Lyapunov theorems for measure functional differential equations via Kurzweil-equations

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Abstract

We consider measure functional differential equations (we write measure FDEs) of the form $Dx = f(x_t, t)Dg$, where f is Perron-Stieltjes integrable, x_t is given by $x_t(\theta) = x(t+\theta), \theta \in [-r, 0]$, with $r > 0$, and Dx and Dg are the distributional derivatives in the sense of the distribution of L. Schwartz, with respect to functions $x : [t_0, \infty) \to \mathbb{R}^n$ and $g : [t_0, \infty) \to \mathbb{R}$, $t_0 \in \mathbb{R}$, and we present new concepts of stability of the trivial solution, when it exists, of this equation. The new stability concepts generalize, for instance, the variational stability introduced by Š. Schwabik and M. Federson for FDEs and yet we are able to establish a Lyapunov-type theorem for measure FDEs via theory of generalized ordinary differential equations (also known as Kurzweil equations).

Keywords: Measure functional differential equations, generalized ordinary differential equations, stability, Kurzweil-Henstock-Stieltjes integral, Lyapunov functionals.

1 Introduction

An initial value problem for a measure functional differential equation (we write measure FDEs, for short) can be given in the form

$$\begin{cases} Dx = f(x_t, t)Dg, \\ x_{t_0} = \phi, \end{cases} \tag{1.1}$$

where x_t is given by $x_t(\theta) = x(t+\theta), \theta \in [-r, 0]$, with $r > 0$ and Dx and Dg are the distributional derivatives, in the sense of the distribution of L. Schwartz, with respect to the functions x and g.

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The integral form corresponding to (1.1) is given by

$$\begin{cases} x(t) = x(t_0) + \displaystyle\int_{t_0}^{t} f(x_s, s)\,\mathrm{d}g(s), \quad t \geq t_0, \\ x_{t_0} = \phi, \end{cases} \tag{1.2}$$

where we consider the integral in the sense of Perron-Stieltjes taken with respect to a nondecreasing function $g : [t_0, \infty) \to \mathbb{R}$.

We introduce some new concepts of stability for the trivial solution of measure FDEs which generalize some concepts presented in the literature as the variational stability for FDEs (see [8] and [9]). We also present a correspondence between measure FDEs and generalized ODEs, using more general conditions than those presented in [6]. Then, we introduce some new concepts of stability for the trivial solution of generalized ODEs and relate these new concepts to the corresponding ones for measure FDEs. Finally, we prove Lyapunov-type theorems for generalized ODEs and, by the correspondence between generalized ODEs and measure FDEs, we prove Lyapunov-type theorems for measure FDEs.

The present paper is organized as follows. The second section is devoted to of the presentation new stability concepts for measure FDEs. In the third section, we present a correspondence between generalized ODEs and measure FDEs, generalizing the results from [6] and also, we present existence-uniqueness theorems for solutions of these equations. The fourth section is devoted to stability concepts for generalized ODEs. In the fifth section, we establish a correspondence between the new stability definitions for the trivial solution of a generalized ODEs and a measure FDE of the above type. The sixth section is devoted to prove some Lyapunov theorems for generalized ODEs. Finally, in the last section, we prove, using the correspondences between solutions, Lyapunov theorems for measure FDEs. The paper also contains two appendixes which describe the basis of Perron integration and the fundamental results of the theory of generalized ODEs.

2 Measure functional differential equations

In this section, we introduce new concepts of stability for measure FDEs. Let $r,\ t_0 \in \mathbb{R}$, with $r > 0$ and consider the following problem

$$Dx = f(x_t, t)Dg, \tag{2.1}$$

where x_t is given by the formula $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$, $r > 0$ and Dx, Dg are the distributional derivatives, in the sense of distributions of L. Schwartz (see [3]), of regulated functions $x : [t_0, \infty) \to \mathbb{R}^n$ and $g : [t_0, \infty) \to \mathbb{R}$. In particular, we consider g nondecreasing and left-continuous.

Let $a \in \mathbb{R}$. By $G([a, \infty), \mathbb{R}^n)$ we denote the space of regulated functions from $[t_0, \infty)$ to $\mathbb{R}^n$ with the topology of uniform convergence. By regulated function we mean that the left and right limits at a point $t \geq a$ exist whenever they can be defined. Then by $BG([a,\infty),\mathbb{R}^n)$ we denote the subspace of $G([a,\infty),\mathbb{R}^n)$ of bounded functions.

Next, we define a special set of functions in $BG([t_0-r,\infty),\mathbb{R}^n)$ with a property which we call the prolongation property.

Definition 2.1. Let O be an open subset of $BG([t_0-r,\infty),\mathbb{R}^n)$. We say that O has the prolongation property, if for every $y \in O$ and every $\bar{t} \in [t_0-r,\infty)$, the function $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), & t_0 - r \leq t \leq \bar{t}, \\ y(\bar{t}), & \bar{t} < t < \infty \end{cases}$$

is also an element of O.

Now, having a set $O \subset BG([t_0-r,\infty),\mathbb{R}^n)$ with the prolongation property, we also consider the set

$$S = \{y_t;\ y \in O,\ t \in [t_0,\infty)\} \subset BG([-r,0],\mathbb{R}^n)$$

and we assume that the function $f : S \times [t_0,\infty) \to \mathbb{R}^n$ on the right-hand side of the measure FDE (2.1) is such that

$$f(0,t) = 0 \text{ for every } t \in [t_0,\infty)$$

so that $y \equiv 0$ is a solution of (2.1) on $[t_0-r,\infty)$, and the following conditions are satisfied:

(H_1) For all $y \in O$, the mapping $t \mapsto f(y_t,t)$ is Perron-Stieltjes integrable with respect to $g : [t_0,\infty) \to \mathbb{R}$ which we assume to be nondecreasing.

(H_2) There exists a function $M : [t_0,\infty) \to \mathbb{R}$ which is locally Lebesgue-Stieltjes integrable with respect to g such that the inequality

$$\left| \int_{\gamma}^{v} f(y_s,s)\mathrm{d}g(s) \right| \leq \int_{\gamma}^{v} M(s)\mathrm{d}g(s)$$

holds for every $y \in O$ and every $\gamma, v \in [t_0,\infty)$

(H_3) There exists a function $L : [t_0,\infty) \to \mathbb{R}$ which is locally Lebesgue-Stieltjes integrable with respect to g such that the inequality

$$\left| \int_{\gamma}^{v} [f(y_s,s) - f(z_s,s)]\mathrm{d}g(s) \right| \leq \int_{\gamma}^{v} L(s)\|y_s - z_s\|_\infty \mathrm{d}g(s)$$

holds for every $y, z \in O$ and every $\gamma, v \in [t_0,\infty)$.

Clearly, the integral form of (2.1) is given by

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\, \mathrm{d}g(s),\ t \geq t_0, \tag{2.2}$$

where the integral on the right-hand side is in the sense of Perron-Stieltjes integral taken with respect to $g : [t_0, \infty) \to \mathbb{R}$ which we are assuming to be nondecreasing and left-continuous.

In the following lines, we recall the classical definitions of Lyapunov stability, uniform (Lyapunov) stability and uniform asymptotic stability of the trivial solution of (2.1). See [11], for instance.

Definition 2.2. The trivial solution of system (2.1) is called *Lyapunov stable*, if for every $\varepsilon > 0$ and $\gamma \in \mathbb{R}$, there exists $\delta = \delta(\varepsilon, \gamma) > 0$ such that if $\phi \in S$ and $\overline{y} : [\gamma, v] \to \mathbb{R}^n$, with $[\gamma, v] \subset [t_0, \infty)$ and $[\gamma, v] \ni t_0$, is a solution of (2.1) such that $\overline{y}_\gamma = \phi$ and

$$\|\phi\|_\infty < \delta,$$

then

$$\|\overline{y}_t(\gamma, \phi)\|_\infty < \varepsilon, \quad t \in [\gamma, v].$$

Definition 2.3. The trivial solution of system (2.1) is called *uniformly stable*, if the number δ in Definition 2.2 is independent of γ.

Definition 2.4. The solution $y \equiv 0$ of (2.1) is called *uniformly asymptotically stable*, if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$, there exists a $T = T(\varepsilon, \delta_0) \geq 0$ such that if $\phi \in S$, and $\overline{y} : [\gamma, v] \to \mathbb{R}^n$, with $[\gamma, v] \subset [t_0, \infty)$ and $[\gamma, v] \ni t_0$, is solution of (2.1) such that $\overline{y}_\gamma = \phi$ and

$$\|\phi\|_\infty < \delta_0,$$

then

$$\|\overline{y}_t(\gamma, \phi)\|_\infty < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, \infty).$$

Now, we consider the perturbed measure FDE

$$Dy = f\,(y_t, t)\, Dg + p(t)Du, \quad t \geq t_0 \tag{2.3}$$

where $p : [t_0, \infty) \to \mathbb{R}$. Again, we consider $g, u : [t_0, \infty) \to \mathbb{R}$ to be nondecreasing and left-continuous.

The solution of (2.3) has to be interpreted as a solution of the integral equation

$$y\,(t) = y(t_0) + \int_{t_0}^{t} f\,(y_s, s)\, \mathrm{d}g(s) + \int_{t_0}^{t} p(s) \mathrm{d}u(s), \quad t \geq t_0 \tag{2.4}$$

where the integrals are in Perron-Stieltjes sense.

We assume that the conditions (H_1), (H_2) and (H_3) are fulfilled and that the function $p : [t_0, \infty) \to \mathbb{R}^n$ satisfies:

(H_4) The Perron-Stieltjes integral $\int_{t_0}^{t} p(s)\mathrm{d}u(s)$ exists for every $t \in [t_0, \infty)$;

(H_5) There exists a function $K : [t_0, \infty) \to \mathbb{R}$ which is locally Lebesgue-Stieltjes integrable with respect to u such that

$$\left| \int_{\gamma}^{v} p(s)\mathrm{d}u(s) \right| \leq \int_{\gamma}^{v} K(s)\mathrm{d}u(s),$$

for all $\gamma, v \in [t_0, \infty)$.

Under conditions (H_1) to (H_5), a solution y of (2.4) is regulated and left continuous, that is, $y \in G^-([t_0, \infty), \mathbb{R}^n)$. See [6] for a proof of this fact.

The following definitions are based on the definitions given by A. Halanay in [10] concerning integral stability.

Definition 2.5. The solution $y \equiv 0$ of (2.1) is said to be *integrally stable*, if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $\psi \in S$ with $\|\psi\|_\infty < \delta$ and

$$\sup_{t \in [\gamma, v]} \left| \int_{\gamma}^{t} p(s)\mathrm{d}u(s) \right| < \delta,$$

where $t_0 \leq \gamma \leq v < \infty$, then

$$\|\bar{y}_t(\gamma, \psi)\|_\infty < \varepsilon, \quad \text{for every} \quad t \in [\gamma, v],$$

where $\bar{y}(t; \gamma, \psi)$ is a solution of the perturbed equation (2.3) with $\bar{y}_\gamma = \psi$.

Definition 2.6. The solution $y \equiv 0$ of (2.1) is called *integrally attracting*, if there is a $\widetilde{\delta} > 0$ and for every $\varepsilon > 0$, there exist a $T = T(\varepsilon) \geq 0$ and a $\rho = \rho(\varepsilon) > 0$ such that if

$$\|\psi\|_\infty < \widetilde{\delta} \qquad \text{and} \qquad \sup_{t \in [\gamma, v]} \left| \int_{\gamma}^{t} p(s)\mathrm{d}u(s) \right| < \rho,$$

where $t_0 \leq \gamma \leq v < \infty$, then

$$\|\bar{y}_t(\gamma, \psi)\|_\infty < \varepsilon \;\text{ for all }\; t \geq \gamma + T,\; t \in [\gamma, v],$$

where $\bar{y}(t; \gamma, \psi)$ is a solution of the equation (2.4) satisfying $\bar{y}_\gamma = \psi$.

Definition 2.7. The solution $y \equiv 0$ of (2.1) is called *integrally asymptotically stable*, if it is integrally stable and integrally attracting.

Remark 2.8. The reader may note that, if the solution $y \equiv 0$ of (2.1) is integrally stable, then it is uniformly stable. An analogue assertion holds for the asymptotic stability, that is, if the solution $y \equiv 0$ of (2.1) is integrally asymptotically stable, then it is uniformly asymptotically stable.

In the next section, we present a correspondence between generalized ODEs and measure FDEs. The results we obtain generalize the corresponding ones from [6].

3 Measure RFDEs regarded as generalized ODEs

Suppose the function $f : S \times [t_0, \infty) \to \mathbb{R}^n$ satisfies conditions (H_1), (H_2) and (H_3) and $p : [t_0, \infty) \to \mathbb{R}^n$ satisfies conditions (H_4) and (H_5). Moreover assume, as before, that the functions $u : [t_0, \infty) \to \mathbb{R}$ and $g : [t_0, \infty) \to \mathbb{R}$ are nondecreasing (not necessarily left-continuous).

Let $BG^{-}([t_0 - r, \infty), \mathbb{R}^n)$ denote the set of functions from $BG([t_0 - r, \infty), \mathbb{R}^n)$ which are left-continuous. Let O a subset of $BG^{-}([t_0 - r, \infty), \mathbb{R}^n)$ with the prolongation property.

