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The Lefschetz Coincidence Class of p Maps

Carlos Biasi

Instituto de Ciências Matemáticas e de Computação. Universidade de São Paulo.
E-mail: biasi@icmc.usp.br

Alice K. M. Libardi

Instituto de Geociências e Ciências Exatas. Universidade Estadual Paulista.
E-mail: alicekml@rc.unesp.br

Thaís F. M. Monis

Instituto de Geociências e Ciências Exatas. Universidade Estadual Paulista.
E-mail: tfmonis@rc.unesp.br

Let X be an arbitrary topological space and let Y be a closed connected oriented n-dimensional manifold. In this work we consider p maps $f_1, \ldots, f_p : X \to Y$, $p \geq 2$, define a Lefschetz class $L(f_1, \ldots, f_p) \in H^{n(p-1)}(X; \mathbb{Q})$ and prove that $L(f_1, \ldots, f_p) \neq 0$ implies $f_1(x) = f_2(x) = \cdots = f_p(x)$ for some $x \in X$. In the particular case where Y is a homology sphere are presented some formulas to calculate $L(f_1, \ldots, f_p)$. October, 2012 ICMC-USP

1. INTRODUCTION

Given $f, g : X \to Y$ two maps between topological spaces, the coincidence set of f and g is defined by:

$$\operatorname{Coin}(f, g) = \{x \in X \mid f(x) = g(x)\}.$$

Obviously, $Coin(f, g) \neq \emptyset$ if and only if the intersection of the set $\{(f(x), g(x)) \mid x \in X\}$ with the diagonal $\Delta_Y = \{(y, y) \mid y \in Y\}$ is a nonempty set.

Let $j : Y \times Y \to (Y \times Y,\ Y \times Y - \Delta_Y)$ be the inclusion map and let $(f, g) : X \to Y \times Y$ be defined by $(f, g)(x) = (f(x), g(x))$, $\forall x \in X$. Thus, if $\operatorname{Coin}(f, g) = \emptyset$, in any cohomology theory, the induced homomorphism $[j \circ (f, g)]^* : H^*(Y \times Y,\ Y \times Y - \Delta_Y) \to H^*(X)$ is trivial. In fact, in this case, $j \circ (f, g)$ is factored by

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$$\begin{array}{ccccc} X & \xrightarrow{(f,g)} & Y\times Y & \xrightarrow{j} & (Y\times Y,\ Y\times Y-\Delta_Y)\\ \downarrow & & & & \uparrow \\ Y\times Y-\Delta_Y & & \longrightarrow & & (Y\times Y-\Delta_Y,\ Y\times Y-\Delta_Y)\end{array}$$

Thus, if $[j\circ(f,g)]^*(u)\neq 0$ for some cohomology class $u\in H^*(Y\times Y,\ Y\times Y-\Delta_Y)$ then $\mathrm{Coin}(f,g)\neq\emptyset$, i.e, there exists $x_0\in X$ such that $f(x_0)=g(x_0)$.

Throughout this paper, homology and cohomology will considered with rational coefficients. If Y is a closed connected oriented n-dimensional manifold, there exists a cohomology class $\mu\in H^n(Y\times Y, Y\times Y-\Delta_Y)$, called the Thom class of the manifold Y, which is related to the fundamental class $\zeta\in H_n(Y)$ of Y. The cohomology class $L(f,g)=[j\circ(f,g)]^*(\mu)\in H^n(X)$ is called the Lefschetz class of the pair (f,g). Also if X is a closed connected oriented n-dimensional manifold the Lefschetz number of the pair (f,g) is defined by

$$\Lambda(f,g)=\sum_i(-1)^i\mathrm{trace}_i(f^*g^!) \tag{1.1}$$

where $g^!=D_Y^{-1}\circ g_*\circ D_X$. Here, D_X and D_Y are the Poincaré duality isomorphisms of X and Y, respectively.

We recall that

$$\Lambda(f,g)=[\zeta',L(f,g)],$$

where $\zeta'\in H_n(X)$ is the fundamental class of X and $[\ ,\ ]$ denotes the Kronecker product (see [3], Theorem 14.4, p. 396). Therefore, if $\Lambda(f,g)\neq 0$ there exists a coincidence of f and g.

