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On the topology of real analytic maps

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We study the topology of the fibers of real analytic maps $\mathbb{R}^n \to \mathbb{R}^p$, $n > p$, in a neighborhood of a critical point. We first prove that every real analytic map germ $f\colon \mathbb{R}^n \to \mathbb{R}^p$, $p \geq 1$, with arbitrary critical set, has a Milnor-Lê type fibration. Now assume also that f has the Thom a_f-property, and its zero-locus has positive dimension. Also consider another real analytic map germ $g\colon \mathbb{R}^n \to \mathbb{R}^k$ with an isolated critical point at the origin. We have Milnor-Lê type fibrations for f and for $(f,g)\colon \mathbb{R}^n \to \mathbb{R}^{p+k}$, and we prove for these the analogous of the classical Lê-Greuel formula, expressing the difference of the Euler characteristics of the fibers F_f and $F_{f,g}$ in terms of an invariant associated to these maps. This invariant can be expressed in various ways: as the index of the gradient vector field of a map $\tilde{g}$ on F_f associated to g; as the number of critical points of $\tilde{g}$ on F_f; or in terms of polar multiplicities. When $p = 1$ and $k = 1$, this invariant can also be expressed algebraically, as the signature of a certain bilinear form. When the germs of f and (f,g) are both isolated complete intersection germs, we exhibit an even deeper relation between the topology of the fibers F_f and $F_{f,g}$, and construct in this setting, an integer-valued invariant, that we call the *curvatura integra* that picks up the Euler characteristic of the fibers. This invariant, and its name, spring from Gauss' theorem, and its generalizations by Hopf and Kervaire, expressing the Euler characteristic of a manifold (with some conditions) as the degree of a certain map. October, 2012 ICMC-USP

1. INTRODUCTION

In this work we study the topology of the fibers of real analytic maps $\mathbb{R}^n \to \mathbb{R}^p$, $n > p$, in a neighborhood of a critical point. This is a fundamental problem that appears naturally in many areas of mathematics. Our point of view is inspired by the classical Lê-Greuel formula for the Milnor number of isolated, complex, complete intersection germs. The idea is that if the map germ is defined by functions $(f_1, ..., f_k, g)$, then we can study the topology of its fibers by comparing it with the topology of the germ we get by dropping down the last defining equation.

The starting point is Milnor's fibration theorem which is one of the main foundational results in the theory of complex singularities. Its counter-part for real singularities $\mathbb{R}^n \to \mathbb{R}^p$, $n > p$, also comes from Milnor's book, and there are several interesting articles on the topic published by various people in the 1970s and 1980s, as for instance by E. Looijenga, P. T. Church and K. Lamotke, N. A'Campo, B. Perron, L. Kauffman, W. Neumann, A. Jacquemard and others. Later, in the mid 1990s, a new wave of interest on the topic arose, and nowadays the study of Milnor fibrations for real singularities is an active field of research (see for instance [44, 5, 38, 9, 39, 35, 36, 6, 1] or the survey article [12]).

Yet, the study of real singularities is in many ways harder than the study of complex singularities, and all the literature on the subject of Milnor fibrations for real mappings restricts to the case where the map germ has an isolated critical value, which is very stringent when $p > 1$. And even in that case, the study of the topology of the corresponding Milnor fibrations is still in its childhood, except for $p = 1$. The purpose of this work is to study the topology of real analytic map-germs $(\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$, $n > p \geq 2$, with zero-locus of positive dimension and with arbitrary critical locus.

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In the case of complex analytic isolated complete intersection germs, in [23] H. Hamm proved that one also has a Milnor fibration. In the late 1960s, B. Teissier, Lê Dũng Tráng and others, used ideas of R. Thom to study the topology of singularities in the vein of S. Lefschetz, by considering "slices" (or the pencil) determined by some general linear form. If instead of a linear form, we slice the singular variety by the level sets of some complex-valued holomorphic function, then we are in the setting envisaged in the classical Lê-Greuel formula. This is a celebrated theorem about the Milnor number of isolated complex analytic complete intersection germs, first proved by Lê Dũng Tráng in 1970 (see [32]) and then by Gert Martin Greuel in [21]. This theorem says that if $f_1, \cdots, f_k$ and g are holomorphic map germs $(\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}, 0)$ such that $f = (f_1, \cdots, f_k)$ and (f, g) define isolated complete intersection germs (ICIS for short), then the Milnor number $\mu(f, g)$ of (f, g) can be computed in terms of that of f by the formula:

$$\mu(f,g) + \mu(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+k,\underline{0}}}{(f, \operatorname{Jac}(f,g))} ,$$

where $\operatorname{Jac}(f, g)$ denotes the ideal generated by the determinants of all the $(k+1)$ minors of the corresponding Jacobian matrix. Since, by [23], the Milnor fibre of an ICIS is a bouquet of μ spheres of middle dimension, where μ is the Milnor number, and the term on the right equals the number of critical points of a Morsification of the restriction of g to the Milnor fibre of f, the Lê-Greuel formula can be re-stated as:

$$\chi(F_f) = \chi(F_{f,g}) + \operatorname{Ind}_{\mathrm{PH}} \overline{\nabla} g|_{F_f} ,$$

where F_f is the corresponding Milnor fibre and the term on the right is the total Poincaré-Hopf index of the vector field $\overline{\nabla} g$, conjugate of the gradient vector field of the restriction of g to the Milnor fibre F_f. That expression for the Lê-Greuel formula was recently used in [8] for extending this formula to the general setting of holomorphic map-germs defined on arbitrary singular varieties, provided the mappings have the Thom property and g has an isolated critical point.

The techniques used in [8] lend themselves to generalization, and this inspired us to study real analytic map germs from the viewpoint of the Lê-Greuel formula.

Consider a real analytic map $f\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$, $n > p \geq 2$, with arbitrary critical locus. Let $\overline{\mathbb{B}}_\varepsilon$ be the closed ball in $\mathbb{R}^n$ centred at $\underline{0}$ of radius ε, $\mathbb{D}^p_\delta$ be the open ball in $\mathbb{R}^p$ centred at 0 of radius δ and Δ_f be the discriminant of f. In Section 2 we prove that f has an associated locally trivial fibration of the Milnor-Lê type

$$f\colon \hat{N}_f(\varepsilon,\delta) = \overline{\mathbb{B}}_\varepsilon \cap f^{-1}(\mathbb{D}^p_\delta \setminus \Delta_f) \to \mathbb{D}^p_\delta \setminus \Delta_f .$$

This is in fact an easy extension of well known results and techniques in singularity theory. The point here is that we do not impose any conditions on the type of critical sets that the map may have. Of course that in this general setting the topology of the Milnor fibre is not always unique: the discriminant of f may split a neighbourhood of the origin in $\mathbb{R}^p$ into several connected components, and one has a topological (actually differentiable) model for the Milnor fibre on each such component (compare [33]).

In Section 3 we further require that the map f satisfies the Thom a_f-property with respect to some Whitney stratification $\{S_\alpha\}$ such that its zero-set $V(f)$ has dimension ≥ 2 and it is union of strata. We also consider another real analytic map germ $g\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^k, 0)$ with an isolated critical point in $\mathbb{R}^n$ with respect to the stratification $\{S_\alpha\}$. By Section 2 the map-germs f and (f,g) have associated locally trivial fibrations of the Milnor-Lê type. Then we prove the corresponding Lê-Greuel formula:

Theorem 1.1. *Let F_f be a Milnor fibre of f (any Milnor fibre, regardless of the discriminant of f). Then one has:*

$$\chi(F_f) = \chi(F_{f,g}) + \mathrm{Ind}_{\mathrm{PH}}\, \nabla\tilde{g}|_{F_f} ,$$

where $\tilde{g}\colon \mathbb{R}^n \to \mathbb{R}$ is given by $\tilde{g}(x) = \|g(x) - t_0\|^2$ with $t_0 \in \mathbb{R}^k$ such that $F_{f,g} = g|_{F_f}^{-1}(t_0)$.

The term on the right, which by definition is the total Poincaré-Hopf index in F_f of the vector field $\nabla\tilde{g}|_{F_f}$, can be expressed also in the following equivalent ways:

1.As a sum of polar multiplicities relative to $\tilde{g}$ on F_f;

2.As the Euler class of the tangent bundle of F_f relative to the vector field $\nabla\tilde{g}|_{F_f}$ on its boundary.