For $y \in O$ and $t \in [t_0, \infty)$, define

$$F(y,t)(\vartheta) = \begin{cases} 0, & t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\int_{t_0}^{\vartheta} f(y_s, s)\mathrm{d}g(s), & t_0 \leq \vartheta \leq t < \infty, \\ \displaystyle\int_{t_0}^{t} f(y_s, s)\mathrm{d}g(s), & t \leq \vartheta < \infty \end{cases} \tag{3.1}$$

and

$$P(t)(\vartheta) = \begin{cases} 0, & t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\int_{t_0}^{\vartheta} p(s)\mathrm{d}u(s), & t_0 \leq \vartheta \leq t < \infty, \\ \displaystyle\int_{t_0}^{t} p(s)\mathrm{d}u(s), & t \leq \vartheta < \infty. \end{cases} \tag{3.2}$$

Then,

$$G(y,t) = F(y,t) + P(t) \tag{3.3}$$

defines an element $G(y,t)$ of $BG^{-}([t_0 - r, \infty), \mathbb{R}^n)$ and $G(y,t)(\vartheta) \in \mathbb{R}^n$ is the value of $G(y,t)$ at a point $\vartheta \in [t_0 - r, \infty)$, that is,

$$G : O \times [t_0, \infty) \to BG^{-}([t_0 - r, \infty), \mathbb{R}^n).$$

In order to carry on with our correspondence results, we now need to consider a special type of generalized ODEs, whose definition is presented in details in the Appendix B (see Definition B.1).

Consider the following generalized ODE

$$\frac{dx}{d\tau} = DG(x,t), \tag{3.4}$$

where the function G is given by (3.3).

Let $h : [t_0, \infty) \to \mathbb{R}$ be defined by

$$h(t) = \int_{t_0}^{t} [M(s) + L(s)]\mathrm{d}g(s) + \int_{t_0}^{t} K(s)\mathrm{d}u(s), \quad t \in [t_0, \infty). \tag{3.5}$$

Clearly the function h is nondecreasing (and left-continuous whenever u and g do so).

Under the above assumptions, it is a matter of routine to prove that the function G given by (3.3) belongs to the class $\mathcal{F}(\Omega, h)$, given by Definition B.3 in Appendix B, with $\Omega = O \times [t_0, \infty)$.

Let $\sigma > 0$ and consider the space $G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ of regulated functions from $[t_0 - r, t_0 + \sigma]$ to $\mathbb{R}^n$. The next result can be carried out as in the proof of Lemma 3.7 from [6], with obvious adaptations.

Lemma 3.1. *Let O be a subset of $G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ with the prolongation property and $S = \{y_t;\ y \in O,\ t \in [t_0, t_0 + \sigma]\}$. Assume that $\phi \in S$, $g, u : [t_0, t_0 + \sigma] \to \mathbb{R}$ are nondecreasing functions, $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ is such that the Perron-Stieltjes integral $\int_{t_0}^t f(y_s, s)\, \mathrm{d}g(s)$ exists for every $y \in O$ and $t \in [t_0, t_0 + \sigma]$ and $p : [t_0, t_0 + \sigma] \to \mathbb{R}^n$ is such that the Perron-Stieltjes integral $\int_{t_0}^t p(s)\, \mathrm{d}u(s)$ exists. Consider G given by (3.3) and assume that $x : [t_0, t_0 + \sigma] \to O$ is a solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DG(x, t)$$

with initial condition $x(t_0)(\vartheta) = \phi(\vartheta - t_0)$ for $\vartheta \in [t_0 - r, t_0]$, and $x(t_0)(\vartheta) = x(t_0)(t_0)$ for $\vartheta \in [t_0, t_0 + \sigma]$. If $v \in [t_0, t_0 + \sigma]$ and $\vartheta \in [t_0 - r, t_0 + \sigma]$, then

$$x(v)(\vartheta) = x(v)(v), \quad \vartheta \geq v, \tag{3.6}$$

and

$$x(v)(\vartheta) = x(\vartheta)(\vartheta), \quad v \geq \vartheta. \tag{3.7}$$

Note that if the generalized ODE (3.4) admits a global solution, then Lemma 3.1 can be extended to an unbounded interval.

The next result, namely Theorem 3.2, gives a correspondence between the solution of an initial value problem for a generalized ODEs and the solution of an initial value problem for a measure FDE. Theorem 3.2 generalizes Theorems 3.8 and 3.9 from [6] which, together with [17] inspired our proof presented here.

Theorem 3.2. *Consider $O \subset G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ with the prolongation property, let $S = \{x_t;\ x \in O,\ t \in [t_0, t_0 + \sigma]\}$ and $\phi \in S$. Assume that $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ and $u : [t_0, t_0 + \sigma] \to \mathbb{R}$ are nondecreasing, $f : S \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions (H_1), (H_2), (H_3) and $p : [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions (H_4) and (H_5). Let G be given by (3.3).*

(i) *Let $y : [t_0 - r, t_0 + \sigma] \to O$ be a solution of the measure functional differential equation*

$$Dy = f(y_t, t)Dg + p(t)Du, \quad t \in [t_0, t_0 + \sigma], \tag{3.8}$$

with initial condition $y_{t_0} = \phi$. *For every* $t \in [t_0 - r, t_0 + \sigma]$, *let*

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in [t_0 - r, t], \\ y(t), & \vartheta \in [t, t_0 + \sigma]. \end{cases}$$

Then the function $x : [t_0, t_0 + \sigma] \to O$ *is a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DG(x, t)$$

with G *given by* (3.3) *and initial condition*

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & \text{for } \vartheta \in [t_0 - r, t_0], \\ x(t_0)(t_0), & \text{for } t_0 \leq \vartheta < t_0 + \sigma. \end{cases} \tag{3.9}$$

(ii) *Reciprocally, if* $x : [t_0, t_0 + \sigma] \to O$ *is a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DG(x, t),$$

with initial condition

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & t_0 - r \leq \vartheta \leq t_0, \\ x(t_0)(t_0), & t_0 \leq \vartheta < t_0 + \sigma, \end{cases}$$

then the function $y : [t_0 - r, t_0 + \sigma] \to O$ *defined by*

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \leq \vartheta \leq t_0 \\ x(\vartheta)(\vartheta), & t_0 \leq \vartheta < t_0 + \sigma \end{cases}$$

is a solution of the measure functional differential equation

$$Dy = f(y_t, t)Dg + p(t)Du, \quad t \in [t_0, t_0 + \sigma], \tag{3.10}$$

with initial condition $y_{t_0} = \phi$.

Proof. We need to show that, for every $v \in [t_0, t_0 + \sigma]$, the integral $\int_{t_0}^{v} DG(x(\tau), t)$ exists and

$$x(v) - x(t_0) = \int_{t_0}^{v} DG(x(\tau), t).$$

Let an arbitrary $\varepsilon > 0$ be given. Define the function

$$h(t) = \int_{t_0}^{t} [L(s) + M(s)]\mathrm{d}g(s) + \int_{t_0}^{t} K(s)\mathrm{d}u(s), \quad t \in [t_0, t_0 + \sigma].$$

Since the functions g and u are nondecreasing, they can have only a finite number of points $t \in [t_0, v]$ such that $\Delta^+ g(t) := \lim_{s \to t_+} g(s) - g(t) = g(t+) - g(t) \geq \varepsilon$ and $\Delta^+ u(t) \geq \varepsilon$. The same statement remains true for the function h. Thus h can have only a finite numbers of points $t \in [t_0, v]$ such that $\Delta^+ h(t) \geq \varepsilon$.

Note that the points which satisfies the inequality for the function h are the union of the points which satisfy the analogous inequalities for functions g and u (it follows directly by the definition). Let us denote these points by $t_1, \ldots, t_m$.

Let us define a function $\delta : [t_0, t_0 + \sigma] \to (0, \infty)$, called gauge of $[t_0, t_0 + \sigma]$. We take *delta* such that

$$\delta(\tau) < \min \left\{ \frac{t_k - t_{k-1}}{2}, \; k = 2, \ldots, m \right\}, \quad \tau \in [t_0, t_0 + \sigma]$$

and

$$\delta(\tau) < \min \{ |\tau - t_k|, |\tau - t_{k-1}|; \; \tau \in (t_{k-1}, t_k), \quad k = 2, \ldots, m \}, \quad \tau \in [t_0, t_0 + \sigma].$$

These conditions assure that if a point-interval pair $(\tau, [c, d])$ is δ-fine, then $[c, d]$ contains at most one of the points $t_1, \ldots, t_m$, and moreover, $\tau = t_k$ whenever $t_k \in [c, d]$.

Since $y_{t_k} = x(t_k)_{t_k}$, it follows from Theorem A.5 in Appendix A that

$$\lim_{s \to t_k +} \int_{t_k}^{s} L(s) \| y_s - x(t_k)_s \|_\infty \, \mathrm{d}g(s) = L(t_k) \| y_{t_k} - x(t_k)_{t_k} \|_\infty \Delta^+ g(t_k) = 0$$

for every $k \in \{1, \ldots, m\}$. Thus the gauge δ might be chosen in such a way that

$$\int_{t_k}^{t_k + \delta(t_k)} L(s) \| y_s - x(t_k)_s \|_\infty \, \mathrm{d}g(s) < \frac{\varepsilon}{2m + 1}, \quad k \in \{1, \ldots, m\}.$$

Using Theorem A.5 again, we have

$$|y(\tau + t) - y(\tau)| = \left| \int_{\tau}^{\tau + t} f(y_s, s) \mathrm{d}g(s) + \int_{\tau}^{\tau + t} p(s) \mathrm{d}u(s) \right| \leq$$

$$\leq \int_{\tau}^{\tau + t} M(s) \mathrm{d}g(s) + \int_{\tau}^{\tau + t} K(s) \mathrm{d}u(s) \leq h(\tau + t) - h(\tau),$$

and, therefore,

$$|y(\tau +) - y(\tau)| \leq \Delta^+ h(\tau) < \varepsilon, \quad \tau \in [t_0, t_0 + \sigma] \backslash \{t_1, \ldots, t_m\}.$$

Thus we can assume that the gauge δ is such that

$$|y(\rho) - y(\tau)| \leq \varepsilon,$$

for every $\tau \in [t_0, t_0+\sigma]\backslash\{t_1,\dots,t_m\}$ and $\rho \in [\tau, \tau+\delta(\tau))$.

Assume, now, that $\{(\tau_i,[s_{i-1},s_i]),\ i=1,\dots,l\}$ is a δ-fine tagged division of the interval $[t_0, v]$. Using the definition of x in statement (i), it can be easily shown that

$$\big[x(s_i)-x(s_{i-1})\big](\vartheta) = \begin{cases} 0, & \vartheta \in [t_0-r, s_{i-1}],\\ \displaystyle\int_{s_{i-1}}^{\vartheta} f(y_s,s)\,\mathrm{d}g(s) + \int_{s_{i-1}}^{\vartheta} p(s)\mathrm{d}u(s), & \vartheta \in [s_{i-1}, s_i,]\\ \displaystyle\int_{s_{i-1}}^{s_i} f(y_s,s)\,\mathrm{d}g(s) + \int_{s_{i-1}}^{s_i} p(s)\mathrm{d}u(s), & \vartheta \in [s_i, t_0+\sigma]. \end{cases}$$

Similarly, it follows from the definition of G that

$$\big[G(x(\tau_i),s_i)-G(x(\tau_i),s_{i-1})\big](\vartheta) =$$

$$= \begin{cases} 0, & \vartheta \in [t_0-r, s_{i-1}],\\ \displaystyle\int_{s_{i-1}}^{\vartheta} f(x(\tau_i)_s,s)\,\mathrm{d}g(s) + \int_{s_{i-1}}^{\vartheta} p(s)\mathrm{d}u(s), & \vartheta \in [s_{i-1}, s_i],\\ \displaystyle\int_{s_{i-1}}^{s_i} f(x(\tau_i)_s,s)\,\mathrm{d}g(s) + \int_{s_{i-1}}^{s_i} p(s)\mathrm{d}u(s), & \vartheta \in [s_i, t_0+\sigma]. \end{cases}$$

Thus

$$\big[x(s_i)-x(s_{i-1})\big](\vartheta) - \big[G(x(\tau_i),s_i)-G(x(\tau_i),s_{i-1})\big](\vartheta) =$$

$$= \begin{cases} 0, & \vartheta \in [t_0-r, s_{i-1}],\\ \displaystyle\int_{s_{i-1}}^{\vartheta} \big(f(y_s,s)-f(x(\tau_i)_s,s)\big)\,\mathrm{d}g(s), & \vartheta \in [s_{i-1}, s_i],\\ \displaystyle\int_{s_{i-1}}^{s_i} \big(f(y_s,s)-f(x(\tau_i)_s,s)\big)\,\mathrm{d}g(s), & \vartheta \in [s_i, t_0+\sigma], \end{cases}$$

and hence

$$\|x(s_i)-x(s_{i-1}) - \big[G(x(\tau_i),s_i)-G(x(\tau_i),s_{i-1})\big]\|_\infty =$$

$$= \sup_{\vartheta\in[t_0-r,t_0+\sigma]} \|\big[x(s_i)-x(s_{i-1})\big](\vartheta) - \big[G(x(\tau_i),s_i)-G(x(\tau_i),s_{i-1})\big](\vartheta)\| =$$

$$= \sup_{\vartheta\in[s_{i-1},s_i]} \left|\int_{s_{i-1}}^{\vartheta} \big[f(y_s,s)-f(x(\tau_i)_s,s)\big]\,\mathrm{d}g(s)\right|.$$

Again, by the definition of x, we get $x(\tau_i)_s = y_s$ whenever $s \le \tau_i$. Thus

$$\int_{s_{i-1}}^{\vartheta} \big[f(y_s,s)-f(x(\tau_i)_s,s)\big]\,\mathrm{d}g(s) = \begin{cases} 0, & \vartheta \in [s_{i-1}, \tau_i],\\ \displaystyle\int_{\tau_i}^{\vartheta} \big[f(y_s,s)-f(x(\tau_i)_s,s)\big]\,\mathrm{d}g(s), & \vartheta \in [\tau_i, s_i]. \end{cases}$$

Then condition (H_3) implies

$$\left| \int_{\tau_i}^{\vartheta} \big[f(y_s, s) - f(x(\tau_i)_s, s) \big] \, \mathrm{d}g(s) \right| \leq$$

$$\leq \int_{\tau_i}^{\vartheta} L(s) \|y_s - x(\tau_i)_s\|_\infty \, \mathrm{d}g(s) \leq \int_{\tau_i}^{s_i} L(s) \|y_s - x(\tau_i)_s\|_\infty \, \mathrm{d}g(s).$$

Given a particular point-interval pair $(\tau_i, [s_{i-1}, s_i])$, there are two possibilities:

(a) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \ldots, t_m\}$ contains a single point $t_k = \tau_i$.

