CADERNOS DE MATEMÁTICA **13**, 1–19 May (2012)
ARTIGO NÚMERO SMA# 359

Simultaneous linearization of a class of pairs of involutions with normally hyperbolic composition

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In this paper we obtain a result on simultaneous linearization for a class of pairs of involutions whose composition is normally hyperbolic. This extends the corresponding result when the composition of the involutions is a hyperbolic germ of diffeomorphism. Inside the class of pairs with normally hyperbolic composition, we obtain a characterization theorem for the composition to be hyperbolic. In addition, related to the class of interest, we present the classification of pairs of linear involutions via linear conjugacy. May, 2012
ICMC-USP

Keywords. Involution, linearization, normal hyperbolicity, transversality.

MSC classification. 58K50, 37D10, 37C80.

1. INTRODUCTION

Involutions have attracted attention of several authors in a variety of contexts. We mention the articles [6] and [10] where the classification of pairs of involutions is considered. It is worth saying that dynamical systems governed by piecewise smooth vector fields have found widespread application in recent years, from control theory and nonlinear oscillations to economics and biology. The motivation of the present work is their appearance

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in reversible dynamical systems. In particular, for discrete dynamical systems with a reversing symmetry and no nontrivial symmetries it is well known that this reversibility is an involution φ_1. Also, the generating diffeomorphism F of such a system is of the form $F = \varphi_1 \circ \varphi_2$, with φ_2 being also an involution. References on reversibility and related problems can be found in [4]. One question raised naturally here regards the problem of local linearization around a fixed point; more specifically, the relation between the linearization of a reversible germ of diffeomorphism and the simultaneous linearization of the corresponding pair of involutions. Now, the simultaneous linearization of φ_1 and φ_2 is a sufficient condition for the linearization of $\varphi_1 \circ \varphi_2$. We refer to [6, Subsection 8.2] for a brief explanation on this topic. In [9] Teixeira proves that pairs of involutions on the plane are simultaneously linearizable provided the composition is hyperbolic. In this case, the fixed-point set of this composition reduces to a point. Here we extend this result for a class in which the composition is normally hyperbolic (Definition 2.3), in whose situation the fixed-point set can now be a local submanifold with positive dimension. This is one of the three main results of this work. It is Theorem 2.1 and Section 4 is devoted to its proof. Inside the class of normally hyperbolics, we also present a characterization of pairs of involutions for the composition to be hyperbolic (Corollary 3.2).

The problem of simultaneous behavior of diffeomorphisms have leaded to several interesting results in different settings. Among such results, we mention the Bochner-Montgomery theorem (see [5]) which is a well-known and useful result about linearization of a compact group of transformations around a fixed point. This theorem is preceded by a related result by Cartan [1]. It then follows that an s-tuple of involutions generating an Abelian group is simultaneously linearizable. In particular, any involution is linearizable, and in Lemma 2.1 we shall explicit a special formula for this linearization. As a consequence, for one involution the normal forms up to conjugacy are given as follows: it is either I or it is equivalent to

$$(x_1, \ldots, x_n) \mapsto (-x_1, \ldots, -x_\ell, x_{\ell+1}, \ldots, x_n),$$

when the codimension of its fixed-point submanifold equals $\ell \neq 0$. Now, two involutions with normally hyperbolic composition generate a non-Abelian group, generally noncompact. In our setting, to establish a linearization result we consider a parametrized foliation which is determined by the linear part of the composition at the origin. This, in turn, restricted to each leaf is a hyperbolic linear isomorphism. The conjugacy is guaranteed by Hartman's lemma (see for example [7]) and derived from a particular case of a result by Pugh and Schub given in [8] in the presence of the parameters (Lemma 4.1). Moreover, some properties are imposed on the involutions. Transversality is assumed throughout and each involution must respect the foliation in consideration. We then remark that, in our setting, the subtle point is the derivation of the extension lemma (Lemma 4.2) for the pair of involutions satisfying the conditions that appear in Hartman's lemma.

In Section 3 we present the classification of pairs of linear involutions with normally hyperbolic composition in the special case for which the fixed-point subspaces are in general position. This classification is performed via linear conjugacy and the normal forms are exhibited in Theorems 3.2 and 3.3. These are the other two of our main results. In the deduction process of the normal forms, we have obtained a series of results that shown to

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be very useful in the treatment of the nonlinear case, giving to this section a particular importance besides the classification itself. We have chosen to present the pairs in their matricial form. This is motivated by the fact that it provides a clear illustration that a pair of transversal involutions with normally hyperbolic composition can be seen as a suspension of a corresponding pair of involutions, defined on a vector space of lower dimension, whose composition is hyperbolic.

2. THE LINEARIZATION THEOREM

DEFINITION 2.1. Let $\varphi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a germ of diffeomorphism. We say that φ is an involution if $\varphi \circ \varphi = I$.

LEMMA 2.1. *For any involution φ on $(\mathbf{R}^n, 0)$, the germ of diffeomorphism $h = \frac{1}{2}(I + d\varphi(0) \circ \varphi)$ of $(\mathbf{R}^n, 0)$ is a conjugacy between φ and the germ of its linear part $d\varphi(0)$ at 0, namely $d\varphi(0) = h \circ \varphi \circ h^{-1}$.*

Given a map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, let $\mathcal{F}(f)$ denote the fixed-point set of f,

$$\mathcal{F}(f) \ = \ \{x \in (\mathbf{R}^n, 0) : f(x) = x\}.$$

Using Lemma 2.1, for any involution φ on $(\mathbf{R}^n, 0)$ we have that $\mathcal{F}(\varphi) = h^{-1}\big(\mathcal{F}(d\varphi(0))\big)$; hence, $\mathcal{F}(\varphi)$ is locally diffeomorphic to the linear subspace $\mathcal{F}(d\varphi(0))$ of $\mathbf{R}^n$. Therefore, $\mathcal{F}(\varphi)$ is a submanifold in $(\mathbf{R}^n, 0)$ such that $T_0\mathcal{F}(\varphi) = \mathcal{F}(d\varphi(0))$, where $T_0\mathcal{F}(\varphi)$ denotes the tangent space to $\mathcal{F}(\varphi)$ at 0.

DEFINITION 2.2. Two pairs (φ_1, φ_2) and (ψ_1, ψ_2) of involutions on $(\mathbf{R}^n, 0)$ are said to be (C^0-equivalent) equivalent if there exists a germ of (homeomorphism) diffeomorphism h of $(\mathbf{R}^n, 0)$ such that $\psi_i = h \circ \varphi_i \circ h^{-1}$, for $i = 1, 2$.

Note that in the situation of Definition 2.2 the map-germ h satisfies

$$h(\mathcal{F}(\varphi_i)) = \mathcal{F}(\psi_i), \ i = 1, 2.$$

Also, h is a conjugacy between $\varphi_1 \circ \varphi_2$ and $\psi_1 \circ \psi_2$:

$$\psi_1 \circ \psi_2 \ = \ h \circ (\varphi_1 \circ \varphi_2) \circ h^{-1}.$$

As mentioned in the introduction, in this work we deal with the linearization problem for a class of pairs of involutions whose composition is normally hyperbolic, according to the following definition:

DEFINITION 2.3. Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a germ of diffeomorphism, $f \neq I$. Suppose that $\mathcal{F}(f)$ is a submanifold in $(\mathbf{R}^n, 0)$ and that $\dim \mathcal{F}(f) = k$. We say that f is

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normally hyperbolic if the spectrum of $df(0)$ has, counting multiplicity, $n-k$ elements out of the unit circle $S^1 \subset \mathbf{C}$.

