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On the continuation of solutions of non-autonomous semilinear parabolic problems

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In this paper we study evolutionary problems which fall into a class of singularly nonautonomous parabolic equations with critical exponents in a Banach scale $\{E_\sigma, \sigma \in [0,\mu)\}$. We obtain existence-uniqueness result concerning a suitable notion of $E_{1+\varepsilon}$-solution and describe continuation properties of this solution. This concerns both a situation when the solution can be continued as $E_{1+\varepsilon}$-solution and a situation when $E_{1+\varepsilon}$-norm of the solution 'blows-up', in which case a piecewise-$E_{1+\varepsilon}$-solution can only be constructed. This extends the results reported in *Journal of Mathematical Analysis and Applications (2005), doi:10.1016/j.jmaa.2005.02.024*, [11], to essentially larger class of parabolic problems. We furthermore show that for singularly nonautonomous problems moduli of continuity of the main part operator can influence the growth of the nonlinear term, for which the problem can be well posed in E_1. May, 2012 ICMC-USP

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1. INTRODUCTION

In the studies of long time behavior of dynamical systems governed by differential equations it is of fundamental importance to establish that the problem is suitably well posed in a large space of initial data. Once this is done questions concerning further regularity properties of solutions and the existence of a suitable notion of attractors can be addressed, which enables then to obtain more relevant information about the behavior of the system.

In an infinite dimensional space well posedness results are far from obvious and the theory is far from complete. For example, if we are interested in questions concerning well posedness of the Cauchy problem for the nonlinear equation of the form

$$u_t + G(t,u) = 0,\ t > \tau, \tag{1.1}$$

where $G : \mathbb{R} \times D \subset E \to E$ is an unbounded map, then even very simple situations can bring a negative answer. Specifying $E = L^2(\Omega)$, $D = H^2(\Omega) \cap H^1_0(\Omega)$ and $G(t,u) = -\Delta u$ one gets a sample problem which is known to be ill-posed. It is thus crucial to investigate some suitable conditions on G which would lead to well posedness of (1.1) in a large space of initial data.

When

$$G(t,u) = Au + f(t,u) \tag{1.2}$$

and $-A$ generates a C^0 analytic semigroup in E, let us say $A = -\Delta$, then such conditions are already known, including the case of critical exponents, as one can consider f as a Lipschitz map satisfying suitable boundedness conditions with respect to A. When

$$G(t,u) = A(t)u + f(t,u), \tag{1.3}$$

that is if the operator A is also allowed to depend on time variable, the theory does not seem to be complete and extensions of the results known from the time-independent case, if they are possible, require some special attention.

To indicate a sort of difficulties that may arise in that latter case and, simultaneously, to exhibit a sort of assumptions that may be needed when G is as in (1.3) let us mention that (1.1) can be then looked upon from another perspective as an equation of the form

$$\dot u(t) + Au(t) = \mathcal{F}(t, u(t), A(t)),\ t > \tau, \tag{1.4}$$

with a linear main part operator A equal now to $A(s)$ for some fixed $s \in \mathbb{R}$, that is being independent of time variable, and the right hand side given by

$$\mathcal{F}(t, u(t), A(t)) = F(t, u(t)) + A(t) - A(s).$$

This indicates that without suitable assumption on boundedness of $A(t) - A(s)$ relative to A, the problem may be easily ill posed.

Following ideas of [3, Chapter V] we will consider next a family of Banach spaces

$$\{E_\sigma, \sigma \in [0, 1+\mu)\} \tag{1.5}$$

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with some $\mu \in (0,1]$. In applications (1.5) can be characterized as an interpolation or a fractional power scale and it will naturally enjoy dense embedding properties, which sometimes will even be compact.

We remark that a situation does arise in applications when only a "portion of a scale" defined by a linear part of the equation is "effectively available" (see [15, 21]), for which we mention a sample nonautonomous fourth order equation in $L^p(\mathbb{R}^N)$

$$u_t + A_m u = f(t,x,u), \quad \text{where } A_m = \Delta^2 - m(x)Id.$$

As shown in [15], for "poorly integrable potentials" $m(\cdot)$ of the class $L^r_U(\mathbb{R}^N)$ with $p, p' > r > \max\{\frac{N}{4}, 1\}$, the associated fractional power spaces can be characterized with the aid of Bessel potentials spaces as $H^{4\sigma}_p(\mathbb{R}^N)$ as long as $\sigma \in [-1 - \frac{N}{4p'} + \frac{N}{4r}, 1 + \frac{N}{4p} - \frac{N}{4r}]$ and that on this portion of the scale the autonomous process $U(t,s) = e^{-(t-s)A_m}$, $\infty > t \geqslant s > -\infty$ governed by $-A_m : D(A_m) \subset L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ satisfies, for some $\omega \in \mathbb{R}$,

$$\|U(t,s)\|_{\mathcal{L}(H^{4\sigma}_p(\mathbb{R}^N), H^{4\xi}_p(\mathbb{R}^N))} \leqslant M \frac{e^{-\omega(t-s)}}{(t-s)^{\xi-\sigma}} \quad t > s,\ \sigma \in [-1 - \frac{N}{4p'} + \frac{N}{4r}, 1 + \frac{N}{4p} - \frac{N}{4r}].$$

Note that the length of the latter interval is equal to $2 + \frac{N}{4} - \frac{N}{2r}$ and that it actually tends to one as $r \searrow \frac{N}{4}$ and $N = 4$.

