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Sufficient conditions on diffusivity for existence and nonexistence of stable equilibria with nonlinear flux on the boundary.

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A reaction-diffusion equation with variable diffusivity and nonlinear flux boundary condition is considered. The goal is to give sufficient conditions on the diffusivity function for nonexistence and also for existence of nonconstant stable stationary solutions. Applications are given for the main result of nonexistence. May, 2012 ICMC-USP

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1. INTRODUCTION

The subject of this work is the following nonlinear boundary value evolution problem

$$\begin{cases} u_t = \operatorname{div}(a(x)\nabla u) + f(u), & (t,x) \in \mathbb{R}^+ \times \Omega \\ a(x)\partial_\nu u = g(u), & (t,x) \in \mathbb{R}^+ \times \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$. For $N \geq 2$ it is a smooth bounded domain, ν the exterior unit normal to $\partial\Omega$, g and f are bistable type nonlinearity and $a \in C^{1,\theta}(\overline{\Omega}, \mathbb{R}^+)$ is the diffusivity function.

Typically (1.1) models the time evolution of the concentration of a diffusing substance or heat in a medium whose diffusivity function is given by a with the flux on the boundary being proportional to a prescribed function of the concentration.

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Roughly speaking, once f and g are fixed, non-constant stable stationary solutions to (1.1) (herein occasionally referred to as patterns, for short) arise from specific properties of the geometry of the domain and/or of the diffusivity function a.

This work should be seen as an attempt to understand the role played by a on existence and nonexistence of patterns to (1.1).

By stationary solutions to (1.1) we mean $C^2(\Omega) \cap C^0(\overline{\Omega})$ solutions to the nonlinear boundary value problem

$$\begin{cases} \operatorname{div}(a(x)\nabla u) + f(u) = 0, & x \in \Omega \\ a(x)\partial_\nu u = g(u), & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Let us define the set of bi-stable functions $\mathcal{B}$ as the class of C^1 functions $h : \mathbb{R} \to \mathbb{R}$ such that

- $\exists\, \alpha, \beta \in \mathbb{R},\ \ \alpha < 0 < \beta : \ h(\alpha) = h(\beta) = h(0) = 0,$
- $h(s) \neq 0$ in $(\alpha, 0) \cup (0, \beta)$,
- $h'(\alpha) < 0,\ h'(\beta) < 0,\ h'(0) > 0.$

Throughout this work it is assumed that $f, g \in \mathcal{B}$ but eventually we need either $f \equiv 0$ or $g \equiv 0$ and this will be explicitly mentioned wherever the case.

Let us briefly state our main results and applications. Suppose that $\exists\, \bar{a} \in C^{1,\theta}(\overline{\Omega})$ such that the only stable stationary solutions to

$$\begin{cases} u_t = \operatorname{div}(\bar{a}(x)\nabla u) + f(u), & (t,x) \in \mathbb{R}^+ \times \Omega \\ \bar{a}(x)\partial_\nu u = g(u), & (t,x) \in \mathbb{R}^+ \times \partial\Omega. \end{cases} \tag{1.3}$$

are $\bar{u} = \alpha$ and $\bar{u} = \beta$.

We prove that if $||a - \bar{a}||_{C^{1,\theta}(\overline{\Omega})}$ is small enough then the only stable stationary solutions to (1.1) are $\bar{u} = \alpha$ and $\bar{u} = \beta$ as well. As applications we manage to conclude that

1. If $\overline{a}(x) = \mathrm{c}$ (c a real positive constant) is large enough and $a \in C^{1,\theta}(\overline{\Omega})$ is a function satisfying $||a - \mathrm{c}||_{C^{1,\theta}(\overline{\Omega})}$ sufficiently small then the only stable stationary solutions to (1.1) are $\bar{u} = \alpha$ and $\bar{u} = \beta$.

2. Suppose that Ω is convex, $f \equiv 0$ and $\overline{a}(x) = \mathrm{c}$ (c a positive constant). If $||a - \mathrm{c}||_{C^{1,\theta}(\overline{\Omega})}$ is sufficiently small then the only stable stationary solutions to (1.1) are $\bar{u} = \alpha$ and $\bar{u} = \beta$.

Note that here, as opposed to (1), it was not required that c be large. This condition is not required in the next application neither.

3. Suppose that Ω is a smooth convex domain, $g \equiv 0$ and $\overline{a}(x) = \mathrm{c}$ (c a positive constant). If $||a - \mathrm{c}||_{C^{1,\theta}(\overline{\Omega})}$ is sufficiently small then the same conclusion of (1) holds.

4. Another interesting application is when the domain is a ball, $\Omega = B_R(0)$ say, $g \equiv 0$, $\overline{a}(x) = \overline{a}(r)$ where $r = |x|$, i.e., $\overline{a}$ is radially symmetric and satisfies $r^2(\sqrt{\overline{a}})'' + (N - 1)r(\sqrt{\overline{a}})' \leq (N-1)a$ for $0 < r < R$. Under these conditions if a (not necessarily radially symmetric) is any smooth function satisfying $||a - \overline{a}||_{C^{1,\theta}(\Omega)}$ small enough then the only stable stationary solutions to (1.1) are the constant ones, i.e., α and β. The same result holds when $N = 1$, i.e., Ω is an interval, under the condition $(\sqrt{\overline{a}})'' < 0$.

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We also present a specific function a so that (1.1) has a pattern for $f \not\equiv 0$ and $g \not\equiv 0$. It turns out that a is uniformly small in a thin region which disconnect Ω in two sets on each of which a is sufficiently large. As expected, from the above results, a is not near any constant function in the topology of $C^{1,\theta}(\overline{\Omega})$.

The conclusion is that in order to create patterns for (1.1) it suffices to have the diffusibility function $a(\cdot)$ sufficiently small around some narrow tubular neighborhood of a compact hyper-surface S (with or without boundary as long as in the former case it holds $\partial S \subset \partial\Omega$) and large outside so that $a(\cdot)$ will satisfy $||a(\cdot) - \bar{a}||_{C^{1,\theta}(\overline{\Omega})}$ large enough. For the sake of illustration let us take S without boundary, $\partial S \subset \partial\Omega$ and splitting Ω into two disjoint regions Ω_α and Ω_β. In this case the underlying physical mechanism allowing for the existence of a stable patterns whose values are close to the stable equilibrium α, say, on Ω_α and close to stable equilibrium β on Ω_β, is that small diffusibility around S works as a barrier for the diffusing substance (it could be heat) preventing an initial condition $u(0,x) = u_0(x)$ starting close to those values (in the H^1 or C^0 topology) from spreading out homogeneously in space and eventually settling down, as time evolves, in a constant concentration (temperature, respectively) over the domain.

The problem of characterizing the class of diffusivity functions for which (1.1) has no patterns has been considered by some authors for one-dimensional domains and $g \equiv 0$. For instance, this condition was found to be $a'' < 0$ in [15] and $(\sqrt{a})'' < 0$ in [14]. Still for in interval and $g \equiv 0$ the authors in [11] and [12] showed existence of pattern for a class of diffusivity function of step type.

These works were generalized in [5] for N-dimensional domains by roughly requiring a to assume a local minimum along a hyper-surface without boundary. Also in [6] for $N = 2$ existence of stable patterns to (1.1) when $g \equiv 0$ was established using $\Gamma-$convergence theory; given a simple closed planar curve $\gamma \subset \Omega$ the hypothesis on a associates the value of its first and second directional derivatives along the normal vector to γ with the curvature of γ.