(b) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \ldots, t_m\}$ is empty.

In case (a), it follows from the definition of the gauge δ that

$$\int_{\tau_i}^{s_i} L(s) \|y_s - x(\tau_i)_s\|_\infty \, \mathrm{d}g(s) \leq \frac{\varepsilon}{2m+1},$$

that is

$$\|x(s_i) - x(s_{i-1}) - \big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1}) \big] \|_\infty \leq \frac{\varepsilon}{2m+1}.$$

In case (b), we have

$$\|y_s - x(\tau_i)_s\|_\infty = \sup_{\rho \in [\tau_i, s]} |y(\rho) - y(\tau_i)| \leq \varepsilon, \quad s \in [\tau_i, s_i]$$

by the definition of the gauge δ. Thus

$$\|x(s_i) - x(s_{i-1}) - \big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1}) \big] \|_\infty \leq \varepsilon \int_{\tau_i}^{s_i} L(s) \, \mathrm{d}g(s).$$

Combining cases (a) and (b) and using the fact that case (a) occurs at most $2m$ times, we obtain

$$\left\| x(v) - x(t_0) - \sum_{i=1}^{l} \big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1}) \big] \right\|_\infty \leq \varepsilon \int_{t_0}^{t_0+\sigma} L(s) \, \mathrm{d}g(s) + \frac{2m\varepsilon}{2m+1}.$$

Then, since ε is arbitrary, it follows that

$$x(v) - x(t_0) = \int_{t_0}^{v} DG(x(\tau), t).$$

Reciprocally, the equality $y_{t_0} = \phi$ follows easily from the definitions of y and $x(t_0)$ in statement (ii). It remains to prove that if $v \in [t_0, t_0 + \sigma]$, then

$$y(v) - y(t_0) = \int_{t_0}^{v} f(y_s, s) \, \mathrm{d}g(s) + \int_{t_0}^{v} p(s) \mathrm{d}u(s).$$

Using Lemma 3.1, we obtain

$$y(v) - y(t_0) = x(v)(v) - x(t_0)(t_0) = x(v)(v) - x(t_0)(v) = \left(\int_{t_0}^{v} DG(x(\tau), t)\right)(v).$$

Thus

$$y(v) - y(t_0) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s) + \int_{t_0}^{v} p(s)\mathrm{d}u(s) =$$

$$= \left(\int_{t_0}^{v} DG(x(\tau), t)\right)(v) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s) - \int_{t_0}^{v} p(s)\mathrm{d}u(s). \tag{3.11}$$

Define a function $h : [t_0, t_0 + \sigma] \to \mathbb{R}$ by

$$h(t) = \int_{t_0}^{t} [L(s) + M(s)]\mathrm{d}g(s) + \int_{t_0}^{t} K(s)\mathrm{d}u(s).$$

Let an arbitrary $\varepsilon > 0$ be given. Since g and u are nondecreasing, they can have only a finite number of points $t \in [t_0, v]$ such that $\Delta^+ g(t) \geq \varepsilon$ and $\Delta^+ u(t) \geq \varepsilon$. The same applies to the function h.

We denote the union of points of discontinuities of the functions g and u by $t_1, \ldots, t_m$. (Recall that these points are the same for the function h).

Consider a gauge $\delta : [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\delta(\tau) < \min\left\{\frac{t_k - t_{k-1}}{2},\ k = 2, \ldots, m\right\},\quad \tau \in [t_0, t_0 + \sigma]$$

and

$$\delta(\tau) < \min\left\{|\tau - t_k|, |\tau - t_{k-1}|;\ \tau \in (t_{k-1}, t_k),\quad k = 2, \ldots, m\right\},\quad \tau \in [t_0, t_0 + \sigma].$$

These conditions assure that if a point-interval pair $(\tau, [c, d])$ is δ-fine, then $[c, d]$ contains at most one of the points $t_1, \ldots, t_m$, and, moreover, $\tau = t_k$ whenever $t_k \in [c, d]$.

Again, the gauge δ might be chosen in such a way that

$$\int_{t_k}^{t_k + \delta(t_k)} L(s)\|y_s - x(t_k)_s\|_\infty\,\mathrm{d}g(s) < \frac{\varepsilon}{2m + 1},\quad k \in \{1, \ldots, m\}.$$

Since the function G given by (3.1) belongs to the class $\mathcal{F}(O \times [t_0, t_0 + \sigma], h)$ and

$$|h(\tau+) - h(\tau)| \leq \varepsilon,\quad \tau \in [t_0, t_0 + \sigma] \backslash \{t_1, \ldots, t_m\},$$

we can assume that the gauge δ satisfies

$$|h(\rho) - h(\tau)| \leq \varepsilon \text{ for every } \rho \in [\tau, \tau + \delta(\tau)).$$

Finally, the gauge δ should be such that

$$\left\| \int_{t_0}^{v} DG(x(\tau),t) - \sum_{i=1}^{l} \big[G(x(\tau_i),s_i) - G(x(\tau_i),s_{i-1})\big] \right\|_\infty < \varepsilon \tag{3.12}$$

for every δ-fine division $\{(\tau_i,[s_{i-1},s_i]), i = 1,\dots,l\}$ of $[t_0,v]$. The existence of such a gauge follows from the definition of the Kurzweil integral (given in Definition A.1 from Appendix A). Choose a particular δ-fine partition $\{(\tau_i,[s_{i-1},s_i]), i = 1,\dots,l\}$ of $[t_0,v]$. By (3.11) and (3.12), we have

$$\left| y(v) - y(t_0) - \int_{t_0}^{v} f(y_s,s)\,\mathrm{d}g(s) - \int_{t_0}^{v} p(s)\mathrm{d}u(s) \right| =$$

$$= \left| \left(\int_{t_0}^{v} DG(x(\tau),t) \right)(v) - \int_{t_0}^{v} f(y_s,s)\,\mathrm{d}g(s) - \int_{t_0}^{v} p(s)\mathrm{d}u(s) \right| <$$

$$< \varepsilon + \left| \sum_{i=1}^{l} \big[G(x(\tau_i),s_i) - G(x(\tau_i),s_{i-1})\big](v) - \int_{t_0}^{v} f(y_s,s)\,\mathrm{d}g(s) - \int_{t_0}^{v} p(s)\mathrm{d}u(s) \right| \le$$

$$\le \varepsilon + \sum_{i=1}^{l} \left| \big[G(x(\tau_i),s_i) - G(x(\tau_i),s_{i-1})\big](v) - \int_{s_{i-1}}^{s_i} f(y_s,s)\,\mathrm{d}g(s) - \int_{s_{i-1}}^{s_i} p(s)\mathrm{d}u(s) \right|.$$

Thus the definition of G yields

$$\big[G(x(\tau_i),s_i) - G(x(\tau_i),s_{i-1})\big](v) = \int_{s_{i-1}}^{s_i} f(x(\tau_i)_s,s)\,\mathrm{d}g(s) + \int_{s_{i-1}}^{s_i} p(s)\mathrm{d}u(s),$$

which implies

$$\left| \big[G(x(\tau_i),s_i) - G(x(\tau_i),s_{i-1})\big](v) - \int_{s_{i-1}}^{s_i} f(y_s,s)\,\mathrm{d}g(s) - \int_{s_{i-1}}^{s_i} p(s)\mathrm{d}u(s) \right| =$$

$$= \left| \int_{s_{i-1}}^{s_i} f(x(\tau_i)_s,s)\,\mathrm{d}g(s) - \int_{s_{i-1}}^{s_i} f(y_s,s)\,\mathrm{d}g(s) \right| = \left| \int_{s_{i-1}}^{s_i} [f(x(\tau_i)_s,s) - f(y_s,s)]\,\mathrm{d}g(s) \right|.$$

By Lemma 3.1 again, for every $i \in \{1,\dots,l\}$, $x(\tau_i)_s = x(s)_s = y_s$ for $s \in [s_{i-1},\tau_i]$ and $y_s = x(s)_s = x(s_i)_s$ for $s \in [\tau_i,s_i]$. Therefore

$$\left| \int_{s_{i-1}}^{s_i} [f(x(\tau_i)_s,s) - f(y_s,s)]\,\mathrm{d}g(s) \right| = \left| \int_{\tau_i}^{s_i} [f(x(\tau_i)_s,s) - f(y_s,s)]\,\mathrm{d}g(s) \right| =$$

$$= \left| \int_{\tau_i}^{s_i} [f(x(\tau_i)_s,s) - f(x(s_i)_s,s)]\,\mathrm{d}g(s) \right| \le \int_{\tau_i}^{s_i} L(s)\|x(\tau_i)_s - x(s_i)_s\|_\infty\,\mathrm{d}g(s),$$

where the last inequality follows from condition (H_3).

Again, we distinguish two cases:

(a) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \dots, t_m\}$ contains a single point $t_k = \tau_i$.

(b) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \dots, t_m\}$ is empty.

In case (a), it follows from the definition of the gauge δ that

$$\int_{\tau_i}^{s_i} L(s)\|y_s - x(\tau_i)_s\|_\infty \,\mathrm{d}g(s) \le \frac{\varepsilon}{2m+1},$$

that is

$$\left|\big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})\big](v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s) - \int_{s_{i-1}}^{s_i} p(s)\mathrm{d}u(s)\right| \le \frac{\varepsilon}{2m+1}.$$

In case (b), we use Lemma B.2 from Appendix B to obtain the estimate

$$\|x(s_i)_s - x(\tau_i)_s\|_\infty \le \|x(s_i) - x(\tau_i)\|_\infty \le h(s_i) - h(\tau_i) \le \varepsilon$$

for every $s \in [\tau_i, s_i]$, and therefore,

$$\left|\big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})\big](v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s) - \int_{s_{i-1}}^{s_i} p(s)\mathrm{d}u(s)\right| \le \varepsilon \int_{\tau_i}^{s_i} L(s)\,\mathrm{d}g(s).$$

Combining cases (i) and (ii) and using the fact that case (i) occurs at most $2m$ times, we obtain

$$\sum_{i=1}^{l}\left|\big[G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1})\big](v) - \int_{s_{i-1}}^{s_i} f(y_s, s)\,\mathrm{d}g(s) - \int_{s_{i-1}}^{s_i} p(s)\mathrm{d}u(s)\right| \le$$

$$\le \varepsilon \int_{t_0}^{t_0+\sigma} L(s)\,\mathrm{d}g(s) + \frac{2m\varepsilon}{2m+1} < \varepsilon\left(1 + \int_{t_0}^{t_0+\sigma} L(s)\,\mathrm{d}g(s)\right),$$

and hence,

$$\left|y(v) - y(t_0) - \int_{t_0}^{v} f(y_s, s)\,\mathrm{d}g(s) - \int_{s_{i-1}}^{s_i} p(s)\mathrm{d}u(s)\right| < \varepsilon\left(2 + \int_{t_0}^{t_0+\sigma} L(s)\,\mathrm{d}g(s)\right),$$

which completes the proof. □

Now, one can use Theorem 3.2 and Theorem B.4 from Appendix B to obtain the next existence-uniqueness result for measure FDEs. A proof of it follows as in Theorem 5.3 from [6].

Theorem 3.3. *Assume that $O \subset G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ has the prolongation property, $P = \{x_t;\ x \in O,\ t \in [t_0, t_0 + \sigma]\}$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ and $u : [t_0, t_0 + \sigma] \to \mathbb{R}$ are left-continuous and nondecreasing functions, $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions (H_1), (H_2), (H_3) and $p : [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions (H_4) and (H_5). Let $G : O \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ be given by (3.3) and assume that $G(x, t) \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ for $x \in O$ and $t \in [t_0, t_0 + \sigma]$. If $\phi \in P$ is such that the function*

$$z(t) = \begin{cases} \phi(t - t_0), & t \in [t_0 - r, t_0], \\ \phi(0) + f(\phi, t_0)\Delta^+ g(t_0) + p(t_0)\Delta^+ u(t_0), & t \in (t_0, t_0 + \sigma] \end{cases}$$

belongs to O, then there exist $\delta > 0$ and a function $y : [t_0 - r, t_0 + \delta] \to \mathbb{R}^n$ which is the unique solution of the measure functional differential equation

$$\begin{cases} Dy = f(y_s, s)Dg + p(s)Du, \\ y_{t_0} = \phi. \end{cases} \tag{3.13}$$

4 New concepts of stability for generalized ODEs

Let X be a Banach space and $B_c = \{x \in X; \|x\| < c\}$, where $c > 0$. Define $\Omega = B_c \times [t_0, \infty)$. Suppose $F : \Omega \to X$ belongs to $\mathcal{F}(\Omega, h_2)$, where $h_2(t) = \int_{t_0}^{t} [M(s) + L(s)]\mathrm{d}g(s)$. Assume further that $F(0, t) - F(0, s) = 0$, for $t, s \in [t_0, +\infty)$. Then, for every $[\gamma, v] \subset [t_0, \infty)$, we have

$$\int_\gamma^v DF(0, t) = F(0, v) - F(0, \gamma) = 0,$$

which implies that $x \equiv 0$ is a solution on $[t_0, \infty)$ of the generalized ODE

$$\frac{dx}{d\tau} = DF(x, t). \tag{4.1}$$

Recall (see [14]) that if $\overline{x} : [\gamma, v] \subset [t_0, \infty) \to X$ is a solution of (4.1), then the following assertions hold:

(a) $\overline{x}$ is of bounded variation on $[\gamma, v]$;

(b) $\overline{x}(s) = \overline{x}(\gamma) + \int_\gamma^s DF(\overline{x}(\tau), t)$, for $s \in [\gamma, v]$, by definition.