It should be noted that a Lê-Greuel type formula for real analytic complete intersection germs with an isolated critical point in the ambient space was recently obtained in [13]. This is the setting we envisage in Section 3. Now one has $f = (f_1, \cdots, f_k)$, and (f,g) which are both ICIS-germs, and a Milnor-Lê fibration

$$(f,g)\colon N(\varepsilon,\delta) \setminus V(f,g) \longrightarrow \mathbb{D}_\delta \setminus \{0\} ,$$

which determines a locally trivial fibre bundle

$$\phi\colon \mathbb{S}_\varepsilon^{n-1} \setminus V(f,g) \longrightarrow \mathbb{S}^p .$$

As noticed in [39, 11], one has that the projection map ϕ can always be taken as $\frac{(f,g)}{\|(f,g)\|}$ in a neighbourhood of the link $L_{(f,g)} := V(f,g) \cap \mathbb{S}_\varepsilon^{n-1}$. Following [11] we say that the map germ (f,g) is d-regular if the projection map ϕ can be taken as $\frac{(f,g)}{\|(f,g)\|}$ everywhere. In this case we notice that there is a relation between the topology of f and that of (f,g), which is much deeper than the one given by the Theorem 1.1. To state this result , it is convenient to write g as f_{k+1}, then we have the following immediate application of Corollary 5.4 and Corollary 5.5 in [11]:

Theorem 1.2. *Let $f = (f_1, \cdots, f_{p+1})\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^{p+1}, 0)$ be a complete intersection germ with an isolated critical point at $\underline{0}$ and which is d-regular. Let $V = f^{-1}(0)$ and for each $i = 1, \cdots, p+1$ let V_i be the singular variety $V_i = (f_1, \cdots, \widehat{f_i}, \cdots, f_{p+1})^{-1}(0)$, where $\widehat{f_i}$ means that we are removing this component and looking at the corresponding map germ* *into* $\mathbb{R}^p$*. Then the topology of* $V_i \setminus V$ *is independent of the choice of* i *and its link, which is a smooth manifold, is diffeomorphic to the disjoint union of two copies of the interior of the Milnor fibre of* f*.*

This motivates the following question: Is the topology of V_i independent of i as in the complex case? (see [10, Thm. 1 (i)]). In this case the link of V_i would be diffeomorphic to the double of the Milnor fibre of f.

In Section 4 we focus in this setting and address a different, related problem. When considering real analytic map-germs, it is important to have invariants that allow their study. So for instance, when $p = 1$ and the singularity is algebraically isolated, one has the local Milnor number at $\underline{0}$, which is an invariant in $\mathbb{Z}/2\mathbb{Z}$ introduced by C.T.C. Wall in [46]. As noticed in [2], when n is even this amounts to considering the usual Euler characteristic $\chi(K)$ of the link of f at $\underline{0}$ and reduce it modulo 2. In particular, in these dimensions $\chi(K)$ equals twice the Euler characteristic of the corresponding Milnor fibre, so it is itself an interesting invariant of f with values in $\mathbb{Z}$. When n is odd the Euler characteristic $\chi(K)$ is always zero and one must do something else to get a topological interpretation of the Milnor number. This can be done using classical work by M. Kervaire [27] about the so-called *curvatura integra* of manifolds, and one finds (see [2]) that Wall's Milnor number essentially coincides with the semi-characteristic of the link with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Yet, in these dimensions there is not an obvious lifting of this invariant to $\mathbb{Z}$.

Here we show that when $p > 1$, if f has a Milnor open-book fibration and the dimension of the zero set $f^{-1}(0)$ has dimension grater than 0, then one does have an invariant of f with values in $\mathbb{Z}$, that we call the *curvatura integra* of f, in analogy with [27]. This invariant is defined as the topological degree of a certain map, and it coincides with the Euler characteristic of the Milnor fibre of f. So this can be regarded as being a topological counterpart for real singularities, of G. Kennedy's integral formula for the Milnor number of complex map-germs [26].

When $n - p$ is odd, so that the link K of f is even-dimensional, this invariant equals $\frac{1}{2}\chi(K)$, the Euler characteristic of the link, and it coincides with the Euler characteristic of the Milnor fibre. When $n - p$ is even this construction only gives an invariant in $\mathbb{Z}/2\mathbb{Z}$, so we modify as follows. Instead of considering f, we consider the composition $\widehat{f} : \mathbb{R}^n \to \mathbb{R}^{p-1}$ of f with the orthogonal projection π of $\mathbb{R}^p$ onto a hyperplane through the origin in $\mathbb{R}^p$. This is a real analytic map-germ with an isolated critical point at the origin, and by [11] its link $\widehat{K}$ is diffeomorphic to the double of the Milnor fibre of f, independently of the choice of hyperplane. Then the curvatura integra of f essentially is one half of the Euler characteristic of $\widehat{K}$, and it coincides with the Euler characteristic of f.

The invariant we introduce here is inspired by the construction in [43, 41, 40, 11] of a pencil $\{V_\ell\}$ canonically associated to every map-germ $(\mathbb{R}^n, \underline{0}) \xrightarrow{f} (\mathbb{R}^p, 0)$, which has strong relations with the theory of Milnor fibrations, as shown in [10, 11, 42].

We remark that the canonical pencil exists naturally for all map-germs with $p > 1$, whether or not the germ has isolated singularities, and therefore similar considerations can be done in a more general setting.

Finally, in Section 5 we consider complete intersections germs defined by only two equations, with algebraically isolated singularities. We use the theory of indices of vector fields on singular varieties developed by X. Gómez-Mont and P. Mardešić in [18, 19], to give an algebraic expression of the Lê-Greuel formula in terms of the signature of a certain quadratic form associated to (f, g). This algebraic expression arises from the Eisenbud-Levine-Khimshiasvili formula for the local Poincaré-Hopf index of real analytic vector fields.

2. MILNOR-LÊ FIBRATION

Let $f\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$, $n > p \geq 2$, be a real analytic map-germ with a critical point at $\underline{0}$. Let $\overline{\mathbb{B}}_\varepsilon$ be a closed ball in $\mathbb{R}^n$ centred at $\underline{0}$ of sufficiently small radius $\varepsilon > 0$. We see $\overline{\mathbb{B}}_\varepsilon$ as a stratified set where the strata are the interior $\mathbb{B}_\varepsilon$ and the boundary $\mathbb{S}_\varepsilon = \partial\overline{\mathbb{B}}_\varepsilon$ of $\overline{\mathbb{B}}_\varepsilon$. Consider the restriction $f|_{\overline{\mathbb{B}}_\varepsilon}$ which to simplify notation we still denote just by f. Denote by $\mathcal{C}_f(\mathbb{B}_\varepsilon)$ the set of critical points of f in $\mathbb{B}_\varepsilon$ and denote by $\mathcal{C}_f(\mathbb{S}_\varepsilon)$ the set of critical points in $\mathbb{S}_\varepsilon$ of the the restriction $f|_{\mathbb{S}_\varepsilon}$. Let $\mathcal{C}_f = \mathcal{C}_f(\mathbb{B}_\varepsilon) \cup \mathcal{C}_f(\mathbb{S}_\varepsilon)$ be the *set of critical points of f in $\overline{\mathbb{B}}_\varepsilon$* and denote by $\Delta_f = f(\mathcal{C}_f)$ the *discriminant* of f. We have the following proposition (see [37, IV.4]):

PROPOSITION 2.1. *The restriction*

$$f\colon E_f(\varepsilon) = \overline{\mathbb{B}}_\varepsilon \cap f^{-1}(\mathbb{R}^p \setminus \Delta_f) \to \mathbb{R}^p \setminus \Delta_f$$

is a locally trivial fibre bundle over its image.

Proof. We have that $\mathcal{C}_f$ is a closed set of $\overline{\mathbb{B}}_\varepsilon$ [37, IV Cor. 4.6] and since f is a continuous map from a compact space to a Haudorff space, it is proper and closed. Thus Δ_f is a closed subset of $\mathbb{R}^p$. It follows that $\mathbb{R}^p \setminus \Delta_f$ is an open submanifold of $\mathbb{R}^p$ and that $E_f(\varepsilon) = \overline{\mathbb{B}}_\varepsilon \cap f^{-1}(\mathbb{R}^p \setminus \Delta_f)$ is also a manifold with boundary, with a stratification given by its interior and its boundary. Therefore the map

$$f\colon E_f(\varepsilon) \to \mathbb{R}^p \setminus \Delta_f$$

is a proper map which restricted to each stratum is a submersion and the result follows by the Thom-Mather First Isotopy Lemma (compare with [37, p. 80]). ∎

Let $\mathbb{D}^p_\delta$ be a open ball in $\mathbb{R}^p$ centred at 0 of radius $0 < \delta \ll \varepsilon$. Let $\hat{N}_f(\varepsilon, \delta) = \overline{\mathbb{B}}_\varepsilon \cap f^{-1}(\mathbb{D}^p_\delta \setminus \Delta_f)$ and $N_f(\varepsilon, \delta) = \overline{\mathbb{B}}_\varepsilon \cap f^{-1}(\partial\mathbb{D}^p_\delta \setminus \Delta_f)$ be the restrictions of the fibre bundle of Proposition 2.1 to $\mathbb{D}^p_\delta \setminus \Delta_f$ and $\partial\mathbb{D}^p_\delta \setminus \Delta_f$ respectively. We call $\hat{N}_f(\varepsilon, \delta)$ a *solid Milnor tube* and $N_f(\varepsilon, \delta)$ a *Milnor tube* for f. Hence we have,

COROLLARY 2.1. *Let $f\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$ be as before. Then the restricitions*

$$f\colon \hat{N}_f(\varepsilon, \delta) \to \mathbb{D}^p_\delta \setminus \Delta_f,$$

and

$$f\colon N_f(\varepsilon, \delta) \to \partial\mathbb{D}^p_\delta \setminus \Delta_f,$$

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are locally trivial fibrations.