This work is divided in four sections. The first one is dedicated to preliminary notions. In section 2, we consider Y a closed connected oriented n-dimensional manifold, X an arbitrary topological space and given p maps $f_1,\dots,f_p : X\to Y$, $p\geq 2$, define a Lefschetz class $L(f_1,\dots,f_p)\in H^{n(p-1)}(X)$ and prove that $L(f_1,\dots,f_p)\neq 0$ implies $\mathrm{Coin}(f_1,\dots,f_p)\neq\emptyset$. In section 3, we calculate the Lefschetz class when Y is a homology sphere space. In section 4, we give formulas for $L(f_1,\dots,f_p)$ which are more computable, but which are applied only in special cases.

2. THE LEFSCHETZ CLASS OF P MAPS

We recall that the product of two pairs of topological spaces (X,A) and (Y,B) is defined by

$$(X,A)\times(Y,B):=(X\times Y,\ X\times B\cup A\times Y).$$

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Inductively, given (X_i, A_i), $i = 1, \dots, k$, k pairs of topological spaces we define

$$\prod_{i=1}^{k}(X_i, A_i) = (\prod_{i=1}^{k} X_i, \bigcup_{i=1}^{k} Y_i),$$

where

$$Y_i = X_1 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \times \cdots \times X_k.$$

Also, given a pair (Z, W), we denote $(Z, W)^k := \underbrace{(Z, W) \times \cdots \times (Z, W)}_{k-\text{times}}$.

From now on, Y denotes a closed connected oriented n-manifold.

Let $\zeta \in H_n(Y)$ be the fundamental class of Y, let $\mu \in H^n(Y \times Y, Y \times Y - \Delta_Y)$ be the Thom class related to such orientation and let $j : Y \times Y \to (Y \times Y, Y \times Y - \Delta_Y)$ be the inclusion map. Given maps $f_1, \dots, f_p : X \to Y$, $p \geq 2$, the coincidence set of $f_1, \dots, f_p$ is defined by

$$\mathrm{Coin}(f_1, \dots, f_p) = \{x \in X \mid f_1(x) = f_2(x) = \cdots = f_p(x)\}.$$

For each $i = 1, \dots, p-1$, let us consider $h_i = (f_i, f_{i+1}) : X \to Y \times Y$. It is clear that

$$\mathrm{Coin}(f_1, f_2, \dots, f_p) = \mathrm{Coin}(h_1, \dots, h_{p-1}).$$

We define the Lefschetz class of $f_1, \dots, f_p$ by

$$L(f_1, f_2, \dots, f_p) = [(\underbrace{j \times \cdots \times j}_{(p-1)-\text{times}}) \circ (h_1, \dots, h_{p-1})]^*(\underbrace{\mu \times \cdots \times \mu}_{(p-1)-\text{times}}) \in H^{n\cdot(p-1)}(X).$$

Theorem 2.1. *If* $L(f_1, f_2, \dots, f_p) \neq 0$ *then* $\mathrm{Coin}(f_1, \dots, f_p) \neq \emptyset$.

Proof. Suppose $\mathrm{Coin}(f_1, \dots, f_p) = \emptyset$. Then, for every $x \in X$, $f_i(x) \neq f_j(x)$, for some $i \neq j$. Therefore, for each $x \in X$,

$$(h_1(x), \dots, h_{p-1}(x)) \in \bigcup_{i=1}^{p-1} Z_i,$$

where $Z_i = \prod_{j=1}^{p-1} A_i^j$, with $A_i^j = Y \times Y - \Delta_Y$ if $j = i$ and $A_i^j = Y \times Y$ if $j \neq i$.

We may see that $(Y \times Y, Y \times Y - \Delta_Y)^{p-1} = ((Y \times Y)^{p-1}, \bigcup_{i=1}^{p-1} Z_i)$.

Thus, the composed map $(j \times \cdots \times j) \circ (h_1, \dots, h_{p-1})$ factors as in the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{(h_1,\ldots,h_{p-1})} & (Y \times Y)^{p-1} & \xrightarrow{j\times\cdots\times j} & (Y \times Y,\ Y \times Y - \Delta_Y)^{p-1} \\ \downarrow & & & & \uparrow \\ \bigcup_{i=1}^{p-1} Z_i & & \longrightarrow & & (\bigcup_{i=1}^{p-1} Z_i, \bigcup_{i=1}^{p-1} Z_i) \end{array}$$

Therefore, $L(f_1, f_2, \ldots, f_p) = [(j \times \cdots \times j) \circ (h_1, \ldots, h_{p-1})]^*(\mu \times \cdots \times \mu) = 0.$ ▮

The Lefschetz class has the following property.

