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Uniform attractors of discontinuous semidynamical systems

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In the present paper, we define the concept of uniform attractor for semidynamical systems subject to variable impulses. We characterize the region of uniform attraction for compact sets and we show an asymptotic stability result for impulsive systems. Also, we present a result of uniform attractor by using Lyapunov functionals. We finish the paper by presenting an application for an impulsive nonlinear autonomous system. May, 2012 ICMC-USP

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1. INTRODUCTION

The theory of impulsive semidynamical systems describes the evolution of systems where the continuous development of a process is interrupted by abrupt changes of state. See [5, 6, 7, 8], [10, 11] and [16] for details of this theory. In the last years, the action of impulses on differential systems has been intensively investigated. The reader may find some important results and applications in [1, 2, 3], [9], [12, 13, 14, 15], for instance.

In this paper, we extend the results of uniform attractors present in [4] for impulsive semidynamical systems. In the first part of this article, we present the basis of the theory of impulsive semidynamical systems. We present basic definitions and notations and then we discuss the continuity of a function which describes the times of reaching the impulsive set. We also present additional useful definitions and results.

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The second part of the paper concerns the main results. Given an impulsive semidynamical system, we start by introducing the concept of the region of uniform attraction of a set in this system. Then we present conditions for the region of uniform attraction to be positively invariant and open in the impulsive system, see Proposition 3.1 and Theorem 3.1. For compact sets, we give a characterization for its region of uniform attraction, see Proposition 3.2. In Theorem 3.3, we establish conditions for a compact uniform attractor set to be asymptotically stable in the impulsive system. In Theorem 3.4, we give conditions for an asymptotically stable set to be uniform attractor. Theorem 3.5 establishes conditions for a set to be uniform attractor by using Lyapunov functionals. Finally we present examples to show how the theory can be employed.

2. IMPULSIVE SEMIDYNAMICAL SYSTEMS

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The pair (X, π) is called a *semidynamical system*, if the function $\pi : X \times \mathbb{R}_+ \to X$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t + s)$, for all $x \in X$ and $t, s \in \mathbb{R}_+$. For every $x \in X$, we consider the continuous function $\pi_x : \mathbb{R}_+ \to X$ given by $\pi_x(t) = \pi(x, t)$ and we call it the *motion* of x.

Let (X, π) be a semidynamical system. Given $x \in X$, the *positive orbit* of x is given by $\pi^+(x) = \{\pi(x, t) : t \in \mathbb{R}_+\}$. For $t \geq 0$ and $x \in X$, we define $F(x, t) = \{y \in X : \pi(y, t) = x\}$ and, for $\Delta \subset [0, +\infty)$ and $D \subset X$, we define

$$F(D, \Delta) = \cup\{F(x, t) : x \in D \text{ and } t \in \Delta\}.$$

Then a point $x \in X$ is called an *initial point* if $F(x, t) = \emptyset$ for all $t > 0$.

An *impulsive semidynamical system* $(X, \pi; M, I)$ consists of a semidynamical system, (X, π), a non-empty closed subset M of X such that for every $x \in M$, there exists $\varepsilon_x > 0$ such that

$$F(x, (0, \varepsilon_x)) \cap M = \emptyset \quad \text{and} \quad \pi(x, (0, \varepsilon_x)) \cap M = \emptyset,$$

and a continuous function $I : M \to X$ whose action we explain below in the description of the impulsive trajectory of an impulsive semidynamical system. The set M is called the *impulsive set* and the function I is called *impulse function*. We also define

$$M^+(x) = (\pi^+(x) \cap M) \setminus \{x\}.$$

Let $(X, \pi; M, I)$ be an impulsive semidynamical system. We define a function $\phi : X \to (0, +\infty]$, which represents the least strict positive time for which the trajectory of x meets M, by

$$\phi(x) = \begin{cases} s, & \text{if } \pi(x, s) \in M \text{ and } \pi(x, t) \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } M^+(x) = \emptyset. \end{cases} \tag{2.1}$$

Thus for each $x \in X$, we call $\pi(x, \phi(x))$ the *impulsive point* of x.

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The *impulsive trajectory* of x in $(X, \pi; M, I)$ is an $X-$valued function $\widetilde{\pi}_x$ defined on the subset $[0, s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such trajectory follows inductively as described in the following lines.

If $M^+(x) = \emptyset$, then $\widetilde{\pi}_x(t) = \pi(x, t)$, for all $t \in \mathbb{R}_+$, and $\phi(x) = +\infty$. However if $M^+(x) \neq \emptyset$, it follows from Lemma 2.1 in [5] that there is the smallest positive number s_0 such that $\pi(x, s_0) = x_1 \in M$ and $\pi(x, t) \notin M$, for $0 < t < s_0$. Then we define $\widetilde{\pi}_x$ on $[0, s_0]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x,t), & 0 \le t < s_0 \\ x_1^+, & t = s_0, \end{cases}$$

where $x_1^+ = I(x_1)$ and $\phi(x) = s_0$. Let us denote x by x_0^+.