Regarding nonexistence of patterns for (1.1) the main tools utilized are the Implicit Function Theorem in a special setting and a careful regularity analysis. As for existence the approach consists of finding an invariant set, say Λ, for the positive flow defined by (1.1) and then showing that it contains the solution we are looking for as long as $\Lambda \neq \emptyset$. This technique seems to have been introduced in [13] and utilized in a different setting in [5], for instance, as well as in many other works.

2. NONEXISTENCE OF PATTERNS

Before proving Theorem 2.1, which is the main result of this section, we need some technical lemmas. Throughout this section we take $\theta = 1/(N+1)$, $\Omega \subset \mathbb{R}^N$ a $C^{2,\theta}$ bounded domain and recall an useful result.

LEMMA 2.1. *([10]) Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and $u \in W^{2,q}(\Omega)$ a solution to*

$$\begin{cases} \Delta u = \varphi(x), & x \in \Omega \\ \partial_\nu u = \psi(x), & x \in \partial\Omega \end{cases}$$

with $\varphi \in L^p(\Omega)$ *and* $\psi \in W^{1-1/p}(\partial\Omega)$, $1 < p < \infty$. *Moreover assume that* $p \leq \frac{Nq}{N-q}$ *with* $N > q$. *Then* $u \in W^{2,p}(\Omega)$.

For a proof the reader is referred to [10], p.114, for instance. Next results, regarding regularity of solutions to (1.2), will also play a important role in the sequel.

LEMMA 2.2. *Assume* $g \in C^2(\mathbb{R})$, $f \in C^1(\mathbb{R})$ *and let* $a \in C^{1,\theta}(\overline{\Omega})$ *be a positive function. If* $u \in H^1(\Omega) \cap L^\infty(\Omega)$ *is a solution to (1.2) then* $u \in W^{2,N+1}(\Omega)$. *Moreover* $u \in C^{2,\theta}(\overline{\Omega})$.

Proof: We start by proving that if $N > 2k$, for some $k \in \mathbb{N}$, then $u \in W^{2,p_k}\Omega)$ where $p_k = \frac{2N}{N-2k}$. The proof is by induction on k.

Let $k = 0$; thus $p_0 = 2$ and by hypothesis on u and a we have $\psi(x) = (\dfrac{g(u)}{a(x)} + u) \in H^1(\Omega)$, $\varphi(x) = -f(u) \in L^2(\Omega)$ and we see that u is also a solution to

$$\begin{cases} \operatorname{div}(a(x)\nabla u) = \varphi(x), & x \in \Omega \\ \partial_\nu u + u = \psi(x), & x \in \partial\Omega. \end{cases} \tag{2.1}$$

But if Ω is a $C^{2,\theta}$ domain, $\psi \in C^{1,\theta}(\partial\Omega)$ and $\varphi \in C^\theta(\overline{\Omega})$, then (2.1) has only one solution in $C^{2,\theta}(\overline{\Omega})$ (cf. [9], Chapter 6, for instance). Hence from regularity of a, f and g as well as density of the inclusions $C^{1,\theta}(\overline{\Omega}) \subset H^1(\Omega) \subset L^2(\Omega)$, one easily proves that $u \in H^2(\Omega)$.

Supposing the result is true for $N > 2k$ let us take $N > 2(k+1)$. By induction hypothesis $u \in W^{2,p_k}(\Omega)$ with $p_k = \frac{2N}{N-2k}$. But $W^{2,p_k}(\Omega) \subset W^{1,p_{k+1}}(\Omega)$ since $2 - \frac{N}{p_k} = 1 - \frac{N}{p_{k+1}}$ for $N > 2(k+1)$.

By hypothesis on f, g, a we have $f(u)/a \in L^{p_{k+1}}(\Omega)$, $a^{-1}\nabla a \cdot \nabla u \in L^{p_{k+1}}(\Omega)$ and $\dfrac{g(u)}{a} \in W^{1-\frac{1}{p_{k+1}},p_{k+1}}(\partial\Omega)$. Moreover $p_{k+1} = \dfrac{Np_k}{N - p_k}$ and $p_k < N$. Therefore Lemma 2.1 yields $u \in W^{2,p_{k+1}}(\Omega)$.

If N is odd then there is $k \geq 0$ such that $N = 2k+1 > 2k$ and as such, from the argument above, $u \in W^{2,p_k}(\Omega)$ with $p_k = 2N > N$. In case N is even $\exists\, k \geq 0$, $N = 2k+2 > 2k$. Again the same argument implies $u \in W^{2,p_k}(\Omega)$ with $p_k = N$. On the account that $W^{2,N}(\Omega) \subset W^{2,(N-1/10)}(\Omega) \cap W^{1,N+1}(\Omega)$, Lemma 2.1 yields once more $u \in W^{2,N+1}(\Omega)$.

Then in any case it follows that $u \in W^{2,N+1}(\Omega)$. But Sobolev continuous imbedding assures us that $W^{2,N+1}(\Omega) \subset C^{1,\theta}(\overline{\Omega})$ for $\theta = \frac{1}{N+1}$. Then u is the solution to (2.1) with $\varphi \in C^\theta(\overline{\Omega})$ and $\psi \in C^{1,\theta}(\partial\Omega)$. Given that Ω is $C^{2,\theta}$ domain we conclude $u \in C^{2,\theta}(\overline{\Omega})$. ■

Before establishing our main results in this section we present an application of the Implicit Function Theorem in a specific setting that suits our purposes; it is a generalization of [1] where the case $a \equiv$ constant and $g \equiv 0$ was treated.

LEMMA 2.3. *Suppose* $g \in C^2(\mathbb{R})$ *and* $u_0 = \alpha$ *or* $u_0 = \beta$. *Then for any positive function* $\bar{a} \in C^{1,\theta}(\overline{\Omega})$ *there are neighborhoods* $\mathcal{V}_{\bar{a}}$ *of* $\bar{a}$ *in* $C^{1,\theta}(\overline{\Omega})$ *and* $\mathcal{U}_{u_0}$ *of* u_0 *in* $W^{2,p}(\Omega)$ $(p > N)$ *such that if* $a \in \mathcal{V}_{\bar{a}}$ *then* u_0 *is the only solution to (1.2) in* $\mathcal{U}_{u_0}$. *Moreover if either* $f \equiv 0$ *and* $g \neq 0$ *or* $g \equiv 0$ *and* $f \neq 0$ *the result is still valid.*

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Proof: First of all for simplicity in notation $\mathcal{H}^N$ stands for the $N-$ dimensional Hausdorff measure which in our case, according to the dimension, corresponds to the usual area or volume measure.

Let us define

$$E_p \stackrel{\text{def}}{=} \left\{(v,w) \in L^p(\Omega) \times W^{1-\frac{1}{p},p}(\partial\Omega);\quad \int_\Omega v = \int_{\partial\Omega} w\right\}$$

and the operator $F : C^{1,\theta}(\overline{\Omega}) \times W^{2,p}(\Omega) \to E^p \times \mathbb{R}$ by

$$F(a,u) = \begin{pmatrix} \operatorname{div}(a(x)\nabla u) + f(u) - \dfrac{1}{\mathcal{H}^N(\Omega)}\left[\displaystyle\int_{\partial\Omega} g(u) + \int_\Omega f(u)\right], \\ a(x)\partial_\nu u - g(u), \\ \dfrac{1}{\mathcal{H}^N(\Omega)}\left(\int_{\partial\Omega} g(u) + \int_\Omega f(u)\right) \end{pmatrix}. \tag{2.2}$$

Note that F is a C^1 operator by regularity of a, f, g and on the account that $p > N$. Moreover $F(a,u) = (0,0,0)$ if and only if u is a solution to (1.2).