Let us observe that if $\mathcal{F}(f) = \{0\}$ then the definition above reduces to the concept of a hyperbolic germ of diffeomorphism. Corollary 3.2 characterizes the hyperbolic composition of involutions inside the class of normally hyperbolic germs of diffeomorphisms. If $\dim \mathcal{F}(f) = k > 0$, then 1 is an eigenvalue of $df(0)$ with same geometric and algebraic multiplicities, equal to k; in addition, we get $T_0\mathcal{F}(f) = \mathcal{F}(df(0))$. It then follows that $df(0)$ is a linear normally hyperbolic isomorphism provided f is.

We shall also require transversality of the two involutions, which we define next:

Definition 2.4. Given two involutions $\varphi_1, \varphi_2 : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, we say that φ_1 and φ_2 are transversal if $\mathcal{F}(\varphi_1)$ and $\mathcal{F}(\varphi_2)$ are in general position at 0, i.e.,

$$\mathbf{R}^n = T_0\mathcal{F}(\varphi_1) + T_0\mathcal{F}(\varphi_2). \tag{1}$$

Under transversality of the involutions φ_1 and φ_2 we remark that, up to equivalence, it is no loss of generality to assume that the pair (φ_1, φ_2) satisfies

$$\mathcal{F}(\varphi_i) = \mathcal{F}(d\varphi_i(0)),\ i = 1, 2 \tag{2}$$

in $(\mathbf{R}^n, 0)$. In fact, being φ_1 and φ_2 transversal, we can take a germ of diffeomorphism h of $(\mathbf{R}^n, 0)$ such that $h(\mathcal{F}(\varphi_1))$ and $h(\mathcal{F}(\varphi_2))$ are linear submanifolds, that is, h linearizes simultaneously the submanifolds $\mathcal{F}(\varphi_1)$ and $\mathcal{F}(\varphi_2)$. If we now consider the pair of involutions (ψ_1, ψ_2), where $\psi_i = h \circ \varphi_i \circ h^{-1}$, $i = 1, 2$, then we have that (ψ_1, ψ_2) is equivalent to (φ_1, φ_2) and

$$\mathcal{F}(\psi_i) = \mathcal{F}(d\psi_i(0)),\ i = 1, 2$$

in $(\mathbf{R}^n, 0)$.

Yet for the description of the structure of the class of pairs of involutions for which our linearization theorem applies, we consider $L : \mathbf{R}^n \to \mathbf{R}^n$ a linear normally hyperbolic isomorphism and take the decomposition

$$\mathbf{R}^n = E^s \oplus E^u \oplus \mathcal{F}(L), \tag{3}$$

where E^s and E^u are respectively the stable and unstable subspaces of L. Let

$$\begin{array}{ccc} \mathbf{R}^n & \xrightarrow{L} & \mathbf{R}^n \\ \downarrow & & \downarrow \\ \mathcal{F}(L) & \xrightarrow{I} & \mathcal{F}(L) \end{array} \tag{4}$$

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be the hyperbolic bundle automorphism covering the identity I, whose fibers are all equal to $E^s \oplus E^u$.

We are now in position to state the theorem:

Theorem 2.1. *Let (φ_1, φ_2) be a pair of transversal involutions on $(\mathbf{R}^n, 0)$ such that $\mathcal{F}(\varphi_i) = \mathcal{F}(d\varphi_i(0))$, $i = 1, 2$, $\varphi_1 \circ \varphi_2$ is normally hyperbolic and locally each φ_i respects the fiber bundle in (4) for $L = d(\varphi_1 \circ \varphi_2)(0)$. Then, this pair is C^0-equivalent to (L_1, L_2), where $L_i = d\varphi_i(0)$, $i = 1, 2$.*

Section 4 is devoted to the proof of the theorem above. Therein we shall also remark about the grounds which the last hypothesis relies on.

3. CLASSIFICATION OF PAIRS (φ_1, φ_2) OF TRANSVERSAL LINEAR INVOLUTIONS WITH $\varphi_1 \circ \varphi_2$ NORMALLY HYPERBOLIC

3.1. Preliminaries

We start with two general results concerning the composition of two linear involutions that are essencial to all that follows. For $L : \mathbf{R}^n \to \mathbf{R}^n$ linear, we denote by $\mathcal{A}(L)$ the antipodal subspace of L in $\mathbf{R}^n$ given by

$$\mathcal{A}(L) \;=\; \{x \in \mathbf{R}^n \;:\; L(x) = -x\}.$$

In the special case of a linear involution φ on $\mathbf{R}^n$, we have the composition

$$\mathbf{R}^n \;=\; \mathcal{F}(\varphi) \oplus \mathcal{A}(\varphi). \tag{5}$$

Proposition 3.1. *Let φ_1, φ_2 be linear involutions on $\mathbf{R}^n$. The following equalities hold:*

(a) $\mathcal{F}(\varphi_1 \circ \varphi_2) \;=\; \mathcal{F}(\varphi_1) \cap \mathcal{F}(\varphi_2) \oplus \mathcal{A}(\varphi_1) \cap \mathcal{A}(\varphi_2)$,

(b) $\mathcal{A}(\varphi_1 \circ \varphi_2) \;=\; \mathcal{F}(\varphi_1) \cap \mathcal{A}(\varphi_2) \oplus \mathcal{A}(\varphi_1) \cap \mathcal{F}(\varphi_2)$.

Proof. Given $x \in \mathbf{R}^n$, having in mind the composition (5) for φ_i, $i = 1, 2$, let $y_i = (x + \varphi_i(x))/2$ be the projection of x on $\mathcal{F}(\varphi_i)$ parallelly to $\mathcal{A}(\varphi_i)$. Since

$$x \in \mathcal{F}(\varphi_1 \circ \varphi_2) \Leftrightarrow \varphi_1(x) = \varphi_2(x),$$

we conclude that

$$x \in \mathcal{F}(\varphi_1 \circ \varphi_2) \Leftrightarrow y_1 \;=\; y_2,$$

which shows part (a) of the proposition. The proof of part (b) is analogous, observing that

$$x \in \mathcal{A}(\varphi_1 \circ \varphi_2) \Leftrightarrow \varphi_1(x) = -\varphi_2(x).$$

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▮

Corollary 3.1. (of part (b) of Proposition 3.1) *Let* φ_1, φ_2 *be linear involutions on* $\mathbf{R}^n$*. If* $\mathcal{A}(\varphi_1 \circ \varphi_2) = \{0\}$, *then* $\dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2)$.

Proof. By hypothesis, $\mathcal{F}(\varphi_1) \cap \mathcal{A}(\varphi_2) = \{0\}$. So $\dim \mathcal{F}(\varphi_1) + (n - \dim \mathcal{F}(\varphi_2)) = \dim(\mathcal{F}(\varphi_1) + \mathcal{A}(\varphi_2)) \leq n$, which implies that $\dim \mathcal{F}(\varphi_1) \leq \dim \mathcal{F}(\varphi_2)$. We can now interchange φ_1 and φ_2 to get $\dim \mathcal{F}(\varphi_2) \leq \dim \mathcal{F}(\varphi_1)$. ▮

The following definition is the corresponding to Definition 2.2 for pairs of linear involutions when the equivalence is realized by a linear isomorphism:

Definition 3.1. Let (φ_1, φ_2) and (ψ_1, ψ_2) be two pairs of linear involutions on $\mathbf{R}^n$. We say that (φ_1, φ_2) and (ψ_1, ψ_2) are linearly equivalent if there exists a linear isomorphim $h : \mathbf{R}^n \to \mathbf{R}^n$ such that $\psi_i = h \circ \varphi_i \circ h^{-1}$, for $i = 1, 2$.