We call the fibrations in Corollary 2.1 the *Milnor-Lê type fibrations*.

Remark 2. 1. As we mentioned in the introduction, if the discriminant of f splits a neighbourhood of the origin in $\mathbb{R}^p$ into several connected components, the topology of the Milnor fibre is not unique, and one has a topological (actually differentiable) model for the Milnor fibre on each such component.

3. REAL LÊ-GREUEL TYPE FORMULA

As in the previous section, consider a real analytic map-germ $f\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$, $n > p \geq 2$ with a critical point at $\underline{0}$ and $V(f) = f^{-1}(0)$ has dimension greater than 2. Let $\{S_\alpha\}_{\alpha\in A}$ be a Whitney stratification of $\mathbb{R}^n$ with $V(f)$ union of strata, and let $\{R_\gamma\}_{\gamma\in G}$ be a Whitney stratification of $\mathbb{R}^p$ such that both stratifications give a *stratification of* f, i. e., for every $\alpha \in A$ there exists $\gamma \in G$ such that f induces a submersion from S_α to R_γ. We further assume that f satisfies the Thom a_f-property with respect to such stratification of f: let S_α and S_β be strata such that $S_\alpha \subset \bar{S}_\beta$, let $x \in S_\alpha$ and let $\{x_i\}$ be a sequence of points in S_β converging to x. Set $f_x^\alpha = f|_{S_\alpha}^{-1}(f(x))$, the fibre of $f|_{S_\alpha}$ which contains x and $f_{x_i}^\beta = f|_{S_\beta}^{-1}(f(x_i))$ the fibre of $f|_{S_\beta}$ which contains x_i. Let T be the limit of the sequence of tangent spaces $T_{x_i} f_{x_i}^\beta$. Then $T_x f_x^\alpha \subset T$.

The following is a technical lemma wich says that the rank of a map $g\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^k, 0)$ restricted to the fibres f_x^α is lower semi-continuous.

LEMMA 3.1. *Let $f\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$ be a real analytic map-germ which satisfies the Thom a_f-property with respect to a stratification of f and let $g\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^k, 0)$ be another real analytic map-germ. Let S_α and S_β two strata of $\mathbb{R}^n$ such that $S_\alpha \subset \bar{S}_\beta$. Let $x \in S_\alpha$ and denote by f_x^α the fibre of $f|_{S_\alpha}$ which contains x. Suppose that* $\operatorname{rank} D_x(g|_{f_x^\alpha}) =$ *r, then there exists an open neighbourhood U_x of x in $\mathbb{R}^n$ so that* $\operatorname{rank} D_{x'}(g|_{f_{x'}^\beta}) \geq r$ *for all $x' \in S_\beta \cap U_x$.*

Proof. Let W be an open neighbourhood of x in $\mathbb{R}^n$ such that $W \cap S_\beta$ is a chart. Consider the tangent bundle $TS_\beta|_{W\cap S_\beta}$ of S_β restricted to $W \cap S_\beta$ which is trivial. Now consider the subbundle TF of $TS_\beta|_{W\cap S_\beta}$ where the fibre of $x' \in W \cap S_\beta$ is $T_{x'} f_{x'}^\beta$ the tanget space at x' of the fibre of $f|_{S_\beta}$ which contains x'. Suppose $\dim f_{x'}^\beta = l$, since TF is trivial we can choose a local frame $\{s_1, \ldots, s_l\}$, that is $\{s_1(x'), \ldots, s_l(x')\}$ is a basis for $T_{x'} f_{x'}^\beta$ for every $x' \in W \cap S_\beta$. By continuity we can extend this frame to the closure of $W \cap S_\beta$, in particular to the points in $W \cap S_\alpha$.

Let $\{x_i\}$ be a sequence of points in $W \cap S_\beta$ converging to x. Consider the sequence of tangent spaces $T_{x_i} f_{x_i}^\beta$, taking a subsequence if it is necessary, suppose that $\lim_{i\to\infty} T_{x_i} f_{x_i}^\beta = T$. Since f has the Thom a_f-property we have that $T_x f_x^\alpha \subset T$. By hypothesis $D_x(g|_{f_x^\alpha}) =$

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$(D_xg)|_{T_xf_x^\alpha}\colon T_xf_x^\alpha \to T_{g(x)}\mathbb{R}^k$ has rank r, and since $T_xf_x^\alpha \subset T$ it follows that $\operatorname{rank}(D_xg)|_T \geq r$. Using the frame $\{s_1,\dots,s_l\}$ we obtain a matrix expresion for $(D_xg)|_T$, and since $\operatorname{rank}(D_xg)|_T \geq r$ it has an $r\times r$-submatrix A_x (which without loss of generality we can suppose it is given by the first r rows and columns) whose determinant is nonzero. Hence the continuous map given by

$$\begin{array}{ccccccc} \overline{W\cap S_\beta} & \longrightarrow & \mathbb{R}^{k\cdot l} & \longrightarrow & \mathbb{R}^{r\cdot r} & \longrightarrow & \mathbb{R} \\ x' & \longmapsto & (D_{x'}g)|_{T_{x'}f_{x'}^\beta} & \longmapsto & A_{x'} & \longmapsto & \det A_{x'} \end{array}$$

does not vanish at x nor in an open neighbourhood U_x of x. Thus $\operatorname{rank} D_{x'}(g|_{f_{x'}^\beta}) \geq r$ for all $x' \in S_\beta \cap U_x$. ∎

Now we further assume that the real analytic map-germ $g\colon(\mathbb{R}^n,\underline{0})\to(\mathbb{R}^k,0)$ has an isolated critical point at $\underline{0}$ in $\mathbb{R}^n$ with respect to the stratification $\{S_\alpha\}$, that is, the restriction of g to any stratum is a submersion, except at $\underline{0}$. This implies that g has $\underline{0}$ as an isolated critical point as a map from $\mathbb{R}^n$ to $\mathbb{R}^k$.

Let $\overline{\mathbb{B}}_\varepsilon$ be a closed ball in $\mathbb{R}^n$ centred at $\underline{0}$ of sufficiently small radius $\varepsilon>0$, such that every stratum of $\{S_\alpha\}$ which contains $\underline{0}$ in its closure meets transversally every sphere in $\overline{\mathbb{B}}_\varepsilon$ centred at $\underline{0}$.

Proposition 3.1. *Let $f\colon(\mathbb{R}^n,\underline{0})\to(\mathbb{R}^p,0)$ and $g\colon(\mathbb{R}^n,\underline{0})\to(\mathbb{R}^k,0)$ be as before. Let $0<\delta\ll\varepsilon$ and consider the Milnor-Lê type fibration given by Corollary 2.1*

$$f\colon \hat{N}_f(\varepsilon,\delta)\to \mathbb{D}^p_\delta\setminus\Delta_f.$$

Let $0<\varepsilon'<\varepsilon$ small enough and consider the closed ball $\overline{\mathbb{B}}_{\varepsilon'}$ in $\mathbb{R}^n$ centred at $\underline{0}$ of radius $\varepsilon'>0$. Then there exists $y_0\in\mathbb{D}^p_\delta\setminus\Delta_f\subset\mathbb{R}^p$ with $\|y_0\|$ sufficiently small such that the restriction of g to the fibre $f^{-1}(y_0)\cap\mathbb{B}_\varepsilon$ of y_0 only has critical points in the interior of $\overline{\mathbb{B}}_{\varepsilon'}$.