Since $s_0 < +\infty$, the process now continues from x_1^+ onwards. If $M^+(x_1^+) = \emptyset$, then we define $\widetilde{\pi}_x(t) = \pi(x_1^+, t - s_0)$, for $s_0 \le t < +\infty$, and $\phi(x_1^+) = +\infty$. When $M^+(x_1^+) \neq \emptyset$, it follows again from Lemma 2.1 in [5] that there is the smallest positive number s_1 such that $\pi(x_1^+, s_1) = x_2 \in M$ and $\pi(x_1^+, t - s_0) \notin M$, for $s_0 < t < s_0 + s_1$. Then we define $\widetilde{\pi}_x$ on $[s_0, s_0 + s_1]$ by $\widetilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - s_0), & s_0 \le t < s_0 + s_1 \\ x_2^+, & t = s_0 + s_1, \end{cases}$ where $x_2^+ = I(x_2)$ and $\phi(x_1^+) = s_1$, and so on. Notice that $\widetilde{\pi}_x$ is defined on each interval $[t_n, t_{n+1}]$, where $t_{n+1} = \sum_{i=0}^{n} s_i$. Hence $\widetilde{\pi}_x$ is defined on $[0, t_{n+1}]$.

The process above ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some n. Or it continues infinitely, if $M^+(x_n^+) \neq \emptyset$, $n = 0, 1, 2, \ldots$, and in this case the function $\widetilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$.

Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Given $x \in X$, the *impulsive positive orbit* of x is defined by the set $\widetilde{\pi}^+(x) = \{\widetilde{\pi}(x, t) : t \in [0, T(x))\}$.

Analogously to the non-impulsive case, an impulsive semidynamical system satisfies the following standard properties: $\widetilde{\pi}(x, 0) = x$ for all $x \in X$ and $\widetilde{\pi}(\widetilde{\pi}(x, t), s) = \widetilde{\pi}(x, t + s)$, for all $x \in X$ and for all $t, s \in [0, T(x))$ such that $t + s \in [0, T(x))$. See [6] for a proof of it.

For details about the structure of these types of impulsive semidynamical systems, the reader may consult [5, 6, 7, 8], [10, 11] and [16].

2.1. Continuity of ϕ

The result of this section is borrowed from [10]. It concerns the continuity of the function ϕ defined in (2.1).

Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing x ($x \in X$) is called a *section* or a λ-*section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that

(a) $F(L, \lambda) = S$;
(b) $F(L, [0, 2\lambda])$ is a neighborhood of x;
(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \le \mu < \nu \le 2\lambda$.

The set $F(L,[0,2\lambda])$ is called a *tube* or a λ-*tube* and the set L is called a *bar*. Let (X,π) be a semidynamical system. We now present the conditions TC and STC for a tube.

Any tube $F(L,[0,2\lambda])$ given by a section S through $x \in X$ such that $S \subset M \cap F(L,[0,2\lambda])$ is called *TC-tube* on x. We say that a point $x \in M$ fulfills the *Tube Condition* and we write (TC), if there exists a TC-tube $F(L,[0,2\lambda])$ through x. In particular, if $S = M \cap F(L,[0,2\lambda])$ we have a *STC-tube* on x and we say that a point $x \in M$ fulfills the *Strong Tube Condition* (we write (STC)), if there exists a STC-tube $F(L,[0,2\lambda])$ through x.

The following theorem concerns the continuity of ϕ which is accomplished outside M for M satisfying the condition TC. See [10, Theorem 3.8].

THEOREM 2.1. *Consider an impulsive semidynamical system $(X,\pi;M,I)$. Assume that no initial point in (X,π) belongs to the impulsive set M and that each element of M satisfies the condition (TC). Then ϕ is continuous at x if and only if $x \notin M$.*

2.2. Additional definitions and results

Let us consider a metric space X with metric ρ. By $B(x,\,\delta)$ we mean the open ball with center at $x \in X$ and radius $\delta > 0$. Given $A \subset X$, let $B(A,\,\delta) = \{x \in X : \rho(x,A) < \delta\}$ where $\rho(x,A) = \inf\{\rho(x,\,y) : y \in A\}$. Throughout this paper, we use the notation ∂A, $\operatorname{int} A$ and $\overline{A}$ to denote respectively the boundary, interior and closure of A in X.

Let $(X,\,\pi;\,M,\,I)$ be an impulsive semidynamical system and $A \subset X$. We define

$$\widetilde{\pi}^+(A) = \bigcup_{x\in A} \widetilde{\pi}^+(x) \quad \text{and} \quad \widetilde{\pi}(A,t) = \bigcup_{x\in A} \widetilde{\pi}(x,t)$$

for each $t \geq 0$. If $\widetilde{\pi}^+(A) \subset A$, we say that A is *positively $\widetilde{\pi}-$invariant*. And, we say that A is *$I-$invariant* in the impulsive system if $I(x) \in A$ for each $x \in A \cap M$. If for every $\varepsilon > 0$ and every $x \in A$, there is $\delta = \delta(x,\varepsilon) > 0$ such that $\widetilde{\pi}(B(x,\delta),[0,+\infty)) \subset B(A,\varepsilon)$, then A is called *$\widetilde{\pi}$-stable*. The set A is *orbitally $\widetilde{\pi}$-stable* if for every neighborhood U of A, there is a positively $\widetilde{\pi}-$invariant neighborhood V of A, $V \subset U$.