In the condition of the definition above, we have, in addition to $h(\mathcal{F}(\varphi_i)) = \mathcal{F}(\psi_i)$, that

$$h(\mathcal{A}(\varphi_i)) = \mathcal{A}(\psi_i), \ \ i = 1, 2.$$

A very useful result, with an immediate proof, is the following:

Proposition 3.2. *Two pairs* (φ_1, φ_2) *and* (ψ_1, ψ_2) *of linear involutions on* $\mathbf{R}^n$ *are linearly equivalent if, and only if, so are* $(-\varphi_1, -\varphi_2)$ *and* $(-\psi_1, -\psi_2)$.

For two linear involutions φ_1 and φ_2 on $\mathbf{R}^n$, the transversality condition reduces to

$$\mathbf{R}^n = \mathcal{F}(\varphi_1) + \mathcal{F}(\varphi_2). \tag{6}$$

Remark 3. 1. Considering the composition $\varphi_1 \circ \varphi_2$ normally hyperbolic, note that if transversality fails for φ_1 and φ_2 but $\mathcal{A}(\varphi_1)$ and $\mathcal{A}(\varphi_2)$ are in general position, then it is still possible to obtain the normal form of the pair (φ_1, φ_2) applying the results to $(-\varphi_1, -\varphi_2)$. This is a consequence of Proposition 3.2 and of the fact that

$$\mathcal{F}(-\varphi_i) = \mathcal{A}(\varphi_i), \ \ \ i = 1, 2.$$

In certain dimensions, this provides the complete classification of pairs of linear involutions with normally hyperbolic composition. These are precisely the cases for which the normal hyperbolicity of $\varphi_1 \circ \varphi_2$ implies that either $\mathcal{F}(\varphi_1)$ and $\mathcal{F}(\varphi_2)$ or $\mathcal{A}(\varphi_1)$ and $\mathcal{A}(\varphi_2)$ are in general position.

We end this subsection with two propositions. The first proposition gives normal forms of pairs of transversal involutions and the other characterizes their equivalence classes.

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Proposition 3.3. *Let φ_1, φ_2 be transversal linear involutions on $\mathbf{R}^n$. Let $r = \dim \mathcal{F}(\varphi_1)$ and $s = \dim \mathcal{F}(\varphi_2)$. Then (φ_1, φ_2) is linearly equivalent to a pair of involutions (ψ_1, ψ_2) such that $\mathcal{F}(\psi_1)$ is given by $x_1 = \ldots = x_{n-r} = 0$ and $\mathcal{F}(\psi_2)$ by $x_{n-r+1} = \ldots = x_{2n-r-s} = 0$. Therefore, ψ_1 and ψ_2 have matrices of the types*

$$\psi_1 = \left(\begin{array}{c|c|c} -I_{n-r} & 0 & \\ \cline{1-2} A_2 & I_{n-s} & 0 \\ \hline A_3 & 0 & I_{r+s-n} \end{array}\right), \quad \psi_2 = \left(\begin{array}{c|c|c} I_{n-r} & B_1 & \\ \cline{1-2} 0 & -I_{n-s} & 0 \\ \hline 0 & B_3 & I_{r+s-n} \end{array}\right). \tag{7}$$

Proof. It is a direct consequence of the transversality condition that (φ_1, φ_2) is linearly equivalent to a pair (ψ_1, ψ_2) such that $\mathcal{F}(\psi_1)$ is given by $x_1 = \ldots = x_{n-r} = 0$ and $\mathcal{F}(\psi_2)$ is given by $x_{n-r+1} = \ldots = x_{2n-r-s} = 0$. Then ψ_1 and ψ_2 have matrices of the types

$$\psi_1 = \left(\begin{array}{c|c|c} A_1 & 0 & \\ \cline{1-2} A_2 & I_{n-s} & 0 \\ \hline A_3 & 0 & I_{r+s-n} \end{array}\right), \quad \psi_2 = \left(\begin{array}{c|c|c} I_{n-r} & B_1 & \\ \cline{1-2} 0 & B_2 & 0 \\ \hline 0 & B_3 & I_{r+s-n} \end{array}\right).$$

Being ψ_1 and ψ_2 involutions, we have that

$$\begin{array}{ll} (a)\ A_1^2 = I_{n-r} & (b)\ A_2 + A_2 A_1 = 0,\ A_3 + A_3 A_1 = 0 \\ (c)\ B_2^2 = I_{n-s} & (d)\ B_1 + B_1 B_2 = 0,\ B_3 + B_3 B_2 = 0. \end{array}$$

We show now that (b) above implies that $\mathcal{F}(A_1) = \{0\}$, which, together with (a), gives $A_1 = -I_{n-r}$. Then, let $(x_1, \ldots, x_{n-r}) \in \mathcal{F}(A_1)$. So

$$\psi_1(x_1, \ldots, x_{n-r}, 0, \ldots, 0) = (x_1, \ldots, x_{n-r}, A_2(x_1, \ldots, x_{n-r}), A_3(x_1, \ldots, x_{n-r})).$$

From (b) we get that $A_2(x_1, \ldots, x_{n-r}) = 0$ and $A_3(x_1, \ldots, x_{n-r}) = 0$. Hence, $(x_1, \ldots, x_{n-r}, 0, \ldots, 0) \in \mathcal{F}(\psi_1)$, which implies that $x_1 = \ldots = x_{n-r} = 0$. Therefore, $\mathcal{F}(A_1) = \{0\}$.

Analogously, using (c) and (d) we get $B_2 = -I_{n-s}$. ∎

The next proposition generalizes [6, Proposition 5.1], for pairs of linear involutions.

Proposition 3.4. *Let (ψ_1, ψ_2) and (ψ_1', ψ_2') be pairs of transversal linear involutions on $\mathbf{R}^n$ with matrices as in (7):*

$$\psi_1 = \left(\begin{array}{c|c|c} -I_{n-r} & 0 & \\ \cline{1-2} A_2 & I_{n-s} & 0 \\ \hline A_3 & 0 & I_{r+s-n} \end{array}\right), \quad \psi_2 = \left(\begin{array}{c|c|c} I_{n-r} & B_1 & \\ \cline{1-2} 0 & -I_{n-s} & 0 \\ \hline 0 & B_3 & I_{r+s-n} \end{array}\right).$$

and

$$\psi_1' = \left(\begin{array}{c|c|c} -I_{n-r} & 0 & \\ \cline{1-2} A_2' & I_{n-s} & 0 \\ \hline A_3' & 0 & I_{r+s-n} \end{array}\right), \quad \psi_2' = \left(\begin{array}{c|c|c} I_{n-r} & B_1' & \\ \cline{1-2} 0 & -I_{n-s} & 0 \\ \hline 0 & B_3' & I_{r+s-n} \end{array}\right).$$

Then (ψ_1, ψ_2) *and* (ψ_1', ψ_2') *are linearly equivalent if, and only if, there exists an invertible matrix*

$$H = \left(\begin{array}{c|c|c} (\alpha_1)_{n-r} & 0 & \\ \cline{1-2} 0 & (\alpha_2)_{n-s} & 0 \\ \hline \delta & \gamma & \beta_{r+s-n} \end{array}\right)$$

such that

$$A_2'\alpha_1 = \alpha_2 A_2,$$

$$B_1'\alpha_2 = \alpha_1 B_1,$$

$$A_3'\alpha_1 = -2\delta + \gamma A_2 + \beta A_3$$

$$B_3'\alpha_2 = \delta B_1 - 2\gamma + \beta B_3.$$

Proof. By a direct computation one shows that a linear isomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ satisfies $\psi_1' = h \circ \psi_i \circ h^{-1}$, $i = 1, 2$, if, and only if, h has matrix H as above. ▮

3.2. The classification

Let (φ_1, φ_2) be a pair of linear involutions on $\mathbf{R}^n$, $n \geq 2$, such that $\varphi_1 \circ \varphi_2$ is a normally hyperbolic isomorphism. As mentioned in the introduction, it is direct from the normal hyperbolicity of the composition that the group $\Lambda = [\varphi_1, \varphi_2]$, generated by φ_1 and φ_2, is non-Abelian, since, otherwise, $\varphi_1 \circ \varphi_2$ would be also an involution. In particular, $\varphi_1, \varphi_2 \neq I, -I$, hence

(NH1) $1 \leq \dim \mathcal{F}(\varphi_i) \leq n - 1, \ i = 1, 2.$

In addition, we also have that

(NH2) $\dim \mathcal{F}(\varphi_1 \circ \varphi_2) \leq n - 2;$

(NH3) $\mathcal{A}(\varphi_1 \circ \varphi_2) = \{0\}.$

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By Corollary 3.1, (NH3) implies that $\dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2)$.