Proof. Let $x\in S_\alpha\subset V(f)\setminus\{\underline{0}\}$. Since $V(f)$ is union of strata we have that $f_x^\alpha=S_\alpha$. Since g has $\underline{0}\in\mathbb{R}^n$ as an isolated critical point with respecto to the stratification $\{S_\alpha\}_{\alpha\in A}$ of $\mathbb{R}^n$, by Lemma 3.1 there exist a neighbourhood U_x such that for any stratum S_β such that $S_\alpha\subset\overline{S}_\beta$ we have

$$\operatorname{rank} D_{x'}(g|_{f_{x'}^\beta})\begin{cases} = k & \text{if } x\neq\underline{0},\\ \geq \operatorname{rank} D_{\underline{0}}(g|_{S_\alpha}) & \text{if } x=\underline{0},\end{cases}$$

for any $x'\in U_x\cap S_\beta$. Let $0<\varepsilon'<\varepsilon$ be small enough so that the open ball $\mathbb{B}_{\varepsilon'}$ centred at $\underline{0}$ is contained in the neighbourhood $U_{\underline{0}}$. The neighbourhoods U_x with $x\in V(f)\setminus\{\underline{0}\}$ and $\mathbb{B}_{\varepsilon'}$ give an open cover of $\mathbb{B}_\varepsilon\cap V(f)$, and since it is compact, there exist a finite

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subcover $\{U_1, \dots, U_m\}$. Notice that one of the elements of such cover must be the ball $\mathbb{B}_{\varepsilon'}$ since for every $x \in V(f) \setminus \{\underline{0}\}$ the neighbourhood U_x contains only points such that $\operatorname{rank}(D_{x'}g)|_{T_{x'}f^{\beta}_{x'}} = k$, so $\underline{0}$ cannot belong to any of them. Consider the intersection $f(U_1) \cap \dots \cap f(U_m) \cap (\mathbb{D}^p_\delta \setminus \Delta_f)$ and take a point $y_0 \neq 0$ in such intersection. The fibre $f^{-1}(y_0) \cap \mathbb{B}_\varepsilon$ of y_0 only has critical points in the interior of $\overline{\mathbb{B}}_{\varepsilon'}$ by the definition of the cover $\{U_1, \dots, U_m\}$. ∎

To simplify notation denote by F_f the fibre given in Proposition 3.1. We have that F_f is a compact manifold with boundary $F_f \cap \mathbb{S}_\varepsilon$.

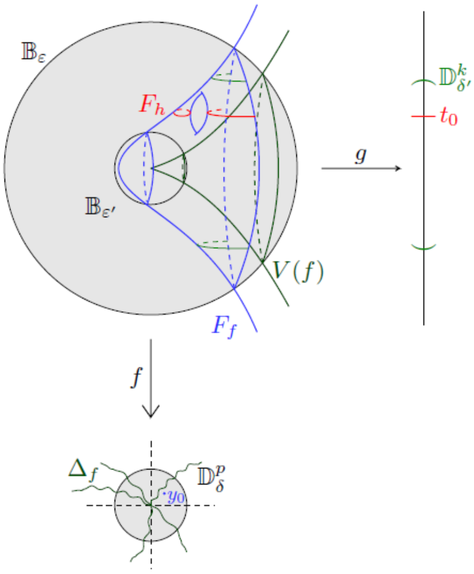
Consider the real analytic map-germ

$$h = (f, g) \colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p \times \mathbb{R}^k, 0)$$
$$h(x) = (f(x), g(x)).$$

Notice that the discriminant of h is given by $\Delta_h = \Delta_f \times \mathbb{R}^k$. Consider the Milnor-Lê type fibration given by Corollary 2.1

$$h \colon \hat{N}_h(\varepsilon, \delta) \to \mathbb{D}^{p+k}_\delta \setminus \Delta_h.$$

Let $\delta' < \delta$ be such that for all $t \in \mathbb{D}^k_{\delta'}$ we have that $(y_0, t) \in \mathbb{D}^{p+k}_\delta$. Choose $t_0 \in \mathbb{D}^k_{\delta'}$ such that the fibre $F_h = h^{-1}(y_0, t_0) \cap \mathbb{B}_\varepsilon$ does not intersect the ball $\mathbb{B}_{\varepsilon'}$ (see Figure 3). Notice that $(y_0, t_0) \notin \Delta_h$ and therefore $F_h \subset \hat{N}_h(\varepsilon, \delta)$. Let $\delta'' > 0$ sufficiently small with respect to δ' such that the open ball $\mathbb{D}^k_{\delta''}(t_0)$ centred at t_0 of radius δ'' is contained in the open ball $\mathbb{D}^k_{\delta'}$ centred at 0 of radius δ' and $h^{-1}(y_0, t) = F_f \cap g^{-1}(\mathbb{D}^k_{\delta''}(t_0))$ does not intersect the ball $\mathbb{B}_{\varepsilon'}$ for every $t \in \mathbb{D}^k_{\delta''}(t_0)$.



The fibre F_f of Proposition 3.1 and the fibre F_h.

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Consider the map

$$\tilde{g}\colon \mathbb{R}^n \to \mathbb{R}$$
$$\tilde{g}(x) = \|g(x) - t_0\|^2.$$

LEMMA 3.2. *Let $\tilde{g}|_{F_f}$ be the restriction of $\tilde{g}$ to F_f. The gradient vector field $\nabla\tilde{g}|_{F_f}$ on $F_f \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$ is the lifting, up to homotopy, of a non-zero vector field on $\mathbb{R}$ via $d\tilde{g}$.*

Proof. Let $x \in F_f \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$ and consider local coordinates of F_f around x. Consider the map $n\colon \mathbb{R}^k \to \mathbb{R}$ given by $n(t) = \|t\|^2$ and the map $m\colon \mathbb{R}^k \to \mathbb{R}^k$ given by $m(t) = t - t_0$. The map $\tilde{g}|_{F_f}$ can be seen as the composition $\tilde{g}|_{F_f} = n \circ m \circ g|_{F_f}$, thus, by the chain rule we have that the differential of $\tilde{g}|_{F_f}$ is given by

$$d(\tilde{g}|_{F_f})_x = dn_{m(g(x))} \circ dm_{g(x)} \circ d(g|_{F_f})_x$$

and as a product of matrices we have

$$d(\tilde{g}|_{F_f})_x = (g(x) - t_0) \cdot I_{k\times k} \cdot d(g|_{F_f})_x = (g(x) - t_0) \cdot d(g|_{F_f})_x.$$

Hence the gradient of $\tilde{g}|_{F_f}$ is given by $\nabla\tilde{g}|_{F_f} = (g(x) - t_0) \cdot d(g|_{F_f})_x$. Since $d(g|_{F_f})_x$ has rank k its column vectors form a basis for $\mathbb{R}^k$, therefore there exist at least one column vector of $d(g|_{F_f})_x$ such that its inner product with $(g(x) - t_0)$ is no zero, otherwise the linear subspace generated by the column vectors of $d(g|_{F_f})_x$ would be contained in the orthogonal subspace of the vector $(g(x) - t_0)$, which contradicts the fact that $d(g|_{F_f})_x$ has rank k. Hence $\nabla\tilde{g}|_{F_f}$ is non zero in $F_f \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$. Moreover, at the points in $F_f \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$ the gradient $\nabla\tilde{g}|_{F_f}$ satisfies

$$d(\tilde{g}|_{F_f})_x(\nabla\tilde{g}|_{F_f}(x)) = \left\|\nabla\tilde{g}|_{F_f}(x)\right\|^2 \in \mathbb{R}^+, \qquad \text{for } x \in F_f \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'}).$$

Hence $\nabla\tilde{g}|_{F_f}$ is the lifting, up to scaling, of a non-zero vector field on $\mathbb{R}$ via $d(\tilde{g}|_{F_f})$. ∎

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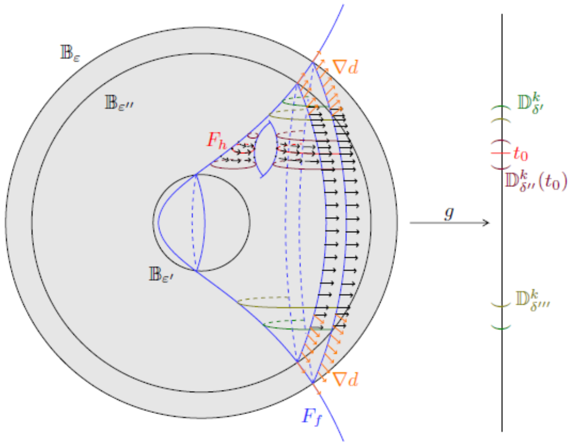

The vector field r of Lemma 3.3.

Let $0 < \delta''' < \delta'$ be such that the open ball $\mathbb{D}^k_{\delta''}(t_0)$ is contained in the open ball $\mathbb{D}^k_{\delta'''}$. In the following lemma we construct a vector field r which is ilustrated in Figure 3.