PROPOSITION 2.1. *If $L(f_1, \ldots, f_p) \neq 0$ then $L(f_i, f_j) \neq 0$, for all $1 \leq i < j \leq p$.*

Proof. Let us to prove the case $p = 3$. In this case,

$$\begin{aligned} L(f_1, f_2, f_3) &= (f_1, f_2)^*(e_n \times e_n) + (-1)^n (f_1, f_3)^*(e_n \times e_n) + (f_2, f_3)^*(e_n \times e_n) \\ L(f_1, f_2) &= f_1^*(e_n) + (-1)^n f_2^*(e_n) \\ L(f_1, f_3) &= f_1^*(e_n) + (-1)^n f_3^*(e_n) \text{ and} \\ L(f_2, f_3) &= f_2^*(e_n) + (-1)^n f_3^*(e_n) \end{aligned}$$

Suppose that $L(f_1, f_2) = 0$. Then,

$$(f_1^*(e_n) + (-1)^n f_2^*(e_n)) \smile f_2^*(e_n) = 0 \quad (1)$$

$$(f_1^*(e_n) + (-1)^n f_2^*(e_n)) \smile f_3^*(e_n) = 0 \quad (2)$$

From equation (1) we conclude that $(f_1, f_2)^*(e_n \times e_n) = 0$ and from equation (2) we conclude that $(-1)^n (f_1, f_3)^*(e_n \times e_n) + (f_2, f_3)^*(e_n \times e_n) = 0$. Therefore,

$$L(f_1, f_2, f_3) = (f_1, f_2)^*(e_n \times e_n) + (-1)^n (f_1, f_3)^*(e_n \times e_n) + (f_2, f_3)^*(e_n \times e_n) = 0.$$

Similarly, $L(f_1, f_3) = 0$ or $L(f_2, f_3) = 0$ implies $L(f_1, f_2, f_3) = 0$.

Therefore,

$$L(f_1, f_2, f_3) \neq 0 \Rightarrow L(f_1, f_2) \neq 0,\ L(f_1, f_3) \neq 0 \text{ and } L(f_2, f_3) \neq 0.$$

For the general case, it is enough to show that $L(f_1, \ldots, f_p) \neq 0$ implies $L(\widehat{f_j}) \neq 0$, where $\widehat{f_j} = (f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_p)$, for all $j = 1, \ldots, p$. In fact to prove this, it is used the same idea of the previous case. ▮

3. THE LEFSCHETZ CLASS FOR A HOMOLOGY SPHERE SPACE Y

Throughout this section Y denotes an n-homology sphere space.

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Let $e_n \in H^n(Y)$ be such that $[\zeta, e_n] = 1$ and let $e_0 \in H^0(Y)$ be the identity element of the $\mathbb{Q}$- algebra $H^*(Y)$. Recall that $j : Y \times Y \to (Y \times Y, Y \times Y - \Delta_Y)$ denotes the inclusion map and $\mu \in H^n(Y \times Y, Y \times Y - \Delta_Y)$ denotes the Thom class. Thus

Proposition 3.1. $j^*(\mu) = e_n \times e_0 + (-1)^n e_0 \times e_n$.

Now, we can describe $L(f_1, \ldots, f_p)$ in the setting that Y is a homology sphere space.

Proposition 3.2. *Let Y be an n-homology sphere. Then*

$$\begin{aligned} L(f_1, \ldots, f_p) &= \widehat{f_p}^*(e) + (-1)^n \widehat{f}_{p-1}^*(e) + (-1)^{2n} \widehat{f}_{p-2}^*(e) + \cdots \\ &\quad \cdots + (-1)^{(p-2)n} \widehat{f_2}^*(e) + (-1)^{(p-1)n} \widehat{f_1}^*(e) \end{aligned}$$

where $\widehat{f_j} = (f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_p)$, $j = 1, \ldots, p$, *and* $e = \underbrace{e_n \times \cdots \times e_n}_{(p-1)-times}$.

