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Some notes on the Euler obstruction of a function

Nicolas Dutertre*

Université de Provence, Centre de Mathématiques et Informatique, 39 rue Joliot-Curie, 13453 Marseille Cedex 13, France.
E-mail: dutertre@cmi.univ-mrs.fr

Nivaldo G. Grulha Jr.†

Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: njunior@icmc.usp.br

In this paper we present an alternative proof for the Brasselet, Massey, Parameswaran and Seade's formula for the Euler obstruction of a function using Ebeling and Gusein-Zade techniques. October, 2012 ICMC-USP

1. INTRODUCTION

The Euler obstruction was defined by MacPherson [10] as a tool to prove the conjecture about existence and unicity of the Chern classes in the singular case. Since that the Euler obstruction was deeply investigated by many authors as Brasselet, Schwartz, Seade, Sebastiani, Gonzalez-Springberg, Lê, Teissier, Sabbah, Dubson, Kato and others. For an overview about the Euler obstruction see [1, 2].

In [8], Ebeling and Gusein-Zade established relations between the local Euler obstruction of a 1-form, radial index and Euler characteristics of complex links. The radial index is a generalization to the singular case of the Poincaré-Hopf index. We use this results to give an alternative proof for the Brasselet, Massey, Parameswaran and Seade's formula for the Euler obstruction of a function (Theorem 2.2).

2. THE EULER OBSTRUCTION

Let us now introduce some objects in order to define the Euler obstruction. Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an equidimensional reduced complex analytic germ of dimension d in an open set $U \subset \mathbb{C}^N$. We consider a complex analytic Whitney stratification $\{V_i\}$ of U adapted

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to X and we assume that $\{0\}$ is a stratum. We choose a small representative of $(X,0)$ such that 0 belongs to the closure of all the strata. We will denote it by X and we will write $X = \cup_{i=0}^{q} V_i$ where $V_0 = \{0\}$ and $V_q = X_{\text{reg}}$, the set of smooth points of X. We will assume that the strata $V_0, \ldots, V_{q-1}$ are connected and that the analytic sets $\overline{V_0}, \ldots, \overline{V_{q-1}}$ are reduced. We will set $d_i = \dim V_i$ for $i \in \{1, \ldots, q\}$ (note that $d_q = d$).

Let $G(d,N)$ denote the Grassmanian of complex d-planes in $\mathbb{C}^N$. On the regular part X_{reg} of X the Gauss map $\phi : X_{\text{reg}} \to U \times G(d,N)$ is well defined by $\phi(x) = (x, T_x(X_{\text{reg}}))$.

Definition 2.1. The *Nash transformation* (or *Nash blow up*) $\widetilde{X}$ of X is the closure of the image $\text{Im}(\phi)$ in $U \times G(d,N)$. It is a (usually singular) complex analytic space endowed with an analytic projection map $\nu : \widetilde{X} \to X$ which is a biholomorphism away from $\nu^{-1}(\text{Sing}(X))$.

The fiber of the tautological bundle $\mathcal{T}$ over $G(d,N)$, at the point $P \in G(d,N)$, is the set of the vectors v in the d-plane P. We still denote by $\mathcal{T}$ the corresponding trivial extension bundle over $U \times G(d,N)$. Let $\widetilde{T}$ be the restriction of $\mathcal{T}$ to $\widetilde{X}$, with projection map π. The bundle $\widetilde{T}$ on $\widetilde{X}$ is called *the Nash bundle* of X.