In particular for any $\bar{a} \in C^{1,\theta}(\overline{\Omega})$ and any constant solution $u_0 \in \{\alpha, \beta\}$ to (1.2) we have $F(\bar{a}, u_0) = (0,0,0)$.

Claim: $D_uF(\bar{a}, u_0) : W^{2,p}(\Omega) \longmapsto E_p \times \mathbb{R}$ is an isomorphism for any positive $\bar{a} \in C^{1,\theta}(\overline{\Omega})$.

Note that this will be the case if for each $(v,w,t) \in E^p \times \mathbb{R}$ there is only one solution $\phi \in W^{2,p}(\Omega)$ to

$$\begin{cases} \operatorname{div}(\bar{a}\nabla\phi) + f'(u_0)\phi - \dfrac{1}{\mathcal{H}^N(\Omega)}\left[\int_{\partial\Omega} g'(u_0)\phi + \int_\Omega f'(u_0)\phi\right] = v,\ x \in \Omega \\ \bar{a}\partial_\nu\phi - g'(u_0)\phi = w,\ x \in \partial\Omega \\ \dfrac{1}{\mathcal{H}^N(\Omega)}\left(\int_{\partial\Omega} g'(u_0)\phi + \int_\Omega f'(u_0)\phi\right) = t \end{cases} \tag{2.3}$$

In order to prove that the application above is an isomorphism it suffices to show that

$$\begin{cases} \operatorname{div}(\bar{a}\nabla\varphi) + f'(u_0)\varphi = v,\ x \in \Omega \\ \bar{a}\partial_\nu\varphi - g'(u_0)\varphi = w + t\dfrac{g'(u_0)}{f'(u_0)},\ x \in \partial\Omega \end{cases} \tag{2.4}$$

has a unique solution φ. Indeed if this is the case then, keeping in mind that $(v,w) \in E_p$, the function $\phi = \varphi + \dfrac{t}{f'(u_0)}$ will be the only solution to (2.3).

In order to prove existence and uniqueness of solutions to (2.4) we start by defining the operator $T : W^{2,p}(\Omega) \to L^p(\Omega) \times W^{1-1/p,p}(\partial\Omega)$ by

$$T(\varphi) = (\operatorname{div}(\bar{a}\nabla\varphi) + f'(u_0)\varphi, \bar{a}\,\partial_\nu\varphi - g'(u_0)\varphi).$$

It is well known that T is a Fredholm operator with index zero. And for $u_0 = \alpha$ or $u_0 = \beta$ we have that $\ker T = \{0\}$ since $f'(u_0) < 0$ and $g'(u_0) < 0$. So T is an isomorphism and hence $D_uF(\bar{a}, u_0)$ is an isomorphism from $W^{2,p}(\Omega)$ to $E_p \times \mathbb{R}$.

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Finally we conclude from the Implicit Function Theorem (see [3] for instance) the existence of a neighborhood $\mathcal{U}_{u_0} \in W^{2,p}(\Omega)$ of u_0 and a neighborhood $\mathcal{V}_{\bar{a}} \in C^1(\overline{\Omega})$ of $\bar{a}$ such that if $a \in \mathcal{V}_{\bar{a}}$, $u \in \mathcal{U}_{u_0}$ and $F(a,u) = (0,0,0)$ then $u = u_0$, i.e., u_0 is the only solution to (1.2) in $\mathcal{U}_{u_0}$.

The cases $g = 0$ and $f \neq 0$ or $f = 0$ and $g \neq 0$ are similar and will be omitted. ■

Now we are ready to show

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a $C^{2,\theta}$ bounded domain, $a \in C^{1,\theta}(\overline{\Omega})$ with $\theta = 1/(N+1)$ and $g \in C^2(\mathbb{R})$.*

Let $\bar{a} \in C^{1,\theta}(\overline{\Omega})$ be a positive function and suppose that $u_0 = \alpha$ and $u_0 = \beta$ are the unique stable stationary solutions to (1.3). Then there is $\rho > 0$ such that whenever $||a - \bar{a}||_{C^{1,\theta}(\overline{\Omega})} < \rho$, any stable stationary solution u to (1.1) satisfying $\alpha \leq u \leq \beta$ in Ω must be constant, i.e., $u = \alpha$ or $u = \beta$.

Moreover if $f \equiv 0$ and $g \neq 0$ or $g \equiv 0$ and $f \neq 0$ the result still holds true.

Proof: Arguing by contradiction we obtain a sequence $\{a_j\}_{j=1}^{\infty}$ satisfying $a_j \to \bar{a}$ in $C^{1,\theta}(\overline{\Omega})$, as $j \to \infty$, and a sequence of corresponding nonconstant stable solutions $\{u_j\}$ to (1.1) satisfying $\alpha \leq u_j(x) \leq \beta$ and

$$\begin{cases} \operatorname{div}(a_j(x)\nabla u) + f(u) = 0, \ x \in \Omega \\ a_j(x)\partial_\nu u = g(u), \ x \in \partial\Omega \end{cases} \tag{2.5}$$

Lemma 2.2 yields $u_j \in C^{2,\theta}(\overline{\Omega})$. But for all $v \in H^1(\Omega)$ we have

$$\int_\Omega a_j(x)\nabla v \nabla u_j - v f(u_j) dx - \int_{\partial\Omega} v g(u_j) d\sigma = 0 \tag{2.6}$$

For j large enough there is $k > 0$ such that $a_j(x) \geq k, \ \forall x \in \overline{\Omega}$. Hence

$$k \int_\Omega |\nabla u_j|^2 dx \leq \int_{\partial\Omega} g(u_j) u_j \, d\sigma + \int_\Omega u_j f(u_j) dx$$

and given that the sequence $\{u_j\}$ is bounded in $L^\infty(\Omega)$ it is also bounded in $H^1(\Omega)$. Extracting a subsequence, still denoted by $\{u_j\}$, there is a function $\overline{u} \in H^1(\Omega)$ such that $u_j \rightharpoonup \bar{u}$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ as well as in $L^2(\partial\Omega)$. Given the uniform convergence of $\{a_j\}$ in $\overline{\Omega}$ we conclude from (2.6) that $\bar{u}$ is a weak solution to

$$\begin{cases} \operatorname{div}(\bar{a}\nabla\bar{u}) + f(\bar{u}) = 0, \ x \in \Omega \\ \bar{a}\partial_\nu \bar{u} = g(\bar{u}), \ x \in \partial\Omega. \end{cases} \tag{2.7}$$

where $\bar{u} \in C^{2,\theta}(\overline{\Omega})$ by Lemma 2.2. Moreover $\alpha \leq \bar{u} \leq \beta$ and $u_j \to \bar{u}$ in $W^{2,p}(\Omega)$. In fact, since u_j and $\bar{u}$ are in $C^2(\overline{\Omega})$ we can utilize the classical Amann estimative,

$$||u_j - \bar{u}||_{H^1(\Omega)} \leq C \left(||\Delta(u_j - \bar{u})||_{L^2(\Omega)} + || \, \partial_\nu(u_j - \bar{u})||_{L^2(\partial\Omega)} \right)$$

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$$\leq C\left(||\frac{1}{a_j}\nabla a_j\nabla u_j-\frac{1}{\bar a}\nabla\bar a\nabla\bar u||_{L^2(\Omega)}+||\frac{f(u_j)}{a_j}-\frac{f(\bar u)}{\bar a}||_{L^2(\Omega)}\right.$$
$$\left.+||\frac{g(u_j)}{a_j}-\frac{g(\bar u)}{\bar a}||_{L^2(\partial\Omega)}\right) \tag{2.8}$$

where C is a constant independent of j, to obtain strong convergence in $H^1(\Omega)$. Using the Agmon-Douglis-Nirenberg inequality (see [2], for instance) and previous convergence we conclude that $u_j\to\bar u$ in $H^2(\Omega)$. Now similarly to the proof of Lemma 2.2 we can prove that $u_j\to\bar u$ in $W^{2,p}(\Omega)$ for $p=N+1$.