Now, we present some new concepts concerning stability of the trivial solution of generalized ODEs.

Definition 4.1. The trivial solution $x \equiv 0$ of (4.1) is called *regularly stable*, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, with $t_0 \leq \gamma < v < \infty$, is a regulated function which satisfies

$$\|\overline{x}(\gamma)\| < \delta \quad \text{and} \quad \sup_{s \in [\gamma, v]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_\gamma^s DF(\overline{x}(\tau), t) \right\| < \delta,$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v].$$

Definition 4.2. The trivial solution $x \equiv 0$ of (4.1) is called *regularly attracting* if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, with $t_0 \leq \gamma < v < \infty$, is a regulated function satisfying

$$\|\overline{x}(\gamma)\| < \delta_0 \quad \text{and} \quad \sup_{s\in[\gamma,v]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_{\gamma}^{s} DF(\overline{x}(\tau), t) \right\| < \rho,$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad \text{for } t \in [\gamma, v] \cap [\gamma + T, \infty) \text{ and } \gamma \geq t_0.$$

Definition 4.3. The trivial solution $x \equiv 0$ of (4.1) is called *regularly asymptotically stable*, if it is regularly stable and regularly attracting.

Note that if the trivial solution $x \equiv 0$ of (4.1) is regularly stable, then it is variational stable in the sense defined by Štefan Schwabik in [15] (see also [14]). Similar statements hold for regular attractivity and regular asymptotic stability.

Besides the generalized ODE (4.1), we consider the perturbed generalized ODE

$$\frac{dx}{d\tau} = D[F(x, t) + P(t)], \tag{4.2}$$

where $F : B_c \times [t_0, \infty) \to X$ and $P : [t_0, \infty) \to X$, and consider the following definitions.

Let $G^{-}([\gamma, v], X)$ denote the space of regulated functions from $[\gamma, v]$ to X which are left-continuous.

Definition 4.4. The trivial solution $x \equiv 0$ of (4.1) is called *regularly stable with respect to perturbations*, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $\|x_0\| < \delta$ and $P \in G^{-}([\gamma, v], X)$ with

$$\sup_{s\in[\gamma,v]} \|P(s) - P(\gamma)\| < \delta,$$

then

$$\|\bar{x}(t, \gamma, x_0)\| < \varepsilon, \quad \text{for every } \ t \in [\gamma, v]$$

where $\bar{x}(t, \gamma, x_0)$ is a solution of the perturbed generalized ODE (4.2), with initial condition $\bar{x}(\gamma, \gamma, x_0) = x_0$ and $[\gamma, v] \subset [t_0, \infty)$.

Definition 4.5. The trivial solution $x \equiv 0$ of (4.1) is called *regularly attracting with respect to perturbations*, if there exists $\widetilde{\delta} > 0$ and for every $\varepsilon > 0$, there exist $T = T(\varepsilon) \geq 0$ and a $\rho = \rho(\varepsilon) > 0$ such that, if

$$\|x_0\| < \widetilde{\delta} \qquad \text{and} \qquad \sup_{s\in[\gamma,v]} \|P(s) - P(\gamma)\| < \rho$$

with $P \in G^{-}([\gamma, v], X)$, then

$$\|\bar{x}(t, \gamma, x_0)\| < \varepsilon, \quad \text{for all } \ t \geq \gamma + T, \ t \in [\gamma, v]$$

where $\bar{x}(t, \gamma, x_0)$ is a solution of the perturbed generalized ODE (4.2) with initial condition $\bar{x}(\gamma, \gamma, x_0) = x_0$ and $[\gamma, v] \subset [t_0, \infty)$.

Definition 4.6. The trivial solution $x \equiv 0$ of (4.1) is called *regularly asymptotically stable with respect to perturbations* if it is both stable and attracting with respect to perturbations.

The following result shows us how all the previous concepts of stability can be related.

Theorem 4.7. *The following statements hold.*

(i) The trivial solution $x \equiv 0$ *of* (4.1) *is regularly stable, if and only if, it is regularly stable with respect to perturbations.*

(ii) The trivial solution $x \equiv 0$ *of* (4.1) *is regularly attracting, if and only if, it is regularly attracting with respect to perturbations.*

(iii) The trivial solution $x \equiv 0$ *of* (4.1) *is regularly asymptotically stable, if and only if, it is regularly asymptotically stable with respect to perturbations.*

Proof. Let us prove (i). Assume that the trivial solution $x \equiv 0$ of (4.1) is regularly stable. Let $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ be given according to Definition 4.4.

Let $x(t) = x(t, \gamma, x_0)$ be a solution of the perturbed generalized equation (4.2) on $[\gamma, v]$. Then, by the definition, we obtain

$$x(s) - x(\gamma) = \int_{\gamma}^{s} DF(x(\tau), t) + P(s) - P(\gamma). \tag{4.3}$$

Also, suppose $\|x(\gamma)\| = \|x(\gamma, \gamma, x_0)\| < \delta$ and $\sup_{s \in [\gamma, v]} \|P(s) - P(\gamma)\| < \delta$. It follows from (4.3) that

$$\sup_{s \in [\gamma, v]} \left\| x(s) - x(\gamma) - \int_{\gamma}^{s} DF(x(\tau), t) \right\| = \sup_{s \in [\gamma, v]} \|P(s) - P(\gamma)\| < \delta.$$

Then, by regularly stability, we have

$$\|x(t)\| = \|x(t, \gamma, x_0)\| < \varepsilon \ \text{ for } \ t \in [\gamma, v]$$

and the trivial solution of (4.1) is regularly stable with respect to perturbations.

Reciprocally, if the trivial solution $x \equiv 0$ of (4.1) is regularly stable with respect to perturbations, let $\overline{x} : [\gamma, v] \to B_c$, $t_0 \leq \gamma < v < \infty$, be regulated function on $[\gamma, v]$ such that

$$\|\overline{x}(\gamma)\| < \delta \quad \text{and} \quad \sup_{s \in [\gamma, v]} \left\| x(s) - x(\gamma) - \int_{\gamma}^{s} DF(x(\tau), t) \right\| < \delta,$$

where $\delta > 0$ corresponds to some $\varepsilon > 0$ from the Definition 4.4.

For $s \in [\gamma, v]$, define

$$P(s) = P(\gamma) + \overline{x}(s) - \overline{x}(\gamma) - \int_{\gamma}^{s} DF(\overline{x}(\tau), t).$$

Then, for all $s_1, s_2 \in [\gamma, v]$,

$$\overline{x}(s_2) - \overline{x}(s_1) = \int_{s_1}^{s_2} DF(\overline{x}(\tau), t) + P(s_2) - P(s_1),$$

which implies that $\overline{x}$ is a solution of (4.2) on $[\gamma, v]$. Moreover,

$$\sup_{s\in[\gamma,v]} \|P(s) - P(\gamma)\| = \sup_{s\in[\gamma,v]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_{\gamma}^{s} DF(\overline{x}(\tau), t) \right\| < \delta.$$

Thus the regular stability with respect to perturbations implies $\|\overline{x}(t)\| = \|\overline{x}(t, \gamma, x_0)\| < \varepsilon$, for all $t \in [\gamma, v]$, which implies that the trivial solution $x \equiv 0$ of (4.1) is regularly stable.

Now, assume that the trivial solution $x \equiv 0$ of (4.1) is regularly attracting. Then there exists $\widetilde{\delta} > 0$ and for every $\varepsilon > 0$, there exist $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, $t_0 \leq \gamma < v < \infty$, is a regulated function such that $\|\overline{x}(\gamma)\| < \widetilde{\delta}$ and

$$\sup_{s\in[\gamma,v]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_{\gamma}^{s} DF(\overline{x}(\tau), t) \right\| < \rho,$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, \infty),\ \gamma \geq t_0.$$

Denote $x(t) = x(t, \gamma, x_0)$ the solution of the perturbed generalized equation (4.2) satisfying $x(\gamma, \gamma, x_0) = x_0$. Suppose there exists $\widetilde{\delta} > 0$ and for every $\varepsilon > 0$, there exists $\rho = \rho(\varepsilon) > 0$ such that $\|x_0\| < \widetilde{\delta}$ and

$$\sup_{s\in[\gamma,v]} \|P(s) - P(\gamma)\| < \rho.$$

Moreover, suppose $P \in G^{-}([\gamma, v], X)$. It follows that $\|x_0\| = \|x(\gamma)\| < \widetilde{\delta}$ and by the definition of a solution of equation (4.2), we obtain

$$\sup_{s\in[\gamma,v]} \left\| x(s) - x(\gamma) - \int_{\gamma}^{s} DF(x(\tau), t) \right\| = \sup_{s\in[\gamma,v]} \|P(s) - P(\gamma)\| < \rho.$$

Hence, since x is regularly attracting, we have

$$\|x(t, \gamma, x_0)\| = \|x(t)\| < \varepsilon, \ \text{ for all } \ t \geq \gamma + T,\ t \in [\gamma, v],$$

that is, the trivial solution $x \equiv 0$ of (4.1) is regularly attracting with respect to perturbations.

Reciprocally, we assume that the trivial solution $x \equiv 0$ of (4.1) is regularly attracting with respect to perturbations. Let $\overline{x} : [\gamma, v] \to B_c$, $t_0 \leq \gamma < v < \infty$, be a regulated and left continuous function on $[\gamma, v]$ satisfying

$$\|\overline{x}(\gamma)\| < \widetilde{\delta} \quad \text{and} \quad \sup_{s \in [\gamma, v]} \left\| \overline{x}(s) - \overline{x}(\gamma) \int_{\gamma}^{s} DF(\overline{x}(\tau), t) \right\| < \rho.$$

Again, for $s \in [\gamma, v]$, let $P(s) = P(\gamma) + \overline{x}(s) - \overline{x}(\gamma) - \int_{\gamma}^{s} DF(\overline{x}(\tau), t)$. Then, $\overline{x}(t)$ is a solution of (4.2) on $[\gamma, v]$. Thus,

$$\sup_{s \in [\gamma, v]} \|P(s) - P(\gamma)\| < \rho.$$

Therefore, Definition 4.5 yields

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, \infty), \ \gamma \geq t_0.$$

which implies the regular attractivity of the trivial solution of (4.1).

Finally, item (iii) follows from itens (i) and (ii) and the proof is complete. □

5 Stability relations between generalized ODEs and measure FDEs

We are now able to present a result which relates the concepts of regular stability and regular attractivity of the trivial solution of the generalized ODE (5.3) and the concepts of integral stability and integral attractivity of the trivial solution of its corresponding measure FDE (5.1).

Consider the measure FDE

$$Dx = f(x_t, t)Dg, \tag{5.1}$$

where $f : S \times [t_0, \infty) \to \mathbb{R}^n$ satisfies the conditions (H_1), (H_2) and (H_3) and $g : [t_0, \infty) \to \mathbb{R}$ is a nondecreasing and left-continuous function. Furthermore, assume that Dx and Dg are the distributional derivatives with respect to x and g in the sense of L. Schwartz and $f(0, t) = 0$ for every $t \in [t_0, \infty)$ so that $y \equiv 0$ is necessarily a solution of (5.1).

Consider the function

$$F : O \times [t_0, \infty) \to BG^{-}([t_0 - r, \infty), \mathbb{R}^n)$$

defined by

$$F(y,t)(\vartheta) = \begin{cases} 0, \quad t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\int_{t_0}^{\vartheta} f(y_s, s)\mathrm{d}g(s), \quad t_0 \leq \vartheta \leq t < \infty, \\ \displaystyle\int_{t_0}^{t} f(y_s, s)\mathrm{d}g(s), \quad t \leq \vartheta < \infty \end{cases} \tag{5.2}$$

Since $f(0,t)=0$, for every $t \in [t_0,\infty)$, then $F(0,t)=0$, for all $t \in [t_0,\infty)$. Hence, as we observed in the previous section, $x \equiv 0$ is a solution, on $[t_0,\infty)$, of the generalized ODE

$$\frac{dx}{d\tau} = DF(x,t). \tag{5.3}$$

We also consider the perturbed measure FDE

$$Dy = f(y_t,t)Dg + p(t)Du, \tag{5.4}$$

where $p : [t_0,\infty) \to \mathbb{R}^n$ satisfies conditions (H_4) and (H_5) and $u : [t_0,\infty) \to \mathbb{R}$ is a nondecreasing and left-continuous function. Moreover, Du is the distributional derivative in the sense of L. Schwartz in respect to u.

By Theorem 3.2, the generalized ODE corresponding to the perturbed FDE (5.4) is given by

$$\frac{dx}{d\tau} = DG(x,t) = D[F(x,t)+P(t)], \tag{5.5}$$

where $F : O \times [t_0,\infty) \to BG^-([t_0-r,\infty),\mathbb{R}^n)$ is given by (5.2) and $P : [t_0,\infty) \to BG^-([t_0-r,\infty),\mathbb{R}^n)$ is given by

$$P(t)(\vartheta) = \begin{cases} 0, & t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\int_{t_0}^{\vartheta} p(s)\mathrm{d}u(s), & t_0 \leq \vartheta \leq t < \infty, \\ \displaystyle\int_{t_0}^{t} p(s)\mathrm{d}u(s), & t \leq \vartheta < \infty. \end{cases} \tag{5.6}$$

Theorem 5.1. *Suppose the function $f : S \times [t_0,\infty) \to \mathbb{R}^n$ satisfies conditions (H_1), (H_2) and (H_3). Assume that the functions $g, u : [t_0,\infty) \to \mathbb{R}$ are nondecreasing and left-continuous. Further, suppose $p : [t_0,\infty) \to \mathbb{R}^n$ satisfies conditions (H_4) and (H_5). Then the following statements hold:*

(i) *The trivial solution $y \equiv 0$ of (5.1) is integrally stable, if and only if, the trivial solution $x \equiv 0$ of (5.3) is regularly stable.*

(ii) *If the trivial solution $x \equiv 0$ of (5.3) is regularly attracting, then the trivial solution $y \equiv 0$ of (5.1) is integrally attracting.*

(iii) *If the trivial solution $x \equiv 0$ of (5.3) is regularly asymptotically stable, then the trivial solution $y \equiv 0$ of (5.1) is integrally asymptotically stable.*

Proof. We start by proving (i). Suppose the trivial solution of (5.1) is integrally stable. Then given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $\phi \in P$ with $\|\phi\| < \delta$ and

$$\sup_{t\in[\gamma,v]} \left| \int_{\gamma}^{t} p(s)\mathrm{d}u(s) \right| < \delta,$$

then

$$|\bar{y}_t(\gamma,\phi)| < \frac{\varepsilon}{2}, \quad t \in [\gamma, v],$$

where $\bar{y}(t;\gamma,\phi)$ is a solution of (5.4) such that $\bar{y}_\gamma = \phi$.

