot{y} = y(-3 + b_{20}x^2 + b_{11}xy + b_{02}y^2).$$

This system has four algebraic invariant curves: $l_1 = x,\ l_2 = y$ and

$$l_3 = 1 + \frac{3x^2}{4b_{02}} + \frac{xy}{2},$$

$$l_4 = 1 + \frac{3x^2}{2b_{02}} + \frac{9x^4}{16b_{02}^2} - 5xy + \frac{9x^3y}{4b_{02}} - \frac{b_{02}y^2}{3} + 3x^2y^2 + \frac{4}{3b_{02}}xy^3,$$

which allow to construct a Darboux integrating factor of the form $\mu = l_2^{-\frac{1}{3}} l_3^{-\frac{2}{3}} l_4^{-\frac{5}{6}}$. By Theorem 4.13 in [4], there exists an analytic integral in the form $\Phi = x^3y^2 + h.o.t.$

Case 4. Here, the corresponding system is

$$\dot{x} = 2x + \frac{21x^3}{8b_{02}} - x^2y - b_{02}xy^2, \qquad \dot{y} = -3y + \frac{105x^2y}{16b_{02}} + \frac{23xy^2}{4} + b_{02}y^3,$$

and it has four algebraic invariant curves: $l_1 = x,\ l_2 = y$ and

$$l_3 = 1 + \frac{21x^2}{16b_{02}} + \frac{xy}{2},$$

$$l_4 = 1 + \frac{21x^2}{8b_{02}} + \frac{441x^4}{256b_{02}^2} - \frac{13xy}{2} + \frac{147x^3y}{32b_{02}} + \frac{49x^2y^2}{16},$$

yielding the integrating factor $\mu = l_1^5 l_2^3 l_3^{-2} l_4^{-\frac{7}{2}}$, and, therefore, a first integral $\Phi = x^3y^2 + h.o.t.$

Case 5. The system of this case is written as

$$\dot{x} = 2x + \frac{12x^3}{b_{02}} - x^2y - b_{02}xy^2, \qquad \dot{y} = -3y + \frac{30x^2y}{b_{02}} - 13xy^2 + b_{02}y^3.$$

It has three algebraic invariant curves: $l_1 = x,\ l_2 = y$ and

$$l_3 = 1 + \frac{12x^2}{b_{02}} + \frac{36x^4}{b_{02}^2} + 16xy - \frac{24x^3y}{b_{02}} + 4x^2y^2$$

and it is possible to compute an integrating factor of the form $\mu = l_1^5 l_2^3 l_3^{-\frac{9}{2}}$. Integration yields the first integral $\Phi = -b_{02}/525000 + (-\frac{y^4}{2} + b_{02}\frac{y^6}{6} + O(y^7))x^6 + O(x^7)$, where by $O(x^k)$ $(O(y^k))$ we mean an analytic function which series expansion starts from terms of order k.

Case 6. The corresponding system is

$$\dot{x} = 2x + \frac{9x^3}{32b_{02}} - x^2y - b_{02}xy^2, \qquad \dot{y} = -3y + \frac{45x^2y}{64b_{02}} + \frac{21xy^2}{8} + b_{02}y^3.$$

It has five algebraic invariant curves: $l_1 = x$, $l_2 = y$,

$$l_3 = 1 + \frac{9x^2}{32b_{02}} + \frac{81x^4}{4096b_{02}^2} + \frac{3xy}{8} + \frac{87x^3y}{512b_{02}} + \frac{3x^2y^2}{32},$$

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$$l_4 = 1 + \frac{9x^2}{32b_{02}} + \frac{81x^4}{4096b_{02}^2} - \frac{11xy}{4} + \frac{81x^3y}{256b_{02}} + \frac{81x^2y^2}{64}$$

$$l_5 = 1 + \frac{9x^2}{32b_{02}} + \frac{81x^4}{4096b_{02}^2} - 3xy + \frac{21x^3y}{64b_{02}} - \frac{b_{02}y^2}{3} + \frac{93x^2y^2}{64} + \frac{3}{4}b_{02}xy^3.$$

The system has a first integral given by $\Phi = l_1^6 l_2^4 l_3^{-2} l_4^{-3} l_5$ or $\Phi = (y^4 - \frac{b_{02}y^6}{3} + O(y^9))x^6 + (\frac{9y^5}{2} - \frac{7b_{02}y^7}{4} + O(y^9))x^7 + (-\frac{9y^4}{8b_{02}} + \frac{963y^6}{64} - \frac{25b_{02}y^8}{4} + O(y^9))x^8 + O(x^9)$.