Proof.

$$\begin{aligned} & L(f_1, \ldots, f_p) = \\ & (h_1, \ldots, h_{p-1})^*(j^*(\mu) \times \cdots \times j^*(\mu)) = \\ & h_1^*(j^*(\mu)) \smile h_2^*(j^*(\mu)) \smile \cdots \smile h_{p-1}^*(j^*(\mu)) = \\ & (f_1^*(e_n) + (-1)^n f_2^*(e_n)) \smile (f_2^*(e_n) + (-1)^n f_3^*(e_n)) \smile \cdots \smile (f_{p-1}^*(e_n) + (-1)^n f_p^*(e_n)). \end{aligned}$$

It is not difficult to see that the above product is exactly the same as

$$\widehat{f_p}^*(e) + (-1)^n \widehat{f}_{p-1}^*(e) + (-1)^{2n} \widehat{f}_{p-2}^*(e) + \cdots + (-1)^{(p-2)n} \widehat{f_2}^*(e) + (-1)^{(p-1)n} \widehat{f_1}^*(e).$$

∎

Corollary 3.1. *Let $f_1, \ldots, f_p : Y^{p-1} \to Y$ be maps, where Y is a n-homology sphere. Then,*

$$L(f_1, \ldots, f_p) = \left(\sum_{i=1}^{p} (-1)^{n(p-i)} deg(\widehat{f_i}) \right) e$$

where $\widehat{f_i} = (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_p)$, $i = 1, \ldots, p$, *and* $e = \underbrace{e_n \times \cdots \times e_n}_{(p-1)-times}$.

Therefore, $L(f_1, \ldots, f_p) \neq 0$ if and only if $\sum_{i=1}^{p} (-1)^{n(p-i)} deg(\widehat{f_i}) \neq 0$. Thus, we define the Lefschetz number for $f_1, \ldots, f_p$ by

$$\Lambda(f_1, \ldots, f_p) = \sum_{i=1}^{p} (-1)^{n(p-i)} deg(\widehat{f_i}).$$

4. RELATED RESULTS

Throughout this section Y denotes an n-homology sphere space.

4.1. Particular case 1

Let $f_1, f_2, f_3 : Y \times Y \to Y$ be maps. By using the induced homomorphisms $f_1^*, f_2^*, f_3^* : H^n(Y) \to H^n(Y \times Y)$, we will calculate the Lefschetz number, $\Lambda(f_1, f_2, f_3)$. From Corollary 3.1,

$$\begin{aligned}\Lambda(f_1, f_2, f_3) &= \deg(f_1, f_2) + (-1)^n \deg(f_1, f_3) + (-1)^{2n} \deg(f_2, f_3) \\ &= \deg(f_1, f_2) + (-1)^n \deg(f_1, f_3) + \deg(f_2, f_3).\end{aligned}$$

Suppose that the homomorphisms $f_1^*, f_2^*, f_3^* : H^n(Y) \to H^n(Y \times Y)$ are given by

$$\begin{aligned}f_1^*(e_n) &= a_{11}e_1 + a_{12}e_2 \\ f_2^*(e_n) &= a_{21}e_1 + a_{22}e_2 \\ f_3^*(e_n) &= a_{31}e_1 + a_{32}e_2\end{aligned}$$

where $e_1 = e_n \times e_0$ and $e_2 = e_0 \times e_n$. We have that $e_1 \smile e_1 = e_2 \smile e_2 = 0$, $e_1 \smile e_2 = e_n \times e_n$ and $e_2 \smile e_1 = (-1)^n e_n \times e_n$. It follows that

$$\begin{aligned}(f_1, f_2)^*(e_n \times e_n) &= f_1^*(e_n) \smile f_2^*(e_n) \\ &= (a_{11}e_1 + a_{12}e_2) \smile (a_{21}e_1 + a_{22}e_2) \\ &= (a_{11}a_{22} + (-1)^n a_{12}a_{21}) e_n \times e_n.\end{aligned}$$

Similarly,

$$(f_1, f_3)^*(e_n \times e_n) = (a_{11}a_{32} + (-1)^n a_{12}a_{31}) e_n \times e_n$$

and

$$(f_2, f_3)^*(e_n \times e_n) = (a_{21}a_{32} + (-1)^n a_{22}a_{31}) e_n \times e_n$$

Thus, we conclude that

$$\Lambda(f_1, f_2, f_3) = (a_{11}a_{22} + (-1)^n a_{12}a_{21}) - (a_{11}a_{32} + (-1)^n a_{12}a_{31}) + (a_{21}a_{32} + (-1)^n a_{22}a_{31}).$$

Note that for n odd, $\Lambda(f_1, f_2, f_3)$ is the determinant of the matrix

$$\begin{pmatrix} 1 & a_{11} & a_{12} \\ 1 & a_{21} & a_{22} \\ 1 & a_{31} & a_{32} \end{pmatrix}.$$

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Thus, we have the following result.