The *limit set*, the *prolongational limit set* and the *prolongation set* of $A \subset X$ in $(X,\pi;M,I)$ is given by $\widetilde{L}^+(A) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \widetilde{\pi}(A,\tau)}$, $\widetilde{J}^+(A) = \bigcap_{\epsilon>0} \bigcap_{t\geq 0} \overline{\bigcup \{\widetilde{\pi}(B(A,\epsilon),\tau) :\, \tau \geq t\}}$ and $\widetilde{D}^+(A) = \bigcap_{\epsilon>0} \overline{\bigcup \{\widetilde{\pi}(B(A,\epsilon),t) :\, t \geq 0\}}$, respectively. Moreover, these sets may be represented by

$$\widetilde{L}^+(A) = \{y \in X : \text{there exist sequences } \{x_n\}_{n\geq 1} \subset A \text{ and } \{t_n\}_{n\geq 1} \subset \mathbb{R}_+$$

$$\text{such that } t_n \overset{n\to+\infty}{\longrightarrow} +\infty \text{ and } \widetilde{\pi}(x_n,t_n) \overset{n\to+\infty}{\longrightarrow} y\},$$

$$\widetilde{J}^+(A) = \{y \in X : \text{there are sequences } \{x_n\}_{n\geq 1} \subset X \text{ and } \{t_n\}_{n\geq 1} \subset \mathbb{R}_+ \text{ such that}$$

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$$\rho(x_n, A) \overset{n\to+\infty}{\longrightarrow} 0,\ t_n \overset{n\to+\infty}{\longrightarrow} +\infty \text{ and } \widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} y\}$$

and

$$\widetilde{D}^+(A) = \{y \in X : \text{ there exist sequences } \{x_n\}_{n\geq 1} \subset X \text{ and } \{t_n\}_{n\geq 1} \subset \mathbb{R}_+ \text{ such that}$$

$$\rho(x_n, A) \overset{n\to+\infty}{\longrightarrow} 0 \text{ and } \widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} y\}.$$

If $A = \{x\}$ we set $\widetilde{L}^+(x) = \widetilde{L}^+(\{x\})$, $\widetilde{J}^+(x) = \widetilde{J}^+(\{x\})$ and $\widetilde{D}^+(x) = \widetilde{D}^+(\{x\})$.

For a compact set $K \subset X$, we consider $\widetilde{D}^+(K) = \cup\{\widetilde{D}^+(x) : x \in K\}$. See Proposition 3.2 in [8].

In [5], it is defined the *region of attraction* of a set $A \subset X$ with respect to $\widetilde{\pi}$ by

$$\widetilde{P}^+(A) = \big\{x \in X : \text{ for every neighborhood } U \text{ of } A, \text{ there is } T \in \mathbb{R}_+ \\ \text{such that } \widetilde{\pi}(x, [T, +\infty)) \subset U\}$$

and the *region of weak attraction* of A with respect to $\widetilde{\pi}$ by

$$\widetilde{P}^+_W(A) = \big\{x \in X : \text{ for every neighborhood } U \text{ of } A, \text{ there is a sequence} \\ \{t_n\} \subset \mathbb{R}_+, t_n \overset{n\to+\infty}{\to} +\infty \text{ such that } \widetilde{\pi}(x, t_n) \in U\}.$$

If $x \in \widetilde{P}^+_W(A)$ or $x \in \widetilde{P}^+(A)$, then we say that x is *$\widetilde{\pi}$-weakly attracted* or *$\widetilde{\pi}-$attracted* to A respectively. A subset $A \subset X$ is called a *weak $\widetilde{\pi}$-attractor*, if $\widetilde{\mathrm{P}}^+_W(A)$ is a neighborhood of A, and it is called a *$\widetilde{\pi}-$attractor*, if $\widetilde{P}^+(A)$ is a neighborhood of A. A set $A \subset X$ is called *asymptotically $\widetilde{\pi}$-stable*, if it is both a weak $\widetilde{\pi}$-attractor and orbitally $\widetilde{\pi}$-stable.

3. THE MAIN RESULTS

Throughout this paper we shall consider an impulsive semidynamical system $(X, \pi;\, M, I)$, where (X, ρ) is a metric space. Moreover, we shall assume the following additional hypotheses:

(H1) The conditions of Theorem 2.1 are satisfied. Hence, ϕ is continuous on $X \setminus M$.
(H2) $M \cap I(M) = \emptyset$.
(H3) For each $x \in X$, the motion $\widetilde{\pi}(x, t)$ is defined for every $t \geq 0$, that is, $[0, +\infty)$ denotes the maximal interval of definition of $\widetilde{\pi}_x$.

Definition 3.1. Let $(X, \pi; M, I)$ be an impulsive semidynamical system and A be a subset of X. The set

$$\widetilde{P}^+_u(A) = \{x \in X : \text{ for every neighborhood } U \text{ of } A, \text{ there is a neighborhood } V \text{ of } x$$

$$\text{and a } T > 0 \text{ such that } \widetilde{\pi}(V, t) \subset U \text{ for all } t \geq T\},$$

is called the *region of uniform attraction* of the set A with respect to $\widetilde{\pi}$. If $x \in \widetilde{P}_u^+(A)$, we say that x is *uniformly $\widetilde{\pi}$-attracted* to A.