The next two lemmas are concerned with the classification of pairs of transversal linear involutions on $\mathbf{R}^n$, $n \geq 2$, under the condition (NH3).

LEMMA 3.1. *Let φ_1, φ_2 be transversal linear involutions on $\mathbf{R}^n$ such that $\mathcal{A}(\varphi_1 \circ \varphi_2) = \{0\}$. Let $r = \dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2)$. Then (φ_1, φ_2) is linearly equivalent to a pair (ψ_1, ψ_2) such that ψ_1 and ψ_2 have matrices of the following forms:*

$$\psi_1 = \left(\begin{array}{cc|c} -I_{n-r} & 0 & \\ \hline A_2 & I_{n-r} & 0 \\ \hline A_3 & 0 & I_{2r-n} \end{array}\right), \quad \psi_2 = \left(\begin{array}{cc|c} I_{n-r} & I_{n-r} & \\ \hline 0 & -I_{n-r} & 0 \\ \hline 0 & B_3 & I_{2r-n} \end{array}\right) \tag{8}$$

with A_2 invertible.

Proof. From Proposition 3.3, (φ_1, φ_2) is linearly equivalent to a pair $(\tilde{\psi}_1, \tilde{\psi}_2)$ such that $\tilde{\psi}_1$ and $\tilde{\psi}_2$ have matrices of the forms

$$\tilde{\psi}_1 = \left(\begin{array}{cc|c} -I_{n-r} & 0 & \\ \hline \tilde{A}_2 & I_{n-r} & 0 \\ \hline \tilde{A}_3 & 0 & I_{2r-n} \end{array}\right), \quad \tilde{\psi}_2 = \left(\begin{array}{cc|c} I_{n-r} & \tilde{B}_1 & \\ \hline 0 & -I_{n-r} & 0 \\ \hline 0 & \tilde{B}_3 & I_{2r-n} \end{array}\right) \tag{9}$$

Since $\mathcal{A}(\varphi_1 \circ \varphi_2) = \{0\}$, then also $\mathcal{A}(\tilde{\psi}_1 \circ \tilde{\psi}_2) = \{0\}$ and, therefore, $\tilde{A}_2$ and $\tilde{B}_1$ are invertible.

Let $h : \mathbf{R}^n \to \mathbf{R}^n$ be the linear isomorphism with matrix

$$H = \left(\begin{array}{cc|c} \tilde{B}_1^{-1} & 0 & \\ \hline 0 & I_{n-r} & 0 \\ \hline 0 & 0 & I_{2r-n} \end{array}\right).$$

Considering the involutions $\psi_1 = h \circ \tilde{\psi}_1 h^{-1}$ and $\psi_2 = h \circ \tilde{\psi}_2 h^{-1}$, by transitivity (φ_1, φ_2) is linearly equivalent to (ψ_1, ψ_2), with ψ_1 and ψ_2 of the form 8. ∎

The next lemma, fundamental for the desired classification, is an immediate consequence of Proposition 3.4.

LEMMA 3.2. *Consider the pairs of transversal linear involutions (ψ_1, ψ_2) and (ψ_1', ψ_2'),*

$$\psi_1 = \left(\begin{array}{c|c|c} -I_{n-r} & 0 & \\ \cline{1-2} A_2 & I_{n-r} & 0 \\ \hline A_3 & 0 & I_{2r-n} \end{array}\right), \quad \psi_2 = \left(\begin{array}{c|c|c} I_{n-r} & I_{n-r} & \\ \cline{1-2} 0 & -I_{n-r} & 0 \\ \hline 0 & B_3 & I_{2r-n} \end{array}\right)$$

and

$$\psi_1' = \left(\begin{array}{c|c|c} -I_{n-r} & 0 & \\ \cline{1-2} A_2' & I_{n-r} & 0 \\ \hline A_3' & 0 & I_{2r-n} \end{array}\right), \quad \psi_2' = \left(\begin{array}{c|c|c} I_{n-r} & I_{n-r} & \\ \cline{1-2} 0 & -I_{n-r} & 0 \\ \hline 0 & B_3' & I_{2r-n} \end{array}\right).$$

Then the two pairs are linearly equivalent if, and only if, there exist invertible matrices $\alpha \in M(n-r)$ and $\beta \in M(2r-n)$ and matrices $\delta, \gamma \in M((2r-n) \times (n-r))$ such that

$$\begin{aligned} A_2' &= \alpha A_2 \alpha^{-1} \\ A_3'\alpha &= -2\delta + \gamma A_2 + \beta A_3 \\ B_3'\alpha &= \delta - 2\gamma + \beta B_3. \end{aligned} \tag{10}$$

We are now in position to characterize the orbits of pairs (φ_1, φ_2) of transversal linear involutions such that the composition $\varphi_1 \circ \varphi_2$ is normally hyperbolic. We first treat the case when the composition is hyperbolic, and this is done in Subsection 3.3. We shall see that for the other possibilities the forms are just suspensions of this case. Before we go into that, we state necessary and sufficient conditions for that, among the normally hyperbolics, the composition of the two involutions to be hyperbolic:

THEOREM 3.1. *Let φ_1 and φ_2 be linear involutions on $\mathbf{R}^n$ with $\varphi_1 \circ \varphi_2$ normally hyperbolic. Then $\varphi_1 \circ \varphi_2$ is hyperbolic if, and only if, n is even, φ_1 and φ_2 are transversal and* $\dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2) = n/2$.

Proof. First we notice that normal hyperbolicity of $\varphi_1 \circ \varphi_2$ implies that φ_1 and φ_2 are transversal if, and only if, $\dim\big(\mathcal{F}(\varphi_1) \cap \mathcal{F}(\varphi_2)\big) = 2r-n$, where $r = \dim \mathcal{F}(\varphi_1) = dim\mathcal{F}(\varphi_2)$.

If $\varphi_1 \circ \varphi_2$ is hyperbolic, then $\mathcal{F}(\varphi_1 \circ \varphi_2) = \{0\}$. So by Proposition 3.1

$$\mathcal{F}(\varphi_1) \cap \mathcal{F}(\varphi_2) = \mathcal{A}(\varphi_1) \cap \mathcal{A}(\varphi_2) = \{0\}.$$

Hence,

$$n \geq \dim\big(\mathcal{F}(\varphi_1) + \mathcal{F}(\varphi_2)\big) = \dim \mathcal{F}(\varphi_1) + \dim \mathcal{F}(\varphi_2) = 2r.$$

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Now, replacing $\mathcal{F}(\varphi_i)$ by $\mathcal{A}(\varphi_i)$, and recalling that $\dim \mathcal{A}(\varphi_i) = n - r$, $i = 1, 2$, we get $n \leq 2r$. Therefore, $n = 2r$.