LEMMA 3.3. *There exists a vector field r on F_f with the following properties:*

1. The restriction of r to $F_f \cap g^{-1}(\mathbb{D}^k_{\delta'''})$ is tangent to all the fibres $h^{-1}(y_0, t)$ with $t \in \mathbb{D}^k_{\delta'''}$.
2. The restriction of r to the fibre F_h has only isolated singularities (zeros).
3. r is transverse to $\mathbb{S}_\varepsilon$, pointing outwards.

Proof. Since F_h is a compact manifold with boundary we can construct r in F_h satisfying (2.): take the unit normal vector field on $F_h \cap \mathbb{S}_\varepsilon$ pointing outwards and extend it to the interior of F_h with isolated zeros. Since the restiction of g to $F_f \cap g^{-1}(\mathbb{D}^k_{\delta''}(t_0))$ does not have critical points and the fibres of g in $F_f \cap g^{-1}(\mathbb{D}^k_{\delta''}(t_0))$ are transverse to $\mathbb{S}_\varepsilon$ we have that $g\colon F_f \cap g^{-1}(\mathbb{D}^k_{\delta''}(t_0)) \to \mathbb{D}^k_{\delta''}(t_0)$ determines a trivial fibration (since $\mathbb{D}^k_{\delta''}(t_0)$ is contractible). Hence the vector field r in F_h can be extended, as a product, to all the other fibres of g in $F_f \cap g^{-1}(\mathbb{D}^k_{\delta''}(t_0))$. Let us choose ε'' with $\varepsilon' \ll \varepsilon'' < \varepsilon$ such that F_f and the fibres of g in F_f are transverse to $\mathbb{S}_{\varepsilon''}$. Then the restriction of g to $(F_f \cap g^{-1}(\mathbb{D}^k_{\delta'}) \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon''})$ is a submersion and its fibres are transverse to $\mathbb{S}_\varepsilon$ and to $\mathbb{S}_{\varepsilon''}$ and therefore it is a trivial fibration. Hence the vector field r restricted to $(F_f \cap g^{-1}(\mathbb{D}^k_{\delta''}(t_0)) \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon''})$ can be extended to $(F_f \cap g^{-1}(\mathbb{D}^k_{\delta'}) \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon''})$ being tangent to the fibres of g and transverse to $\mathbb{S}_\varepsilon$ pointing outwards. Consider the map $d\colon \mathbb{R}^n \to \mathbb{R}$ given by $d(x) = \|x\|^2$. We have that F_f is transverse to the spheres $\mathbb{S}_{\varepsilon'''}$ for all $\varepsilon'' < \varepsilon''' < \varepsilon$, therefore the restriction of the gradient ∇d to $(F_f \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon''})) \setminus g^{-1}(\overline{\mathbb{D}^k_{\delta'''}})$, where $\overline{\mathbb{D}^k_{\delta'''}}$ is the closed ball, is a non-zero vector field transverse to $\mathbb{S}_\varepsilon$ pointing outwards. Using a partition of unity we glue this vector field to the vector field constructed on $(F_f \cap g^{-1}(\mathbb{D}^k_{\delta'}) \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon''})$ getting a vector field on $F_f \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon''})$ which is transverse to $\mathbb{S}_\varepsilon$ pointing outwards. Using a partition of unity, we can extend r to F_f as zero in the complement of a neighbourhood of $(F_f \cap \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon''})) \cup (F_f \cap g^{-1}(\mathbb{D}^k_{\delta''}(t_0))$. ∎

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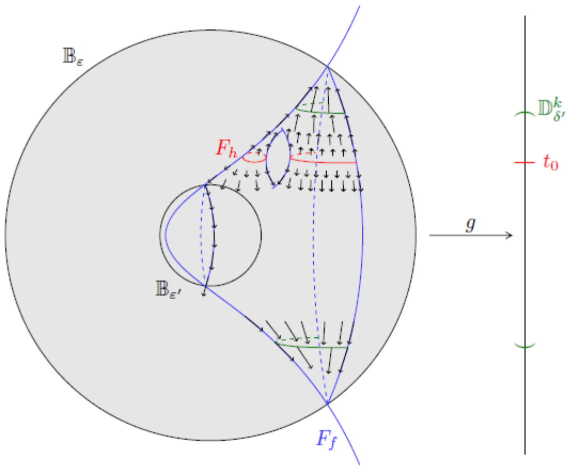
Lemma 3.4. *There exists a vector field u defined on $F_f \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$ satisfying the following:*

[(i)]u is tangent to $\mathbb{S}_\varepsilon$. Its zero set is F_h, and u is transversally radial to F_h. u is transverse to $F_f \cap g^{-1}(\partial \mathbb{D}^k_{\delta''}(t_0))$.

Proof. The restriction of g to $(F_f \cap g^{-1}(\mathbb{D}^k_{\delta'}) \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$ is a submersion and its fibres are transverse to the sphere $\mathbb{S}_\varepsilon$. Thus, we can lift any vector field on $\mathbb{D}^k_{\delta'}$ to a vector field on $(F_f \cap g^{-1}(\mathbb{D}^k_{\delta'}) \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$ which is orthogonal to the fibres and tangent to $\mathbb{S}_\varepsilon$. Let $\bar{u}$ be the vector field on $\mathbb{D}^k_{\delta'}$ radial from t_0 which is given by $\bar{u}(t) = t - t_0$. Let u be a lifting of $\bar{u}$ by g to $(F_f \cap g^{-1}(\mathbb{D}^k_{\delta'}) \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$. Then u satisfies the properties stated in the lemma in $(F_f \cap g^{-1}(\mathbb{D}^k_{\delta'}) \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$. Using a partition of unity we extend u to $F_f \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$ as zero in the complement of a neighbourhood of $(F_f \cap g^{-1}(\mathbb{D}^k_{\delta'}) \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$. ∎

The vector field u of Lemma 3.4 is ilustrated in Figure 3.



Vector field u of Lemma 3.3.

1. 2. 3. Lemma 3.5. *There exists a vector field w on $F_f \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$ with the following properties:*

1.It is tangent to F_h.

2.w has only a finite number of zeros, and they are all contained in F_h.

3.At each zero w is transversally radial, that is, it is transverse to the boundary of a tubular neighborhood of F_h in F_f.

4.w is transverse to $\mathbb{S}_\varepsilon$, pointing outwards.

5.w coincides with $\nabla \tilde{g}|_{F_f}$ on $F_f \cap \mathbb{S}_{\varepsilon'}$.

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Proof. We use a partition of unity to define w on $F_f \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$ as the sum of the vector fields r and u constructed in Lemma 3.3 and Lemma 3.4. We get a vector field which satisfies properties (1.), (2.), (3.) and (4.). Property (5.) follows from Lemma 3.2 and the fact that the restriction of w to $F_f \cap \mathbb{S}_{\varepsilon'}$ coincides with the vector field u of Lemma 3.4 which is the lifting of the vector field $\bar{u}$ which is non-zero on $\mathbb{R}^k \setminus \mathbb{D}^k_{\delta''}(t_0)$. Let $n\colon \mathbb{R}^k \to \mathbb{R}^k$ be the map given by $n(t) = \|t\|^2$ and $m\colon \mathbb{R}^k \to \mathbb{R}^k$ the map given by $m(t) = t - t_0$ as in the proof of Lemma 3.2. It remains to see that the image of the vector field $\bar{u}$ on $\mathbb{R}^k \setminus \mathbb{D}^k_{\delta''}(t_0)$ under the differential of the map $n \circ m$ is non zero. Therefore the vector field w on $F_f \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'})$ is the lift of a non zero vector field on $\mathbb{R}$. To see this, given $t \in \mathbb{R}^k \setminus \mathbb{D}^k_{\delta''}(t_0)$ we have that

$$d(n \circ m)_t(u(t)) = (t - t_0) \cdot (t - t_0) = \|t - t_0\|^2 \neq 0.$$

∎

Proof (Proof of Theorem 1.1). We can extend the vector field w given in Lemma 3.5 to $F_f \cap \mathrm{Int}(\mathbb{B}_{\varepsilon'})$ with a finite number of isolated zeros. By the Poincaré-Hopf Theorem, the sum of the indices of the zeros of w give the Euler-Poincaré characteristic $\chi(F_f)$. By construction, w only has zeros inside $\mathbb{B}_{\varepsilon'}$ and in F_h. The sum of the indices of the zeros in F_h give the Euler-Poincaré characteristic $\chi(F_h)$ of F_h and the sum of the indices of the zeros in $\mathbb{B}_{\varepsilon'}$ give the total Poincaré-Hopf index of the vector field $\nabla\tilde{g}|_{F_f}$. Hence we have