It is easy to see that the following condition holds for all subset A of X,

$$\widetilde{P}_u^+(A) \subset \widetilde{P}^+(A) \subset \widetilde{P}_W^+(A).$$

Given a semidynamical system (X, π), it is proved in [4] that a region of uniform attraction is positively invariant if π admits the following property:

$$\pi(G, t) \text{ is an open subset of } X,$$

for all open subset G of X and for all $t \geq 0$. In the impulsive case, $\widetilde{\pi}(G, t)$ may not be an open set even if G is open. In this way, we need to impose some condition to the impulsive system in order to show that the region of uniform attraction is positively $\widetilde{\pi}$-invariant. Next, we define a special condition for an impulsive semidynamical system in order to obtain invariance of the region of uniform attraction.

Definition 3.2. A point $x \in X$ satisfies the condition $\mathcal{C}$ in $(X, \pi; M, I)$ when every neighborhood of x contains an open set $\mathcal{O} \subset X \setminus M$ such that $x \in \mathcal{O}$ and $\widetilde{\pi}(\mathcal{O}, t)$ is open for all $t \neq \displaystyle\sum_{j=0}^{k} \phi(x_j^+)$ with $k = 0, 1, 2, \ldots$.

By the definition above, it is clear that points in M do not satisfy the condition $\mathcal{C}$.

Given a set $A \subset X$, we establish in our first result a sufficient condition for the positive $\widetilde{\pi}-$invariance of the set $\widetilde{P}_u^+(A) \setminus M$, see the next proposition.

Proposition 3.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and A be a subset of X. Suppose every point in $X \setminus M$ satisfies the condition $\mathcal{C}$. Then $\widetilde{P}_u^+(A) \setminus M$ is positively $\widetilde{\pi}$-invariant.*

Proof. Let $x \in \widetilde{P}_u^+(A) \setminus M$ and $s \geq 0$. Given a neighborhood U of A, there are a neighborhood V of x and $T > 0$ such that $\widetilde{\pi}(V, [T, +\infty)) \subset U$.

If $s \neq \displaystyle\sum_{i=0}^{k} \phi(x_i^+)$ for all $k \geq 0$, it follows by the condition $\mathcal{C}$ of the point x that there is an open set $\mathcal{O} \subset V \setminus M$ which contains x such that $\widetilde{\pi}(\mathcal{O}, s)$ is open. Then

$$\widetilde{\pi}(\mathcal{O}, [T, +\infty)) \subset \widetilde{\pi}(V, [T, +\infty)) \subset U.$$

Let $\widetilde{T} = \max\{0, T - s\}$. Hence, $\widetilde{\pi}(x, s) \in \widetilde{\pi}(\mathcal{O}, s)$ and $\widetilde{\pi}(\widetilde{\pi}(\mathcal{O}, s), [\widetilde{T}, +\infty)) \subset U$. Since $I(M) \cap M = \emptyset$ we have $\widetilde{\pi}(x, s) \notin M$, then $\widetilde{\pi}(x, s) \in \widetilde{P}_u^+(A) \setminus M$.

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On the other side, if $s = \sum_{i=0}^{k} \phi(x_i^+)$ for some $k \geq 0$, then there is $0 < \epsilon < \phi(x_{k+1}^+)$ such that $s + \epsilon \neq \sum_{i=0}^{\ell} \phi(x_i^+)$ for all $\ell \geq 0$. By the previous case, there is an open set W which contains $\widetilde{\pi}(x, s+\epsilon)$ and $T_1 > 0$ such that

$$\widetilde{\pi}(W, [T_1, +\infty)) \subset U.$$

Since $\widetilde{\pi}(x, s) \notin M$, because $x \notin M$ and $I(M) \cap M = \emptyset$, there is an open set Z which contains $\widetilde{\pi}(x, s)$ such that $Z \cap M = \emptyset$ and $\widetilde{\pi}(Z, \epsilon) \subset W$. Hence,

$$\widetilde{\pi}(Z, [\epsilon + T_1, +\infty)) \subset U$$

and it also shows that $\widetilde{\pi}(x, s) \in \widetilde{P}_u^+(A) \setminus M$.

Therefore, $\widetilde{P}_u^+(A) \setminus M$ is positively $\widetilde{\pi}-$invariant. ▮

The next definition presents the concept of a uniform attractor for a subset A in $(X, \pi; M, I)$.

Definition 3.3. Let $(X, \pi; M, I)$ be an impulsive semidynamical system. We say that a subset $A \subset X$ is a *uniform $\widetilde{\pi}$-attractor* if $\widetilde{P}_u^+(A)$ is a neighborhood of A.

Theorem 3.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and A be a subset of X. Suppose A is a uniform $\widetilde{\pi}$-attractor. Then*

a)$\widetilde{P}_u^+(A) \setminus M$ is an open set in X;
b)$\widetilde{P}_u^+(A)$ is positively $\widetilde{\pi}$-invariant.

Proof. a) Let $x \in \widetilde{P}_u^+(A) \setminus M$. Since A is a uniform $\widetilde{\pi}$-attractor, there are a neighborhood V of x and $T > 0$ such that

$$\widetilde{\pi}(V, [T, +\infty)) \subset \mathrm{int}\widetilde{P}_u^+(A).$$

We may assume without loss of generality that $V \cap M = \emptyset$.

Now, we are going to show that $V \subset \widetilde{P}_u^+(A) \setminus M$. In fact, let $w \in V$ and U be an arbitrary neighborhood of A. Since $\widetilde{\pi}(w, T) \in \mathrm{int}\widetilde{P}_u^+(A)$, there are an open set W and $s > 0$ such that

$$\widetilde{\pi}(w, T) \in W \subset \mathrm{int}\widetilde{P}_u^+(A) \quad \text{and} \quad \widetilde{\pi}(W, [s, +\infty)) \subset U.$$

We have two cases to consider: when $T \neq \sum_{j=0}^{k} \phi(w_j^+)$ for all $k = 0, 1, 2, \ldots$, and when $T = \sum_{j=0}^{k} \phi(w_j^+)$ for some $k = 0, 1, 2, \ldots$.