In this paper, given a family of Banach spaces $\{E_\sigma, \sigma \in [0, 1+\mu)\}$, and given a family of unbounded operators $A(t) : E_1 \subset E_0 \longrightarrow E_0$, $t \in \mathbb{R}$, we consider a Cauchy problem of the form

$$\begin{cases} \dot{u}(t) + A(t)u(t) = F(t, u(t)),\ t > \tau, \\ u(\tau) = u_\tau, \end{cases} \tag{1.6}$$

with F satisfying suitable Lipschitz and growth conditions and with u_τ lying in some large "phase" space of admissible initial data.

Our main concern is the situation when the linear main part operator essentially depends on the time variable, that is the problem is in this sense *singularly nonautonomous* (see [13]) in contrary to the case when the time dependence enters merely the nonlinear term F. On the other hand our concern will be *critically growing nonlinearities*, that is roughly speaking we will allow $F(t, u(t))$ to exhibit the same order of magnitude as the linear main part operator $A(t)$ (see [7, 8, 13, 11]. As mentioned above the latter leads to essential difficulties as even for globally Lipschitz perturbations, like $F(t,u) = 2A(t)u$, the problem may be ill posed. On the other hand note that in the singularly nonautonomous case some previous results concerning continuation and asymptotic properties of solutions, see [11], cannot be directly applied and require essential modifications. This will be our main goal in the present paper.

We remark that even very simple problems, like

$$\begin{cases} u_t + \nabla(\theta(t,x)\nabla u) = f(u),\ t > \tau,\ x \in \Omega, \\ u(t,x) = 0,\ t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), x \in \Omega, \end{cases} \tag{1.7}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, can still bring questions that are not immediate to answer. Namely, assuming that there exists some $\rho > 1$ such that $|f(u)| = O(|u|^\rho)$ and $\theta, \theta'_{x_1}, \dots, \theta'_{x_N} \in C^\mu_{loc}(\mathbb{R}\times\overline{\Omega}, \mathbb{R})$ for some $\mu > 0$, is (1.7) well posed in $L^2(\Omega)$? Without being to exhaustive we merely remark here that the value of μ influences the smoothness property of the process associated with the linear main part, which in turn determines possibility of a satisfactory treatment of the critical exponents. This will be described in larger capacity in the main body of the paper, including also more general applications than (1.7). From this it will be seen that the exponent ρ for which (1.7) is well posed in $L^2(\Omega)$ actually depends on μ. Namely, when $\mu > \frac{1}{7}$, local well posedness in $L^2(\Omega)$ can be obtained for $\rho \in (1, 1 + \frac{4}{N}]$ including the critical case $\rho = 1 + \frac{4}{N}$ and, with some additional conditions on f, continuation result holds as well even in an almost critical case. In contrary, for $\mu < \frac{1}{7}$, $N = 3$, one can observe that local well posedness in the critical case $\rho = \frac{7}{3}$ can hardly be proved.

In what follows, to describe our results better, we will need some assumptions and definitions. For this it seems reasonable to look first at the variation of constants formula for (1.6)

$$u(t) = U(t,\tau)u_\tau + \int_\tau^t U(t,s)F(s,u(s))ds,$$

where

$$U(t,\tau) = e^{-(t-\tau)A(\tau)} + \int_\tau^t U(t,s)(A(\tau)-A(s))e^{-(s-\tau)A(\tau)}ds$$

(see [23, (1.5), (1.8)]). From these formulas one gets the expression

$$u(t) = e^{-(t-\tau)A(\tau)}u_\tau + \int_\tau^t U(t,s)(A(\tau)-A(s))e^{-(s-\tau)A(\tau)}u_\tau ds + \int_\tau^t U(t,s)F(s,u(s))ds$$

and hence it becomes clear that, above all, we need to ensure suitable integrability properties of the functions under the above integrals, which in turn leads to a number of technical conditions concerning both linear and nonlinear part of the equation.

We thus turn our attention towards a singularly nonautonomous linear problem

$$\begin{cases} \dot{u}(t) + A(t)u(t) = 0, \ t > \tau, \\ u(\tau) = u_\tau, \end{cases} \tag{1.8}$$

for which we will assume that it defines a process in E_0, which possesses suitable smoothing properties. Recall that a two parameter family $\{U(t,\tau) : \ (t,\tau) \in \mathbb{R}^2, \ t \geqslant \tau\}$ of maps $U(t,\tau) : V \to V$ is an *evolution process* in V (process for short) provided that

$$U(\tau,\tau) = Id \quad \text{and} \quad U(t,\sigma)U(\sigma,\tau) = U(t,\tau) \ \text{ for any } \ t \geqslant \sigma \geqslant \tau \in \mathbb{R}.$$

If, in addition, V is a topological space and $\{(t,s) \in \mathbb{R}^2 : t \geqslant s\} \times V \ni (t,\tau,v) \mapsto U(t,\tau)v \in V$ is a continuous map then we say that the process $\{U(t,\tau) : \ (t,\tau) \in \mathbb{R}^2, \ t \geqslant \tau\}$ is continuous.

Concerning the linear part of (1.6) we will need the following assumption.

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Assumption 1.1. *Given a family of Banach spaces $\{E_\alpha, \alpha \in [0,\mu)\}$ there exists a continuous process $\{U(t,\tau): (t,\tau) \in \mathbb{R}^2,\ t \geqslant \tau\} \subset L(E_0)$ defined by (1.8) in E_0. Furthermore, given any point $\tau \in \mathbb{R}$ there is a certain "time" interval $I \subset \mathbb{R}$ centered at τ such that for any $0 \leqslant \zeta \leqslant \sigma < 1+\mu$ one can choose a positive constant M for which we have*

$$\|U(t,\tau)\|_{L(E_\zeta,E_\sigma)} \leqslant M(t-\tau)^{\zeta-\sigma},\ t,\tau \in I,\ t > \tau. \tag{1.9}$$

Concerning the nonlinear part of (1.6) we will assume that for a certain $\varepsilon \in (0,\mu)$ the right hand side term F belongs to a class of maps satisfying a suitable Lipschitz type condition relative to $\{E_\alpha, \alpha \in [0,\mu)\}$; see Definition 1.1 below. Note that any such F falls in particular into the class of ε-regular maps originally considered in [7].