Claim: $\bar u$ is a stable stationary solution to (1.3).

Indeed since $\lambda_1(a_j,u_j)$ is the first eigenvalue of the linearized problem

$$\begin{cases}\operatorname{div}(a_j(x)\nabla\phi)+f'(u_j)\phi=\lambda\phi,\ x\in\Omega\\ a_j(x)\partial_\nu\phi=g'(u_j)\phi,\ x\in\partial\Omega\end{cases} \tag{2.9}$$

then

$$\lambda_1(a_j,u_j)=$$
$$\sup_{\phi\in H^1(\Omega),\ \phi\neq 0}\left\{\frac{\int_\Omega -a_j|\nabla\phi|^2+\int_\Omega f'(u_j)\phi^2+\int_{\partial\Omega}g'(u_j)\phi^2}{\int_\Omega\phi^2}\right\}$$

and $\lambda_1(a_j,u_j)\leq 0$ on the account that u_j is stable. Since $u_j\to\bar u\in W^{2,p}(\Omega)$ and $a_j\to\bar a\in C^{1,\theta}(\overline{\Omega})$ we can pass to the limit to obtain

$$0\geq\lambda_1(\bar a,\bar u)=$$
$$\sup_{\phi\in H^1(\Omega),\ \phi\neq 0}\left\{\frac{\int_\Omega -\bar a|\nabla\phi|^2+\int_\Omega f'(\bar u)\phi^2+\int_{\partial\Omega}g'(\bar u)\phi^2}{\int_\Omega\phi^2}\right\}.$$

This implies that $\bar u$ is a stable stationary solution to (1.3), which is the evolutionary equation corresponding to (2.7). Indeed if $\lambda_1(\bar a,\bar u)<0$ this result is very well-known. If $\lambda_1(\bar a,\bar u)=0$ the result still holds (see [17], Theorem 6.2.1). Roughly speaking in this case 0 is a simple eigenvalue (having Ω smooth is crucial here) and therefore there is a local one-dimensional critical invariant manifold $W(\bar u)$ tangent to the principal eigenfunction such that if $\bar u$ is stable in $W(\bar u)$ then it also stable in $H^1(\Omega)$. As for the stability of $\bar u$ in $W(\bar u)$ it follows from the existence of a Lyapunov functional and the fact the $W(\bar u)$ is one-dimensional.

Summing up, $\bar u$ is a stable stationary solution to (1.3) but by hypothesis the only stable stationary solution to (1.3) are $\bar u=\alpha$ or $\bar u=\beta$.

Thus $u_j\to\bar u$ in $W^{2,p}(\Omega)$ for $p>N$ where $\bar u\equiv$ constant. But according to Lemma 2.3, if j large enough, this cannot happen given that from the contradiction hypothesis each u_j is a nonconstant function. ■

Aiming at future applications we now prove the following result.

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Theorem 2.2. *In addition to the hypotheses mentioned in the Introduction assume $g \in C^2(\mathbb{R})$ and that $f(s)g(s) > 0$ for $s \neq 0, \alpha, \beta$. Then for $\lambda > 0$ small enough any stable solution to*

$$\begin{cases} u_t = \Delta u + \lambda f(u) = 0, & t > 0, \quad x \in \Omega \\ \partial_\nu u = \lambda g(u), & x \in \partial\Omega. \end{cases} \tag{2.10}$$

with $\alpha \leq u \leq \beta$ satisfies $u = \alpha$ or $u = \beta$.

Proof: The proof is similar but simpler than those given in Lemma 2.3 and Theorem 2.1 and hence the details will be omitted. First we define a C^1 operator $T : \mathbb{R} \times W^{2,p}(\Omega) \to E^p \times \mathbb{R}$, with $p > N$, and a set of functions E_p as in Lemma 2.3,

$$T(\lambda, u) = \begin{pmatrix} \Delta u - \lambda f(u) + \dfrac{\lambda}{\mathcal{H}^N(\Omega)} \left[\int_{\partial\Omega} g(u) d\sigma + \int_\Omega f(u) dx \right], \\ \partial_\nu u - \lambda g(u), \\ \dfrac{1}{\mathcal{H}^N(\Omega)} \left[\int_{\partial\Omega} g(u) d\sigma + \int_\Omega f(u) dx \right] \end{pmatrix}$$

We see that for $\lambda \neq 0$, $T(\lambda, u) = (0, 0, 0)$ if and only if u is a solution to (2.10) and $T(0, u) = (0, 0, 0)$ if and only if $u = \alpha$ or $u = \beta$ or $u = 0$ since $f(s)g(s) > 0$ for $s \neq 0$.

It is easy to verify that for a constant function $u_0 \in \{\alpha, \beta, 0\}$, the operator $D_u T(0, u_0) : W^{2,p}(\Omega) \to E_p \times \mathbb{R}$, where

$$D_u T(0, u_0)\phi = \left(\Delta\phi, \ \partial_\nu \phi, \ \frac{1}{\mathcal{H}^N(\Omega)} \left[\int_{\partial\Omega} g'(u_0)\phi + \int_\Omega f'(u_0)\phi \right] \right),$$

is an isomorphism. Indeed the problem

$$\begin{cases} \Delta\phi = v \in L^p(\Omega) \\ \partial_\nu \phi = w \in W^{1-1/p,p}(\partial\Omega) \end{cases}$$

has a family of solutions $\{\phi_c = \varphi + c, c \in \mathbb{R}\}$ and then given $t \in \mathbb{R}$ there exists only one c such that

$$t = \frac{1}{\mathcal{H}^N(\Omega)} \left[\int_{\partial\Omega} g'(u_0)\phi_c + \int_\Omega f'(u_0)\phi_c \right].$$

Again as in Lemma 2.3, since $T(0, u_0) = (0, 0, 0)$, we conclude from the Implicit Function Theorem the existence of a neighborhood $\mathcal{U}_{u_0} \in W^{2,p}(\Omega)$ of u_0 and $\lambda_0 > 0$ such that if $|\lambda| < \lambda_0$, $u \in \mathcal{U}_{u_0}$ and $T(\lambda, u) = (0, 0, 0)$ then $u = u_0$, i.e., u_0 is the only solution to (2.10) in $\mathcal{U}_{u_0}$.

Now arguing by contradiction let us suppose that there is a sequence $\lambda_j \to 0$ and a corresponding sequence $\{u_j\}$ of nonconstant stable stationary solutions to (2.10) satisfying

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$\alpha \leq u_j \leq \beta$ and

$$\begin{cases} \Delta u_j = -\lambda_j f(u_j), \ \ t > 0, \quad x \in \Omega \\ \partial_\nu u_j = \lambda_j g(u_j), \quad x \in \partial\Omega. \end{cases} \tag{2.11}$$

First of all as in Theorem 2.1 we can show the existence of $\bar{u}$ and a subsequence of non-constant functions, still denoted by $\{u_j\}$, such that $u_j \rightharpoonup \bar{u} \in H^1(\Omega)$. Moreover once $\lambda_j \to 0$ we have $|\nabla u_j|_{L^2(\Omega)} \to 0$ and then the convergence is strong and $\bar{u}$ is constant. Now using Agmon-Douglis-Niremberger inequality we conclude that $u_j \to \bar{u}$ in $W^{2,p}(\Omega)$ for some $p > N$.