We will prove that the trivial solution of generalized equation (5.3), with F given by (3.1), is regularly stable with respect to perturbations, then the result will follow by Proposition 4.7.

Let $x(t;\gamma,x_0)$ be a solution of the perturbed generalized ODE (5.5) with the initial condition $x(\gamma;\gamma,x_0) = x_0$, where F and P are given by (3.1) and (3.2), respectively. Let $\varepsilon > 0$ and suppose that there exists a $\delta = \delta(\varepsilon) > 0$ such that $\delta < \varepsilon/2$ and moreover, assume that

$$\|x_0\|_\infty < \delta \quad \text{and} \quad \sup_{t\in[\gamma,v]} \|P(t) - P(\gamma)\| < \delta,$$

where $x_0 \in BG^-([t_0 - r, \infty), \mathbb{R}^n)$ and $P \in G^-([\gamma, v], \mathbb{R}^n)$.

We have $\|x(\gamma)\| = \|x(\gamma;\gamma,x_0)\| = \|x_0\| < \delta$ which means that $\sup_{\theta\in[\gamma-r,\infty)} |x(\gamma)(\theta)| < \delta$ and therefore $\sup_{\theta\in[\gamma-r,\gamma]} |\phi(\theta-\gamma)| < \delta/2$. Thus,

$$\|\phi\|_\infty < \frac{\delta}{2}.$$

Since x is a solution of the perturbed generalized ODE on $[\gamma, v]$, we have

$$x(u_2) - x(u_1) = \int_{u_1}^{u_2} DF(x(\tau), t) + P(u_2) - P(u_1),$$

for $u_1, u_2 \in [\gamma, v]$.

Therefore,

$$\sup_{t\in[\gamma,v]} \left\| x(t) - x(t_0) - \int_{t_0}^{t} DF(x(\tau), s) \right\| = \sup_{t\in[\gamma,v]} \|P(s) - P(t_0)\| < \delta/2.$$

Then, it follows that

$$\begin{aligned} \|P(t) - P(\gamma)\| &= \sup_{\vartheta\in[\gamma-r,t]} |P(t)(\vartheta) - P(\gamma)(\vartheta)| \\ &\geq \sup_{\vartheta\in[\gamma,t]} |P(t)(\vartheta) - P(\gamma)(\vartheta)| \\ &= \sup_{\vartheta\in[\gamma,t]} \left| \int_\gamma^\vartheta p(s)\mathrm{d}u(s) \right| \geq \left| \int_\gamma^t p(s)\mathrm{d}u(s) \right| \end{aligned} \tag{5.7}$$

and we get

$$\sup_{t\in[\gamma,v]} \left| \int_\gamma^t p(s)\mathrm{d}u(s) \right| \leq \frac{\delta}{2}.$$

Thus, by the integral stability of the trivial solution of (5.1), we have

$$|y(t)| < \varepsilon/2, \quad \text{for all} \quad t \in [\gamma, v].$$

Finally, for $t \in [\gamma, v]$, we have

$$\begin{aligned} \|x(t)\|_\infty &= \sup_{\theta\in[\gamma-r,\infty)} |x(t)(\theta)| = \sup_{\theta\in[\gamma-r,t]} |y(\theta)| \\ &\leq \|\phi\|_\infty + \sup_{\theta\in[\gamma,t]} |y(\theta)| \leq \frac{\delta}{2} + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

since $\delta < \varepsilon$, and we have the sufficiency of item (i).

Now, using (i) from Proposition 4.7, we assume that the trivial solution of (5.3) is regularly stable with respect to perturbations. Thus, given $\varepsilon > 0$, let $\delta = \delta(\varepsilon) > 0$ be the quantity from Definition 4.4.

Let $y(t; \gamma, \phi)$ be a solution of the perturbed measure FDE (5.4). Suppose $\phi \in P$ with $\|\phi\|_\infty < \delta$ and

$$\sup_{t\in[\gamma,v]} \left| \int_\gamma^t p(s)\mathrm{d}u(s) \right| < \delta/2.$$

We want to prove that $y \equiv 0$ is integrally stable, that is, $|y(t; \gamma, \phi)| < \varepsilon$, $t \in [\gamma, v]$.

Let $x(t; \gamma, x_0)$ be the solution of the perturbed generalized ODE (5.5) with F given by (3.1) and P given by (3.2), that is, x is the solution corresponding to y obtained according to Proposition 3.2.

By the definition of function P and from the fact that $\sup_{t\in[\gamma,v]} \left| \int_\gamma^t p(s)\mathrm{d}u(s) \right| < \delta/2$, we obtain

$$\sup_{t\in[\gamma,v]} \|P(t) - P(\gamma)\| < \delta/2.$$

Thus, from the regular stability with respect to perturbations of the trivial solution of (5.5), we obtain $\|x(t)\| < \varepsilon$, which implies

$$\sup_{\theta\in[\gamma-r,v]} |x(t)(\theta)| < \varepsilon, \quad t \in [\gamma, v].$$

Therefore, the relation in Proposition 3.2 implies

$$\sup_{\theta\in[\gamma-r,t]} |y(\theta)| < \varepsilon, \quad t \in [\gamma, v].$$

In particular,

$$\sup_{\theta\in[\gamma,v]} |y(\theta)| \leq \sup_{\theta\in[\gamma-r,v]} |y(\theta)| < \varepsilon.$$

Now, we will prove (ii). Suppose the trivial solution of generalized ODE (5.3) is regularly attracting with respect to perturbations. Then there exists $\widetilde{\delta} > 0$ and given $\varepsilon > 0$, let $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ be as in Definition 4.5.

Let $y(t;\gamma,\phi)$ be a solution of the perturbed retarded equation (5.4). Suppose $\|\phi\|_\infty < \widetilde{\delta}$ and $\sup_{t\in[\gamma,v]}\left|\int_\gamma^t p(s)\mathrm{d}u(s)\right| < \rho$. Then, it follows that it implies that

$$\sup_{t\in[\gamma,v]} \|P(t)-P(\gamma)\| < \rho.$$

By Proposition 3.2, it follows that $\|x_0\|_\infty = \|\phi\|_\infty < \widetilde{\delta}$. Thus the regularly attractivity with respect to perturbations of the trivial solution of (5.3) implies

$$\|x(t)\|_\infty = \|x(t;\gamma,x_0)\|_\infty < \varepsilon, \quad t \geq \gamma+T,\ t\in[\gamma,v].$$

Therefore, for $t\geq\gamma+T$ and taking $T(\varepsilon) > r$, we have by Proposition 3.2, for $t\in[\gamma,v]$,

$$|y(t)| = |y(t;\gamma,\phi)| = |x(t)(t)| \leq \|x(t)\|_\infty < \varepsilon.$$

Assertion (iii) follows from (i), (ii) and from Proposition 4.7. □

6 Lyapunov theorems for generalized ODEs

In this section, we prove some Lyapunov-type theorems for generalized ODEs using the concepts introduced before. We start by considering the following set

$$\overline{B}_\rho = \{x\in O; \|x\|\leq\rho\},$$

where $O\subset BG([t_0-r,\infty),\mathbb{R}^n)$ with the prolongation property.

In what follows, we present a result which will be essential to our purposes. The proof is inspired in Lemma 10.12 from [14].

Lemma 6.1. *Let $G\in\mathcal{F}(\Omega,h)$. Suppose $V:[t_0,\infty)\times\overline{B}_\rho\to\mathbb{R}$ is such that $V(\cdot,x):[t_0,\infty)\to\mathbb{R}$ is continuous from the left on (t_0,∞) for $x\in\overline{B}_\rho$ and satisfies*

$$|V(t,z)-V(t,y)| \leq K\|z-y\|, \quad z,y\in\overline{B}_\rho,\ t\in[t_0,\infty), \tag{6.1}$$

where K is a positive constant. Furthermore, suppose there exists a function $\Phi:\overline{B}_\rho\to\mathbb{R}$ such that for every solution $x:[a,b]\to\overline{B}_\rho$, $[a,b]\subset[t_0,\infty)$, of (5.3), we have

$$\dot{V}(t,x(t)) = \limsup_{\eta\to0+}\frac{V(t+\eta,x(t+\eta))-V(t,x(t))}{\eta} \leq \Phi(x(t)), \quad t\in[a,b]. \tag{6.2}$$

If $\overline{x}:[\gamma,v]\to\overline{B}_\rho$, $t_0\leq\gamma<v<\infty$, is left-continuous on $(\gamma,v]$ and of bounded variation on $[\gamma,v]$, then

$$V(v,\overline{x}(v))-V(\gamma,\overline{x}(\gamma)) \leq K\sup_{s\in[\gamma,v]}\left\|\overline{x}(s)-\overline{x}(\gamma)-\int_\gamma^s DG(\overline{x}(\tau),t)\right\| + M(v-\gamma), \tag{6.3}$$

where $M=\sup_{t\in[\gamma,v]}\Phi(\overline{x}(t))$.

Proof. Let $\overline{x} : [\gamma, v] \to \overline{B}_\rho$ be a left-continuous function on $(\gamma, v]$ and regulated on $[\gamma, v] \subset [t_0, \infty)$. By [14], Corollary 3.16, the integral $\int_\gamma^v DG(\overline{x}(\tau), t)$ exists.

Take $\sigma \in [\gamma, v]$. By Theorem B.4 from Appendix B, equation (5.3) has a local solution, say, $x : [\sigma, \sigma + \eta_1(\sigma)] \to \overline{B}_\rho$ on $[\sigma, \sigma + \eta_1(\sigma)]$, satisfying the initial condition $x(\sigma) = \overline{x}(\sigma)$. It is clear that the integral $\int_\sigma^{\sigma+\eta_1(\sigma)} DG(x(\tau), t)$ exists.

Let $\eta_2 > 0$ be arbitrarily small such that $\eta_2 \leq \eta_1(\sigma)$ and $\sigma + \eta_2 \leq v$. Then the integral $\int_\sigma^{\sigma+\eta_2} DG(x(\tau), t)$ exists and the integral $\int_\sigma^{\sigma+\eta_2} D[G(\overline{x}(\tau), t) - G(x(\tau), t)]$ also exists by the property of integrability on subintervals. Therefore, given $\varepsilon > 0$, there exists a gauge δ on $[\sigma, \sigma+\eta_2]$ corresponding to ε in the definition of the last integral and we can assume, without loss of generality, that $\eta_2 < \delta(\sigma)$. By (6.2), we can take $0 < \eta \leq \eta_2$ such that the inequality

$$V(\sigma + \eta, x(\sigma + \eta)) - V(\sigma, x(\sigma)) \leq \eta \Phi(x(\sigma)) \tag{6.4}$$

holds, and we can assume, by Corollary A.7(i) from Appendix A, that

$$\left\| G(\overline{x}(\sigma), \sigma + \eta) - G(\overline{x}(\sigma), \sigma) - \int_\sigma^{\sigma+\eta} DG(\overline{x}(\tau), t) \right\| < \frac{\eta \epsilon}{2K} \tag{6.5}$$

and

$$\left\| G(x(\sigma), \sigma + \eta) - G(x(\sigma), \sigma) - \int_\sigma^{\sigma+\eta} DG(x(\tau), t) \right\| < \frac{\eta \epsilon}{2K}. \tag{6.6}$$

Notice that

$$\begin{aligned}
&\left\| \int_\sigma^{\sigma+\eta} D[G(\overline{x}(\tau), t) - G(x(\tau), t)] \right\| \\
&- \| G(\overline{x}(\sigma), \sigma + \eta) - G(\overline{x}(\sigma), \sigma) - G(x(\sigma), \sigma + \eta) + G(x(\sigma), \sigma) \| \\
&\leq \left\| \int_\sigma^{\sigma+\eta} D[G(\overline{x}(\tau), t) - G(x(\tau), t)] \right. \\
&\left. -(G(\overline{x}(\sigma), \sigma + \eta) - G(\overline{x}(\sigma), \sigma) - G(x(\sigma), \sigma + \eta) + G(x(\sigma), \sigma)) \right\| \\
&\leq \left\| G(\overline{x}(\sigma), \sigma + \eta) - G(\overline{x}(\sigma), \sigma) - \int_\sigma^{\sigma+\eta} DG(\overline{x}(\tau), t) \right\| \\
&+ \left\| G(x(\sigma), \sigma + \eta) - G(x(\sigma), \sigma) - \int_\sigma^{\sigma+\eta} DG(x(\tau), t) \right\|.
\end{aligned}$$

Also

$$\begin{aligned}
&\| G(\overline{x}(\sigma), \sigma + \eta) - G(\overline{x}(\sigma), \sigma) - G(x(\sigma), \sigma + \eta) + G(x(\sigma), \sigma) \| \\
&\leq \| \overline{x}(\sigma) - x(\sigma) \| |h(\sigma + \eta) - h(\sigma)| = 0,
\end{aligned}$$

since $\overline{x}(\sigma) = x(\sigma)$ and (A.2) from the Appendix is fulfilled. Then, we have by (6.5) and (6.6),

$$\left\| \int_\sigma^{\sigma+\eta} D[G(\overline{x}(\tau), t) - G(x(\tau), t)] \right\| \leq \frac{\eta \epsilon}{K}. \tag{6.7}$$