For the converse, let $n \geq 2$ be an even integer number and φ_1 and φ_2 transversal with $\dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2) = n/2 = r$. From Lemma 3.1 the pair (φ_1, φ_2) is linearly equivalent to a pair (ϕ_1, ϕ_2) such that ϕ_1 and ϕ_2 have matrices

$$\phi_1 = \left(\begin{array}{c|c} -I_r & 0 \\ \hline A & I_r \end{array}\right), \quad \phi_2 = \left(\begin{array}{c|c} I_r & I_r \\ \hline 0 & -I_r \end{array}\right), \tag{11}$$

with A invertible. From Lemma 3.2, the pair (ϕ_1, ϕ_2) is linearly equivalent to a pair (ϕ_1', ϕ_2') of the same type

$$\phi_1' = \left(\begin{array}{c|c} -I_r & 0 \\ \hline A' & I_r \end{array}\right), \quad \phi_2' = \left(\begin{array}{c|c} I_r & I_r \\ \hline 0 & -I_r \end{array}\right)$$

(A' invertible) if, and only if, A and A' are similar. So the matrix A can be considered in its canonical Jordan form.

Now we observe that

$$\dim \mathcal{F}(\varphi_1 \circ \varphi_2) = \dim \ker(A - 4I_r)$$

and that the characteristic polynomial of $\varphi_1 \circ \varphi_2$ is given by

$$p_{\varphi_1 \circ \varphi_2}(\lambda) = \det(\lambda^2 I_r - \lambda(A - 2I_r) + I_r). \tag{12}$$

Yet from the normal hyperbolicity of $\varphi_1 \circ \varphi_2$, we cannot encounter 4 as an eigenvalue of A. Otherwise, the algebraic multiplicity of 4 in the characteristic polynomial of A would contribute with twice this number in the algebraic multiplicity of 1 in $p_{\varphi_1 \circ \varphi_2}$, which can be easily seen by taking A in its Jordan form. This would imply that the algebraic and geometric multiplicities of 1 in $\varphi_1 \circ \varphi_2$ would be distinct, which contradicts the normal hyperbolicity. But 4 not being an eigenvalue of A is equivalent to $\mathcal{F}(\varphi_1 \circ \varphi_2) = \{0\}$, which gives hyperbolicity. ▌

The theorem above, concerned with pairs of linear involutions, generalizes to nonlinear pairs:

Corollary 3.2. *Let φ_1 and φ_2 be involutions on $(\mathbf{R}^n, 0)$ with $\varphi_1 \circ \varphi_2$ normally hyperbolic. Then $\varphi_1 \circ \varphi_2$ is hyperbolic if, and only if, n is even, φ_1 and φ_2 are transversal and* $\dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2) = n/2$.

Proof. The proof follows directly from the previous theorem applied to the involutions $d\varphi_1(0)$ and $d\varphi_2(0)$, recalling that $T_0\mathcal{F}(g) = \mathcal{F}(dg(0))$ when g is an involution or g is normally hyperbolic. ▌

3.3. The hyperbolic case

Let φ_1 and φ_2 be linear involutions on $\mathbf{R}^n$ such that $\varphi_1 \circ \varphi_2$ is hyperbolic. By Theorem 3.1, n is even, φ_1 and φ_2 are transversal and $\dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2) = n/2$ $(= r)$. Also, (φ_1, φ_2) is linearly equivalent to a pair (ϕ_1, ϕ_2) as in (11), with A invertible. As already observed in the proof of Theorem 3.1, the pair (ϕ_1, ϕ_2) is linearly equivalent to a pair (ϕ_1', ϕ_2') of the same type if, and only if, the corresponding matrices A and A' are similar. Therefore, in the hyperbolic case, the classification of pairs envolves the classification of $r \times r$ invertible matrices by similarity. The study shall then proceed considering the matrices A in their Jordan form with an analysis of which of them lead to hyperbolic compositions.

Hence, it remains to investigate the spectrum of A such that (12) does not have roots on $\mathbf{S}^1 \subset \mathbf{C}$. But this is a simple calculation and all the discussion above can now be summarized in the following theorem:

THEOREM 3.2. *Let φ_1 and φ_2 be linear involutions on $\mathbf{R}^n$ such that $\varphi_1 \circ \varphi_2$ is hyperbolic and $r = \dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2) = n/2$. Then, the pair (φ_1, φ_2) is linearly equivalent to a pair (ϕ_1, ϕ_2),*

$$\phi_1 = \left(\begin{array}{c|c} -I_r & 0 \\ \hline A & I_r \end{array}\right), \quad \phi_2 = \left(\begin{array}{c|c} I_r & I_r \\ \hline 0 & -I_r \end{array}\right) \tag{13}$$

for some invertible matrix A such that its possible real eigenvalues ξ satisfy $\xi < 0$ or $\xi > 4$, with no restrictions on occurrence of non-real eigenvalues.

We end this subsection presenting the explicit classification relatively to Theorem 3.2 for $n = 2, 4$.

• The case $n = 2$. Here $\dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2) = 1$ and $\mathrm{tr}(\varphi_1 \circ \varphi_2) > 2$ or < -2. The normal form for the pair (φ_1, φ_2) is give by (ϕ_1, ϕ_2),

$$\phi_1 = \begin{pmatrix} -1 & 0 \\ 2 + \mathrm{tr}(\varphi_1 \circ \varphi_2) & 1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

We remark that this normal form can also be obtained directly from the classification that appears in [6, Theorem 6.2], as follows. First, the group $\Lambda[\varphi_1, \varphi_2]$ generated by φ_1 and φ_2 is non-Abelian. Also, since $\mathcal{A}(\varphi_1 \circ \varphi_2) = \{0\}$, then $\mathcal{F}(\varphi_1) \cap \mathcal{A}(\varphi_2) = \{0\}$, which is the same as $\mathrm{Im}(\varphi_2 - I) \neq \mathcal{F}(\varphi_1)$, for $\mathrm{Im}(\varphi_2 - I) = \mathcal{A}(\varphi_2)$.

• The case $n = 4$. Here $\dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2) = 2$. The normal form of the pair is presented by taking the order-2 matrix A in (13) in its Jordan form, which can be obtained via the original pair (φ_1, φ_2) as follows. First, we observe that the characteristic polynomial of A is given by

$$p_A(\lambda) = \lambda^2 - (\mathrm{tr}(\varphi_1 \circ \varphi_2) + 4)\lambda + \det(\varphi_1 \circ \varphi_2 + I_4).$$

Now, if A has a real eigenvalue ξ with algebraic multiplicity 2, in order to decide between the two possible Jordan forms of A, we use the fact that for both cases the characteristic

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polynomial of the composition $\varphi_1 \circ \varphi_2$ is

$$p_{\varphi_1 \circ \varphi_2}(\lambda) = q^2(\lambda),$$

with $q(\lambda) = \lambda^2 - (\xi - 2)\lambda + 1$. The geometric multiplicity of ξ is 2 if q is the minimal polynomial of $\varphi_1 \circ \varphi_2$, and 1 otherwise.