Case 7. The corresponding system in this case is

$$\dot{x} = 2x + \frac{4x^3}{b_{02}} - x^2y - 5b_{02}xy^2, \qquad \dot{y} = -3y + \frac{10x^2y}{b_{02}} + 11xy^2 + b_{02}y^3. \tag{1.6}$$

For system (1.6) we are not able to find more invariant curves except of the coordinate axes and using these curves $l_1 = x$ and $l_2 = y$ we are not able to find a Darboux first integral or a Darboux integrating factor. We use another method to prove the sufficiency in this case. We apply to system (1.6) the following transformation $Y = y/x$, $X = (x + b_{02}y)^3/x$ and we get the system

$$\dot{X} = 4b_{02}X + 8X^2 - 11b_{02}^2XY, \qquad \dot{Y} = -5b_{02}Y + 6XY - 5b_{02}^2Y^2, \tag{1.7}$$

after scaling time by the factor $b_{02}(1 + b_{02}Y)$. System (1.7) which is an integrable Lotka-Volterra system according to Theorem 13 in [5]. In [5] the quadratic Lotka-Volterra systems are written in the form $\dot{x} = x(1 + ax + by)$, $\dot{y} = y(-\lambda + cx + dy)$. Hence system (1.7) has $a = 2/b_{02}$, $b = -11b_{02}/4$, $c = 3/(2b_{02})$, $d = -5b_{02}/4$ and $\lambda = 5/4$ and satisfies statement (A_n) of Theorem 13 in [5] because $\lambda + c/a = 2 \in \mathbb{N}$. Therefore system has a first integral of the form $\Phi = X^5Y^4 + h.o.t.$ However the transformation $Y = y/x$, $X = (x + b_{02}y)^3/x$ does not give the correct first integral of the original system. Therefore we use monodromy arguments to obtain the result. For proving statement (A_n) of Theorem 13 in [5] it is showed that the monodromy of the finite critical points $P_1 = (-1/a, 0)$ and $P_2 = (0, \lambda/d)$ on the X-axis (resp. Y-axis) is the identity if $\lambda + c/a = n$ (resp. $(b/d + 1/\lambda = n)$ with $n \in \mathbb{N}$, $n \geq 2$. Moreover in the study of the singular point at infinity, it is considered the chart $(u, z) \to (Y/X, 1/X)$ and the intersection of the line at infinity with the X-axis gives the point $P_X = (0, 0)$. This critical point at infinity is a node which is always linearizable since there are two analytic separatrices. Hence applying Theorem 9 in [5] we obtain that the origin of system (1.7) is integrable. In our case the other two ratio of eigenvalues on the X axis are $\lambda + c/a = 2$ for P_1 and $(c-a)/(-a) = 1/4$ (at infinity of P_X). The first gives identity monodromy and the second gives a linearizable monodromy since there are two analytic separatrices passing though the critical point at infinity. Finally the monodromy around a small curve in the x-axis surrounding the origin of system (1.6) gets taken to the monodromy in the X-axis about the origin of system (1.7). The latter is integrable and hence so must be the former.

Case 9. In this case the system (0.1) takes the form

$$\dot{x} = 2x - \frac{3x^3}{2b_{02}} - x^2y - b_{02}xy^2, \qquad \dot{y} = -3y + \frac{3x^2y}{4b_{02}} + \frac{7xy^2}{2} + b_{02}y^3.$$

It has five algebraic invariant curves: $l_1 = x$, $l_2 = y$,

$$l_{3,4} = \pm\frac{6\sqrt{b_{02}} + 3\sqrt{3}x + 2\sqrt{3}b_{02}y}{6\sqrt{b_{02}}},$$

$$l_5 = -\frac{-4b_{02} + 3x^2 + 4b_{02}xy}{4b_{02}},$$

which allow to construct a Darboux integrating factor of the form $\mu = l_1^{-\frac{5}{2}} l_2^{-2} (l_3 l_4)^{-\frac{5}{4}} l_5^{\frac{3}{4}}$. By Theorem 4.13 in [4], there exists an analytic integral of the form $\Phi = x^3y^2 + h.o.t.$

Case 10. Here, the corresponding system is

$$\dot{x} = 2x - \frac{24x^3}{49b_{02}} - x^2y - b_{02}xy^2, \qquad \dot{y} = -3y + \frac{72x^2y}{49b_{02}} + 3xy^2 + b_{02}y^3,$$

and it has four algebraic invariant curves: $l_1 = x,\ l_2 = y$ and

$$l_3 = 1 - \frac{12x^2}{49b_{02}} - \frac{2xy}{7},$$

$$l_4 = 1 - \frac{24xy}{7} + \frac{384x^3y}{343b_{02}} - \frac{b_{02}y^2}{3} + \frac{96x^2y^2}{49} + \frac{6}{7}b_{02}xy^3,$$

yielding the integrating factor $\mu = l_2^{-\frac{1}{3}} l_3^{-\frac{1}{2}} l_4^{-\frac{5}{6}}$, and, therefore, a first integral $\Phi = x^3y^2 + h.o.t.$