Definition 1.1. We say that a continuous function $F : \mathbb{R} \times E_{1+\varepsilon} \to E_{\gamma(\varepsilon)}$ is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ of Lipschitz maps relative to $\{E_\alpha, \alpha \in [0, 1+\mu)\}$ with constants $\rho > 1$, $0 < \varepsilon < \min\{\frac{1}{\rho}, \mu\}$, $\gamma(\varepsilon) \in [\rho\varepsilon, 1)$, $\eta > 0$ and $C_\eta > 0$ if and only if for any bounded time interval $I \subset \mathbb{R}$ there exists $c > 0$ such that for each $v, w \in E_{1+\varepsilon}$, $t \in I$ we have

$$\|F(t,v) - F(t,w)\|_{E_{\gamma(\varepsilon)}} \leqslant c\|v-w\|_{E_{1+\varepsilon}}(\eta\|v\|^{\rho-1}_{E_{1+\varepsilon}} + \eta\|w\|^{\rho-1}_{E_{1+\varepsilon}} + C_\eta) \tag{1.10}$$

and

$$\|F(t,v)\|_{E_{\gamma(\varepsilon)}} \leqslant c(\eta\|v\|^{\rho}_{E_{1+\varepsilon}} + C_\eta). \tag{1.11}$$

It may be sometimes useful to consider constant C_η of F depending on time as well. If I is a fixed bounded time interval, let us say $I = [0,T]$, then one can replace C_η in (1.10) and (1.11) by $C_\eta(t)t^{\varepsilon-\gamma(\varepsilon)}$ and $C_\eta(t)t^{-\gamma(\varepsilon)}$ respectively, requiring then that $C_\eta(t)$ is a certain nonnegative function converging to zero as $t \to 0^+$ (see [7, 13]). Nonetheless, since we do not restrict our consideration to any particular time interval, we will not pursue this generalization here.

Let us single out for special attention the case when in (1.10)-(1.11) one can only have $\gamma(\varepsilon) = \rho\varepsilon$ and not $\gamma(\varepsilon) \in (\rho\varepsilon, 1)$ as this exhibits the case of *critical exponents* according to [7], that is the largest growth allowed for the nonlinear term in a given class of initial data (see [7]).

Definition 1.2. In the case when for a certain $\eta > 0$ (1.10)-(1.11) hold with $\gamma(\varepsilon) \in (\rho\varepsilon, 1)$ we say that F is *subcritical*. When for a certain $\eta > 0$ (1.10)-(1.11) hold with $\gamma(\varepsilon) = \rho\varepsilon$ but not with $\gamma(\varepsilon) \in (\rho\varepsilon, 1)$, F is called *critical* and ρ is then called a *critical exponent*.

For critical F, we also single out for special attention the situation when $\eta > 0$ can be chosen arbitrarily small as the solutions will then enjoy much of the properties specific for a subcritical case.

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Definition 1.3. In the case when F is critical relative to (E_1, E_0) and (1.10)-(1.11) hold with any $\eta > 0$ we say that F is an *almost critical* map.

Let us make a technical remark that when F is subcritical, then without loss of generality one can always assume that the parameter $\eta > 0$ in (1.10)-(1.11) can be chosen arbitrarily small, which may not be the case when F is critical. Indeed, given $\rho > 1$, $\varepsilon \in (0, \frac{1}{\rho})$, $\varepsilon < \mu$, $\gamma(\varepsilon) \in (\rho\varepsilon, 1)$ we can choose $\tilde{\rho} > \rho$ in such a way that we will have $\varepsilon \in (0, \frac{1}{\tilde{\rho}})$ and $\gamma(\varepsilon) \in (\tilde{\rho}\varepsilon, 1)$. On the other hand the term $\eta \|w\|^{\rho-1}_{E_{1+\varepsilon}}$ can be then estimated with the aid of Hölder's inequality by $\tilde{\eta}\|w\|^{\tilde{\rho}-1}_{E_{1+\varepsilon}} + c_{\tilde{\eta},\eta}$, which yields (1.10)-(1.11) with parameters $\varepsilon, \gamma(\varepsilon)$ as before, ρ being replaced by $\tilde{\rho} > \rho$ chosen suitably close to ρ and η replaced by $\tilde{\eta}$, which we can fix as small as we wish. Note that after this change the parameters ε, $\gamma(\varepsilon)$ and c in (1.10)-(1.11) remain the same and the only difference will come from the replacement of C_η by $C_\eta + c_{\tilde{\eta},\eta}$, which will not influence the heart of our consideration.

We will introduce next a suitable notion of solution of (1.6) (see [13]; also [7, 8, 11]).