Since $\int_\Omega f(u_j)dx + \int_{\partial\Omega} g(u_j)d\sigma = 0$ it holds that

$$\mathcal{H}^N(\Omega)f(\bar{u}) + \mathcal{H}^{N-1}(\partial\Omega)g(\bar{u}) = 0,$$

and on the account that we have $f(v)g(v) > 0$ for $v \neq \alpha, 0, \beta$ then we must have $\bar{u} \in \{\alpha, 0, \beta\}$.

Given that $u_j \to \bar{u}$ in $W^{2,p}(\Omega)$ $(p > N)$ we conclude from the first part of the proof that for small λ_j the corresponding solution to (2.11) satisfies one of the following cases; $u_j = \alpha$, $u_j = \beta$ or $u_j = 0$. This is a contradiction since each u_j is a non-constant function.

Then for λ small enough any stationary stable solution to (2.10) must be $u = \alpha$, $u = \beta$ or $u = 0$. But if $u = 0$ the first eigenvalue $\mu(\lambda, 0)$ corresponding to the linearized problem

$$\begin{cases} \Delta\phi + \lambda f'(0)\phi = \lambda\phi, \ x \in \Omega \\ \partial_\nu \phi = \lambda g'(0)\phi, \ x \in \partial\Omega \end{cases} \tag{2.12}$$

satisfies

$$\mu(\lambda, 0) = \sup_{\phi \in H^1(\Omega), \ \phi \neq 0} \frac{\int_\Omega -|\nabla\phi|^2 + \int_\Omega \lambda f'(0)\phi^2 + \int_{\partial\Omega} \lambda g'(0)\phi^2}{\int_\Omega \phi^2} > 0$$

due to the fact that $f'(0), g'(0) > 0$. Hence we must have $u = \alpha$ or $u = \beta$. ■

The next result is direct consequence of the last two ones.

THEOREM 2.3. *Let $\Omega \in \mathbb{R}^N$ be a $C^{2,\theta}$ bounded domain, $a \in C^{1,\theta}(\overline{\Omega})$ with $\theta = 1/(N+1)$, $f, g \in \mathcal{B}$, $f(s)g(s) > 0$ for $s \neq \alpha, 0, \beta$ and $g \in C^2(\mathbb{R})$.*

Then given any real number $\bar{a}$ large enough there is $\rho > 0$ such that whenever $||a(\cdot) - \bar{a}||_{C^{1,\theta}(\overline{\Omega})} < \rho$, any stable stationary solution u to (1.1) satisfying $\alpha \leq u \leq \beta$ in Ω must be constant, i.e., $u = \alpha$ or $u = \beta$.

3. APPLICATION TO SOME SPECIFIC CASES

In this section we illustrate how the above version of the Implicit Function theorem can be used to draw conclusions on non-existence of non-constant stable stationary solutions to some specific cases of (1.1).

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COROLLARY 3.1. *In addition to the hypotheses of Theorem 2.1 suppose*

•Ω *is convex and*
•$f \equiv 0$.

Then given any real number $\bar{a} > 0$ *(not necessarily large) there is* $\rho > 0$ *such that whenever* $||a(\cdot) - \bar{a}||_{C^{1,\theta}(\overline{\Omega})} < \rho$, *any stable stationary solution* u *to (1.1) satisfying* $\alpha \leq u \leq \beta$ *in* Ω *must be constant, i.e.,* $u = \alpha$ *or* $u = \beta$.

Proof: Since Ω is convex we know (see [4], for instance) that if $\bar{u}$ is a stable solution to

$$\begin{cases} u_t = \bar{a}\triangle u, \ x \in \Omega \\ \partial_\nu u = \bar{a}^{-1} g(u), \ x \in \partial\Omega \end{cases} \tag{3.1}$$

then $\bar{u}$ must be a constant function. Hence, since $u \equiv 0$ is a unstable equilibrium, we conclude $\bar{u} = \alpha$ or $\bar{u} = \beta$ and Theorem 2.1 can be applied to complete the proof. ■

A similar nonexistence result can be obtained for $g \equiv 0$ as long as Ω is smooth and convex.

COROLLARY 3.2. *In addition to the hypotheses of Theorem 2.1 suppose*

•$\Omega \subset \mathbb{R}^N$ *is smooth and convex and*
•$g \equiv 0$.

Then given any real number $\bar{a} > 0$ *there is* $\rho > 0$ *such that whenever* $||a(\cdot) - \bar{a}||_{C^{1,\theta}(\overline{\Omega})} < \rho$, *any stable stationary solution* u *to (1.1) satisfying* $\alpha \leq u \leq \beta$ *in* Ω *must be constant, i.e.,* $u = \alpha$ *or* $u = \beta$.

Proof: If $\bar{u}$ is a stable stationary solution to

$$\begin{cases} u_t = \bar{a}\triangle u + f(u), \ x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, \ x \in \partial\Omega \end{cases} \tag{3.2}$$

since Ω is smooth and convex, we can resort to [16] or [13] to conclude that $\bar{u}$ must be constant. Hence, since $u \equiv 0$ is a unstable equilibrium, we conclude $\bar{u} = \alpha$ or $\bar{u} = \beta$. Now the result is a immediate consequence of Theorem 2.1. ■

COROLLARY 3.3. *Consider the problem*

$$\begin{cases} u_t = div(a(x)\nabla u) + f(u), & (t,x) \in \mathbb{R}^+ \times B_R(0) \\ \partial_\nu u = 0, & (t,x) \in \mathbb{R}^+ \times \partial B_R(0). \end{cases} \tag{3.3}$$

where $f \in C^1$ *and* $B_R(0)$ *stands for the* N*-dimensional ball of radius* R *and center at the origin. Let* $r = \|x\|$ *and suppose that* $\overline{a} \in C^2(0,R)$ *is a positive radially symmetric function satisfying*

$$r^2(\sqrt{\overline{a}})'' + (N-1)r(\sqrt{\overline{a}})' \leq (N-1)a$$

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for $0 < r < R$. If $a \in C^2(B_R(0))$ is a positive function (not necessarily radial symmetric) such that $||a - \overline{a}||_{C^{1,\theta}(B_R(0))}$ is small enough then any stable stationary solution to (3.3) is a constant function and equals either α or β.

The same conclusion holds for $N = 1$, i.e., when $\Omega = (0\,,1)$ say, as long as

$$(\sqrt{\overline{a}})'' < 0.$$

Proof Indeed under the hypothesis on $\overline{a}(r)$ it follows from [7], Lemma 2.1, and [8], Theorem 5.2, that any stable stationary solution to

$$\begin{cases} u_t = \operatorname{div}(\overline{a}(r)\nabla u) + f(u), & (t,x) \in \mathbb{R}^+ \times B_R(0) \\ \partial_\nu u = 0, & (t,x) \in \mathbb{R}^+ \times \partial B_R(0). \end{cases} \tag{3.4}$$

is constant. The result now follows from an application of Theorem 2.1.