Let us recall the original definition of the Euler obstruction, due to MacPherson [10]. Let $z = (z_1, \ldots, z_N)$ be local coordinates in $\mathbb{C}^N$ around $\{0\}$, such that $z_i(0) = 0$. We denote by B_ε and S_ε the ball and the sphere centered at $\{0\}$ and of radius ε in $\mathbb{C}^N$. Let us consider the norm $\|z\| = \sqrt{z_1\overline{z}_1 + \cdots + z_N\overline{z}_N}$. Then the differential form $\omega = d\|z\|^2$ defines a section of the real vector bundle $T(\mathbb{C}^N)^*$, cotangent bundle on $\mathbb{C}^N$. Its pull back restricted to $\widetilde{X}$ becomes a section denoted by $\widetilde{\omega}$ of the dual bundle $\widetilde{T}^*$. For ε small enough, the section $\widetilde{\omega}$ is nonzero over $\nu^{-1}(z)$ for $0 < \|z\| \leq \varepsilon$. The obstruction to extend $\widetilde{\omega}$ as a nonzero section of $\widetilde{T}^*$ from $\nu^{-1}(S_\varepsilon)$ to $\nu^{-1}(B_\varepsilon)$, denoted by $Obs(\widetilde{T}^*, \widetilde{\omega})$ lies in $H^{2d}(\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon); \mathbb{Z})$. Let us denote by $\mathcal{O}_{\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon)}$ the orientation class in $H_{2d}(\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon); \mathbb{Z})$.

Definition 2.2. The local Euler obstruction of X at 0 is the evaluation of $Obs(\widetilde{T}^*, \widetilde{\omega})$ on $\mathcal{O}_{\nu^{-1}(\mathbb{B}_\varepsilon), \nu^{-1}(\mathbb{S}_\varepsilon)}$, i.e:

$$\text{Eu}_X(0) = \langle Obs(\widetilde{T}^*, \widetilde{\omega}), \mathcal{O}_{\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon)} \rangle.$$

An equivalent definition of the Euler obstruction was given by Brasselet and Schwartz in the context of vector fields [5].

The idea of studying the Euler obstruction using hyperplane sections appears in the works of Dubson and Kato, but the approach we follow here come from [3, 4].

Theorem 2.1 ([3]). *Let $(X,0)$ and $\{V_i\}$ be given as before, then for each generic linear form l, there is ε_0 such that for any ε with $0 < \varepsilon < \varepsilon_0$ and $t_0 \neq 0$ sufficiently small, the Euler obstruction of $(X,0)$ is equal to:*

$$\text{Eu}_X(0) = \sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon \cap l^{-1}(\delta)\big) \cdot \text{Eu}_X(V_i),$$

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where χ denotes the Euler-Poincaré characteristic, $\mathrm{Eu}_X(V_i)$ *is the value of the Euler obstruction of X at any point of V_i, $i = 1, \ldots, q$, and* $0 < |\delta| \ll \varepsilon \ll 1$.

We define now an invariant introduced by Brasselet, Massey, Parameswaran and Seade in [4], which measures in a way how far the equality given in Theorem 2.1 is from being true if we replace the generic linear form l with some other function on X with at most an isolated stratified critical point at 0. So let $f : X \to \mathbb{C}$ be a holomorphic function which is the restriction of a holomorphic function $F : U \to \mathbb{C}$. A point x in X is a critical point of f if it is a critical point of $F_{|V(x)}$, where $V(x)$ is the stratum containing x. We will assume that f has an isolated singularity (or an isolated critical point) at 0, i.e that f has no critical point in a punctured neighborhood of 0 in X. In order to define the new invariant the authors constructed in [4] a stratified vector field on X, denoted by $\overline{\nabla}_X f$. This vector field is homotopic to $\overline{\nabla} F|_X$ and one has $\overline{\nabla}_X f(x) \neq 0$ unless $x = 0$.

Let $\tilde{\zeta}$ be the lifting of $\overline{\nabla}_X f$ as a section of the Nash bundle $\widetilde{T}$ over $\widetilde{X}$ without singularity over $\nu^{-1}(X \cap S_\varepsilon)$. Let $\mathcal{O}(\tilde{\zeta}) \in H^{2n}\big(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon)\big)$ be the obstruction cocycle to the extension of $\widetilde{\zeta}$ as a nowhere zero section of $\widetilde{T}$ inside $\nu^{-1}(X \cap B_\varepsilon)$.