Moreover, (6.1) implies

$$
\begin{aligned}
&V(\sigma+\eta,\overline{x}(\sigma+\eta))-V(\sigma+\eta,x(\sigma+\eta))\leq\\
&\leq K\|\overline{x}(\sigma+\eta)-x(\sigma+\eta)\|=K\|\overline{x}(\sigma+\eta)-\overline{x}(\sigma)+x(\sigma)-x(\sigma+\eta)\|\\
&=K\left\|\overline{x}(\sigma+\eta)-\overline{x}(\sigma)-\int_{\sigma}^{\sigma+\eta}DG(x(\tau),t)\right\|. \qquad (6.8)
\end{aligned}
$$

Then (6.4), (6.7) and (6.8) imply

$$
\begin{aligned}
&V(\sigma+\eta,\overline{x}(\sigma+\eta))-V(\sigma,\overline{x}(\sigma))\\
&=V(\sigma+\eta,\overline{x}(\sigma+\eta))-V(\sigma+\eta,x(\sigma+\eta))+V(\sigma+\eta,x(\sigma+\eta))-V(\sigma,x(\sigma))\\
&\leq K\left\|\overline{x}(\sigma+\eta)-\overline{x}(\sigma)-\int_{\sigma}^{\sigma+\eta}DG(x(\tau),t)\right\|+\eta\Phi(x(\sigma))\\
&\leq K\left\|\overline{x}(\sigma+\eta)-\overline{x}(\sigma)-\int_{\sigma}^{\sigma+\eta}DG(x(\tau),t)\right\|+\eta M\\
&\leq K\left\|\overline{x}(\sigma+\eta)-\overline{x}(\sigma)-\int_{\sigma}^{\sigma+\eta}DG(\overline{x}(\tau),t)\right\|\\
&+K\left\|\int_{\sigma}^{\sigma+\eta}D[G(\overline{x}(\tau),t)-G(x(\tau),t)]\right\|+\eta M\\
&\leq K\left\|\overline{x}(\sigma+\eta)-\overline{x}(\sigma)-\int_{\sigma}^{\sigma+\eta}DG(\overline{x}(\tau),t)\right\|+\eta\varepsilon+\eta M. \qquad (6.9)
\end{aligned}
$$

Given $s\in[\gamma,v]$, we define

$$P(s)=\overline{x}(s)-\int_{\gamma}^{s}DG(\overline{x}(\tau),t).$$

Since $\overline{x}$ is regulated on $[\gamma,v]$, we have by [14], Corollary 3.16 that P is also regulated on $[\gamma,v]$. Then, we have

$$
\begin{aligned}
P(\sigma+\eta)-P(\sigma) &= \overline{x}(\sigma+\eta)-\overline{x}(\sigma)-\int_{\gamma}^{\sigma+\eta}DG(\overline{x}(\tau),t)+\int_{\gamma}^{\sigma}DG(\overline{x}(\tau),t)\\
&= \overline{x}(\sigma+\eta)-\overline{x}(\sigma)-\int_{\sigma}^{\sigma+\eta}DG(\overline{x}(\tau),t).
\end{aligned}
$$

Now, we define

$$f(t)=K\|P(t)-P(\sigma)\|+\varepsilon t+Mt.$$

Notice that

$$\begin{aligned}
V(\sigma+\eta,\overline{x}(\sigma+\eta))-V(\sigma,\overline{x}(\sigma)) &\leq K\|P(\sigma+\eta)-P(\sigma)\|+\eta\varepsilon+\eta M \\
&= K(\|P(\sigma+\eta)-P(\sigma)\|-\|P(\sigma)-P(\sigma)\|) \\
&+ \eta\varepsilon+\eta M = f(\sigma+\eta)-f(\sigma). \qquad (6.10)
\end{aligned}$$

By (6.10) and by [14], Proposition 10.11, we have

$$\begin{aligned}
V(v,\overline{x}(v))-V(\gamma,\overline{x}(\gamma)) &\leq f(v)-f(\gamma) \\
&= K(\|P(v)-P(\sigma)\|-\|P(\gamma)-P(\sigma)\|)+\varepsilon(v-\gamma) \\
&+ M(v-\gamma) \leq K\|P(v)-P(\sigma)-[P(\gamma)-P(\sigma)]\| \\
&+ \varepsilon(v-\gamma)+M(v-\gamma) = K\|P(v)-P(\gamma)\|+\varepsilon(v-\gamma) \\
&+ M(v-\gamma) \leq K\left(\left\|\overline{x}(v)-\int_\gamma^v DG(\overline{x}(\tau),t)-\overline{x}(\gamma)\right\|\right) \\
&+ \varepsilon(v-\gamma)+M(v-\gamma) \\
&\leq K\left\|\overline{x}(v)-\overline{x}(\gamma)-\int_\gamma^v DG(\overline{x}(\tau),t)\right\| \\
&+ \varepsilon(v-\gamma)+M(v-\gamma).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the result follows as well. □

In what follows, we remind the reader the definition of Lyapunov functional for generalized ODEs.

Definition 6.2. We say that $V : [t_0,\infty)\times\overline{B}_\rho \to \mathbb{R}$ is a Lyapunov functional (with respect to the generalized ODE (5.3)), if the following conditions are satisfied:

(i) $V(\cdot,x) : [t_0,\infty) \to \mathbb{R}$ is continuous from the left on (t_0,∞), for every $x \in \overline{B}_\rho$;

(ii) There exists a continuous and strictly increasing function $b : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $b(0) = 0$ (we say that such function is of Hahn class), such that

$$V(t,x) \geq b(\|x\|),$$

for every $t \in [t_0,\infty)$ and $x \in \overline{B}_\rho$;

(iii) For every solution $\bar{x} : [\gamma,v] \to \overline{B}_\rho$ of (5.3), with $[\gamma,v] \subset [t_0,\infty)$, we have

$$\dot{V}(t,\bar{x}(t)) = \limsup_{\eta\to 0^+} \frac{V(t+\eta,\bar{x}(t+\eta))-V(t,\bar{x}(t))}{\eta} \leq 0, \quad t \in [\gamma,v],$$

that is, the right-derivative of V is non-positive along the solutions of (5.3).

The next result gives us conditions such that the trivial solution of (5.3) is regularly stable. We inspire the proof in Theorem 10.13 from [14].

Theorem 6.3. *Let $V : [t_0, \infty) \times \overline{B_\rho} \to \mathbb{R}$ be a Lyapunov functional, where $\overline{B_\rho} = \{y \in X : \|y\| \leq \rho\}$, $0 < \rho < c$. Suppose V satisfies the following conditions:*

(i) $V(t, 0) = 0$, $t \in [t_0, \infty)$;

(ii) There exists a constant $K > 0$ such that

$$|V(t, z) - V(t, y)| \leq K\|z - y\|, \quad t \in [t_0, \infty), \ z, y \in \overline{B_\rho}.$$

Then the trivial solution $x \equiv 0$ of (5.3) is regularly stable.

Proof. Let $\overline{x} : [\gamma, v] \to X$ be a regulated function on $[\gamma, v] \subset [t_0, \infty)$. By condition (iii) from Definition 6.2 and by Lemma 6.1, we have

$$V(t, \overline{x}(t)) \leq V(\gamma, \overline{x}(\gamma)) + K \sup_{s \in [\gamma, t]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_\gamma^s DG(\overline{x}(\tau), t) \right\|, \tag{6.11}$$

for every $t \in [\gamma, v]$.

Since V is a Lyapunov functional, there exists a function $b : \mathbb{R}^+ \to \mathbb{R}^+$ of Hahn class such that

$$V(t, x) \geq b(\|x\|), \quad (t, x) \in [\gamma, v] \times \overline{B}_\rho. \tag{6.12}$$

Let $\varepsilon > 0$ and $b(\varepsilon) > 0$. Let $\delta(\varepsilon) > 0$ be such that $2K\delta(\varepsilon) < b(\varepsilon)$. If

$$\sup_{s \in [\gamma, t]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_\gamma^s DG(\overline{x}(\tau), t) \right\| < \delta(\varepsilon),$$

then

$$\begin{aligned} V(t, \overline{x}(t)) &\leq V(\gamma, \overline{x}(\gamma)) + +K \sup_{s \in [\gamma, t]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_\gamma^s DG(\overline{x}(\tau), t) \right\| \\ &< |V(\gamma, \overline{x}(\gamma))| + K\delta(\varepsilon) \\ &\leq 2K\delta(\varepsilon) < b(\varepsilon), \ t \in [\gamma, v]. \end{aligned} \tag{6.13}$$

On the other hand, suppose there exists $u \in [\gamma, v]$ such that $\|\overline{x}(u)\| \geq \varepsilon$. Then, by (6.12), we have

$$V(u, \overline{x}(u)) \geq b(\|\overline{x}(u)\|) \geq b(\varepsilon),$$

which contradicts (6.13). Thus, $\|\overline{x}(t)\| < \varepsilon$ for $t \in [\gamma, v]$ and the result follows. □

The next result shows that, with an additional hypothesis on the Lyapunov functional, the trivial solution of generalized ODE (5.3) is regularly asymptotically stable. The proof is inspired in Theorem 10.14 from [14].

Theorem 6.4. *Let $V : [t_0, \infty) \times \overline{B_\rho} \to \mathbb{R}$ be a Lyapunov functional, where $\overline{B_\rho} = \{y \in X : \|y\| \leq \rho\}$, $0 < \rho < c$. Suppose V satisfies conditions (i) and (ii) from Theorem 6.3. Moreover, suppose there exists a continuous function $\Phi : X \to \mathbb{R}$ satisfying $\Phi(0) = 0$ and $\Phi(x) > 0$ for $x \neq 0$, such that for every solution $x : [\gamma, v] \to B_\rho$ of (5.3), with $[\gamma, v] \subset [t_0, \infty)$, we have*

$$\dot{V}(t, x(t)) \leq -\Phi(x(t)), \quad t \in [\gamma, v]. \tag{6.14}$$

Then the trivial solution $x \equiv 0$ of (5.3) is regularly asymptotically stable.

Proof. Since all hypotheses from Theorem 6.3 are satisfied, the trivial solution $x \equiv 0$ of (5.3) is regularly stable. It remains to prove that the solution $x \equiv 0$ of (5.3) is regularly attracting.

Since the solution $x \equiv 0$ of (5.3) is regularly stable, we have

(I) There exists $\widetilde{\delta} \in (0, \rho)$ such that if $\overline{x} : [\gamma, v] \to B_c$, $[\gamma, v] \subset [t_0, \infty)$, is a regulated function on $[\gamma, v]$, such that

$$\|\overline{x}(\gamma)\| < \frac{\widetilde{\delta}}{2}$$

and

$$\sup_{s \in [\gamma, v]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_\gamma^s DG(\overline{x}(\tau), t) \right\| < \widetilde{\delta},$$

then

$$\|\overline{x}(t)\| < \rho, \quad t \in [\gamma, v].$$

(II) For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, $\delta < \varepsilon$ such that, if $\overline{x} : [\overline{\gamma}, \overline{v}] \to B_c$, $[\overline{\gamma}, \overline{v}] \subset [t_0, \infty)$, is a regulated function on $[\overline{\gamma}, \overline{v}]$, such that

$$\|\overline{x}(\overline{\gamma})\| < \delta$$

and

$$\sup_{s \in [\overline{\gamma}, \overline{v}]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_{\overline{\gamma}}^s DG(\overline{x}(\tau), t) \right\| < \delta,$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\overline{\gamma}, \overline{v}].$$

Let

$$N = \sup\{-\Phi(y) : \delta(\varepsilon) \leq \|y\| \leq \varepsilon\} = -\inf\{\Phi(y) : \delta(\varepsilon) \leq \|y\| \leq \varepsilon\} < 0.$$

Define

$$T(\varepsilon) := \min\left\{v - \gamma, -K\frac{\delta_0 + \delta(\varepsilon)}{N}\right\} > 0. \tag{6.15}$$

Suppose $\overline{x} : [\gamma, v] \to B_c$, $[\gamma, v] \subset [t_0, \infty)$, is a regulated function on $[\gamma, v]$, such that

$$\|\overline{x}(\gamma)\| < \frac{\widetilde{\delta}}{2} \tag{6.16}$$

and

$$\sup_{s\in[\gamma,v]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_{\gamma}^{s} DG(\overline{x}(\tau), t) \right\| < \delta(\varepsilon). \tag{6.17}$$

We want to prove that

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T(\varepsilon), \infty),\ \gamma \geq t_0.$$

Let us prove that there exists a $t^* \in [\gamma, \gamma + T(\varepsilon)]$ such that $\|\overline{x}(t^*)\| < \delta(\varepsilon)$. Suppose the contrary, that is, $\|\overline{x}(s)\| \geq \lambda(\varepsilon)$, for every $s \in [\gamma, \gamma + T(\varepsilon)]$. By Lemma 6.1 and conditions (i) and (ii) from Theorem 6.3, (6.15), (6.16) and (6.17), we obtain

$$\begin{aligned} V(\gamma + T(\varepsilon), \overline{x}(\gamma + T(\varepsilon))) &\leq V(\gamma, \overline{x}(\gamma)) + K \sup_{s\in[\gamma,\gamma+T(\varepsilon)]} \left\| \overline{x}(s) - \overline{x}(\gamma) - \int_{\gamma}^{s} DG(\overline{x}(\tau), t) \right\| \\ &+ N\,T(\varepsilon) < K\frac{\widetilde{\delta}}{2} + K\,\delta(\varepsilon) + N\left(-K\frac{\widetilde{\delta} + \delta(\varepsilon)}{N}\right) \\ &\leq K\widetilde{\delta} + K\delta(\varepsilon) + N\left(-K\frac{\widetilde{\delta} + \delta(\varepsilon)}{N}\right) = K\widetilde{\delta} - K\widetilde{\delta} = 0. \end{aligned} \tag{6.18}$$

On the other hand, since V is a Lyapunov functional, there exists a function $b : \mathbb{R}^+ \to \mathbb{R}^+$ of Hahn class such that

$$V(t, x) \geq b(\|x\|), \quad \text{para } (t, x) \in [t_0, \infty) \times \overline{B_\rho}.$$

Thus,

$$V(\gamma + T(\varepsilon), \overline{x}(\gamma + T(\varepsilon))) \geq b\left(\|\overline{x}(\gamma + T(\varepsilon))\|\right) \geq b(\lambda(\varepsilon)) > 0,$$

which contradicts (6.18). Then, we conclude that there exists a $t^* \in [\gamma, \gamma + T(\varepsilon)]$ such that $\|\overline{x}(t^*)\| < \delta(\varepsilon)$. Therefore $\|\overline{x}(t)\| < \varepsilon$ for $t \in [t^*, v]$, since (II) holds for $\overline{\gamma} = t^*$ and $\overline{v} = v$. Also, $\|\overline{x}(t)\| < \varepsilon$ for $t > \gamma + T(\varepsilon)$, since $t^* \in [\gamma, \gamma + T(\varepsilon)]$ and, thus, the solution $x \equiv 0$ of equation (5.3) is regularly attracting and the result follows. □

7 Lyapunov theorems for measure FDEs

In this section, we prove some Lyapunov theorems for measure FDEs, using Lyapunov theorems for generalized ODEs presented in the previous section and the correspondence between the solutions.