Case 1: Suppose that $T \neq \sum_{j=0}^{k} \phi(w_j^+)$ for all $k = 0, 1, 2, \ldots$. Since $w \notin M$, it follows from the continuity of π and I that there is an open set $\mathcal{O}$ of X which contains w such that

$$\widetilde{\pi}(\mathcal{O}, T) \subset W.$$

Hence, $\widetilde{\pi}(\mathcal{O}, [T + s, +\infty)) \subset U$ and $w \in \widetilde{P}_u^+(A) \setminus M$.

Case 2: Suppose that $T = \sum_{j=0}^{k} \phi(w_j^+)$ for some $k \geq 0$. Then there is $\beta > 0$, $\beta < \phi(w_{k+1}^+)$, such that $\widetilde{\pi}(w, T + \beta) \in W \setminus M$. Thus we may obtain an open set $\mathcal{U}$ of X such that $w \in \mathcal{U}$ and

$$\widetilde{\pi}(\mathcal{U}, T + \beta) \subset W.$$

Hence, $\widetilde{\pi}(\mathcal{U}, [T + \beta + s, +\infty)) \subset U$ and $w \in \widetilde{P}_u^+(A) \setminus M$.

Therefore, $x \in V \subset \widetilde{P}_u^+(A) \setminus M$ and $\widetilde{P}_u^+(A) \setminus M$ is open.

b) Let $x \in \widetilde{P}_u^+(A)$. Then there is $T > 0$ such that

$$\widetilde{\pi}(x, [T, +\infty)) \subset \text{int}\widetilde{P}_u^+(A).$$

We need to prove that $\widetilde{\pi}(x, (0, T)) \subset \widetilde{P}_u^+(A)$. Indeed, let $s \in (0, T)$ and U be an arbitrary neighborhood of A. Since $\widetilde{\pi}(x, T) \in \text{int}\widetilde{P}_u^+(A)$, there are a neighborhood W of $\widetilde{\pi}(x, T)$ and $r > 0$ such that $\widetilde{\pi}(W, [r, +\infty)) \subset U$. We have two cases to consider: when $T \neq \sum_{j=0}^{k} \phi(x_j^+)$ for all $k = 0, 1, 2, \ldots$, and when $T = \sum_{j=0}^{k} \phi(x_j^+)$ for some $k = 0, 1, 2, \ldots$.

Case 1: Suppose that $T \neq \sum_{j=0}^{k} \phi(x_j^+)$ for all $k = 0, 1, 2, \ldots$. Since $\widetilde{\pi}(x, s) \notin M$ because $s > 0$ and $I(M) \cap M = \emptyset$, it follows that there is an open set $\mathcal{O}$ in X such that $\widetilde{\pi}(x, s) \in \mathcal{O}$ and

$$\widetilde{\pi}(\mathcal{O}, T - s) \subset W.$$

Hence, $\widetilde{\pi}(\mathcal{O}, [T + r - s, +\infty)) \subset U$ and $\widetilde{\pi}(x, s) \in \widetilde{P}_u^+(A)$.

Case 2: Suppose that $T = \sum_{j=0}^{k} \phi(x_j^+)$ for some $k \geq 0$. Then there is a $\beta > 0$, $\beta < \phi(x_{k+1}^+)$, such that $\widetilde{\pi}(x, T + \beta) \in W \setminus M$. Thus there is an open set W' such that $\widetilde{\pi}(x, T + \beta) \in W' \subset W$. Then there is an open set $\mathcal{U}$ which contains $\widetilde{\pi}(x, s)$ such that

$$\widetilde{\pi}(\mathcal{U}, T + \beta - s) \subset W' \subset W.$$

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Hence, $\widetilde{\pi}(\mathcal{U}, [T+\beta+r-s, +\infty)) \subset U$ and $\widetilde{\pi}(x,s) \in \widetilde{P}_u^+(A)$.

In conclusion, $\widetilde{\pi}^+(x) \subset \widetilde{P}_u^+(A)$ and the result is proved. ▮

If A is a compact subset of X, we can give a characterization for the region of uniform $\widetilde{\pi}-$attraction $\widetilde{P}_u^+(A)$. See the next result.

Proposition 3.2. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and A be a compact subset of X. Suppose X is locally compact. Then*

$$\widetilde{P}_u^+(A) = \{x \in X : \ \widetilde{J}^+(x) \neq \emptyset \ \text{ and } \ \widetilde{J}^+(x) \subset A\}.$$