Definition 1.4. Given F of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha, \alpha \in [0, \mu)\}$, $\tau > 0$, $u_\tau \in E_0$ and given an interval $[\tau, T]$, we say that $u : [\tau, T] \to E_{1+\varepsilon} \cup E_0$ is a mild $E_{1+\varepsilon}$-solution ($E_{1+\varepsilon}$-solution for short) of (1.6) on the interval $[\tau, T]$ if and only if
(i) $u \in L^\infty_{loc}((\tau, T], E_{1+\varepsilon})$,
(ii) there exists the limit $\lim_{t\to\tau^+}(t-\tau)^\varepsilon \|u(t)\|_{E_{1+\varepsilon}} = 0$,
(iii) $u(\tau) = u_\tau$ and u satisfies on for a.e. $t \in [\tau, T]$ the variation of constants formula

$$u(t) = U(t,\tau)u_\tau + \int_\tau^t U(t,s)F(s,u(s))ds. \tag{1.12}$$

If, given $a \in (\tau, \infty]$, u is an $E_{1+\varepsilon}$-solution of (1.6) on $[\tau, T]$ for any $T \in (\tau, a)$, then we say that u is an $E_{1+\varepsilon}$-solution on the interval $[\tau, a)$.

To describe a possibly large set of initial data for which (1.6) will possess the unique $E_{1+\varepsilon}$-solution we will consider under Assumption 1.1 a linear subspace $\mathfrak{E}^\tau_\varepsilon$ of E_0

$$\mathfrak{E}^\tau_\varepsilon = \{\varphi \in E_0 : \text{ there exists } \lim_{t\to\tau^+} (t-\tau)^\varepsilon \|U(t,\tau)\varphi\|_{E_{1+\varepsilon}} = 0\}. \tag{1.13}$$

Then the function $\|\cdot\|^{\mathfrak{E}^\tau_\varepsilon}_\delta : \mathfrak{E}^\tau_\varepsilon \times \mathfrak{E}^\tau_\varepsilon \to [0, \infty)$ below is well defined for any $\delta > 0$,

$$\|\varphi\|^{\mathfrak{E}^\tau_\varepsilon}_\delta = \sup_{t\in(\tau,\tau+\delta]} (t-\tau)^\varepsilon \|U(t,\tau)\varphi\|_{E_{1+\varepsilon}}, \ \varphi \in \mathfrak{E}^\tau_\varepsilon, \tag{1.14}$$

and for any $\mathfrak{E}^\tau_\varepsilon$ we have

$$\|\varphi\|^{\mathfrak{E}^\tau_\varepsilon}_\delta \to 0 \ \text{ as } \ \delta \to 0^+.$$

Remark 1. 1. Note that with the Assumption 1.1, given $\tau \in \mathbb{R}$, $\varepsilon < \mu$ and assuming the inclusion $E_{1+\varepsilon} \subset E_0$, we have that $E_{1+\varepsilon} \subset \mathfrak{E}^\tau_\varepsilon$ and $\|\cdot\|^{\mathfrak{E}^\tau_\varepsilon}_\delta$ is the norm in $\mathfrak{E}^\tau_\varepsilon$. If $E_{1+\varepsilon}$ is dense in E_1 and $E_1 \subset E_0$ it is easy to see that $E_1 \subset \mathfrak{E}^\tau_\varepsilon$ and, following [13, Lemma 3.2],

$$\lim_{t\to\tau^+}(t-\tau)^\varepsilon \|U(t,\tau)\varphi\|_{E_{1+\varepsilon}} = 0 \text{ uniformly for } \varphi \text{ in compact sets of } E_1. \tag{1.15}$$

In what follows, given $w_0 \in \mathfrak{E}^\tau_\varepsilon$ and $\delta \in (0, \delta_{w_0}]$ we define

$$B^\delta_{\mathfrak{E}^\tau_\varepsilon}(w_0, r) = \{\varphi \in \mathfrak{E}^\tau_\varepsilon : \|\varphi - w_0\|^{\mathfrak{E}^\tau_\varepsilon}_\delta < r\}. \tag{1.16}$$

We also let $B(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1}ds$, $a, b > 0$, denote Euler's Beta function and define

$$B_{\varepsilon,\rho} := \max\{B(1-\rho\varepsilon, \gamma(\varepsilon)-\varepsilon), B(\gamma(\varepsilon)-\varepsilon, 1-\varepsilon)\}. \tag{1.17}$$

Remark 1. 2. If Assumption 1.1 holds then $\sup_{s\in(\tau,\tau+\delta]}(s-\tau)^\varepsilon\|U(s,\tau)\varphi\|_{E_{1+\varepsilon}} \leqslant M\|\varphi\|_{E_1}$ for any $\varphi \in E_1$. If in addition $E_{1+\varepsilon}$ is dense in E_1 and $E_1 \subset E_0$, combining this with Remark 1.1, we obtain that $B^\delta_{\mathfrak{E}^\tau_\varepsilon}(w_0, r)$ contains a ball in E_1 centered at w_0 of radius $\frac{r}{M}$, where M is as in (1.9).

With the above set-up we state the following local well posedness result, which complements earlier consideration of [7, Theorem 1], [11, Theorem 2.1] and [13, Theorem 3.1].