Case 11. The system of this case is written as

$$\dot{x} = 2x - \frac{6x^3}{7b_{02}} - x^2y - b_{02}xy^2, \qquad \dot{y} = -3y + \frac{18x^2y}{7b_{02}} + \frac{30xy^2}{7} + b_{02}y^3.$$

It has four algebraic invariant curves: $l_1 = x, l_2 = y$,

$$l_3 = 1 - \frac{3x^2}{7b_{02}} - \frac{2xy}{7},$$

$$l_4 = 1 - \frac{30xy}{7} + \frac{120x^3y}{49b_{02}} + \frac{15x^2y^2}{7},$$

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and it is possible to compute an integrating factor of the form $\mu = l_1^5 l_2^3 l_3^2 l_4^{-\frac{7}{2}}$. By Theorem 4.13 in [4], there exists an analytic first integral of the form $\Phi = x^3y^2 + h.o.t.$

Case 12. Without loss of generality we can assume that $b_{02} = 1$ obtaining the system

$$\dot{x} = 2x - \frac{27x^3}{32} - x^2y - xy^2, \quad \dot{y} = -3y + \frac{45x^2y}{64} + \frac{21xy^2}{8} + y^3. \tag{1.8}$$

System (1.8) admits an invariant quadratic curve $f = -\frac{27x^2}{64} - \frac{3xy}{4} - \frac{y^2}{3} + 1$. Using the substitution $X = x/f^{1/2}$, $Y = y/f^{1/2}$ and after a rescaling of time we obtain the system

$$\dot{X} = 2X + \frac{27X^3}{32} + \frac{X^2Y}{8} - \frac{4XY^2}{3}, \quad \dot{Y} = -3Y + \frac{9X^2Y}{32} - Y^3.$$

Now doing the linear transformation $u = -X/8,\ v = Y$ we obtain a particular case of case 5 (β) of Theorem 3 of [13], which is an integrable system. Therefore system (1.8) is integrable as well.

Case 14. As above we can set $b_{02} = 1$. Then conditions of this case yield the system

$$\dot{x} = 2x - \frac{42x^3}{121} - x^2y - xy^2, \qquad \dot{y} = -3y + \frac{210x^2y}{121} + \frac{34xy^2}{11} + y^3. \tag{1.9}$$

In addition to the invariant lines $l_1 = x, l_2 = y$ system (1.9) admits the algebraic invariant curve of degree two

$$l_3 = -\frac{21x^2}{121} - \frac{2xy}{11} + 1,$$

and of degree six

$$l_4 = -\frac{28224x^5y}{161051} - \frac{7056x^4y^2}{14641} - \frac{588x^3y^3}{1331} + \frac{2352x^3y}{1331} - \frac{49x^2y^4}{363} + \frac{329x^2y^2}{121} + \frac{226xy^3}{231} - \frac{42xy}{11} - \frac{y^2}{3} + 1,$$

which provide the Darboux integrating factor $\mu = l_2^{-\frac{1}{3}} l_3^{\frac{1}{6}} l_4^{-\frac{5}{6}}$ yielding a first integral of the required form (1.2).

Case 15. As above we can assume that $b_{02} = 1$. Then the system is written as

$$\dot{x} = 2x + \frac{21x^3}{32} - x^2y - xy^2, \quad \dot{y} = -3y + \frac{189x^2y}{64} + \frac{27xy^2}{8} + y^3.$$

It has the Darboux integrating factor of the form $\mu = l_2^{-\frac{1}{3}} l_3^{-\frac{1}{2}} l_4^{-\frac{5}{6}}$, where $l_2 = y$,

$$l_3 = \frac{21x^2}{64} + \frac{xy}{4} + 1,$$

and

$$l_4 = \frac{9261x^6}{262144} + \frac{1323x^5y}{8192} + \frac{567x^4y^2}{2048} + \frac{1323x^4}{4096} + \frac{27x^3y^3}{128} + \frac{105x^3y}{64} + \frac{27x^2y^4}{448} + \frac{159x^2y^2}{64} + \frac{63x^2}{64} + \frac{33xy^3}{28} - \frac{9xy}{2} - \frac{y^2}{3} + 1,$$

and, therefore, a first integral of the form (1.2).