As for the one-dimensional case it was proven in [14] that if $(\sqrt{\overline{a}})'' < 0$ (in [15] the more restrictive hypothesis $a'' < 0$ was found) then any stable stationary solution to

$$\begin{cases} u_t = (\overline{a}(x)u_x)_x + f(u), & (t,x) \in \mathbb{R}^+ \times (0,L) \\ u_x(t,0) = u_x(t,L) = 0, & t \in \mathbb{R}^+ \end{cases} \tag{3.5}$$

is constant. Again the proof can be established by an application of Theorem 2.1. ■

4. EXISTENCE OF PATTERNS

Our goal in this section is to give sufficient condition on the diffusivity function a for existence of patterns to (1.1). It will be clear that the diffusivity function a must be sufficiently far (in the $C^{1,\theta}(\overline{\Omega})$ topology) from any constant function. Let $f, g \in \mathcal{B}$ satisfy

(H) $0 \leq sg(s) \leq s^2$ for $\alpha \leq s \leq \beta$

and set $G(u) = \int_0^u g$ and $F(u) = \int_0^u f$. Assume without loss of generality that $G(\alpha) \leq G(\beta)$, $F(\alpha) \leq F(\beta)$. Also for $p > N$ define the twice continuously differentiable energy functional $E : W^{1,p}(\Omega) \longmapsto \mathbb{R}$ by

$$E(u) = \frac{1}{2}\int_\Omega a(x)\,|\nabla u|^2\;dx - \int_\Omega F(u)\;dx - \int_{\partial\Omega} G(u)\;d\sigma.$$

Before to establish the next result we remember that the eigenvalues of the Steklov problem defined in a set $D \subset \mathbb{R}^N$

$$\begin{cases} \Delta\varphi = 0, & x \in D \\ \dfrac{\partial\varphi}{\partial\eta} = \mu\varphi, & x \in D \end{cases} \tag{4.1}$$

satisfy $0 = \mu_0 < \mu_1 \leq \mu_2 \cdots \to \infty$ and we recall the following well-know result which can be proved using variational characterization of the eigenvalues. The following well-known result will be used.

LEMMA 4.1. *Let $D \subset \mathbb{R}^N$ be a domain with Lipschitz continuous boundary. Then for any $v \in W^{1,2}(D)$ it holds that*

$$\int_{\partial D} v^2 d\sigma \leq \frac{1}{\mu_1} \int_D |\nabla v|^2 \, dx + \frac{1}{\mathcal{H}^{N-1}(\partial D)} \left(\int_{\partial D} v d\sigma \right)^2 .$$

Moreover if $S \subset \partial D$ with $\mathcal{H}^{N-1}(S) \neq 0$ then

$$\int_{\partial S} v^2 d\sigma \leq \frac{1}{\mu_1} \int_D |\nabla v|^2 \, dx + \frac{1}{\mathcal{H}^{N-1}(S)} \left(\int_S v d\sigma \right)^2 . \tag{4.2}$$

A proof of the next lemma can be found in [13], where the case $a \equiv 1$ and $g \equiv 0$ was treated. However given that this technique have since then been used in the related literature we decided to present here a much simpler and entirely variational proof.

LEMMA 4.2. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a smooth bounded domain, Ω_l and Ω_r two disjoint sub-domains of Ω with smooth boundaries and $S_j = \partial\Omega \cap \partial\Omega_j$, $\mathcal{H}^{N-1}(S_j) > 0$ $(j = l, r)$. For $p > N$, we define the set*

$$\Lambda(\Omega_l, \Omega_r) = \left\{ \begin{array}{l} v \in W^{1,p}(\Omega) : \ \alpha \leq v(x) \leq \beta, \quad x \in \overline{\Omega}, \\ \int_{S_l} v \ d\sigma < 0, \ \int_{S_r} v \ d\sigma > 0, \\ E(v) < \varepsilon_0 - G(\beta)\mathcal{H}^{N-1}(\partial\Omega) - F(\beta)\mathcal{H}^N(\Omega) \end{array} \right\}$$

where

$$\varepsilon_0 = G(\beta) \min \left\{ \mathcal{H}^{N-1}(S_l) \min \left\{ 1, \mu_1(\Omega_l) a_m^l \right\}, \mathcal{H}^{N-1}(S_r) \min \left\{ 1, \mu_1(\Omega_r) a_m^r \right\} \right\},$$

$a_m^j = \min_{x \in \Omega_j} a(x)$ $(j = l, r)$ and $\mu_1(\Omega_j)$ is the first positive eigenvalue of Steklov Problem (4.1) defined in Ω_j $(j = l, r)$.

If $\Lambda \neq \emptyset$ then problem (1.1) has at least one nonconstant stationary solution $u \in \Lambda$ which is stable in $W^{1,p}(\Omega)$.

Proof: Let $T(t)u_0 = u(t, x)$ be the solution to (1.1) with $u(0, x) = u_0$. The proof consists in to show that Λ is invariant under $T(t)$ for $t \geq 0$ and then to use this fact to conclude that there is a stable stationary solution in the interior of Λ.

Let us consider $u_0 \in \Lambda$. Since $f, g \in \mathcal{B}$ an application of Maximum Principle yields $\alpha \leq T(t)u_0 \leq \beta$. Moreover $\frac{d}{dt} E(u(t, x)) = -\int_\Omega (u_t(t, x)^2) dx$ and hence $E(u(t, x)) \leq E(u_0) < \varepsilon_0$.

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Let us show that $\int_{S_l} T(t)u_0\,d\sigma < 0$ for $t \geq 0$. By contradiction let $t_1 > 0$ be such that $w_1 = T(t_1)u_0$ and$\int_{S_l} w_1\,d\sigma = 0$. But

$$\int_{S_l} w_1^2\,d\sigma \leq \frac{1}{\mu_1(\Omega_l)}\int_{\Omega_l}|\nabla w_1|^2dx + \left(\int_{S_l} w_1\,d\sigma\right)^2 = \frac{1}{\mu_1(\Omega_l)}\int_{\Omega_l}|\nabla w_1|^2dx$$

and because $0 \leq sg(s) \leq s^2$ for $s \in [\alpha, \beta]$ we have $0 \leq G(s) \leq s^2/2$ and then

$$\int_{\Omega_l}\frac{a}{2}|\nabla w_1|^2dx \geq \int_{\Omega_l}\frac{a_m^l}{2}|\nabla w_1|^2dx \geq \int_{S_l} a_m^l\mu_1(\Omega_l)G(w_1)d\sigma.$$

Since $f, g \in \mathcal{B}$ we have $F(w_1) \leq F(\beta)$ as well as $G(w_1) \leq G(\beta)$ and then

$$E(w_1) \geq a_m^l\mu_1(\Omega_l)\int_{S_l} G(w_1)d\sigma - \int_{S_l} G(w_1)\,d\sigma - G(\beta)\mathcal{H}^{N-1}(\partial\Omega\backslash S_l) - F(\beta)\mathcal{H}^N(\Omega).$$

But we also have $E(w_1) \leq E(u_0) < \varepsilon_0 - G(\beta)\mathcal{H}^{N-1}(\partial\Omega) - F(\beta)\mathcal{H}^N(\Omega)$. Hence $\varepsilon_0 > (a_m^l\mu_1(\Omega_l){-}1)\int_{S_l} G(w_1)\,d\sigma{+}G(\beta)\mathcal{H}^{N-1}(S_l)$. If $a_m^l\mu_1(\Omega_l) < 1$ we have $\varepsilon_0 > a_m^l\mu_1(\Omega_l)G(\beta)\mathcal{H}^{N-1}(S_l)$. And if $a_m^l\mu_1(\Omega_l) \geq 1$ we have $\varepsilon_0 > G(\beta)\mathcal{H}^{N-1}(S_l)$. In both cases we have a contradiction, so $\int_{S_l} T(t)u_0\,d\sigma < 0$ for $t \geq 0$. Analogously we have $\int_{S_r} T(t)u_0\,d\sigma > 0$ for $t \geq 0$. So we conclude that Λ is invariant under $T(t)$.