THEOREM 1.2. *Suppose that Assumption 1.1 holds and F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha, \alpha \in [0, \mu)\}$.*
Then,
i) given $t_0 \in \mathbb{R}$ and $w_0 \in \mathfrak{E}^\tau_\varepsilon$ and given τ in a certain interval $\mathcal{J} \subset \mathbb{R}$ centered at t_0 there exist $\bar\delta_0 \in (0, 1]$ such that for any initial condition u_τ satisfying

$$u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r) \tag{1.18}$$

with

$$\delta_0 \in (0, \bar\delta_0] \quad \text{and} \quad r := \frac{1}{4(8c\eta M B_{\varepsilon,\rho})^{\frac{1}{\rho-1}}}, \tag{1.19}$$

where $M = M(1+\varepsilon, \gamma(\varepsilon), \mathcal{J})$ and $B_{\varepsilon,\rho}$ are as in (1.9) and (1.17) respectively, there exists the unique $E_{1+\varepsilon}$-solution $u = u(\cdot, \tau, u_\tau)$ of (1.6) on $[\tau, \tau + \delta_0]$,
ii) when F is subcritical or F is almost critical, the time of existence δ_0 can be chosen uniformly with respect to initial condition $u_\tau \in B_{\mathfrak{E}^\tau_\varepsilon}(w_0, r)$ for arbitrarily large r,
iii) in the case when either $w_0 \in E_{1+\varepsilon} \subset E_0$ or $w_0 \in E_1 \subset E_0$ and $E_{1+\varepsilon}$ is dense in E_1, the time of existence δ_0 can be chosen uniformly with respect to $\tau \in \mathcal{J}$.
Furthermore, for any $0 \leqslant \theta < \min\{\gamma(\varepsilon), \mu\}$ we have

$$\lim_{t\to\tau^+}(t-\tau)^\theta\|u(t,\tau,u_\tau)\|_{E_{1+\theta}} = 0, \ u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r) \cap \mathfrak{E}^\tau_\theta, \tag{1.20}$$

$$\|u(t,\tau,u^1_\tau) - u(t,\tau,u^2_\tau)\|^{\mathfrak{E}^\tau_\theta}_{\delta_0} \leqslant C(\theta)\big(\|u^1_\tau - u^2_\tau\|^{\mathfrak{E}^\tau_\theta}_{\delta_0} + \|u^1_\tau - u^2_\tau\|^{\mathfrak{E}^\tau_\epsilon}_{\delta_0}\big), \ u^1_\tau, u^2_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r) \cap \mathfrak{E}^\tau_\theta \tag{1.21}$$

and the right hand side of (1.21) *is bounded by a multiple of* $\|u_\tau^1 - u_\tau^2\|_{E_1}$ *provided that* $u_\tau^1, u_\tau^2 \in E_1 \subset E_0$.

Finally,

$$\text{if } \lim_{t\to\tau^+}(t-\tau)^\theta \|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}} = 0 \text{ then } \lim_{t\to\tau^+}(t-\tau)^\theta \|u(t,\tau,u_\tau) - u_\tau\|_{E_{1+\theta}} = 0, \tag{1.22}$$

whenever $0 \leqslant \theta < \min\{\gamma(\varepsilon), \mu\}$ *and* $u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r)$.

Remark 1. 3. Under the assumptions of Theorem 1.2 if $u(\cdot,\tau,u_\tau)$ is the unique $E_{1+\varepsilon}$-solution from Theorem 1.2 then

$$u(\cdot,\tau,u_\tau) \in L^\infty((\tau,\tau+\delta_0], E_{1+\varepsilon}) \text{ whenever } u_\tau \in E_{1+\varepsilon}. \tag{1.23}$$

Note that the above consideration generalizes the concept of ε-regular solution introduced in [7] (see also [8, 11, 13]).

On the other hand note that more smoothing properties of the linear process will translate into more smoothing properties of $E_{1+\varepsilon}$-solutions. In particular, with Assumption 1.3 below, $E_{1+\varepsilon}$-solutions will “immediately” be Hölder continuous in $E_{1+\varepsilon}$.

Assumption 1.3. *Given any point* $\tau \in \mathbb{R}$ *there is a certain time interval* $I \subset \mathbb{R}$ *centered at* τ *such that whenever* $1+\mu > \zeta > \sigma \geqslant 0$, $1 \geqslant \zeta - \sigma > 0$ *one can choose a positive constant* M *for which we have*

$$\|U(t,\tau) - Id\|_{L(E_\zeta, E_\sigma)} \leqslant M(t-\tau)^{\zeta-\sigma}, \; t,\tau \in I, \; t > \tau. \tag{1.24}$$

Theorem 1.4. *Suppose that Assumption 1.1 is satisfied,* F *is of the class* $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ *relative to* $\{E_\alpha, \alpha \in [0,\mu)\}$, $\tau \in \mathbb{R}$, $u_\tau \in \mathfrak{E}^\tau_\varepsilon$ *and* $u = u(\cdot,\tau,u_\tau)$ *is the unique* $E_{1+\varepsilon}$*-solution of* (1.6) *on* $[\tau,\tau+\delta_0]$.

If, in addition, Assumption 1.3 holds then $u \in C^\nu_{loc}((\tau,\tau+\delta_0], E_{1+\theta})$ *for any* $0 < \theta < \min\{\gamma(\varepsilon),\mu\}$ *and* $\nu \in (0,\nu^*)$, *where* $\nu^* = \min\{\gamma(\varepsilon),\mu\} - \theta$.

Given $\tau \in \mathbb{R}$ and $u_\tau \in E_1$ we next define

$$I(u_\tau) := \{T \in (\tau,\infty) : \text{there exists the unique } E_{1+\varepsilon}\text{-solution of (1.6) on } [\tau,T]\}$$

and

$$T_{u_\tau} := \sup I(u_\tau). \tag{1.25}$$

Due to Theorem 1.2, $I(u_\tau) \neq \emptyset$. Thus, given $\tau \in \mathbb{R}$ and $u_\tau \in \mathfrak{E}^\tau_\varepsilon$, the problem (1.6) possesses the unique $E_{1+\varepsilon}$-solution u on $[\tau, T_{u_\tau})$. Note also that, due to (1.25), u cannot

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be continued as $E_{1+\varepsilon}$-solution of (1.6) on any time interval I_τ larger than $[\tau, T_{u_\tau})$. The interval $[\tau, T_{u_\tau})$ will be called *maximal interval of existence* of $E_{1+\varepsilon}$-solution and u on $[\tau, T_{u_\tau})$ will be called *maximally defined* $E_{1+\varepsilon}$-solution.