We consider the following measure FDE

$$Dy = f(y_t, t)Dg, \tag{7.1}$$

where Dy and Dg are distributional derivatives in the sense of L. Schwartz and $f : S \times [t_0, \infty) \to \mathbb{R}^n$, where $S = \{x_t;\ x \in O,\ t \in [t_0, \infty)\}$ and $O \subset BG([t_0 - r, \infty), \mathbb{R}^n)$ has the prolongation property.

We also consider that $g : [t_0, \infty) \to \mathbb{R}$ is a nondecreasing and left-continuous function and $f(0, t) = 0$ for all $t \in [t_0, \infty)$. Thus $y \equiv 0$ is a solution of (7.1). Suppose f satisfies conditions (H_1), (H_2) and (H_3) introduced previously. We also assume that F from equation (5.3) is given by the relation described by equation (3.1), which means that for $y \in O$ and $t \in [t_0, \infty)$, we have

$$F(y,t)(\vartheta) = \begin{cases} 0, & t_0 - r \le \vartheta \le t_0, \\ \displaystyle\int_{t_0}^{\vartheta} f(y_s, s)dg(s), & t_0 \le \vartheta \le t < \infty, \\ \displaystyle\int_{t_0}^{t} f(y_s, s)dg(s), & t \le \vartheta < \infty. \end{cases}$$

Then $F : O \times [t_0, \infty) \to BG^-([t_0 - r, \infty), \mathbb{R}^n)$ and by Theorem 3.2, we have a relation between the initial value problems for the equations (5.3) and (7.1).

Given $t \ge t_0$ and a function $\psi \in G^-([-r, 0], \mathbb{R}^n)$, we consider the measure FDE (7.1) with initial condition $y_t = \psi$. We also consider that the GODE (5.3) is subject to the initial condition $x(t) = \widetilde{x}$, where $\widetilde{x}(\vartheta) = \psi(\vartheta - t)$, $t - r \le \vartheta \le t$, e $\widetilde{x}(\vartheta) = \psi(0)$, $\vartheta \ge t$.

By Theorem 3.3, there exists a unique solution $y : [t - r, v] \to \mathbb{R}^n$ of the measure FDE (7.1) which satisfies $y_t = \psi$.

By Theorem 3.2, we can find a solution $x : [t, v] \to G^-([t, v], \mathbb{R}^n)$ of the GODE (5.3), with initial condition $x(t) = \widetilde{x}$. Then $x(t)(t + \theta) = y(t + \theta)$ for every $\theta \in [-r, 0]$ and hence, $(x(t))_t = y_t$. In this case, we write $y_{t+\eta} = y_{t+\eta}(t, \psi)$ for every $\eta \ge 0$. Then, given a function $U : [t_0, \infty) \times G^-([-r, 0], \mathbb{R}^n) \to \mathbb{R}$, we define

$$D^+U(t, \psi) = \limsup_{\eta \to 0+} \frac{U(t + \eta, y_{t+\eta}(t, \psi)) - U(t, y_t(t, \psi))}{\eta}, \quad t \ge t_0.$$

On the other hand, given $t \ge t_0$, if $\widetilde{x} \in G^-([t - r, \infty), \mathbb{R}^n)$ is such that $\widetilde{x}(\vartheta) = \psi(\vartheta - t)$, $t - r \le \vartheta \le t$, and $\widetilde{x}(\vartheta) = \psi(0)$, $\tau \ge t$, then, by Theorem B.4 from Appendix B, there exists an unique solution $x : [t, \overline{v}] \to G^-([t, \overline{v}], \mathbb{R}^n)$ of the GODE (5.3) such that $x(t) = \widetilde{x}$, with $[t, \overline{v}] \subset [t_0, \infty)$.

By Theorem 3.2 again, we can find a solution $y : [t - r, \overline{v}] \to \mathbb{R}^n$ of (7.1) which satisfies $y_t = \psi$ and it is described in terms of x. In this case, we write $x_\psi(t)$ instead of simply $x(t)$. Then we have $y_t(t, \psi) = (x_\psi(t))_t = \psi$. Thus $(t, x_\psi(t)) \mapsto (t, y_t(t, \psi))$ is a one-to-one mapping and we can define a function $V : [t_0, \infty) \times O \to \mathbb{R}$ by

$$V(t, x_\psi(t)) = U(t, y_t(t, \psi)). \tag{7.2}$$

Therefore

$$D^+U(t,\psi) = \limsup_{\eta\to 0^+} \frac{V(t+\eta, x_\psi(t+\eta)) - V(t, x_\psi(t))}{\eta}, \quad t \geq t_0. \tag{7.3}$$

Thus, given $t \geq t_0$, we have $\|y_t(t,\psi)\| = \|x_\psi(t)\|$. Indeed,

$$\begin{aligned}\|y_t(t,\psi)\| &= \|y_t\| = \sup_{-r\leq\theta\leq 0} |y(t+\theta)| = \sup_{t-r\leq\tau\leq t} |y(\tau)| = \sup_{t-r\leq\tau\leq t} |x_\psi(t)(\tau)| \\ &= \sup_{t-r\leq\tau<\infty} |x_\psi(t)(\tau)| = \|x_\psi(t)\|,\end{aligned}$$

where the fourth equality follows from Theorem 3.2

Now, we present a concept of Lyapunov functional $U : [t_0, \infty) \times G^-([-r,0], \mathbb{R}^n) \to \mathbb{R}$ with respect to the measure FDE (7.1).

Definition 7.1. We say that $U : [t_0, \infty) \times G^-([-r,0], \mathbb{R}^n) \to \mathbb{R}$ is a Lyapunov functional (with respect to the measure FDE (7.1)), if the following conditions are satisfied:

(i) $U(\cdot,\psi) : [t_0,\infty) \to \mathbb{R}$ is left-continuous on (t_0,∞), for every $\psi \in G^-([-r,0],\mathbb{R}^n)$;

(ii) There exists a function of Hahn class $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$U(t,\psi) \geq b(\|\psi\|),$$

for each $t \geq t_0$ and for each $\psi \in G^-([-r,0],\mathbb{R}^n)$;

(iii) The inequality

$$D^+U(t,\psi) \leq 0$$

holds for each $t \geq t_0$ and for each $\psi \in G^-([-r,0],\mathbb{R}^n)$.

We assert that if $U : [t_0,\infty) \times G^-([-r,0],\mathbb{R}^n) \to \mathbb{R}$ is a Lyapunov functional, then $V : [t_0,\infty) \times O \to \mathbb{R}$ given by the relation (7.2) satisfies conditions (i), (ii) and (iii) from Definition 6.2, meaning that V is also a Lyapunov functional. Indeed, by (7.2) and (7.3), it is easy to see that V satisfies conditions (i) and (iii) from Definition 6.2. Therefore, we just have to check that V satisfies (ii). Thus, if $t \geq t_0$ and $z \in O$, then there exist a solution x of the GODE (5.3) and a function $\psi \in G^-([-r,0],\mathbb{R}^n)$ such that $z = x_\psi(t)$, $(x_\psi(t))_t = y_t(t,\psi)$, where y is a solution of the measure FDE (7.1) corresponding to x. But

$$\|\psi\| = \|y_t(t,\psi)\| = \|x_\psi(t)\| = \|z\|.$$

Since U is a Lyapunov functional, by condition (ii) of Definition 6.2, we obtain

$$V(t,z) = V(t,x_\psi(t)) = U(t,y_t(t,\psi)) = U(t,\psi) \geq b(\|\psi\|) = b(\|z\|),$$

which shows (ii).

Now, let us consider the set

$$\overline{E}_\rho = \{y \in G^-([-r,0],\mathbb{R}^n); \|y\| \leq \rho\}.$$

The next result can be carried out as in [2], Lemma 4.4 with obvious adaptations.

Lemma 7.2. *Consider the measure functional differential equation* (7.1)*. Suppose the function* $U : [t_0, \infty) \times \overline{E}_\rho \to \mathbb{R}$ *satisfies the following conditions:*

(i) $U(t, 0) = 0$*,* $t \in [t_0, \infty)$*;*

(ii) There exists a constant $K > 0$ *such that*

$$|U(t, \psi) - U(t, \overline{\psi})| \leq K\|\psi - \overline{\psi}\|, \quad t \in [t_0, \infty), \ \psi, \overline{\psi} \in \overline{E}_\rho.$$

Then the function $V : [t_0, \infty) \times \overline{B}_\rho \to \mathbb{R}$ *defined by* (7.2) *satisfies* $V(t, 0) = 0$ *for every* $t \in [t_0, \infty)$*, and*

$$|V(t, z) - V(t, \overline{z})| \leq K\|z - \overline{z}\|,$$

for $t \geq t_0$ *e* $z, \overline{z} \in \overline{B}_\rho$*.*

With Lemma 7.2 at hand, we can prove the next stability result for measure FDEs.

Theorem 7.3. *Consider the measure functional differential equation* (7.1)*. Suppose the function* $f : S \times [t_0, \infty) \to \mathbb{R}^n$ *satisfies the conditions* (H_1)*,* (H_2) *and* (H_3) *and* $U : [t_0, \infty) \times \overline{E}_\rho \to \mathbb{R}$ *is a Lyapunov functional. Moreover, assume that the following conditions are satisfied:*

(i) $U(t, 0) = 0$*,* $t \in [t_0, \infty)$*;*

(ii) There exists a constant $K > 0$ *such that*

$$|U(t, \psi) - U(t, \overline{\psi})| \leq K\|\psi - \overline{\psi}\|, \quad t \in [t_0, \infty), \ \psi, \overline{\psi} \in \overline{E}_\rho.$$

Then the trivial solution $y \equiv 0$ *of* (7.1) *is uniformly stable.*

Proof. Since f satisfies conditions (H_1), (H_2) and (H_3), the function F in the generalized ODE (5.3) belongs to the class $\mathcal{F}(\Omega, h)$, with $\Omega = O \times [t_0, \infty)$ and $h : [t_0, \infty) \to \mathbb{R}$ given by

$$h(t) = \int_{t_0}^{t} [M(s) + L(s)]dg(s), \quad t \in [t_0, \infty).$$

Let $V : [t_0, \infty) \times \overline{B}_\rho \to \mathbb{R}$ be given by (7.2). By Lemma 7.2,

$$V(t, 0) = 0, \quad t \in [t_0, \infty)$$

and

$$|V(t, z) - V(t, \bar{z}) \leq K\|z - \bar{z}\|, \quad t \in [t_0, \infty), \ z, \bar{z} \in \overline{B}_\rho.$$

Since U is a Lyapunov functional, it follows that V is a Lyapunov functional. Therefore, all the hypotheses from Theorem 6.3 are satisfied. Then the solution $x \equiv 0$ of (5.3) is regularly stable. Thus, by Theorem 5.1, the solution $y \equiv 0$ of (7.1) is integrally stable and, therefore, uniformly stable. □

Theorem 7.4. *Consider the measure functional differential equation* (7.1)*. Suppose* $U : [t_0, \infty) \times \overline{E_\rho} \to \mathbb{R}$ *is a Lyapunov functional and satisfies conditions* (*i*) *and* (*ii*) *from Theorem 7.3. Furthermore, suppose there exists a continuous function* $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\Lambda(0) = 0$ *and* $\Lambda(x) > 0$ *if* $x \neq 0$*, and such that, for every* $\psi \in \overline{E}_\rho$*, we have*

$$D^+U(t, \psi) \leq -\Lambda(\|\psi\|), \quad t \geq t_0. \tag{7.4}$$

Then, the trivial solution $y \equiv 0$ *of* (7.1) *is uniformly asymptoticaly stable.*

Proof. Let $V : [t_0, \infty) \times \overline{B}_\rho \to \mathbb{R}$ be given by (7.2). Then all the hypotheses from Theorem 6.3 are satisfied.

Let us define a function $\Phi : \overline{B}_\rho \to \mathbb{R}$ by

$$\Phi(z) = \Lambda(\|z\|).$$

It is clear from the definition that Φ is continuous and satisfies $\Phi(0) = 0$ and $\Phi(z) > 0$, $z \neq 0$.