Proof. First, let us show the inclusion $\widetilde{P}_u^+(A) \subset \{x \in X : \ \widetilde{J}^+(x) \neq \emptyset \ \text{ and } \ \widetilde{J}^+(x) \subset A\}$. In fact, let $y \in \widetilde{P}_u^+(A)$ and U be a neighborhood of A such that $\overline{U}$ is compact. Then there are a neighborhood V of y and $T > 0$ such that

$$\widetilde{\pi}(V, [T, +\infty)) \subset \overline{U}. \tag{3.1}$$

Let $\{t_n\}_{n \geq 1} \subset \mathbb{R}_+$ be a sequence such that $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$. By (3.1), there is $n_0 \in \mathbb{N}$ such that

$$\widetilde{\pi}(y, t_n) \in \overline{U}$$

for all $n \geq n_0$. By the compactness of $\overline{U}$, we may assume without loss of generality, that there is $a \in \overline{U}$ such that

$$\widetilde{\pi}(y, t_n) \overset{n\to+\infty}{\longrightarrow} a.$$

Thus $a \in \widetilde{L}^+(y)$ and it shows that $\widetilde{L}^+(y) \neq \emptyset$. Since $\widetilde{L}^+(y) \subset \widetilde{J}^+(y)$, it follows that $\widetilde{J}^+(y) \neq \emptyset$. Now, we note that

$$\widetilde{J}^+(y) \subset \widetilde{\pi}(V, [T, +\infty)) \subset \overline{U}.$$

Since U can be taken arbitrary, we have

$$\widetilde{J}^+(y) \subset \cap\{\overline{U} : U \text{ is a neighborhood of } A\} = \overline{A} = A.$$

Hence $y \in \{x \in X : \ \widetilde{J}^+(x) \neq \emptyset \ \text{ and } \ \widetilde{J}^+(x) \subset A\}$.

Now, let us show the other inclusion. Let $x \in X$ such that $\widetilde{J}^+(x) \neq \emptyset$ and $\widetilde{J}^+(x) \subset A$. Let U be an arbitrary neighborhood of A. Since

$$\widetilde{J}^+(x) = \bigcap_{\epsilon > 0} \bigcap_{t \geq 0} \overline{\bigcup \{\widetilde{\pi}(B(x, \epsilon), \tau) : \ \tau \geq t\}},$$

there are $\varepsilon_0 > 0$ and $t_0 > 0$ such that

$$\overline{\bigcup \{\widetilde{\pi}(B(x, \epsilon_0), \tau) : \ \tau \geq t_0\}} \subset U.$$

Therefore, $\widetilde{\pi}(B(x,\epsilon_0),[t_0,+\infty)) \subset U$. Then $x \in \widetilde{P}_u^+(A)$ and the proof is complete. ▮

The next result shows some properties of the sets $\widetilde{J}^+(x)$ and $\widetilde{D}^+(x)$, $x \in X$. The reader may consult [8] for a proof.

THEOREM 3.2. *Let $(X,\pi;M,I)$ be an impulsive semidynamical system. Suppose X is locally compact. The following properties hold:*

i)$\overline{\widetilde{\pi}^+(x)} \subset \widetilde{D}^+(x)$ *for all* $x \in X$*;*
ii)$\widetilde{D}^+(x) = \overline{\widetilde{\pi}^+(x)} \cup \widetilde{J}^+(x)$ *for all* $x \in X \setminus M$*;*
iii)$\overline{\widetilde{\pi}^+(I(x))} \subset \widetilde{D}^+(x)$ *for all* $x \in M$*;*
iv)$\widetilde{D}^+(x) = \overline{\widetilde{\pi}^+(x)} \cup \overline{\widetilde{\pi}^+(I(x))} \cup \widetilde{J}^+(x)$ *for all* $x \in M$*.*

The next result establishes conditions for a compact uniform $\widetilde{\pi}$-attractor set to be asymptotically $\widetilde{\pi}$-stable.

THEOREM 3.3. *Let $(X,\pi;M,I)$ be an impulsive semidynamical system and A be a compact subset of X. Suppose X is locally compact. If A is positively $\widetilde{\pi}$-invariant, $I-$invariant and uniform $\widetilde{\pi}$-attractor then A is $\widetilde{\pi}$-stable. And, consequently, A is asymptotically $\widetilde{\pi}$-stable.*

Proof. By Ura's Theorem we need to show that $\widetilde{D}^+(A) = A$, see Theorem 4.1 in [11]. Since $A \subset \widetilde{D}^+(A)$, we need to prove the other inclusion. Let $x \in \widetilde{D}^+(A)$. Then there is $a \in A$ such that $x \in \widetilde{D}^+(a)$. Since A is positively $\widetilde{\pi}$-invariant and closed, it follows that

$$\overline{\widetilde{\pi}^+(a)} \subset A. \tag{3.2}$$

Since A is a uniform $\widetilde{\pi}$-attractor we have $A \subset \widetilde{P}_u^+(A)$. Thus, by Proposition 3.2, we get

$$\widetilde{J}^+(a) \subset A. \tag{3.3}$$

We have two cases to consider: when $a \in M$ and when $a \notin M$.

Case 1: $a \notin M$.

It follows from item *ii*) of Theorem 3.2, (3.2) and (3.3) that

$$\widetilde{D}^+(a) = \overline{\widetilde{\pi}^+(a)} \cup \widetilde{J}^+(a) \subset A.$$

Case 2: $a \in M$.

Since A is I-invariant, closed and $I(M) \cap M = \emptyset$ we have $I(a) \in A \setminus M$ and then $\overline{\widetilde{\pi}^+(I(a))} \subset A$. Thus, by this last inclusion, Theorem 3.2 item *iv*), (3.2) and (3.3), we obtain

$$\widetilde{D}^+(a) = \overline{\widetilde{\pi}^+(a)} \cup \overline{\widetilde{\pi}^+(I(a))} \cup \widetilde{J}^+(a) \subset A.$$

In both cases, we show that $\widetilde{D}^+(a) \subset A$. Hence, $x \in A$. Therefore $\widetilde{D}^+(A) = A$ and A is $\widetilde{\pi}$-stable. ▮

Next, we present an auxiliary result. The proof can be found in [8], Lemma 3.13.