3.4. The general case

Let φ_1, φ_2 be transversal involutions on $\mathbf{R}^n$, $n \geq 2$, with $\varphi_1 \circ \varphi_2$ normally hyperbolic and $r = \dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2)$. The transversality and the normal hyperbolicity imply that $n/2 \leq r \leq n-1$. Moreover, from Lemma 3.1, (φ_1, φ_2) is linearly equivalent to a pair (ψ_1, ψ_2) such that ψ_1 and ψ_2 have matrices

$$\psi_1 = \left(\begin{array}{cc|c} \begin{array}{c|c} -I_{n-r} & 0 \\ \hline A_2 & I_{n-r} \end{array} & & 0 \\ \hline A_3 & 0 & I_{2r-n} \end{array}\right), \qquad \psi_2 = \left(\begin{array}{cc|c} \begin{array}{c|c} I_{n-r} & I_{n-r} \\ \hline 0 & -I_{n-r} \end{array} & & 0 \\ \hline 0 & B_3 & I_{2r-n} \end{array}\right)$$

for a certain invertible matrix A_2. Let us put

$$\phi_1 = \left(\begin{array}{c|c} -I_{n-r} & 0 \\ \hline A_2 & I_{n-r} \end{array}\right), \qquad \phi_2 = \left(\begin{array}{c|c} I_{n-r} & I_{n-r} \\ \hline 0 & -I_{n-r} \end{array}\right).$$

We then have that

$$\dim \mathcal{F}(\phi_1 \circ \phi_2) = \dim \ker(A_2 - 4I_{n-r}),$$

$$2r - n \leq \dim \mathcal{F}(\varphi_1 \circ \varphi_2) \leq \dim \mathcal{F}(\phi_1 \circ \phi_2) + (2r - n)$$

and that the characteristic polynomial of $\varphi_1 \circ \varphi_2$ is given by

$$\begin{aligned} p_{\varphi_1 \circ \varphi_2}(\lambda) &= (\lambda - 1)^{2r-n} p_{\phi_1 \circ \phi_2}(\lambda) \\ &= (\lambda - 1)^{2r-n} \det(\lambda^2 I_{n-r} - \lambda(A_2 - 2I_{n-r}) + I_{n-r}). \end{aligned}$$

Now, the discussion of the preceding subsection allows us to conclude that, since $\varphi_1 \circ \varphi_2$ is normally hyperbolic, 4 is not eigenvalue of A_2. Furthermore, 4 not being an eigenvalue of A_2 is the same as saying that $\dim \mathcal{F}(\phi_1 \circ \phi_2) = 0$, which gives

$$\dim \mathcal{F}(\varphi_1 \circ \varphi_2) = 2r - n. \tag{14}$$

Let us remark that with the pre-normal form (ψ_1, ψ_2) of the pair (φ_1, φ_2) in hand, we have a characterization for normal hyperbolicity condition. More precisely, the fact that $\varphi_1 \circ \varphi_2$ is normally hyperbolic is equivalent to $\phi_1 \circ \phi_2$ being hyperbolic.

It is then natural to ask whether A_3 and B_3 can be taken to be the zero matrices in the pre-normal form. Our next aim is to show that in fact they can; therefore, as mentioned previously, considering the decomposition $\mathbf{R}^n = \mathbf{R}^{2(n-r)} \times \mathbf{R}^{2r-n}$, the forms ψ_1 and ψ_2 in the present case are just suspensions of the forms ϕ_1 and ϕ_2 in $\mathbf{R}^{2(n-r)}$.

We now turn to Lemma 3.2. In view of the relations (10), we proceed as follows. For $\alpha \in M(n-r)$ a fixed invertible matrix, let

$$L_\alpha : M((2r-n)\times(n-r))\times M((2r-n)\times(n-r)) \to M((2r-n)\times(n-r))\times M((2r-n)\times(n-r))$$

be the linear operator defined by

$$L_\alpha(\delta,\gamma) \;=\; ((-2\delta+\gamma A_2)\alpha^{-1}, (\delta-2\gamma)\alpha^{-1}).$$

For each $\beta \in M(2r-n)$ invertible, let $v_{\alpha\beta}$ denote the pair of matrices in $M((2r-n)\times(n-r))$ given by

$$v_{\alpha\beta} \;=\; (\beta A_3 \alpha^{-1}, \beta B_3 \alpha^{-1}).$$

Hence, (ψ_1,ψ_2) and (ψ_1',ψ_2') are linearly equivalent if, and only if, $A_2' = \alpha A_2 \alpha^{-1}$ and $(A_3', B_3') \in \mathrm{Im}(\tau_{v_{\alpha\beta}} \circ L_\alpha)$ for some $\alpha \in M(n-r)$ invertible and for some $\beta \in M(2r-n)$ also invertible, where $\tau_{v_{\alpha\beta}}$ is the translation in the $v_{\alpha\beta}$-direction on $M((2r-n)\times(n-r))\times M((2r-n)\times(n-r))$.

Returning to our purposes, since 4 is not an eigenvalue of A_2, we have that L_α is an isomorphism. So

$$\mathrm{Im}(T_{v_{\alpha\beta}} \circ L_\alpha) \;=\; \mathrm{Im}(L_\alpha) \;=\; M((2r-n)\times(n-r))\times M((2r-n)\times(n-r))$$

for any β.

Then (ψ_1,ψ_2) and (ψ_1',ψ_2') are linearly equivalent if, and only if, (ϕ_1,ϕ_2) and (ϕ_1',ϕ_2') are linearly equivalent, where

$$\phi_1 \;=\; \left(\begin{array}{c|c} -I_{n-r} & 0 \\ \hline A_2 & I_{n-r} \end{array}\right), \qquad \phi_2 \;=\; \left(\begin{array}{c|c} I_{n-r} & I_{n-r} \\ \hline 0 & -I_{n-r} \end{array}\right)$$

and

$$\phi_1' \;=\; \left(\begin{array}{c|c} -I_{n-r} & 0 \\ \hline A_2' & I_{n-r} \end{array}\right), \qquad \phi_2' \;=\; \left(\begin{array}{c|c} I_{n-r} & I_{n-r} \\ \hline 0 & -I_{n-r} \end{array}\right).$$

Hence, according to the above, we can in fact take $A_3 = B_3 = 0$ in our initial form.

Therefore, we have established the following classification result:

Theorem 3.3. *Let φ_1, φ_2 be transversal linear involutions on $\mathbf{R}^n$, $n \geq 2$, with $\varphi_1 \circ \varphi_2$ normally hyperbolic and let $r = \dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2)$. Then $n/2 \leq r \leq n-1$ and the*

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pair (φ_1, φ_2) *is linearly equivalent to a pair* (ψ_1, ψ_2) *such that* ψ_1 *and* ψ_2 *have matrices*

$$\psi_1 = \left(\begin{array}{cc|c} \begin{array}{c|c} -I_{n-r} & 0 \\ \hline A & I_{n-r} \end{array} & & 0 \\ \hline \begin{array}{c|c} 0 & 0 \end{array} & & I_{2r-n} \end{array}\right), \quad \psi_2 = \left(\begin{array}{cc|c} \begin{array}{c|c} I_{n-r} & I_{n-r} \\ \hline 0 & -I_{n-r} \end{array} & & 0 \\ \hline \begin{array}{c|c} 0 & 0 \end{array} & & I_{2r-n} \end{array}\right),$$

with the submatrices

$$\phi_1 = \left(\begin{array}{c|c} -I_{n-r} & 0 \\ \hline A & I_{n-r} \end{array}\right), \quad \phi_2 = \left(\begin{array}{c|c} I_{n-r} & I_{n-r} \\ \hline 0 & -I_{n-r} \end{array}\right)$$

in the conditions of Theorem 3.2.