Corollary 4.1. *Let n be an odd number and let $f_1, f_2, f_3 : Y \times Y \to Y$ be maps with $f_1^*(e_n) = a_{11}e_1 + a_{12}e_2$, $f_2^*(e_n) = a_{21}e_1 + a_{22}e_2$ and $f_3^*(e_n) = a_{31}e_1 + a_{32}e_2$. If*

$$det \begin{pmatrix} 1 & a_{11} & a_{12} \\ 1 & a_{21} & a_{22} \\ 1 & a_{31} & a_{32} \end{pmatrix} \neq 0$$

then there exists $x_0 \in Y \times Y$ such that $f_1(x_0) = f_2(x_0) = f_3(x_0)$.

We recall that for any map $f : X \to Z$ between topological spaces and for any k, q positive integers given, we have that

$$f_*(u) \frown \beta = f_*(u \frown f^*(\beta)),$$

for every $u \in H_{k+q}(X)$ and $\beta \in H^k(Z)$.

Let us consider now $f : Y \times Y \to Y \times Y$ a map and let $d \in \mathbb{Z}$ be the degree of f, i.e., $f_*(\zeta \times \zeta) = d \cdot \zeta \times \zeta$, where $\zeta \in H_n(Y)$ is the fundamental class of Y. Thus

$$f_*(\zeta \times \zeta \frown f^*(\beta)) = d \cdot (\zeta \times \zeta) \frown \beta, \text{ for all } \beta \in H^k(Y \times Y).$$

Denoting by Δ_k the determinant of $f_* : H_k(Y \times Y) \to H_k(Y \times Y)$, which is equal to the determinant of $f^* : H^k(Y \times Y) \to H^k(Y \times Y)$, we have that

$$d^{\beta_k} = \Delta_k \Delta_{2n-k},$$

where β_k is the k-th Betti number of $Y \times Y$. In particular, for $k = n$,

$$d^2 = \Delta_n^2,$$

Therefore, $d = \pm\Delta_n$.

Example 4.1. Let n be an even integer and let $f_1, f_2 : Y \times Y \to Y$ be maps. Let $e_1 = e_n \times e_0$ and $e_2 = e_0 \times e_n$ be generators of $H^n(Y \times Y)$. Suppose

$$\begin{aligned} f_1^*(e_n) &= a_{11}e_1 + a_{12}e_2 \\ f_2^*(e_n) &= a_{21}e_1 + a_{22}e_2 \end{aligned}$$

By the previous remark we have that $(f_1, f_2)^*(e_n \times e_n) = \pm(a_{11}a_{22} - a_{12}a_{21})\ e_n \times e_n$. On the other hand,

$$(f_1, f_2)^*(e_n \times e_n) = f_1^*(e_n) \smile f_2^*(e_n) = (a_{11}a_{22} + (-1)^n a_{12}a_{21})\ e_n \times e_n.$$

Since n is an even number,

$$(f_1, f_2)^*(e_n \times e_n) = (a_{11}a_{22} + a_{12}a_{21})\ e_n \times e_n,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is the matrix of $(f_1, f_2)^* : H^n(Y \times Y) \to H^n(Y \times Y)$.

Moreover, since $e_n \smile e_n = 0$,

$$0 = f_1^*(e_n \smile e_n) = f_1^*(e_n) \smile f_1^*(e_n) = (a_{11}a_{12} + (-1)^n a_{11}a_{12})\ e_n \times e_n = 2a_{11}a_{12}\ e_n \times e_n,$$

which implies $2a_{11}a_{12} = 0$. Therefore, $a_{11} = 0$ or $a_{12} = 0$. In a similar way we have

$$0 = f_2^*(e_n \smile e_n) = f_2^*(e_n) \smile f_2^*(e_n) = (a_{21}a_{22} + (-1)^n a_{21}a_{22})\ e_n \times e_n = 2a_{21}a_{22}\ e_n \times e_n,$$

which implies that $a_{21} = 0$ or $a_{22} = 0$.

Then the matrix of $(f_1, f_2)^* : H^n(Y \times Y) \to H^n(Y \times Y)$ is one of the following type:

$$A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ or } A = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \text{ or } A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \text{ or } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix}.$$

We remark that in the first and fourth cases we have

$$(f_1, f_2)^*(e_n \times e_n) = -\det(A)\ e_n \times e_n$$

and in the second and third cases we have

$$(f_1, f_2)^*(e_n \times e_n) = \det(A)\ e_n \times e_n.$$

From the above remarks, it follows that:

Corollary 4.2. *Let $f_1, f_2 : Y \times Y \to Y$ be maps with $f_1^*(e_n) = a_{11}e_1 + a_{12}e_2$ and $f_2^*(e_n) = a_{21}e_1 + a_{22}e_2$. If*

$$det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0$$

then, for all $p \in Y$, there exists $x_p \in Y \times Y$ such that $f_1(x_p) = f_2(x_p) = p$.