Now if $v \in \Lambda$ we have $\gamma(v) = \{T(t)v,\ t \geq 0\} \subset \Lambda$. Because the system is gradient $\gamma(v)$ is compact and then the set $\omega(v) = \{u = \lim_{t_n\to\infty} T(t_n)v$ for some real sequence $(t_n)\}$ is not empty. Moreover if $\mathcal{E}$ is the set of all equilibrium solutions to (1.1) then $\omega(v) \subset \mathcal{E}$.

So if $u \in \omega(v)$ it is an equilibrium solution to (1.1), $\alpha \leq u \leq \beta$, $E(u) \leq E(v) < \varepsilon_0 - G(\beta)\mathcal{H}^{N-1}(\partial\Omega)F(\beta)\mathcal{H}^N(\Omega)$ and as before we conclude by contradiction that $\int_{S_l} u\,d\sigma < 0$ and $\int_{S_r} u\,d\sigma > 0$ and then $\omega(v) \subset \Lambda$.

Hence if $v \in \Lambda$ then $\omega(v) \subset \overline{\Lambda} \cap \mathcal{E}$ which is a compact set. Since E is continuous there is $e_0 \in \overline{\Lambda} \cap \mathcal{E}$ such that $E(e_0) \leq E(v)$ for any $v \in \overline{\Lambda} \cap \mathcal{E}$. But in reality e_0 is a minimum of E in Λ since otherwise there would be $v_1 \in \Lambda$ such that $E(v_1) < E(e_0)$ and as before $\omega(v_1) \subset \Lambda$. Then for all $v \in \omega(v_1)$ we have $E(v) \leq E(v_1) < E(e_0)$ which is a contradiction.

Claim: e_0 is a interior point of Λ and thus a local minimizer of E in $W^{1,p}(\Omega)$.

This will follow by proving that the sets Λ_i $(i = 1, 2, 3, 4)$ given by

$$\Lambda_1 = \{u \in W^{1,p}(\Omega):\ \alpha < u < \beta \text{ a.e. in } \Omega\}$$

$$\Lambda_2 = \{u \in W^{1,p}(\Omega):\int_{S_l} u\,d\sigma < 0\}$$

$$\Lambda_3 = \{u \in W^{1,p}(\Omega):\int_{S_r} u\,d\sigma > 0\} \text{ and}$$

$$\Lambda_4 = \{u \in W^{1,p}(\Omega): E(u) < \varepsilon_0 - \lambda G(\beta)\mathcal{H}^{N-1}(\partial\Omega)\}$$

are open in $W^{1,p}(\Omega)$ and that

$$e_0 \in \bigcap_{j=1,\ldots,4} \Lambda_j\,.$$

We have

- Λ_4 is open in $W^{1,p}(\Omega)$ since E is continuous in $W^{1,p}(\Omega)$.
- Λ_3 and Λ_2 are open by the continuity of the functionals $I_2(u) = \int_{S_l} u\, d\sigma$ and $I_3(u) = \int_{S_r} u\, d\sigma$, defined in $W^{1,p}(\Omega)$.
- Finally using that $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ $(p > N)$, one can easily check that Λ_1 is also open in $W^{1,p}(\Omega)$.

As for the inclusion we have:

- Clearly $E(e_0) \leq \varepsilon_0 - G(\beta)\mathcal{H}^{N-1}(\partial\Omega) - F(\beta)\mathcal{H}^N(\Omega)$ and equality can be ruled out since in case it occurred we would have for any $w \in \Lambda$ (by hypothesis $\Lambda \neq \emptyset$),

$$E(w) < \varepsilon_0 - \lambda G(\beta)\mathcal{H}^{N-1}(\partial\Omega) - F(\beta)\mathcal{H}^N(\Omega) = E(e_0),$$

which contradicts $E(e_0) \leq E(v), \;\; \forall\, v \in \Lambda$.
- We have $\int_{S_l} e_0\, d\sigma \leq 0$ and equality can be ruled out by contradicting the definition of ε_0, as it was given before. The other case is similar.
- Since $e_0 \in \mathcal{E}$ an application of Maximum Principle yields $\alpha < e_0 < \beta$ a.e. in $\overline{\Omega}$.

Summing up: e_0 is an interior point of Λ and therefore a local minimizer of E in $W^{1,p}(\Omega)$. Once our claim is proved we conclude, from the variational characterization of the eigenvalues, that the first eigenvalue of the corresponding linearized problem at e_0 is non-positive. If it is negative we conclude as usual by using the principle of linearized stability. In case it is zero, we conclude as in the proof of Theorem 2.4.

This establishes the proof that e_0 is a stable (in the Lyaponov sense) nonconstant stationary solution to (1.1). ■

As mentioned before the goal in this section is to give sufficient conditions for the existence of patterns for (1.1), and this will be accomplished by giving conditions on $a(x)$ so that Λ is not empty.

LEMMA 4.3. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a smooth bounded domain and suppose the equal-area condition $G(\alpha) = G(\beta)$ and $F(\alpha) = F(\beta)$ holds. Then there is a positive smooth function $a : \overline{\Omega} \longmapsto \mathbb{R}$ such that (1.1) has a nonconstant stable equilibrium solution.*

Proof: According to Lemma 4.2 it suffices to show that $\Lambda \neq \emptyset$. Let us take two separate balls B_l and B_r, centered at points of $\partial\Omega$, such that $\Omega_l = B_l \cap \Omega, \Omega_r = B_r \cap \Omega$ are nonempty connected smooth open sets in Ω satisfying $\overline{\Omega}_l \cap \overline{\Omega}_r = \emptyset$, $S_j = \partial\Omega \cap \partial\Omega_j$ and $\mathcal{H}^{N-1}(S_j) \neq 0$ $(j = l, r)$.

Then there is an hyperplane S which separates $\mathbb{R}^N$ in two disjoint regions, denoted by $\mathbb{R}^N_l$ and $\mathbb{R}^N_r$, with the following properties:

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i) $B_l \subset \mathbb{R}^N_l$ and $B_r \subset \mathbb{R}^N_r$,
ii) $\exists\, m > 0$ such that $\text{dist}(\Omega_j, S) \geq m$ $(j = l, r)$.
We define the signed distance function in $\mathbb{R}^N$ by

$$d(x,S) = \begin{cases} \text{dist}(x,S) & \text{if } x \in \mathbb{R}^N_r, \\ -\text{dist}(x,S) & \text{if } x \in \mathbb{R}^N_l. \end{cases}$$

and, for $\delta > 0$, the tubular neighborhood of S by

$$Q_\delta = \left\{ x \in \overline{\Omega} \mid |d(x,S)| < \delta \right\}.$$

For S_l, S_r as in Lemma 4.2 we suppose $\mathcal{H}^{N-1}(S_l) \leq \mathcal{H}^{N-1}(S_r)$ and choose $\delta < m$ small enough such that

$$G(\beta)\mathcal{H}^{N-1}(\partial Q_\delta \cap \partial\Omega) + F(\beta)\mathcal{H}^N(Q_\delta) < G(\beta)\mathcal{H}^{N-1}(S_l) \tag{4.3}$$

Consider a function $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\xi(t) = \begin{cases} \alpha, & \text{if } t \leq -\delta \\ \alpha + \beta + \dfrac{(\beta-\alpha)}{\delta}t, & \text{if } -\delta < t < \delta \\ \beta, & \text{if } t \geq \delta. \end{cases}$$

Then $w_0(x) = \xi(d(x,S))$ is a Lipschitz function in $\mathbb{R}^N$ and consequently its restriction to Ω is in $W^{1,p}(\Omega)$. We will show that under certain conditions on $a(x)$ we have $w_0 \in \Lambda$, with Λ defined as in Lemma 4.2.