We now apply Theorem 3.3 to present the explicit classification of pairs in certain specific dimensions:

- $n \geq 3$ and $r = n - 1$

We first notice that in these dimensions the transversality is an implicit property from the normal hyperbolicity. We have that $\operatorname{tr}(\varphi_1 \circ \varphi_2) > n$ or $< n - 4$, and the normal form for the pair (φ_1, φ_2) is given by the pair (ψ_1, ψ_2),

$$\psi_1 = \left(\begin{array}{cc|c} -1 & 0 & \\ a & 1 & 0 \\ \hline & 0 & I_{n-2} \end{array}\right), \quad \psi_2 = \left(\begin{array}{cc|c} 1 & 1 & \\ 0 & -1 & 0 \\ \hline & 0 & I_{n-2} \end{array}\right), \tag{15}$$

with $a = 4 - n + \operatorname{tr}(\varphi_1 \circ \varphi_2)$.

- $n \geq 3$ and $r = 1$

Here, the transversality fails for φ_1 and φ_2. However, with Remark 3.1 in mind, we observe that the pair $(-\varphi_1, -\varphi_2)$ is under the conditions of the case above and, therefore, the normal form of the pair (φ_1, φ_2) is given by $(-\psi_1, -\psi_2)$ with ψ_1 and ψ_2 as in (15).

- $n \geq 5$ and $r = n - 2$

For such dimensions, transversality may not occur. Hence, we assume this condition to apply Theorem 3.3. Therefore, the normal form of transversal pairs is given by (ψ_1, ψ_2),

$$\psi_1 = \left(\begin{array}{cc|c} \begin{array}{c|c} -I_2 & 0 \\ \hline A & I_2 \end{array} & & 0 \\ \hline \begin{array}{c|c} 0 & 0 \end{array} & & I_{n-4} \end{array}\right), \quad \psi_2 = \left(\begin{array}{cc|c} \begin{array}{c|c} I_2 & I_2 \\ \hline 0 & -I_2 \end{array} & & 0 \\ \hline \begin{array}{c|c} 0 & 0 \end{array} & & I_{n-4} \end{array}\right), \tag{16}$$

where A is a Jordan matrix that can be taken in terms of the original pair (φ_1, φ_2) in the way that has been done in the end of subsection 3.3 for dimension 4, with appropriate adaptations for the dimensions considered here.

Remark 3. 2. For linear involutions φ_1 and φ_2 such as in the beginning of this section, we conclude from equality (14) that

$$\mathcal{F}(\varphi_1 \circ \varphi_2) = \mathcal{F}(\varphi_1) \cap \mathcal{F}(\varphi_2), \tag{17}$$

since the intersection also has dimension $2r - n$. This is precisely what Proposition 3.1(a) reduces to under transversality and normal hyperbolicity.

Moreover, this result generalizes provided φ_1 and φ_2 are transversal involutions on $(\mathbf{R}^n, 0)$ with normally hyperbolic composition. In fact, we can assume

$$\mathcal{F}(\varphi_i) = \mathcal{F}(d\varphi_i(0)), \quad i = 1, 2$$

in $(\mathbf{R}^n, 0)$. Hence, locally we have

$$\mathcal{F}(\varphi_1) \cap \mathcal{F}(\varphi_2) = \mathcal{F}(d(\varphi_1 \circ \varphi_2)(0)) = T_0\mathcal{F}(\varphi_1 \circ \varphi_2),$$

where the first equality is obtained from (17). So $\mathcal{F}(\varphi_1) \cap \mathcal{F}(\varphi_2)$ is a submanifold of $\mathcal{F}(\varphi_1 \circ \varphi_2)$ of same dimension and, therefore, $\mathcal{F}(\varphi_1 \circ \varphi_2) = \mathcal{F}(\varphi_1) \cap \mathcal{F}(\varphi_2)$ in $(\mathbf{R}^n, 0)$.

4. THE PROOF OF THE LINEARIZATION THEOREM

In this section we prove Theorem 2.1. We start with a remark about the grounds for the last hypothesis of this theorem. When φ_1 and φ_2 are linear, this assumption is already a consequence of the normal hyperbolicity of the composition $\varphi_1 \circ \varphi_2$. In fact, in this case, either of them takes the stable subspace $E^s(\varphi_1 \circ \varphi_2)$ of $\varphi_1 \circ \varphi_2$ to the unstable subspace $E^u(\varphi_1 \circ \varphi_2)$, and vice-versa, so leaving the sum $E^s(\varphi_1 \circ \varphi_2) \oplus E^u(\varphi_1 \circ \varphi_2)$ invariant. Also, from (17), we have

$$\mathcal{F}(\varphi_1 \circ \varphi_2) = \mathcal{F}(\varphi_1) \cap \mathcal{F}(\varphi_2) \subseteq \mathcal{F}(\varphi_i). \quad i = 1, 2.$$

Before we go into the proof itself, we need two lemmas.

Let $C_b^0(\mathbf{R}^n)$ denote the space of bounded continuous mappings $\mathbf{R}^n \to \mathbf{R}^n$. The first lemma is a particular case of the assertion that appears in [8, Theorem 2.1], for the hyperbolic bundle automorphism (4):

LEMMA 4.1. *Let $L : \mathbf{R}^n \to \mathbf{R}^n$ be a linear normally hyperbolic isomorphism. There exists $\epsilon > 0$ such that if $g \in C_b^0(\mathbf{R}^n)$ has Lipschitz constant bounded by ϵ, $L + g$ covers the identity $I : \mathcal{F}(L) \to \mathcal{F}(L)$ and $\mathcal{F}(L + g) \supseteq \mathcal{F}(L)$, then there is a unique homeomorphim $h : \mathbf{R}^n \to \mathbf{R}^n$ also covering $I : \mathcal{F}(L) \to \mathcal{F}(L)$, of the form $h = I + \eta$, with $\eta \in C_b^0(\mathbf{R}^n)$ and $\eta|_{\mathcal{F}(L)} \equiv 0$, which is a conjugacy between $L + g$ and L.*

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LEMMA 4.2. *Let* (φ_1, φ_2) *be a pair of involutions on* $(\mathbf{R}^n, 0)$ *under the assumptions of Theorem 2.1. Given* $\epsilon > 0$*, there exist involutory extensions* $\tilde{\varphi}_1, \tilde{\varphi}_2 : \mathbf{R}^n \to \mathbf{R}^n$ *of* φ_1, φ_2 *in such a way that* $\tilde{\varphi}_1 \circ \tilde{\varphi}_2 = d(\varphi_1 \circ \varphi_2)(0) + g$*, where* $g \in C_b^0(\mathbf{R}^n)$ *has Lipschitz constant bounded by* ϵ*,* $\tilde{\varphi}_1 \circ \tilde{\varphi}_2$ *covers the identity* $I : \mathcal{F}(d(\varphi_1 \circ \varphi_2)(0)) \to \mathcal{F}(d(\varphi_1 \circ \varphi_2)(0))$ *and* $\mathcal{F}(\tilde{\varphi}_1 \circ \tilde{\varphi}_2) \supseteq \mathcal{F}(d(\varphi_1 \circ \varphi_2)(0))$.

Proof. The extension process for each involution φ_i, $i = 1, 2$, is the same. In a neighbourhood V_i of the origin we take the C^∞ coordinate system h_i defined by $h_i = I + k_i$, where $k_i = \frac{1}{2}\big(d\varphi_i(0) \circ \varphi_i - I\big)$. We then have that $dk_i(0) = 0$ and, by Lemma 2.1, $\varphi_i = h_i^{-1} \circ d\varphi_i(0) \circ h_i$. For any $\delta_i > 0$, the neighbourhood V_i can be considered in such a way that k_i has Lipschitz constant equal to δ_i.