We now give the results that involve characterization of the maximal time of existence of $E_{1+\varepsilon}$-solutions in terms of E_1 or $E_{1+\varepsilon}$-norm. Note that $E_{1+\varepsilon}$ is the space on which a nonlinear term is well defined (see Definition 1.1) whereas E_1 plays a role of the "phase" space in which (1.6) defines a continuous process (see Remark 3.1).

Theorem 1.5. *Suppose that Assumption 1.1 holds, F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha, \alpha \in [0, \mu)\}$, $\tau \in \mathbb{R}$, $u_\tau \in \mathfrak{E}^\tau_\varepsilon$ and $u(\cdot) = u(\cdot, \tau, u_\tau)$ is $E_{1+\varepsilon}$-solution of* (1.6) *on a maximal interval of existence $[\tau, T_{u_\tau})$. Suppose additionally that $E_{1+\varepsilon}$ is dense in E_1 and $E_1 \subset E_0$.*
i) If F is subcritical or F is almost critical, then

$$T_{u_\tau} < \infty \quad \textit{implies that} \quad \limsup_{t \to T^-_{u_\tau}} \|u(t, \tau, u_\tau)\|_{E_1} = \infty. \tag{1.26}$$

ii) In either case when F is subcritical, almost critical, or F is critical, $T_{u_\tau} < \infty$ implies that there does not exist even one sequence $t_n \to T^-_{u_\tau}$, for which $\{u(t_n, \tau, u_\tau)\}$ converges in E_1; in particular the map $[\tau, T_{u_\tau}) \ni t \to u(t) \in E_1$ cannot be uniformly continuous.

Proposition 1.1. *Suppose that Assumption 1.1 holds, F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha, \alpha \in [0, \mu)\}$, $\tau \in \mathbb{R}$, $u_\tau \in \mathfrak{E}^\tau_\varepsilon$ and $u(\cdot) = u(\cdot, \tau, u_\tau)$ is $E_{1+\varepsilon}$-solution of* (1.6) *on a maximal interval of existence $[\tau, T_{u_\tau})$. Suppose additionally that $E_{1+\varepsilon} \subset E_0$.*
Then

$$T_{u_\tau} < \infty \quad \textit{implies that} \quad \limsup_{t \to T^-_{u_\tau}} \|u(t)\|_{E_{1+\varepsilon}} = \infty. \tag{1.27}$$

Due to Theorem 1.5 i), an a priori bound of $E_{1+\varepsilon}$-solution in E_1 actually implies that the solution exists for all $t \geqslant \tau$ even in a certain critical case. This is significant from the point of view of applications as in many problems $E_{1+\varepsilon}$-estimate required in Proposition 1.1 may be impossible to find, whereas the estimate in E_1 may follow from the natural properties of phenomena described by the equation like, for example, energy dissipation.

A situation may arise that F is critical, not subcritical or almost critical, and boundedness of $E_{1+\varepsilon}$-norm of the solution can hardly be ensured. Hence it is reasonable to consider another notion of solution of (1.1) (see Definition 1.5 below) and investigate a possibility of continuing an $E_{1+\varepsilon}$-solution even though its $E_{1+\varepsilon}$-norm may blow up as $t \to T_{u_\tau}$.

Definition 1.5. Suppose that F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha, \alpha \in [0, \mu)\}$, $\tau > 0$, $v_0 \in \mathfrak{E}^\tau_\varepsilon$ and $I_\tau \subset \mathbb{R}$ is an interval of the form $[\tau, a)$ or $[\tau, \infty)$.

We say that $\mathcal{U} : I_\tau \to E_{1+\varepsilon} \cup E_0$ is a *piecewise-$E_{1+\varepsilon}$-solution* of (1.6) on I_τ if and only if $\mathcal{U}(\tau) = u_\tau$ and, for each $T \in I_\tau \setminus \{\tau\}$, there exist a number $N_T \in \mathbb{N}$ and partition $\tau = \tau_0 < \tau_1 < ... < \tau_{N_T} < T = \tau_{N_T+1}$ of $[\tau, T]$ such that

$$\mathcal{U}(t) \to \mathcal{U}(\tau_{i-1}) \text{ in } E_0 \text{ as } t \to \tau^-_{i-1}, \quad i = 2, ..., N_T + 1, \tag{1.28}$$

$$\mathcal{U} \in L^{\infty}_{loc}((\tau_{i-1}, \tau_i), E_{1+\varepsilon}), \quad i = 1, ..., N_T + 1, \tag{1.29}$$

$$(t - \tau_{i-1})^{\varepsilon} \|u(t)\|_{E_{1+\varepsilon}} \overset{t \to \tau_{i-1}^{+}}{\longrightarrow} 0, \quad i = 1, ..., N_T + 1, \tag{1.30}$$

and

$$\mathcal{U}(t) = U(t, \tau_{i-1}) u_{\tau_{i-1}} + \int_{\tau_{i-1}}^{t} U(t,s) F(s, \mathcal{U}(s)) ds \text{ a. e. in } [\tau_{i-1}, \tau_i], \ i = 1, ..., N_T + 1. \tag{1.31}$$

If the interval $I_\tau = [\tau, a)$ is finite, $\mathcal{U} : [\tau, a) \to E_{1+\varepsilon} \cup E_0$ is a piecewise-$E_{1+\varepsilon}$-solution of (1.6) on $I_\tau = [\tau, a)$ and a is a limit of a strictly increasing sequence $\{\tau_i, i \in \mathbb{N}\}$ of times such that $\limsup_{t \to \tau_i^-} \|\mathcal{U}(t)\|_{E_{1+\varepsilon}} = \infty$, then a is called an accumulation time of singular times.