Case 16. In this case the system (0.1) takes the form

$$\dot{x} = 2x - a_{20}x^3 - x^2y, \qquad \dot{y} = -3y + b_{20}x^2y + 2xy^2. \tag{1.10}$$

Now we do the substitutions $v = xy$ and $w = x^2$ and in these new coordinates system (1.10) becomes

$$\dot{v} = v(1 - v - (b_{20} - a_{20})w), \qquad \dot{w} = w(-4 + 2v + 2a_{20}w). \tag{1.11}$$

System (1.11) is a quadratic Lotka-Volterra system. These systems (as we mentioned in the introduction) were studied in Theorem 7.1 of [4] where the linearizable and integrable quadratic Lotka-Volterra systems written into the form

$$\dot{x} = x(1 + c_{20}x + c_{11}y), \qquad \dot{y} = y(-\lambda + d_{11}x + d_{02}y). \tag{1.12}$$

for $\lambda \in \mathbb{N} \setminus \{1\}$ are classified. In fact system (1.11) corresponds to a particular case of the case (A_m) which satisfy the condition $mc_{20} + d_{11} = 0$, for $m = 0, \ldots, \lambda - 2$. In fact, in system (1.11) we have $c_{20} = -1$ and $d_{11} = 2$ and therefore corresponds to the case with $m = 2$. Consequently, system (1.11) has a linearizable saddle at the origin, i.e., there exist an analytic change of coordinates

$$v_1 = v(1 + O(v, w)) = xy(1 + O(x, y)), \qquad w_1 = w(1 + O(v, w)) = x^2(1 + O(x, y)),$$

which brings the system into a linear saddle. This linear saddle has the first integral $v_1^4 w_1$ which pulls back to a first integral of the form $x^6y^4(1 + O(x, y))$. Extracting the root of this first integral we obtain a first integral of the form $x^3y^2(1 + O(x, y))$ for system (1.10).

Case 17. The corresponding system is

$$\dot{x} = 2x - a_{20}x^3 - x^2y, \qquad \dot{y} = -3y + b_{20}x^2y + 3xy^2. \tag{1.13}$$

Following the same substitutions $v = xy$ and $w = x^2$ we arrive to the system

$$\dot{v} = v(1 - 2v - (b_{20} - a_{20})w), \qquad \dot{w} = w(-4 + 2v + 2a_{20}w). \tag{1.14}$$

Now system (1.14) corresponds to a particular case of the case (A_m) which satisfy the condition $mc_{20} + d_{11} = 0$ with $m = 1$. A similar reasoning as in case 16 shows the existence of a first integral (1.2).

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Case 19. Here, the corresponding system is

$$\dot{x} = 2x - a_{20}x^3 - x^2y - \frac{4b_{02}xy^2}{3(-1+b_{11})} + \frac{2b_{02}b_{11}xy^2}{3(-1+b_{11})},$$

$$\dot{y} = -3y + \frac{3a_{20}x^2y}{2(-1+b_{11})} - \frac{a_{20}b_{11}x^2y}{2(-1+b_{11})} + b_{11}xy^2 + b_{02}y^3,$$

and it has three algebraic invariant curves: $l_1 = x,\ l_2 = y$ and $l_3 = 1 - \frac{a_{20}x^2}{2} + xy - b_{11}xy - \frac{b_{02}y^2}{3}$. Integration yields the first integral

$$\Phi = l_1^3 l_2^2 l_3^{\frac{3-2b_{11}}{b_{11}-1}} = x^3y^2\left(1 - \frac{a_{20}x^2}{2} - \frac{1}{3}y(3(-1+b_{11})x + b_{02}y)\right)^{-2+\frac{1}{b_{11}-1}}.$$

∎

Although Theorem 1.1 gives 19 sufficient conditions for integrability of system (0.1) with $a_{11} = 1$ we have the following conjecture:

Conjecture 1.1.
The conditions 1)–19) of Theorem 1.1 are the necessary and sufficient conditions for integrability of system (0.1) with $a_{11} = 1$.

To check this statement it is sufficient to prove the inclusion, which is opposite to (1.5), that is, to show that

$$\mathbf{V}(\mathcal{B}_{12}) \subset \mathbf{V}(J), \tag{1.15}$$

where the ideal J is defined by (1.4). To this end we have to check that for each polynomial $j \in J$ a reduced Groebner basis of $\langle 1 - jw, \mathcal{B}_{12}\rangle$ (where w is a new variable) is equal to $\{1\}$.

We were not able to complete computations over the field of rationals numbers using the available computational facilities, however calculations over fields $\mathbb{Z}_p$ with randomly chosen characteristics 83, 32003 and 179595127 always yield $\{1\}$. It indicates that with very high probability (see e.g. [1]) (1.15) holds and, therefore, the conjecture is correct.

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