Definition 2.3. The local Euler obstruction $\mathrm{Eu}_{f,X}(0)$ is the evaluation of $\mathcal{O}(\tilde{\zeta})$ on the fundamental class of the pair $(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$.

The following result is the Brasselet, Massey, Parameswaran and Seade's formula that compares the Euler obstruction of the space X with that of a function on X [4].

Theorem 2.2. *Let $(X, 0)$ and $\{V_i\}$ given as before and let $f : (X, 0) \to (\mathbb{C}, 0)$ be a function with an isolated singularity at 0. For $0 < |\delta| \ll \varepsilon \ll 1$ we have:*

$$\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0) = \left(\sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon \cap f^{-1}(\delta)\big) \cdot \mathrm{Eu}_X(V_i) \right).$$

In this paper we present an alternative proof for this result using Ebeling and Gusein-Zade's work. In order to do that, let us consider the Nash bundle $\widetilde{T}$ on $\widetilde{X}$. The corresponding dual bundles of complex and real 1-forms are denoted, respectively, by $\widetilde{T}^* \to \widetilde{X}$ and $\widetilde{T}^*_{\mathbb{R}} \to \widetilde{X}$.

Definition 2.4. Let $(X, 0)$ and $\{V_\alpha\}$ as before. Let ω be a (real or complex) 1-form on X, *i.e.*, a continuous section of either $T^*_{\mathbb{R}}\mathbb{C}^N|_X$ or $T^*\mathbb{C}^N|_X$. *A singularity of ω in the stratified sense* means a point x where the kernel of ω contains the tangent space of the corresponding stratum.

This means that the pull back of the form to V_α vanishes at x. Given a section η of $T^*_{\mathbb{R}}\mathbb{C}^N|_A$, $A \subset V$, there is a canonical way of constructing a section $\tilde{\eta}$ of $\widetilde{T}^*_{\mathbb{R}}|_{\tilde{A}}$, $\tilde{A} = \nu^{-1}A$, such that if η has an isolated singularity at the point $0 \in X$ (in the stratified sense), then we have a never-zero section $\widetilde{\eta}$ of the dual Nash bundle $\widetilde{T}^*_{\mathbb{R}}$ over $\nu^{-1}(S_\varepsilon \cap X) \subset \widetilde{X}$. Let

$o(\eta) \in H^{2d}(\nu^{-1}(B_\varepsilon \cap X), \nu^{-1}(S_\varepsilon \cap X); \mathbb{Z})$ be the cohomology class of the obstruction cycle to extend this to a section of $\widetilde{T}^*_{\mathbb{R}}$ over $\nu^{-1}(B_\varepsilon \cap X)$. Then we can define (c.f. [8]):

Definition 2.5. The *local Euler obstruction* of the real differential form η at an isolated singularity is the integer $\mathrm{Eu}_{X,0}\,\eta$ obtained by evaluating the obstruction cohomology class $o(\eta)$ on the orientation fundamental cycle $[\nu^{-1}(B_\varepsilon \cap X), \nu^{-1}(S_\varepsilon \cap X)]$.

MacPherson's local Euler obstruction $\mathrm{Eu}_X(0)$ corresponds to taking the differential $\omega = d\|z\|^2$ of the square of the function distance to 0.