Proof. Given $p \in Y$, let us consider the constant map $f_3(x) = p$, for all $x \in Y \times Y$. In this case,

$$\Lambda(f_1, f_2, f_3) = \deg(f_1, f_2) = \pm\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

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It follows that if $a_{11}a_{22} - a_{12}a_{22} \neq 0$, then there exists $x_0 \in Y \times Y$ such that

$$f_1(x_0) = f_2(x_0) = p.$$

∎

4.2. Particular case 2

Let $f_1, f_2, f_3, f_4 : Y \times Y \times Y \to Y$ be maps. Let us calculate the Lefschetz number, $\Lambda(f_1, f_2, f_3, f_4)$, in terms of the induced homomorphisms $f_1^*, f_2^*, f_3^*, f_4^* : H^n(Y) \to H^n(Y \times Y \times Y)$. From Corollary 3.1,

$$\begin{aligned}\Lambda(f_1, f_2, f_3) &= \deg(f_1, f_2, f_3) + (-1)^n \deg(f_1, f_2, f_4) \\ &\quad + (-1)^{2n} \deg(f_1, f_3, f_4) + (-1)^{3n} \deg(f_2, f_3, f_4) \\ &= \deg(f_1, f_2, f_3) + (-1)^n \deg(f_1, f_2, f_4) \\ &\quad + \deg(f_1, f_3, f_4) + (-1)^n \deg(f_2, f_3, f_4).\end{aligned}$$

Suppose that the homomorphisms $f_1^*, f_2^*, f_3^*, f_4^* : H^n(Y) \to H^n(Y \times Y \times Y)$ are given by

$$\begin{aligned}f_1^*(e_n) &= a_{11}e_1 + a_{12}e_2 + a_{13}e_3 \\ f_2^*(e_n) &= a_{21}e_1 + a_{22}e_2 + a_{23}e_3 \\ f_3^*(e_n) &= a_{31}e_1 + a_{32}e_2 + a_{33}e_3 \\ f_4^*(e_n) &= a_{41}e_1 + a_{42}e_2 + a_{43}e_3\end{aligned}$$

where $e_1 = e_n \times e_0 \times e_0$, $e_2 = e_0 \times e_n \times e_0$ and $e_3 = e_0 \times e_0 \times e_n$. Then, we have

$$\begin{aligned}\deg(f_1, f_2, f_3) &= a_{11}(a_{22}a_{33} + (-1)^n a_{23}a_{32}) + a_{12}((-1)^n a_{21}a_{33} + a_{23}a_{31}) \\ &\quad + a_{13}(a_{21}a_{32} + (-1)^n a_{22}a_{31})\end{aligned}$$

$$\begin{aligned}\deg(f_1, f_2, f_4) &= a_{11}(a_{22}a_{43} + (-1)^n a_{23}a_{42}) + a_{12}((-1)^n a_{21}a_{43} + a_{23}a_{41}) \\ &\quad + a_{13}(a_{21}a_{42} + (-1)^n a_{22}a_{41})\end{aligned}$$

$$\begin{aligned}\deg(f_1, f_3, f_4) &= a_{11}(a_{32}a_{43} + (-1)^n a_{33}a_{42}) + a_{12}((-1)^n a_{31}a_{43} + a_{33}a_{41}) \\ &\quad + a_{13}(a_{31}a_{42} + (-1)^n a_{32}a_{41})\end{aligned}$$

$$\begin{aligned}\deg(f_2, f_3, f_4) &= a_{21}(a_{32}a_{43} + (-1)^n a_{33}a_{42}) + a_{22}((-1)^n a_{31}a_{43} + a_{33}a_{41}) \\ &\quad + a_{23}(a_{31}a_{42} + (-1)^n a_{32}a_{41})\end{aligned}$$

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Therefore, if n is odd then

$$\Lambda(f_1, f_2, f_3, f_4) = \det \begin{pmatrix} 1 & a_{11} & a_{12} & a_{13} \\ 1 & a_{21} & a_{22} & a_{23} \\ 1 & a_{31} & a_{32} & a_{33} \\ 1 & a_{41} & a_{42} & a_{43} \end{pmatrix}.$$

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