The following result describes some natural situation allowing for continuation of an $E_{1+\varepsilon}$-solution of (1.6) as a piecewise-$E_{1+\varepsilon}$-solution.

Theorem 1.6. *Suppose that Assumption 1.1 holds and F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha, \alpha \in [0, \mu)\}$. Suppose additionally that $E_{1+\varepsilon}$ is dense in $E_1 \subset E_0$, E_1 is reflexive and, given any $\tau \in \mathbb{R}$, $u_\tau \in E_1$,*

$$\sup_{t \in [\tau, T)} \|u(t)\|_{E_1} < \infty \tag{1.32}$$

whenever $T \in (\tau, \infty)$ and $E_{1+\varepsilon}$-solution $u(\cdot) = u(\cdot, \tau, u_\tau)$ of (4.14) exists for all $t \in [\tau, T)$.
Finally suppose that

$$\begin{array}{l} \textit{whenever } \tau \in \mathbb{R},\ u_\tau \in E_1 \textit{ and } T_{u_\tau} < \infty, \textit{ the map} \\ [\tau, T_{u_\tau}) \ni t \longrightarrow u(t) \in E_0, \textit{ where } u(\cdot) = u(\cdot, \tau, u_\tau) \\ \textit{is } E_{1+\varepsilon}\textit{-solution of (1.6), is uniformly continuous.} \end{array} \tag{1.33}$$

Under these assumptions, given $\tau \in \mathbb{R}$, $u_\tau \in E_1$ and the unique $E_{1+\varepsilon}$-solution $u(\cdot) = u(\cdot, \tau, T_{u_\tau})$ of (1.6) for which $T_{u_\tau} < \infty$, there exist $a \in (T_{u_\tau}, \infty]$ and the unique extension $\mathcal{U} : [\tau, a) \to E_1$ of $u(\cdot)$ such that $\mathcal{U}$ is a piecewise-$E_{1+\varepsilon}$-solution of (1.6) on $[\tau, a)$ and either $a = \infty$ or a is an accumulation time of singular times.

Note that in Theorem 1.6 the continuation of $E_{1+\varepsilon}$-solution $u(\cdot, \tau, u_\tau)$ is defined on a larger interval than its maximal interval of existence $[\tau, T_{u_\tau})$. Such a continuation will thus possess singularities in finitely or infinitely many points in which $E_{1+\varepsilon}$-norm will blow-up.
The proofs of the results stated above will be given in Section 2.

In Section 3 there are included some of the related technical results, crucial for the consideration of this paper. Therein we consider families $\{A(t) : t \in \mathbb{R}\}$ of unbounded positive operators generating C^0 analytic semigroups being all of the same type. We discuss conditions guaranteeing that the associated fractional power spaces $D(A^{\alpha}(t))$ will

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coincide and the graph norms will be equivalent uniformly for t in bounded time intervals. In particular we ensure validity of Assumption 1.1 and Assumption 1.3 in an appropriate portion of the extrapolated fractional power scale generated by $(X, A(t))$, $t \in \mathbb{R}$ (see Theorem 3.1).

In Section 4 we show how the abstract results work in applications. This includes in particular a singularly nonautonomous wave equation with a structural damping and a singularly nonautonomous semilinear parabolic initial boundary value problem of order $2m$.

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2. ABSTRACT RESULTS

In this section we prove the abstract results stated in the Introduction.

2.1. Local well posedness result: proof of Theorem 1.2.

The proof will be divided into three main parts. Part I will be devoted to the existence of $E_{1+\varepsilon}$-solution. In Part II we will obtain the uniqueness of $E_{1+\varepsilon}$-solution. In Part III we will show that, under some additional assumptions on F and $\{E_\sigma, \sigma \in [0, 1+\mu)\}$, the time of existence of $E_{1+\varepsilon}$-solution can be chosen uniformly with respect to suitable sets of initial data u_τ or/and initial times τ. Finally, in Part IV, we will prove some additional properties of $E_{1+\varepsilon}$-solution stated in (1.20), (1.21) and (1.22).

2.1.1. Part I: the existence of $E_{1+\varepsilon}$-solution

The existence result will follow in three steps.

Step 1. Let us fix an interval I around t_0 such that (1.9) holds with $\zeta = \gamma(\varepsilon)$ and $\sigma = 1+\varepsilon$. Let us also choose an interval $\mathcal{J}$ centered at t_0 such that $I \setminus \mathcal{J}$ is the union of two intervals of length $l > 0$.