Since locally each φ_i respects the fiber bundle in (4) for $L = d(\varphi_1 \circ \varphi_2)(0)$ and $d\varphi_i(0)$ has also this property, it follows that the image of k_i is a subset of $E^s \oplus E^u$. Moreover, k_i vanishes on $\mathcal{F}(\varphi_i)$, for $\mathcal{F}(\varphi_i) = \mathcal{F}(d\varphi_i(0))$ by hypothesis. We now consider $B[0, r_i] \subseteq V_i$ be the closed ball with center at the origin and ratio $r_i > 0$ and define $K_i : \mathbf{R}^n \to \mathbf{R}^n$ by

$$K_i(x) = \begin{cases} k_i(x), & ||x|| \leq r_i \\ k_i(r_i \frac{x}{||x||}), & ||x|| > r_i \end{cases}$$

K_i is a bounded C^0-extension of k_i, which is zero on $\mathcal{F}(d\varphi_i(0))$ and have Lipschitz constant equal to $2\delta_i$. Taking $\delta_i < 1/2$, we have that K_i is a contraction, so $H_i = I + K_i$ is a homeomorphism on $\mathbf{R}^n$ by the perturbation of the identity theorem. Its inverse is written as $H_i^{-1} = I + \tilde{K}_i$, with $\tilde{K}_i = -K_i \circ H_i^{-1}$, and so $\tilde{K}_i \in C_b^0(\mathbf{R}^n)$. Furthermore, $\tilde{K}_i$ is Lipschitzian with Lipschitz constant $2\delta_i/(1 - 2\delta_i)$.

Next we define

$$\tilde{\varphi}_i = H_i^{-1} \circ d\varphi_i(0) \circ H_i,$$

which is obviously of the form $\tilde{\varphi}_i = d\varphi_i(0) + g_i$, with $dg_i(0) = 0$, and satisfies $\tilde{\varphi}_i \circ \tilde{\varphi}_i = I$. In addition, we can compute g_i to get

$$g_i = d\varphi_i(0) \circ K_i + \tilde{K}_i \circ d\varphi_i(0) \circ H_i. \tag{18}$$

Hence, $g_i \in C_b^0(\mathbf{R}^n)$.

Now, $\tilde{\varphi}_1 \circ \tilde{\varphi}_2 = d(\varphi_1 \circ \varphi_2)(0) + g$, where

$$g = d\varphi_1(0) \circ g_2 + g_1 \circ \tilde{\varphi}_2.$$

It then follows that g is bounded and, based on (18), for a given ϵ we choose δ_1 and δ_2 so that g is Lipschitzian with Lipschitz constant bounded by ϵ.

Yet, it is easy to see that each $\tilde{\varphi}_i$, $i = 1, 2$, covers the identity $I : \mathcal{F}(d(\varphi_1 \circ \varphi_2)(0)) \to \mathcal{F}(d(\varphi_1 \circ \varphi_2)(0))$, so does $\tilde{\varphi}_1 \circ \tilde{\varphi}_2$. Finally, since

$$\mathcal{F}(\tilde{\varphi}_i) = H_i(\mathcal{F}\big(d\varphi_i(0)\big) = \mathcal{F}(d\varphi_i(0)),$$

then

$$\mathcal{F}(\tilde{\varphi}_1 \circ \tilde{\varphi}_2) \supseteq \mathcal{F}(\tilde{\varphi}_1) \cap \mathcal{F}(\tilde{\varphi}_2) = \mathcal{F}\big(d(\varphi_1(0)\big) \cap \mathcal{F}\big(d(\varphi_2(0)\big) = \mathcal{F}\big(d(\varphi_1 \circ \varphi_2)(0)\big).$$

▮

Proof of Theorem 2.1. Let $\epsilon > 0$ be as in Lemma 4.1 for $L = L_1 \circ L_2$. Given this ϵ, we can take involutory extensions $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ of φ_1 and φ_2 as in the proof of Lemma 4.2.

The composition $\tilde{\varphi}_1 \circ \tilde{\varphi}_2 = L_1 \circ L_2 + g$ is in the hypothesis of Lemma 4.1. Hence, there exists a unique homeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ covering the identity $I : \mathcal{F}(L_1 \circ L_2) \to \mathcal{F}(L_1 \circ L_2)$, of the form $h = I + \eta$, with $\eta \in C_b^0(\mathbf{R}^n)$ and $\eta|_{\mathcal{F}(L_1 \circ L_2)} \equiv 0$, which is a conjugacy between $\tilde{\varphi}_1 \circ \tilde{\varphi}_2$ and $L_1 \circ L_2$, i.e.,

$$h \circ (\tilde{\varphi}_1 \circ \tilde{\varphi}_2) \circ h^{-1} = L_1 \circ L_2. \tag{19}$$

The last step is to show that h realizes the desired C^0-equivalence. From (19) we have

$$L_1(h \circ \tilde{\varphi}_1 \circ \tilde{\varphi}_2 \circ h^{-1}) \circ L_1^{-1} = L_2 \circ L_1, \tag{20}$$

then,

$$(L_1 \circ h \circ \tilde{\varphi}_1) \circ (\tilde{\varphi}_2 \circ \tilde{\varphi}_1) \circ (L_1 \circ h \circ \tilde{\varphi}_1)^{-1} = L_2 \circ L_1.$$

Hence, the homeomorphism $L_1 \circ h \circ \tilde{\varphi}_1$ is a conjugacy between $\tilde{\varphi}_2 \circ \tilde{\varphi}_1$ and $L_2 \circ L_1$ and, therefore, between $\tilde{\varphi}_1 \circ \tilde{\varphi}_2$ and $L_1 \circ L_2$. But $L_1 \circ h \circ \tilde{\varphi}_1$ satisfies the same other conditions of h described above. By uniqueness, this implies that

$$h = L_1 \circ h \circ \tilde{\varphi}_1,$$

that is,

$$L_1 = h \circ \tilde{\varphi}_1 \circ h^{-1}.$$

In the same way

$$L_2 = h \circ \tilde{\varphi}_2 \circ h^{-1}.$$

REFERENCES

1. H. Cartan, Sur les groupes de transformations analytiques. *Actualits Scientifiques et Indutrielles.* Paris, **198**, 1935.
2. A. Jacquemard, M.A. Teixeira, Reversible normal forms of degenerate cusps for planar diffeomorphisms. *Bull. Sci. Math.***130** (1998) 409-425.
3. A. Jacquemard, M.A. Teixeira, A note on rigid decompositions of reversible mappings. *Fiels Inst. Comm.* **31** (2002) 189-199.
4. J.S.W. Lamb, M. Roberts, Time-reversal symmetry in dynamical systems: a survey, *Physica D*, **112** (1998) 1-39.
5. D. Montgomery, L. Zippin, *Topological Transformation Groups*, Interscience, New York, 1955.

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6. S. Mancini, M. Manoel, M.A. Teixeira, Divergent diagrams of folds and simultaneous conjugacy of involutions. *Disc. Cont. Dyn. Sys.* **4** (2005) 657-674.
7. J. Palis Jr, W. Melo, *Introdução aos sistemas dinâmicos*, Projeto Euclides (1978).
8. C. Pugh, M. Schub, Linearization of normally hyperbolic diffeomorphisms and flows. *Inventiones Math.* **10** (1970) 187–198.
9. M.A. Teixeira, Local and simultaneous structural stability of certain diffeomorphisms. In: *Proc. of Dyn. Sys., Stab. and Turb.*, Warwick. Lect. Notes in Math. **898** (1981) 382–390.
10. S.M. Voronin, Analytic classification of pairs of involutions and its applications. *Funct. Anal. Appl.* **2** 16 (1982) 21-29.

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