$$\chi(F_f) = \chi(F_h) + \mathrm{Ind}_{\mathrm{PH}}\, \nabla\tilde{g}|_{F_f}.$$

∎

Remark 3. 1. For the case $k = 1$, that is when $g\colon \mathbb{R}^n \to \mathbb{R}$, instead of $\tilde{g}$ one can take g itself. So in this case we get the formula

$$\chi(F_f) = \chi(F_h) + \mathrm{Ind}_{\mathrm{PH}}\, \nabla g|_{F_f}.$$

4. CURVATURA INTEGRA AND *D*-REGULARITY

Let $f\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$ be a real analytic map-germ. We say that f satisfies *the Milnor condition* if it has an isolated critical point at zero. In this case by [34] we have a locally trivial fibration

$$f\colon f^{-1}(\partial\mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon \to \partial D_\delta \cong \mathbb{S}^{p-1},$$

where $\mathbb{B}_\varepsilon$ is a sufficiently small ball around $\underline{0}$ in $\mathbb{R}^n$ and $\mathbb{D}_\delta$ is a ball around 0 in $\mathbb{R}^p$ of radius $\delta > 0$, sufficiently small with respect to ε. Milnor also points out that the tube $f^{-1}(\partial\mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$ can always be inflated to the sphere $\mathbb{S}_\varepsilon$ and one gets a locally trivial fibration

$$\phi\colon \mathbb{S}_\varepsilon \setminus K_\varepsilon \longrightarrow \mathbb{S}^{p-1},$$

for sufficiently small spheres, where $K_\varepsilon = \mathbb{S}_\varepsilon \cap f^{-1}(0)$ is the link. Furthermore, the projection ϕ can always be taken as $\phi = \frac{f}{\|f\|}$ in a neighborhood of the set K_ε (which can be empty).

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The following concept was introduced in [41] for $p = 2$; the extension for $p > 2$ is straightforward.

Definition 4.1. Let $f : (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$, $p \geq 2$, be a real analytic map-germ with an isolated critical point at 0. We say that f has *the strong Milnor property* if for every sufficiently small sphere $\mathbb{S}_\varepsilon$ the map:

$$\phi = \frac{f}{\|f\|} : \mathbb{S}_\varepsilon \setminus (\mathbb{S}_\varepsilon \cap f^{-1}(0)) \longrightarrow \mathbb{S}^{p-1}\,,$$

is a locally trivial fibration.

In this case we also say that f defines a Milnor open-book fibration. For instance, every holomorphic map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ has the strong Milnor property, by [34], and by [43, 41] so does every *twisted Pham-Brieskorn polynomial*:

$$h(z_1, \cdots, z_n) = z_1^{a_1}\overline{z}_{\sigma(1)} + \cdots + z_n^{a_n}\overline{z}_{\sigma(n)}\,,$$

where all $a_i > 0$ and σ is a permutation of $\{1, \ldots, n\}$ (see also [44]), and more generally, every polar-weighted homogeneous singularity (see [9]).

Let us describe now the canonical pencil of a map f as above. This appeared first in [43, 41, 40] for $p = 2$ and later in [11] for $p \geq 2$. An equivalent pencil appears also in [42].

Definition 4.2. Given a map-germ $f \colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$, $n > p$, its *canonical pencil* is the family $\{V_\ell\}$, parametrized by the real projective space $\mathbb{R}\mathbb{P}^{p-1}$, so that for each line ℓ through 0 in $\mathbb{R}^p$ this set is

$$V_\ell = \{x \in \mathbb{R}^n \mid f(x) \in \ell\}.$$

Notice that the union of all V_ℓ is the whole ambient space and their intersection is $f^{-1}(0)$. Moreover, if f is real analytic and it has an isolated singularity, then each V_ℓ is non-singular away from $\underline{0}$.

The following definition is taken from [10, 11].

Definition 4.3. We say that the map-germ f is *d-regular* at zero if there exists ε such that every sphere $\mathbb{S}_\varepsilon$, centered at zero, of radius $\leq \varepsilon$, meets transversally every element of the pecil.

Theorem 4.1 ([10, 11]). *Let $f \colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, \underline{0})$, $n > p > 1$, be a real analytic map-germ with an isolated critical point at $\underline{0}$ and such that* $\dim f^{-1}(0) > 0$. *Then f has the strong Milnor property if and only if it is d-regular, and in that case the two fibrations:*

$$\phi = \frac{f}{\|f\|} : \mathbb{S}_\varepsilon \setminus (\mathbb{S}_\varepsilon \cap f^{-1}(0)) \longrightarrow \mathbb{S}^{p-1}\,,$$

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and

$$N(\varepsilon,\delta) \xrightarrow{f} \partial\Delta_\delta \cong \mathbb{S}^{p-1}\,,$$

are equivalent. Furthermore, let $V_\ell := \{x \in \mathbb{R}^n \mid f(x) \in \ell\}$ *be an element in the pencil of* f*. Then the fibers of* ϕ *over the two points in* $\ell \cap \mathbb{S}^{p-1}$ *are naturally glued together along the link* $K = f^{-1}(0) \cap \mathbb{S}_\varepsilon$ *forming the link of* V_ℓ*.*

Notice that we can always take the line ℓ to be the first coordinate axis in $\mathbb{R}^p$. In that case, if $p = 2$ and we identify $\mathbb{R}^2 \cong \mathbb{C}$, then the variety V_ℓ is the set of zeroes of the real part of f. We remark also that the statement in this theorem saying that f has the strong Milnor property if and only if it is d-regular is proved independently in [42], and so does the equivalence of the two fibrations in the case when f is weighted homogeneous.

Remark 4. 1. Notice that the fibers of ϕ are open manifolds, while the fibers of f in the tube $N(\varepsilon,\delta)$ are compact manifolds. Thus we must be careful when we say that these two fibrations are equivalent: the actual equivalence comes when we consider the open tube $N(\varepsilon,\delta) \setminus (\mathbb{S}_\varepsilon \cap N(\varepsilon,\delta))$.

Let us recall now the concept of the *curvatura integra*. If M is an oriented $(n-1)$-dimensional closed submanifold of Euclidean space $\mathbb{R}^n$, then its normal bundle is necessarily trivial. If ν is a section of the normal bundle "pointing outwards" everywhere, then ν determines a map from M into the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$, the Gauss map, whose degree is the *curvatura integra* of M.

In [24] H. Hopf generalized the classical theorem of C. F. Gauss by proving that if $(n-1)$ is even then its *curvatura integra* is half the Euler characteristic of M, independently of the embedding. This theorem was generalized by M. Kervaire in [27] to submanifolds of Euclidean spaces with arbitrary codimensions, provided these are embedded with a trivial normal bundle. This is as follows.

Let M be an m-dimensional closed manifold embedded in $\mathbb{R}^n$ with trivial normal bundle $\nu(M)$. Set $k = n - m$ and let us denote by $V(k,n)$ the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$. This manifold is $(m-1)$-connected and its first homology group is in dimension m, where its homology is $\mathbb{Z}$ if either m is even or $k = 1$, or $\mathbb{Z}/2\mathbb{Z}$ when m is odd and $k > 1$. A trivialization of $\nu(M)$ defines, up to homotopy, a smooth map

$$M \to V(k, m+k)\,,$$

and one has an induced homomorphism

$$\varphi_*\colon H_m(M;\mathbb{Z}) \to H_m(V(k+1, m+k;\mathbb{Z})).$$

If m is even, the map ϕ has a degree, an integer, and this is by definition the *curvatura integra* of M. Kervaire and H. Hopf, denoted by $\varphi(M)$. When $m = 2$ and $k = 1$ this is the usual Gauss map.

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Theorem 4.2 ([27]). *Let M be a manifold of even dimension embedded in some Euclidean space $\mathbb{R}^n$ with trivial normal bundle. Then its curvatura integra is independent of the embedding and it equals one half of its Euler characteristic:*

$$\varphi(M) = \frac{1}{2}\chi(M).$$

Remark 4. 2. When the dimension of M is odd, its *curvatura integra* is an invariant in $\mathbb{Z}/2\mathbb{Z}$ and in this case Kervaire's theorem (completed in [2], Prop. 3.4) says that if M bounds a parallelizable manifold, then this invariant equals the semi-characteristic of M.

Consider now a real analytic map-germ $f : (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$, $p > 1$, with an isolated critical point at $\underline{0}$. We set $V := f^{-1}(0)$, so this is a real analytic variety with an isolated singularity at $\underline{0}$. If we assume that the germ of V at $\underline{0}$ has dimension greater than 0, then in a neighbourhood of $\underline{0}$, every point in $V^* := V \setminus \{\underline{0}\}$ is a regular point of f. Hence V^* is a smooth submanifold of $\mathbb{R}^n$ of dimension $n-p$, embedded with trivial normal bundle. Furthermore, this manifold meets transversally every sufficiently small sphere $\mathbb{S}_e$ around $\underline{0}$ (by [34] or by Verdier's Bertini-Sard theorem in general). Hence its link $K := V \cap \mathbb{S}_e$ is a smooth submanifold of $\mathbb{R}^n$ of dimension $n-p-1$, embedded with trivial normal bundle.