In the complex case, one can perform the same construction, using the corresponding complex bundles. If ω is a complex differential form, section of $T^*\mathbb{C}^N|_A$ with an isolated singularity, one can define the local Euler obstruction $\mathrm{Eu}_{X,0}\,\omega$. Notice that, as explained in [6] p.151, it is equal to the local Euler obstruction of its real part up to sign:

$$\mathrm{Eu}_{X,0}\,\omega \,=\, (-1)^d \mathrm{Eu}_{X,0} \mathrm{Re}\,\omega.$$

This is an immediate consequence of the relation between the Chern classes of a complex vector bundle and those of its dual. Remark also that when we consider the differential of a function f, we have the following equality (see [8]):

$$\mathrm{Eu}_{X,0}\; df = (-1)^d \mathrm{Eu}_{f,X}(0).$$

We note that the idea to consider the (complex) dual Nash bundle was already present in [11], where Sabbah introduces a local Euler obstruction $\check{\mathrm{Eu}}_X(0)$ that satisfies $\check{\mathrm{Eu}}_X(0) = (-1)^d \mathrm{Eu}_X(0)$. See also [12, sec. 5.2].

3. THE RADIAL INDEX END THE EULER OBSTRUCTION

In order to define the complex radial index, let us consider first the real case. Let $Z \subset \mathbb{R}^n$ be a closed subanalytic set equipped with a Whitney stratification $\{S_\alpha\}_{\alpha\in\Lambda}$. Let ω be a continuous 1-form defined on $\mathbb{R}^n$. We say that a point P in Z is a zero (or a singular point) of ω on Z if it is a zero of $\omega_{|S}$, where S is the stratum that contains P. In the sequel, we will define the radial index of ω at P, when P is an isolated zero of ω on Z. We can assume that $P = 0$ and we denote by S_0 the stratum that contains 0.

Definition 3.1. A 1-form ω is radial on Z at 0 if, for an arbitrary non-trivial subanalytic arc $\varphi : [0, \nu[\to Z$ of class C^1, the value of the form ω on the tangent vector $\dot\varphi(t)$ is positive for t small enough.

Definition 3.2. The radial index $ind^{\mathbb{R}}_{Z,0}\,\omega$ of the 1-form ω on Z at 0 is the sum:

$$1 + \sum_{i=0}^{r} \sum_{Q|\tilde\omega_{|S_i}(Q)=0} ind_{PH}(\tilde\omega, Q, S_i),$$

where $ind_{PH}(\tilde{\omega}, Q, S_i)$ is the Poincaré-Hopf index of the from $\tilde{\omega}_{|S_i}$ at Q and where the sum is taken over all zeros of the 1-form $\tilde{\omega}$ on $(Z \setminus \{0\}) \cap B_\varepsilon$. If 0 is not a zero of ω on Z, we put $ind^{\mathbb{R}}_{Z,0}\ \omega = 0$.

Let us go back to the complex case. As before, $(X, 0) \subset (\mathbb{C}^N, 0)$ is an equidimensional reduced complex analytic germ of dimension d in an open set $U \subset \mathbb{C}^N$. Let ω be a complex 1-form on U with an isolated singular point on X at the origin.

Definition 3.3. The complex radial index $\mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega$ of the complex 1-form ω on X at the origin is $(-1)^d$ times the index of the real 1-form given by the real part of ω.

Theorem 3.1. *[8] Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be the germ of a reduced complex analytic space at the origin, with a Whitney stratification $\{V_i\}$, $i = 0, \dots, q$, where $V_0 = \{0\}$ and V_q is the regular part of X. Then*

$$\mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega = \sum_{i=0}^{q} n_i \cdot \mathrm{Eu}_{\overline{V_i},0}\ \omega.$$

3.1. Alternative proof for the Theorem 2.2

Proof. Let us consider $(x_1, x_2, \dots, x_N)$ as complex coordinates of $\mathbb{C}^N$, where $x_k = u_k + \sqrt{-1}v_k$. This implies that $(u_1, v_1, \dots, u_N, v_N)$ are real coordinates of $\mathbb{R}^{2N}$. Let ω be a 1-form defined by $\omega = \sum \overline{x_k} dx_k$, it means that:

$$\omega = \sum (u_k - \sqrt{-1}v_k)(du_k + \sqrt{-1}dv_k),$$

and so that:

$$\omega = \sum (u_k du_k + v_k dv_k) + \sqrt{-1} \sum (u_k dv_k - v_k dv_k).$$

In this case, the real 1-form $\mathrm{Re}\ \omega = \sum (u_k du_k + v_k dv_k)$ is also a radial 1-form, and $\mathrm{ind}^{\mathbb{R}}_{X,0}\ \mathrm{Re}\ \omega = 1$. Since $\mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega = (-1)^d \mathrm{ind}^{\mathbb{R}}_{X,0} \mathrm{Re}\ \omega$, we find that:

$$\mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega = (-1)^d \mathrm{ind}^{\mathbb{R}}_{X,0} \mathrm{Re}\ \omega = (-1)^d.$$

As it was remarked before,

$$\mathrm{Eu}_{X,0}\ \omega = (-1)^d \mathrm{Eu}_{X,0} \mathrm{Re}\ \omega,$$

for more details see [7].

Following the notations of [8], let N_i be the normal slice in the variety X to the stratum V_i at a point of the stratum V_i. The strata V_i of X are partially ordered: $V_i \prec V_j$ if $V_i \subset \overline{V_j}$

and $V_i \neq V_j$. For two strata V_i and V_j with $V_i \preceq V_j$, let N_{ij} be the normal slice of the variety $\overline{V_j}$ to the stratum V_i at a point of it. For more details about this theory see [9]. In the sequel we will denote $\chi(Z) - 1$ by $\overline{\chi}(Z)$.

From Corrolary 1 of [8] we know that

$$\mathrm{Eu}_{X,0}\ \omega = (-1)^d \sum_{i=0}^{q} \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}}),$$

and that

$$\mathrm{Eu}_{X,0} df = (-1)^{d-1} \times$$

$$\Big(\overline{\chi}(M_{f|_X}) + \sum_{i=0}^{q-1} \overline{\chi}(M_{f|_{\overline{V}_i}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}})\Big).$$

It means that,

$$-\mathrm{Eu}_{f,X}(0) = \overline{\chi}(M_{f|_X}) +$$

$$\sum_{i=0}^{q-1} (\chi(M_{f|_{\overline{V}_i}}) - 1) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}}) =$$

$$= \overline{\chi}(M_{f|_X}) +$$

$$\sum_{i=0}^{q-1} \chi(M_{f|_{\overline{V}_i}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}})$$

$$- \sum_{i=0}^{q-1} \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}}) =$$

$$= \chi(M_{f|_X}) +$$

$$\sum_{i=0}^{q-1} \chi(M_{f|_{\overline{V}_i}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}})$$

$$- \sum_{i=0}^{q} \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}}) =$$

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$$= \chi(M_{f|_X}) +$$

$$\Big(\sum_{i=0}^{q-1} \chi(M_{f|_{\overline{V}_i}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}})\Big) - \mathrm{Eu}_{X,0}\omega$$

Therefore,

$$\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0) =$$

$$= \chi(M_{f|_X}) + \sum_{i=0}^{q-1} \chi(M_{f|_{\overline{V}_i}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}}) =$$

$$= \chi(M_{f|_X}) + \sum_{i=0}^{q-1} \chi(M_{f|_{V_i}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}}) +$$

$$+ \sum_{i=0}^{q-1} \chi(M_{f|_{\partial \overline{V}_i}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}}) =$$

$$= \chi(M_{f|_X}) + \sum_{i=0}^{q-1} \chi(M_{f|_{V_i}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}}) +$$

$$+ \sum_{i=0}^{q-1} \sum_{j \prec i, j \neq i} \chi(M_{f|_{V_j}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_l|_{N_{k_0 k_1}}) \cdots \overline{\chi}(M_l|_{N_{k_{r-1} k_r}}) = (*).$$

From Corrolary 1 of [8] we have,

$$(*) = \sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon \cap f^{-1}(\delta)\big) \cdot \mathrm{Eu}_X(V_i)$$

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