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Lemma 3.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Let $x \notin M$ and $y \in \widetilde{L}^+(x)$. Then $\widetilde{J}^+(x) \subset \widetilde{J}^+(y)$.*

Theorem 3.4. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and A be a compact subset of X such that $A \cap M = \emptyset$. Suppose X is locally compact. If A is asymptotically $\widetilde{\pi}$-stable then A is a uniform $\widetilde{\pi}$-attractor.*

Proof. Since A is asymptotically $\widetilde{\pi}$-stable we have A is a $\widetilde{\pi}$-attractor, see Theorem 3.1 in [5]. Thus $\widetilde{P}^+(A)$ is a neighborhood of A, that is, there exists an open set V such that $A \subset V \subset \widetilde{P}^+(A)$ and $V \cap M = \emptyset$.

We claim that $\widetilde{P}^+(A) \setminus M \subset \widetilde{P}^+_u(A)$. Indeed, let $x \in P^+(A) \setminus M$ and let U be a neighborhood of A such that $\overline{U}$ is compact. Then there is $\tau \geq 0$ such that

$$\widetilde{\pi}(x, [\tau, +\infty)) \subset \overline{U}.$$

Hence $\widetilde{L}^+(x) \neq \emptyset$ (because $\overline{U}$ is compact) and consequenlty $\widetilde{J}^+(x) \neq \emptyset$.

Let $a \in \widetilde{L}^+(x)$. Since $x \notin M$, it follows from Lemma 3.1 and from the stability of A the following inclusions

$$\widetilde{J}^+(x) \subset \widetilde{J}^+(a) \subset \widetilde{D}^+(a) \subset A.$$

Then, by Proposition 3.2, we have $x \in \widetilde{P}^+_u(A)$ and then $\widetilde{P}^+(A) \setminus M \subset \widetilde{P}^+_u(A)$.

Thus, $A \subset V \subset \widetilde{P}^+(A) \setminus M \subset \widetilde{P}^+_u(A)$. Therefore A is uniform $\widetilde{\pi}-$attractor. ∎

By Theorem 3.14 in [7] and Theorem 3.4 above, we have the following result.

Theorem 3.5. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system, X be locally compact and $A \subset X$ be a compact set such that $A \cap M = \emptyset$. If there exists a functional $\psi : X \to \mathbb{R}_+$ with the following properties:*

a)ψ is continuous in $X \setminus (M \setminus A)$;
b)for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \leq \varepsilon$ wherever $\rho(x, A) \leq \delta$;
c)for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \geq \delta$ whenever $\rho(x, A) \geq \varepsilon$ and $x \notin M$;
d)$\psi(\pi(x, t)) \leq \psi(x)$ if $x \in X \setminus M$ and $0 \leq t \leq \phi(x)$, and $\psi(I(x)) \leq \psi(x)$ if $x \in M$;
e)there is a $\delta > 0$ such that if $x \in B(A, \delta) \setminus A$, then $\psi(\widetilde{\pi}(x, t)) \to 0$ as $t \to +\infty$.

Then A is uniform $\widetilde{\pi}-$attractor.

Next, we analyse the equilibrium state of an autonomous system subject to variable impulses.

Example 3.1. Consider the semidynamical system $(\mathbb{R}^n, \pi)$ given by the system

$$x' = f(x),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1 and $f(0) = 0$. Suppose there are constants $c_1, \ldots, c_n \in \mathbb{R}_+ \setminus \{0\}$ such that $\sum_{i=1}^n c_i x_i f_i(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Now, consider the associated impulsive semidynamical system $(\mathbb{R}^n, \pi; M, I)$ where $M \subset \mathbb{R}^n$ is an impulsive set and the impulse function I satisfies the condition $\|I(x)\| \leq \sqrt{\alpha}\|x\|$ for all $x \in M$, where $\alpha = \dfrac{\min\limits_{1\leq i\leq n} c_i}{\max\limits_{1\leq i\leq n} c_i}$. We assume that $0 \notin M$.

Let $A = \{0\}$. Let us show that A is uniform $\widetilde{\pi}-$attractor. In fact, define the function $\psi : \mathbb{R}^n \to \mathbb{R}_+$ by

$$\psi(x) = \psi(x_1, \ldots, x_n) = \sum_{i=1}^n c_i x_i^2.$$

Since $\left(\min\limits_{1\leq i\leq n} c_i\right) \|x\|^2 \leq \psi(x) \leq \left(\max\limits_{1\leq i\leq n} c_i\right) \|x\|^2$, $x \in \mathbb{R}^n$, it is easy to verify that the conditions $a)$, $b)$ and $c)$ of Theorem 3.5 hold. Let us show that items $d)$ and $e)$ hold.