Definition 4.4. [*Curvatura integra*] Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $p > 1$, be real analytic, with $\dim V > 0$ and having the strong Milnor property. Then its *curvatura integra* $\Psi(f) \in \mathbb{Z}$ is defined as follows:

1. If $n-p$ is odd, so the link K is even-dimensional, then we set $\Psi(f) := \varphi(K)$.
2. If $n-p$ is even, let V_ℓ be an element in the canonical pencil of f. Denote by K_ℓ the link of V_ℓ, so this is a closed, even-dimesnional manifold with trivial normal bundle. Then we set: $\Psi(f) := \varphi(K_\ell)$.

The fact that this invariant is well defined, *i.e.*, that for $n-p$ even it does not depend on the choice of element in the pencil, is a consequence of Theorem 4.3 below.

Let us now relate the invariant $\Psi(f)$ with the Euler characteristic of the Milnor fiber of f:

Theorem 4.3. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, with $p > 1$ be a real analytic map-germ with the strong Milnor property. Suppose also that $\dim f^{-1}(0) > 0$. Then we have:*

$$\Psi(f) = \chi(F),$$

where F denotes the Milnor fiber of f.

Proof. Notice that the Milnor fiber has dimension $(n-p)$ and its boundary is isotopic to the link K. We split this proof in two parts, the first being when $(n-p)$ is odd, so the link is even-dimensional and it has trivial normal bundle. By definition $\Psi(f) := \varphi(K)$, so it follows from Theorem 4.2 that $\varphi(K) = \frac{1}{2}\chi(K)$. The result now follows from a general theorem in differential topology: if a closed oriented manifold M bounds a compact, oriented manifold X and M is even-dimensional, then $\chi(M) = \frac{1}{2}\chi(X)$. To prove this, consider the double of X and denote it DX. This is a closed oriented manifold of odd dimension. Hence $\chi(DX) = 0$. But one also has that $\chi(DX) = 2\chi(X) - \chi(M)$, so $\chi(M) = 2\chi(X)$. We conclude that

$$\Psi(f) = \varphi(K) = \frac{1}{2}\chi(K) = \frac{1}{2}(2\chi(F)) = \chi(F).$$

Now consider the case when $(n-p)$ is even. By definition we have $\Psi(f) = \varphi(K_\ell)$, where K_ℓ is the link of an element V_ℓ of the canonical pencil of f. In this case K_ℓ is even dimensional and it has trivial normal bundle, so it is orientable, while the link K is odd-dimensional and also orientable. Theorem 4.1 says that K_ℓ is the union of two Milnor fibers of $\phi = f/\|f\|$ glued along the link K:

$$K_\ell = \phi^{-1}(y_\ell) \cup K \cup \phi^{-1}(-y_\ell).$$

Theorem 4.1 also says that the fibration ϕ is equivalent with the fibration on a Milnor tube (see Remark 4.1). Thence, using as above well known properties of the Euler characteristic, we have:

$$\chi(K_\ell) = \chi(\phi^{-1}(y_\ell)) - \chi(K) + \chi(\phi^{-1}(-y_\ell)) = 2\chi(F),$$

where F denotes de Milnor fiber of f.

By Theorem 4.2 we have $\varphi(K_\ell) = \frac{1}{2}\chi(K_\ell)$, which implies $\Psi(f) = \varphi(K_\ell) = \frac{1}{2}\chi(K_\ell)$, so $\Psi(f) = \chi(F)$. Thus we conclude that $\Psi(f) = \chi(F)$ in all cases. ▮

5. AN ALGEBRAIC VIEWPOINT

In this section we consider the case of $f\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}, 0)$ and $g\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}, 0)$, both with an algebraically isolated critical point at $\underline{0}$ (se below for the definition), which define a complete intersection in $\mathbb{R}^n$. We give an algebraic expresion of the Lê-Greuel formula in terms of the signature of a certain quadratic form asociated to (f, g).

It is well known result that the Poincaré-Hopf index of a holomorphic vector field v with isolated singularity at 0 can be computed as the dimension of the algebra obtained from the ring of germs of holomorphic functions at 0, dividing it by the ideal generated by the coordinate functions of v. This fact, together with a certain "law of conservation of the number" under appropriate types of flat deformations, is what leads to the expression on the right hand side of the classical Lê-Greuel in the holomorphic setting:

$$\mu(f,g) + \mu(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+k,\underline{0}}}{(f, \mathrm{Jac}(f,g))}\,.$$

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It is thus natural to ask what happens in the real case. This is a subject that has been explored in the work of X. Gómez-Mont and P. Mardešić [18, 19], and also by W. Ebeling and S. Gusein-Zade, for instance in [15].

The starting point is the celebrated formula of Eisenbud-Levine-Khimshiashvili [16, 28]. To explain this, let us denote by $A_{\mathbb{R}^n,0}$ the local ring of germs of real analytic real-valued functions, and consider a germ of a vector field $v = (a_1, \cdots, a_n)$ at $0 \in \mathbb{R}^n$, where the components are elements in $A_{\mathbb{R}^n,0}$. We let B_v be the local algebra of v:

$$B_v = A_{\mathbb{R}^n,0}/(a_1, \cdots, a_n),$$

where $(a_1, \cdots, a_n)$ denotes the ideal generated by components of v. The dimension of B_v as an algebra over $\mathbb{R}$ is the multiplicity $\lambda(v)$ of v at 0. If $\lambda(v) < \infty$, then v necessarily has an isolated zero at 0, but the converse is not always true: $\lambda(v) < \infty$ is equivalent to saying that the complexification $v_{\mathbb{C}}$ of v has an isolated singularity at 0. In this case we say that the singularity of v at $0 \in \mathbb{R}^n$ is *algebraically isolated*. The signature formula of [16] and [28] deals with such vector fields (in fact the theorem in [16] is proved for C^∞ vector fields with an isolated singularity of finite multiplicity).

For this, given an analytic vector field v with finite multiplicity, let J_v be the Jacobian of v, *i.e.*, the (local) function whose value at each point is the determinant of the Jacobian matrix:

$$\begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \cdots & \frac{\partial a_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_n}{\partial x_1} & \cdots & \frac{\partial a_n}{\partial x_n} \end{pmatrix}$$

For simplicity we also denote by J_v the residue class of the Jacobian in the local ring B_v. Since B_v is an algebra, given a linear functional $\phi : B_v \to \mathbb{R}$ one can define a map $\langle\ ,\ \rangle_\phi \colon B_v \times B_v \to \mathbb{R}$ by:

$$\langle f, g\rangle_\phi = \phi(fg).$$

This is the composition of the product in the algebra B_v followed by ϕ. We let $\mathrm{Sgn}(v)$ denote its signature, *i.e.*, the number of positive eigenvalues minus the number of negative eigenvalues. Then one has the index formula of Eisenbud-Levine-Khimshiashvili [16, 28] says:

THEOREM 5.1. *One can always choose the linear form ϕ so that $\phi(J_v) > 0$, and in this case one has:*

$$\mathrm{Ind}_{\mathrm{PH}}(v, 0) = \mathrm{Sgn}(v),$$

independently of the choice of ϕ.

The next question now is to consider analytic vector fields defined on real analytic singular varieties, and this is what the authors study in [18, 19, 15]. We recall now some of the main ideas in [18, 19].

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Recall that given an analytic map-germ $(\mathbb{R}^{n+1},0) \stackrel{f}{\to} (\mathbb{R},0)$ with an isolated critical point at 0, in general there is not "a" Milnor fibre, but there are fibres to left and right of $0 \in \mathbb{R}$, with possibly different topology. Therefore one can not define in general an index in the spirit of the GSV-index [20], *i.e.*, a well-defined integer associated to each vector field on the hypersurface $V = f^{-1}(0)$ with an isolated singularity, which measures the number of zeroes of an extension of the vector field to a Milnor fibre. In this case the number that we get may depend on the choice of Milnor fiber (right fibers or left fibers).

If the hypersurface has odd dimension, things became simpler from topological point of view, as we already know from the previous section. In this case the Euler-Poincaré characteristic of the Milnor fiber is well defined, and therefore so is the corresponding GSV-index of vector fields (see [2]). However for even dimensions, in general this is only well defined modulo 2.