Suppose $x : [t_0, \infty) \to \overline{B}_\rho$ is a solution of (5.3) such that $(x(t))_t = \psi$, $t \in [t_0, \infty)$ and $\psi \in \overline{E}_\rho$. Let $y : [t_0 - r, \infty) \to \mathbb{R}^n$ be the solution of (7.1), given by Theorem 3.2, which satisfies $y_t = \psi$. By equation (7.4), we have

$$\limsup_{\eta \to 0+} \frac{V(t+\eta, x_\psi(t+\eta)) - V(t, x_\psi(t))}{\eta} = D^+U(t, y_t(t, \psi)) =$$

$$= D^+U(t, \psi) \leq -\Lambda(\|\psi\|) = -\Lambda(\|y_t\|).$$

We also have

$$\|y_t\| = \|x_\psi(t)\|.$$

Therefore,

$$\limsup_{\eta \to 0+} \frac{V(t+\eta, x_\psi(t+\eta)) - V(t, x_\psi(t))}{\eta} \leq -\Lambda(\|y_t\|) = -\Lambda(\|x_\psi(t)\|) = -\Phi(x_\psi(t))$$

and all the assumptions from Theorem 6.4 are satisfied. Thus, the solution $x \equiv 0$ of (5.3) is regularly asymptotically stable. Finally, by Theorem 5.1, the solution $y \equiv 0$ of (7.1) is integrally asymptotically stable and, hence, uniformly asymptoticaly stable. □

A Perron integration

Let us recall some basic concepts from the non-absolute Perron integration theory and from the theory of generalized ODEs. For more details, the reader may want to consult [14].

A *tagged division* of a compact interval $[a,b] \subset \mathbb{R}$ is a finite collection of point-interval pairs $(\tau_i, [s_{i-1}, s_i])$, where $a = s_0 \leq s_1 \leq \ldots \leq s_k = b$ is a division of $[a,b]$ and $\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \ldots, k$.

A *gauge* on a set $E \subset [a,b]$ is any function $\delta : E \to (0, \infty)$. Given a gauge δ on $[a,b]$, a tagged division $d = (\tau_i, [s_{i-1}, s_i])$ is *δ-fine* if, for every i, we have

$$[s_{i-1}, s_i] \subset \{t \in [a,b]\,;\, |t - \tau_i| < \delta(\tau_i)\}\,.$$

Definition A.1. Let X be a Banach space. A function $U : [a,b] \times [a,b] \to X$ is called *Kurzweil integrable* over $[a,b]$, if there is a unique element $I \in X$ such that given $\varepsilon > 0$, there is a gauge δ on $[a,b]$ such that

$$\left\| \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})] - I \right\| < \varepsilon$$

for every δ-fine tagged division $(\tau_i, [s_{i-1}, s_i])$ of $[a,b]$. In this case, I is called the Kurzweil integral of U over $[a,b]$ and it is denoted by $\int_a^b DU(\tau, t)$.

Remark A.2. In particular, the Perron-Stieljtes integral of a function $f : [a,b] \to X$ with respect to a given function $g : [a,b] \to \mathbb{R}$ corresponds to the choice $U(\tau,t) = f(\tau)g(t)$. Clearly, thus, the choice $U(\tau,t) = f(\tau)t$ corresponds to the well-known Perron integral which we denote by $\int_a^b f(s)ds$.

The Perron and Perron-Stieltjes integrals have the usual properties of linearity, additivity with respect to adjacent intervals and integrability on subintervals.

In order to present a result on the existence of the Perron-Stieltjes integral $\int_a^b f(s)dg(s)$, we need the concept of regulated functions.

Recall that a function $f : [a,b] \to X$ is *regulated*, if the lateral limits

$$\lim_{s \to t-} f(s) = f(t-) \in X, \quad t \in (a,b], \quad \text{and} \quad \lim_{s \to t+} f(s) = f(t+) \in X, \quad t \in [a,b)$$

exist. The space of all regulated functions $f : [a,b] \to X$ is denoted by $G([a,b],X)$ and it is a Banach space under the usual supremum norm $\|f\|_\infty = \sup_{a \leq t \leq b} \|f(t)\|$. We denote by $G^-([a,b],X)$ the subspace of $G([a,b],X)$ of left-continuous functions.

A proof of the next result can be found in [14], Corollary 1.34.

Theorem A.3. *If $f : [a,b] \to X$ is a regulated function and $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, then the Perron-Stieltjes integral $\int_a^b f(s)\,\mathrm{d}g(s)$ exists.*

There is also an existence result (see [14], Corollary 3.16) for the Perron integral $\int_a^b DU(\tau,t)$, in the particular case, where $U(\tau,t) = G(x(\tau),t)$, $x : [a,b] \to O$, $O \subset X$, is a regulated function and $G : O \times [a,b] \to X$ satisfies

$$\|G(x,s_2) - G(x,s_1)\| \leq |h(s_2) - h(s_1)| \tag{A.1}$$

and

$$\|G(x,s_2)-G(x,s_1)-G(y,s_2)+G(y,s_1)\| \leq \|x-y\|_\infty |h(s_2)-h(s_1)| \tag{A.2}$$

for all $(x,s_1),(x,s_2),(y,s_1),(y,s_2) \in O\times[a,b]$ and some nondecreasing function $h:[a,b]\to\mathbb{R}$.

Theorem A.4. *If G satisfies the equations (A.1) and (A.2) and $x:[\alpha,\beta]\to O$ is a regulated function such that $(x(s),s)\in O\times[\alpha,\beta]$, for every $s\in[\alpha,\beta]$, where $[\alpha,\beta]\subset(a,b)$, then the integral $\int_\alpha^\beta DG(x(\tau),t)$ exists.*

The following result, which describes the properties of the indefinite Perron-Stieljtes integral, is a special case of Theorem 1.16 from [14].

Theorem A.5. *Let $f:[a,b]\to\mathbb{R}^n$ and $g:[a,b]\to\mathbb{R}$ be a pair of functions such that g is regulated and $\int_a^b f(s)\,\mathrm{d}g(s)$ exists. Then the function*

$$h(t)=\int_a^t f(s)\,\mathrm{d}g(s),\quad t\in[a,b],$$

is regulated and satisfies

$$\begin{aligned} h(t+) &= h(t)+f(t)\Delta^+g(t),\quad t\in[a,b),\\ h(t-) &= h(t)-f(t)\Delta^-g(t),\quad t\in(a,b], \end{aligned}$$

where $\Delta^+g(t)=g(t+)-g(t)$ and $\Delta^-g(t)=g(t)-g(t-)$.

The next lemma is a fundamental result in the non-absolute integration theory, known as Saks-Henstock Lemma. A proof of this lemma can be found in [14], Lemma 1.13, when $X=\mathbb{R}^n$. The proof for an arbitrary Banach space X follows analogously.

Lemma A.6. *Let $U:[a,b]\times[a,b]\to X$ be Perron integrable over $[a,b]$. Given $\varepsilon>0$, let δ a gauge in $[a,b]$ such that*

$$\left\|\sum_{j=1}^{k}[U(\tau_j,s_j)-U(\tau_j,s_{j-1})]-\int_a^b DU(\tau,t)\right\|<\varepsilon \tag{A.3}$$

for every δ-fine tagged division $d=\{(\tau_j,[s_{j-1},s_j]),j=1,2,\dots,k\}$ of $[a,b]$. If

$$a\leq\beta_1\leq\xi_1\leq\gamma_1\leq\beta_2\leq\xi_2\leq\gamma_2\leq\ldots\leq\beta_m\leq\xi_m\leq\gamma_m\leq b$$

represents a partial delta-fine tagged division $\{(\xi_j,[\beta_j,\gamma_j]),j=1,2,\dots,m\}$ of $[a,b]$, which means,

$$\xi_j\in[\beta_j,\gamma_j]\subset(\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j)),\quad j=1,2,\dots,m,$$

then

$$\left\|\sum_{j=1}^{m}\left[U(\xi_j,\beta_j)-U(\xi_j,\gamma_j)-\int_{\beta_j}^{\gamma_j} DU(\tau,t)\right]\right\|<\varepsilon. \tag{A.4}$$

As a consequence of the Saks-Henstock Lemma we have the following result

Corollary A.7. *Let $U : [a,b] \times [a,b] \to X$ be Perron integrable over $[a,b]$. Given $\varepsilon > 0$, let δ be a gauge in $[a,b]$. Let $[\gamma, v]$ be a closed subinterval of $[a,b]$. Then,*

(i) $(v - \gamma) < \delta(\gamma)$ *implies*

$$\left\| U(\gamma, v) - U(\gamma, \gamma) - \int_{\gamma}^{v} DU(\tau, t) \right\| < \varepsilon;$$

(ii) $(v - \gamma) < \delta(v)$ *implies*

$$\left\| U(v, v) - U(v, \gamma) - \int_{\gamma}^{v} DU(\tau, t) \right\| < \varepsilon.$$

B Generalized ODEs

Now, we recall the definition of a generalized ODE.

Definition B.1. Let X be a Banach space. Consider a set $O \subset X$, an interval $[a,b] \subset \mathbb{R}$ and a function $G : O \times [a,b] \to X$. A function $x : [a,b] \to O$ is called a solution of the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG(x,t) \tag{B.1}$$

on the interval $[a,b]$, if

$$x(d) - x(c) = \int_{c}^{d} DG(x(\tau), t)$$

for every $c, d \in [a,b]$.

When the right-hand side of a generalized ODE satisfies (A.1), we have the following information about its solutions (see Lemma 3.12 from [14]).

Lemma B.2. *Consider a set $O \subset X$, an interval $[a,b] \subset \mathbb{R}$ and a function $G : O \times [a,b] \to X$. If $x : [a,b] \to O$ is a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DG(x,t)$$

and G satisfies (A.1)*, then x is a function of bounded variation (and hence regulated).*

Under the assumptions (A.1) and (A.2) on $G : O \times [a,b] \to X$, it is possible to obtain an existence-uniqueness result for an initial value problem for equation (B.1). See Theorem B.4 in the sequel. As a matter of fact, the class of functions G fulfilling (A.1) and (A.2) is important to several results. We now define this class of functions.

Definition B.3. Given a nondecreasing function $h : [a,b] \to \mathbb{R}$, we say that a function $G : O \times [a,b] \to X$ belongs to the class $\mathcal{F}(\Omega, h)$, if for all $x, y \in G([a,b], X)$, we have

$$\|G(x, s_2) - G(x, s_1)\| \leq |h(s_2) - h(s_1)| \tag{B.2}$$

and

$$\|G(x, s_2) - G(x, s_1) - G(y, s_2) + G(y, s_1)\| \leq \|x - y\|_\infty |h(s_2) - h(s_1)| \tag{B.3}$$

for all s_1, $s_2 \in [a,b]$ and all $x, y \in O$.

For a proof of the next theorem, see [7], Theorem 2.15.

Theorem B.4. *Let $G : \Omega \to X$ be an element of the class $\mathcal{F}(\Omega, h)$, where the function h is left continuous (i.e. $h(t-) = h(t)$). Then for every $(\widetilde{x}, t_0) \in \Omega$ such that for $\widetilde{x}_+ = \widetilde{x} + G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0)$, we have $(\widetilde{x}_+, t_0) \in \Omega$ and there exists a $\Delta > 0$ such that, on the interval $[t_0, t_0 + \Delta]$, there exists a unique solution $x : [t_0, t_0 + \Delta] \to X$ of the generalized ordinary differential equation* (B.1) *for which $x(t_0) = \widetilde{x}$.*

References

[1] S. M. Afonso, E. M. Bonotto, M. Federson, L. P. Gimenes, Boundedness of solutions of functional differential equations with variable impulses via generalized ordinary differential equations, *Mathematische Nachrichten*, v. 285, p. 545-561, 2012.

[2] S. M. Afonso; E. Bonotto; M. Federson; L. P. Gimenes, Stability of functional differential equations with variable impulsive perturbations via generalized ordinary differential equations, *Bulletin des Sciences Mathématiques (Paris, 1885)* v. 137 (2013), 189-214.

[3] P. C. Das, R. R. Sharma, Existence and stability of measure differential equations, *Czech. Math. Journal* 22 (97) (1972), 145–158

[4] S. G. Deo and S. R. Joshi, On abstract measure delay differential equations, *An. Stiint. Univ. Al. I. Cuza Iasi*, N. Ser., Sect. Ia 26, 327–335 (1980).

[5] S. G. Deo and S. G. Pandit, *Differential systems involving impulses*, Lecture Notes in Mathematics, vol. 954, Springer-Verlag, Berlin, 1982.

[6] M. Federson, J. G. Mesquita, A. Slavik, Measure functional differential equations and functional dynamic equations on time scales, *J. Diff. Equations*, 252 (2012), 3816-3847.

[7] M. Federson; Š. Schwabik, Generalized ODEs approach to impulsive retarded differential equations, *Differential and Integral Equations* 19(11), (2006), 1201-1234.

[8] M. Federson; Š. Schwabik, A new approach to impulsive retarded differential equations: stability results. *Functional Differential Equations*, 16(4) (2009), 583-607.

[9] M. Federson, Š. Schwabik, Stability for retarded functional differential equations *Ukrainian Math J.*, 60(1), (2008), 107-126.

[10] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags.* Academic Press, New York, 1966.

[11] J. K. Hale; S. M. Verduyn Lunel, *Introduction to Functional Differential Equations.* Applied Mathematical Sciences, 99. Springer-Verlag, New York, 1993.

[12] S. R. Joshi, A system of abstract measure delay differential equations, *J. Math. Phys. Sci.*, 13 (1979), 497–506.

[13] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter, *Czech. Math. J.*, 7 (82), (1957), 418-448.

[14] Š. Schwabik, *Generalized Ordinary Differential Equations*, World Scientific, Series in Real Anal., vol. 5, 1992.

[15] Š. Schwabik, Variational stability for generalized ordinary differential equations, *Časopis Pěst. Mat.* 109(4), (1984), 389-420.

[16] R. R. Sharma, An abstract measure differential equation, *Proc. Amer. Math. Soc.* 32 (1972), 503–510.

[17] A. Slavík, Measure functional differential equations with infinite delay, *Nonlinear Anal.*, 79 (2013), 140-155.

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