First, we prove that item $d)$ of Theorem 3.5 holds. In fact, let $x_0 \in \mathbb{R}^n \setminus M$ and $x(t) = x(t; 0, x_0)$ be the solution of system

$$\begin{cases} x' = f(x) \\ x(0) = x_0. \end{cases} \tag{4}$$

Now, consider the semidynamical system $\pi(x_0, t) = x(t)$. Then

$$\psi'(\pi(x_0, t)) = 2\sum_{i=1}^n c_i x_i(t) f_i(x(t)) \leq 0$$

for all $t \geq 0$ (if $x_0 \neq 0$ the inequality is strict). By integration from 0 to t we get $\psi(\pi(x_0, t)) \leq \psi(x_0)$ for all $t \geq 0$. By the property of I we have

$$\psi(I(x)) \leq \left(\max_{1\leq i\leq n} c_i\right) \|I(x)\|^2 \leq \left(\max_{1\leq i\leq n} c_i\right) \alpha\|x\|^2 = \left(\min_{1\leq i\leq n} c_i\right) \|x\|^2 \leq \psi(x)$$

for all $x \in M$.

Now, let us show that item $e)$ of Theorem 3.5 holds. By Theorem 3.12 in [7], the set A is uniformly $\widetilde{\pi}$-stable (because ψ satisfies the conditions $a), b), c)$ and $d)$ of Theorem 3.5). Then, given $\epsilon > 0$ such that $\overline{B(A, \epsilon)} \cap M = \emptyset$ there is $\delta > 0$ such that

$$\widetilde{\pi}(B(A, \delta), [0, +\infty)) \subset B\left(A, \frac{\epsilon}{2}\right),$$

where $\widetilde{\pi}$ is the impulsive semidynamical system associated to the continuous semidynamical system π of system $x' = f(x)$. Since $\overline{B(A, \epsilon)} \cap M = \emptyset$, we have

$$\pi(B(A, \delta), [0, +\infty)) = \widetilde{\pi}(B(A, \delta), [0, +\infty)) \subset B\left(A, \frac{\epsilon}{2}\right). \tag{5}$$

Let $x_0 \in B(A,\delta) \setminus A$. By equation (5), we have $\widetilde{\pi}(x_0,t) = \pi(x_0,t)$ for all $t \geq 0$. Since $\psi'(\pi(x_0,t)) < 0$ for all $t \geq 0$, it follows that the limit $\lim\limits_{t\to+\infty} \psi(\pi(x_0,t))$ exists. Suppose by contradiction that

$$\lim_{t\to+\infty} \psi(\pi(x_0,t)) = \ell > 0. \tag{6}$$

Now, we choose a sequence $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$. Then

$$\psi(\pi(x_0,t_n)) \overset{n\to+\infty}{\longrightarrow} \ell.$$

Since $\{\pi(x_0,t_n)\}_{n\geq 1} \subset \overline{B(A,\frac{\epsilon}{2})}$, which is compact, we may assume without loss of generality that

$$\pi(x_0,t_n) \overset{n\to+\infty}{\longrightarrow} y.$$

Then $\psi(\pi(x_0,t_n)) \overset{n\to+\infty}{\longrightarrow} \psi(y)$ and we conclude that $y \in \psi^{-1}(\ell) \cap B(A,\epsilon)$. Also, since

$$\psi(\pi(x_0,t_n+t)) \overset{n\to+\infty}{\longrightarrow} \psi(\pi(y,t))$$

for each $t \geq 0$, we have $\pi(y,[0,+\infty)) \subset \psi^{-1}(\ell)$. Then $\widetilde{\pi}(y,[0,\phi(y))) \subset \psi^{-1}(\ell)$ but this contradicts the fact that $\psi(\widetilde{\pi}(y,t)) < \psi(y)$ for all $t > 0$ (because $\psi'(\pi(y,t)) < 0$ for all $t \geq 0$). Therefore

$$\psi(\widetilde{\pi}(x_0,t)) = \psi(\pi(x_0,t)) \overset{t\to+\infty}{\longrightarrow} 0.$$

Therefore, by Theorem 3.5, the equilibrium is a uniform $\widetilde{\pi}-$attractor. By Theorem 3.14 in [7] the equilibrium is asymptotically $\widetilde{\pi}-$stable.

EXAMPLE 3.2. Consider the impulsive differential system

$$\begin{cases} x' = -\alpha y + yz - x^3 \\ y' = x - \beta xz - \gamma y^3 \\ z' = \theta xy - z^3 \\ I: M \to \mathbb{R}^3, \end{cases}$$

where $\alpha,\beta,\gamma,\theta > 0$, $\beta\alpha > 1$ and the impulsive set M and the impulse function I satisfy hypotheses $(H1)$ and $(H2)$ of Section 3. Let $\lambda_1 = \min\left\{1, \alpha, \dfrac{(\alpha\beta-1)}{\theta}\right\}$, $\lambda_2 = \max\left\{1, \alpha, \dfrac{(\alpha\beta-1)}{\theta}\right\}$ and $\lambda = \dfrac{\lambda_1}{\lambda_2}$. Suppose $\|I(w)\| \leq \sqrt{\lambda}\|w\|$ for all $w \in M$ and $(0,0,0) \notin M$. Then the equilibrium $(0,0,0)$ is uniform $\widetilde{\pi}-$attractor and asymptotically $\widetilde{\pi}-$stable. In fact, since

$$x(-\alpha y + yz - x^3) + \alpha y(x - \beta xz - \gamma y^3) + \frac{(\alpha\beta-1)}{\theta} z(\theta xy - z^3) \leq 0,$$

for all $(x, y, z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, the result follows by Example 3.1.

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