Even though, the formulae by Arnold in [3] show that for gradient vector fields, the algebra behind the function determines the Euler characteristic of the fibers in all cases. This motivated the search for an algebraic formula for the index of vector fields on real analytic hypersurface singularities, and the work of Gómez-Mont and Mardešić [18, 19] is very close to the Eisenbud-Levine-Khimshiashvili formula.

To explain this case, let U be an open neighborhood around $0 \in \mathbb{R}^{n+1}$, and let $f : (U,0) \to (\mathbb{R},0)$ be real analytic and set $V = f^{-1}(0)$. Let A be the local ring of f at 0:

$$A = \frac{A_{\mathbb{R}^{n+1},0}}{\left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_{n+1}}\right)},$$

where $(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_{n+1}}) = J_f$ is the jacobian ideal of f. We say that V has an *algebraically isolated singularity at* 0 if the algebra A is finite dimensional and it is equivalet to saying that the complexification $V(f)_{\mathbb{C}}$ of $V(f)$ has an isolated singularity. The algebra A has a distinguished element: The class of the Hessian,

$$\text{Hess}(f) := \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) \in A.$$

This class generates an ideal in A which is minimal in the sense that it is contained in every nonzero ideal of A; This is called the socle of the corresponding algebra [19].

Now consider a real analytic vector field v on $V \cap U$, tangent to V and with an algebraically isolated singularity at 0. Thus v is the restriction to V of a real analytic vector field $\hat{v}$ on a ball $\mathbb{B} \subset \mathbb{R}^{n+1}$, such that $df(v)(x) = 0$ for each $x \in V$. Since the ideal of functions vanishing on V is generated by f, one has that $df(v)$ is a multiple of f, so the assumption of v being tangent to V is actually equivalent to saying that there exist $h_v \in A_{\mathbb{R}^{n+1},0}$ so that $df(v) = fh_v$.

Following [16, 28], we consider the local algebra of $v = (a_1, \cdots, a_n)$:

$$B_v = A_{\mathbb{R}^n,0}/(a_1, \cdots, a_n).$$

This algebra has also finite dimension because v has an algebraically isolated singularity, and it also has a distinguished element: The class of the Jacobian of v:

$$J_v := \det(\frac{\partial a_i}{\partial x_j}) \in B_v.$$

We know that in the Eisenbud-Levine-Khimshiashvili formula for the index, when the space is smooth, the Jacobian J_v and the signature of certain quadratic form, determine the index. In the singular case we must consider the relative Jacobian $J_f(v)$ and the relative Hessian $\mathrm{Hess}_{\mathrm{rel}}(f)$, that we now introduce. The relative Jacobian $J_f(v)$ is the element,

$$J_f(v) := \frac{J_v}{h_v} \in B_v / \operatorname{Ann}_{B_v}(h_v),$$

where $\operatorname{Ann}_{B_v}(h_v)$ is the annihilator, h_v being as above. This element is a well-defined, and we know from [18] that there is a linear map $l\colon B_v/\operatorname{Ann}(h) \to \mathbb{R}$ such that $l(J_f(v)) > 0$. The product in $B_v/\operatorname{Ann}(h)$ together with l defines a bilinear form on $B_v/\operatorname{Ann}(h)$. Let $\mathrm{Sgn}_{V,0}$ denote the signature of this bilinear form.

It is proved in [18] that the function $\mathrm{Sgn}_{V,0}$ behaves like an index in the sense that for n even it satisfies the law of conservation of number:

$$\mathrm{Sgn}_{V,0}(v) = \mathrm{Sgn}_{V,0}(v_t) + \sum_{x \in V\setminus\{0\}/v_t(x)=0} \mathrm{Ind}_{PH}(v_t, x, V \setminus \{0\})$$

for x close to 0 and v_t tangent to V and close to 0 and v_t tangent to V and close to v. The same formula holds for n odd under certain additional hypotheses.

Similarly, the relative Hessian is defined in [19] by:

$$\mathrm{Hess}_{\mathrm{rel}}(f) := \frac{\mathrm{Hess}(f)}{h_v} \in A/\operatorname{Ann}_A(h_v).$$

It is shown in [19] that one can also construct a linear functional l on $A/\operatorname{Ann}_A(h_v)$ such that:

$$l(\mathrm{Hess}_{\mathrm{rel}}(f)) > 0;$$

the construction of this functional follows the ideas developed in [4, 16, 28]. As before, we may use this functional to define a bilinear form. Let $\mathrm{Sgn}_A(h_v)$ denote the signature of this bilinear form.

Now, if V has odd dimension $n \geq 1$, the Euler-Poincaré characteristic of the fibers $V_t = f^{-1}(t) \cap \mathbb{D}_\varepsilon$, $t \neq 0$ is independent of the choice of t. Recall that $\mathrm{Ind}_{\mathrm{GSV}}(v)$ the GSV-index of v by definition equals the total Poincaré-Hopf index of an extension of the vector field to a local Milnor fiber. One has:

THEOREM 5.2. *Let $n > 1$ be an even integer, let $V = f^{-1}(0) \subset \mathbb{R}^n$ be a real analytic hypersurface with an algebraically isolated singularity at 0, and let v be a real analytic vector field on V with an algebraically isolated singularity at 0. Then:*

$$\mathrm{Ind}_{\mathrm{GSV}}(v) = \mathrm{Sgn}_{(V,0)}(v) - \mathrm{Sgn}_A(h_v),$$

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where $h_v = df(v)/f \in A$.

Notice that if V is regular at 0, then this formula reduces to the on in [16, 28]. There is in [18] the analogous formula for the GSV-index when n is even. In that case, just as in Arnold's work [3], the formula depends on whether one considers a right Milnor fiber or a left one. Yet, in all cases what the formulae of [18, 19] prove is:

THEOREM 5.3 (Gómez-Mont, Mardešić). *Let* $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *be a real analytic map-germ such that* $V = f^{-1}(0)$ *has an algebraically isolated singularity at* 0*, let* v *be a real analytic vector field on* V *with an algebraically isolated singularity at* 0*, and let* $F_t := f^{-1}(t) \cap \mathbb{B}_\varepsilon$ *be a local Milnor fiber of* f *(for* ε *and* t *sufficiently small). Then the total Poincaré-Hopf index of an extension of* v *to* F_t *is fully determined by the local algebra of* f *and* v *at* 0*. In fact, if* $n > 1$ *is an even integer, this index is given by the theorem above (otherwise it is given by the corresponding formula in [18]).*

We now return to the Lê-Greuel formula for real analytic maps and use all the previous information to obtain:

THEOREM 5.4. *Let* f, g *be real analytic map-germs* $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *such that* $V(f) = f^{-1}(0)$ *and* $V(g) = g^{-1}(0)$ *have respectively an algebraically isolated singularity at* 0*, which define a complete intersection germ in* $\mathbb{R}^n$*. Suppose (for simplicity) that* $n > 1$ *is even. Let* $\widehat{g}$ *be the restriction of* g *to* $V(f)$ *and suppose that it also has an algebraically isolated singularity at* 0*. Let* $v := \nabla\widehat{g}$ *be the gradient vector field of* $\widehat{g}$*. Then one has:*

$$\chi(F_f) = \chi(F_{f,g}) + \mathrm{Sgn}_{(V(f),0)}(v) - \mathrm{Sgn}_A(h_v),$$

where F_f *and* $F_{f,g}$ *are as in Theorem 1.1 and the term* $\mathrm{Sgn}_{(V(f),0)}(v) - \mathrm{Sgn}_A(h_v)$ *is determined as above by the local algebra of* f *and* g*.*

Proof. Since $V(f)$ has an algebraically isolated singularity its complexification $V(f)_{\mathbb{C}}$ has an isolated singularity. That $\widehat{g}$ has an algebraically isolated singularity at 0 is equivalent to the fact that the complexification $\hat{g}_{\mathbb{C}}$ of g has an isolated critical point at 0 in $V(f)_{\mathbb{C}}$. Thus, the gradient vector field $v_{\mathbb{C}} = \nabla\hat{g}_{\mathbb{C}}$ has an isolated singularity at 0. Hence the vector field $v = \nabla\hat{g}$ has an algebraically isolated singularity at 0 and we can apply Theorem 5.3. ∎

So the term $[\mathrm{Sgn}_{(V(f),0)}(v) - \mathrm{Sgn}_A(h_v)]$ in this formula replaces the corresponding term $\dim_{\mathbb{C}} \frac{\mathcal{O}_{n+k,0}}{(f,Jac(f,g))}$, in the classical Lê-Greuel formula in the holomorphic setting. Of course it would be interesting to generalize Theorem 5.4 to the case when f is a complete intersection map-germ into $\mathbb{R}^p$ for $p \geq 1$. This means extending the results of [18, 19] from the case of hypersurfaces to complete intersections, and that is in itself a very interesting problem which is being studied by various people.

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