Clearly $\alpha \leq w_0 \leq \beta$, $\int_{S_l} w_0\ d\sigma < 0$ and $\int_{S_r} w_0\ d\sigma > 0$.

Let S^- and S^+ be portions of $\partial\Omega$ defined by $\partial\Omega \backslash (\partial Q_\delta \cap \partial\Omega) = S^- \cup S^+$. Then $S^- \cap S^+ = \emptyset$, $S^- \cap S_l \neq \emptyset \neq S^+ \cap S_r$.

Since w_0 is constant on each connected component of $\Omega \backslash Q_\delta$, $G(\alpha) = G(\beta)$ and $F(\alpha) = F(\beta)$ we obtain

$$\begin{aligned} E(w_0) &\leq \frac{1}{2}\int_{Q_\delta} a(x)|\nabla w_0|^2\ dx - \int_{\partial Q_\delta \cap \partial\Omega} G(w_0)\ d\sigma \\ &- G(\beta)\mathcal{H}^{N-1}(\partial\Omega \backslash (\partial Q_\delta \cap \partial\Omega)) - F(\beta)\mathcal{H}^N(\Omega \backslash Q_\delta). \end{aligned}$$

Given that $\displaystyle\int_{\partial Q_\delta \cap \partial\Omega} G(w_0) d\sigma \geq 0$ in order to have

$$E(w_0) < \varepsilon_0 - G(\beta)\mathcal{H}^{N-1}(\partial\Omega) - F(\beta)\mathcal{H}^N(\Omega), \tag{4.4}$$

where

$$\varepsilon_0 = G(\beta)\min\left\{\mathcal{H}^{N-1}(S_l)\min\left\{1,\mu_1(\Omega_l)a_m^{\Omega_l}\right\},\mathcal{H}^{N-1}(S_r)\min\left\{1,\mu_1(\Omega_r)a_m^{\Omega_r}\right\}\right\}$$

it suffices to require

$$\varepsilon_0 + F(\beta)\mathcal{H}^N(\Omega\backslash Q_\delta) >$$

$$\frac{1}{2}\int_{Q_\delta} a(x)|\nabla w_0|^2\ dx + G(\beta)\mathcal{H}^{N-1}(\partial Q_\delta\cap\partial\Omega) + F(\beta)\mathcal{H}^N(\Omega). \tag{4.5}$$

Since the diffusivity function a is the parameter to be chosen we set $a_m^{\Omega_j} = \min_{x\in\Omega_j} a(x)$ $(j = l, r)$ and take

$$a_m^{\Omega_l} > \frac{1}{\mu_1(\Omega_l)} \quad \text{and} \quad a_m^{\Omega_r} > \frac{1}{\mu_1(\Omega_r)}. \tag{4.6}$$

Hence

$$\varepsilon_0 = G(\beta)\mathcal{H}^{N-1}(S_l). \tag{4.7}$$

Moreover setting

$$a_M^\delta = \max_{x\in Q_\delta} a(x)$$

we have

$$\frac{1}{2}\int_{Q_\delta} a(x)|\nabla w_0|^2\ dx \le \frac{a_M^\delta}{2}\frac{(\beta-\alpha)^2}{\delta^2}\mathcal{H}^N(Q_\delta).$$

Therefore (4.5), and consequently (4.4), will be realized provided

$$0 < a_M^\delta <$$

$$\frac{2\delta^2\left[G(\beta)\mathcal{H}^{N-1}(S_l) - G(\beta)\mathcal{H}^{N-1}(\partial Q_\delta\cap\partial\Omega) - F(\beta)\mathcal{H}^N(Q_\delta)\right]}{(\beta-\alpha)^2\mathcal{H}^N(Q_\delta)}. \tag{4.8}$$

Note that in view of (4.3) the righthand side of (4.8) is positive and does not depend on a. Therefore (4.8) can clearly be satisfied by taking a_M^δ small enough. Therefore $\Lambda \neq \emptyset$ and Lemma 4.2 completes the proof. ■

REFERENCES

1. Ball, J.M., Taheri, A. and Winter, M.; *Local minimizers in micromagnetics and related problems*, Calculus of Variations, **14** (2002), 1-27.
2. Agmon, S. , Douglis, A. and Nirenberg, L.; Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I, Comm. Pure. Appl. Math. **v.12**, p. 623-727 1959.
3. Ambroseti, A. and Prodi, G.; *A primer of nonlinear analysis*, Cambridge University Press, Cambridge, 1990.
4. do Nascimento, A. S.; *Instability results for the heat equation with nonlinear boundary flux* (submitted)

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Sob a supervisão da CPq/ICMC

5. do Nascimento, A. S.; *On the role of diffusivity in some stable equilibria of a diffusion equation*, J. Diff. Eqns., **155**, No. 2 (1999), 231-244.
6. do Nascimento, A. S. and de Moura, R. J.; *Layered stable equilibria of a reaction-diffusion equation with nonlinear Neumann boundary condition*, J. Math. Anal. Appl. **347** (2008), 123-135.
7. do Nascimento, A.S.; *Reaction-diffusion induced stability of a spatially inhomogeneous equilibrium with boundary layer formation J. Diff. Eqts.*, vol. 108, No. 2 (1994), 296-325.
8. do Nascimento, A.S.; *Bifurcation and stability of radially symmetric equilibria of a parabolic equation with variable diffusion J. Diff. Eqts.*, vol. 77, No. 1 (1989), 84-103.
9. Gilbard, D. and Trudinger, N. S.; *Elliptic partial difeferential equations of second order*, Springer - Verlag New York, 1983.
10. Grisvard, P.; *Elliptic problems in nonsmooth domains* Pitman, London, 1985.
11. Fusco, G. and Hale, J.K.; *Stable equilibria in a scalar parabolic equation with variable diffusion*, *SIAM J. Math. Anal.* 16 (1985), 1154-1164.
12. Hale, J.K. and Rocha, C.; *Bifurcations in a parabolic equation with variable diffusion*, *Nonlinear Anal.* 9, n. 5 (1985), 479-494.
13. Matano, H.; *Asymptotic behavior and stability of solutions of semilinear diffusion equations*, Publ. Res. Inst. Math. Sci., **15** (2) (1979), 401-454.
14. Yanagida, E.; *Stability of stationary distributions in a space-dependent population growth process*, J. Math. Biology, 15 (1982), 37-50.
15. Chipot, M. and Hale, J. K.; *Stable equilibria with variable diffusion*, Contemp. Math.,17 (1983), 209-213.
16. Casten, R.G. and Holland, C.J.; *Instability results for reaction diffusion equations with Neumann boundary conditions*, J. of Diff. Eqns.,**27** (1978) 266-273.
17. Henry, D.; *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin/New York (1981).

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