We first note that if $\delta^* \in (0, l)$, $\tau \in \mathcal{J}$, $\delta \in (0, 1] \cap (0, \delta^*)$, $v \in L^\infty_{loc}((\tau, \tau+\delta], E_{1+\varepsilon})$, $\lambda(v,t) := \sup_{s\in(\tau,t]}\{(s-\tau)^\varepsilon \|v(s)\|_{E_{1+\varepsilon}}\}$, $R > 0$, $t \in (\tau, \tau+\delta]$ and $\lambda(v,t) \leqslant R$ then, by Assumption 1.1 and (1.11), we have

$$\begin{aligned}\|U(t,s)F(s,v(s))\|_{E_{1+\varepsilon}} &\leqslant \|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\varepsilon})}\|F(s,v(s))\|_{E_{\gamma(\varepsilon)}} \\ &\leqslant M(t-s)^{-1+\gamma(\varepsilon)-\varepsilon} c(\eta\|v(s)\|^\rho_{E_{1+\varepsilon}} + C_\eta)\end{aligned} \tag{2.1}$$

and consequently

$$
\begin{aligned}
(t-\tau)^{\varepsilon}\|\int_{\tau}^{t}U(t,s)F(s,v(s))ds\|_{E_{1+\varepsilon}} &\leqslant cC_{\eta}M(t-\tau)^{\varepsilon}\int_{\tau}^{t}(t-s)^{-1+\gamma(\varepsilon)-\varepsilon}ds\\
&+c\eta M(t-\tau)^{\varepsilon}\int_{\tau}^{t}(t-s)^{-1+\gamma(\varepsilon)-\varepsilon}(s-\tau)^{-\rho\varepsilon}[(s-\tau)^{\varepsilon}\|v(s)\|_{E_{1+\varepsilon}}]^{\rho}ds\\
&\leqslant cMB(1-\rho\varepsilon,\gamma(\varepsilon)-\varepsilon)[C_{\eta}(t-\tau)^{\gamma(\varepsilon)}+\eta\lambda^{\rho}(v,t)]\leqslant cMB_{\varepsilon,\rho}[C_{\eta}(t-\tau)^{\gamma(\varepsilon)}+\eta R^{\rho}].
\end{aligned}
\tag{2.2}
$$

Also, if $v,\tilde{v}\in L^{\infty}_{loc}((\tau,\tau+\delta],E_{1+\varepsilon})$, $t\in(\tau,\tau+\delta]$ and $\lambda(v,t)\leqslant R$, $\lambda(\tilde{v},t)\leqslant R$ then, with a similar usage of Assumption 1.1 and (1.10), we get

$$
\begin{aligned}
&(t-\tau)^{\varepsilon}\|\int_{\tau}^{t}U(t,s)[F(s,v(s))-F(s,\tilde{v}(s))]ds\|_{E_{1+\varepsilon}}\\
&\leqslant cC_{\eta}M(t-\tau)^{\varepsilon}\int_{\tau}^{t}(t-s)^{-1+\gamma(\varepsilon)-\varepsilon}(s-\tau)^{-\varepsilon}ds\sup_{\tau\leqslant s\leqslant t}\{(s-\tau)^{\varepsilon}\|v(s)-\tilde{v}(s)\|_{E_{1+\varepsilon}}\}\\
&+c\eta M(t-\tau)^{\varepsilon}\int_{\tau}^{t}(t-s)^{-1+\gamma(\varepsilon)-\varepsilon}(s-\tau)^{-\rho\varepsilon}\Big(((s-\tau)^{\varepsilon}\|v(s)\|_{E_{1+\varepsilon}})^{\rho-1}\\
&\qquad+((s-\tau)^{\varepsilon}\|\tilde{v}(s)\|_{E_{1+\varepsilon}})^{\rho-1}\Big)(s-\tau)^{\varepsilon}\|v(s)-\tilde{v}(s)\|_{E_{1+\varepsilon}}ds.
\end{aligned}
$$

Hence, letting

$$
\Gamma_{\varepsilon}(t):=cMB_{\varepsilon,\rho}[C_{\eta}(t-\tau)^{\gamma(\varepsilon)-\rho\varepsilon}+2\eta R^{\rho-1}],
\tag{2.3}
$$

we conclude that

$$
\begin{aligned}
(t-\tau)^{\varepsilon}\|\int_{\tau}^{t}U(t,s)[F(s,v(s))-F(s,\tilde{v}(s))]ds\|_{E_{1+\varepsilon}}&\\
\leqslant\Gamma_{\varepsilon}(t)\sup_{s\in(\tau,t]}\{(s-\tau)^{\varepsilon}\|v(s)-\tilde{v}(s)\|_{E_{1+\varepsilon}}\}.&
\end{aligned}
\tag{2.4}
$$

We now choose

$$
R_0\geqslant R>0\ \text{ and }\ \delta>0
$$

such that

$$
c\eta MB_{\varepsilon,\rho}R_0^{\rho-1}=\frac{1}{8}\quad\text{ and }\quad cC_{\eta}MB_{\varepsilon,\rho}\delta^{\gamma(\varepsilon)-\rho\varepsilon}=\min\Big\{\frac{R}{8},\frac{1}{4},\delta^{*},1\Big\}.
\tag{2.5}
$$

We also define

$$
r:=\frac{R}{4}\leqslant\frac{R_0}{4}=\frac{1}{4(8c\eta MB_{\varepsilon,\rho})^{\frac{1}{\rho-1}}}
$$

and, since $\lim_{t\to\tau^+}\|(t-\tau)^\varepsilon U(t,\tau)w_0\|_{E_{1+\varepsilon}}=0$ we choose $\bar\delta_0=\bar\delta_0(R)\in(0,\delta]$ such that

$$\|(t-\tau)^\varepsilon U(t,\tau)w_0\|_{E_{1+\varepsilon}}\leqslant\frac{R}{2},\qquad \tau<t\leqslant\tau+\bar\delta_0. \tag{2.6}$$

Step 2. For arbitrarily fixed $\delta_0\in(0,\bar\delta_0]$ consider the set

$$K(R,\tau)=\{v\in L^\infty_{loc}((\tau,\tau+\delta_0],E_{1+\varepsilon}):\sup_{t\in(\tau,\tau+\delta_0]}\{(t-\tau)^\varepsilon\|v(t)\|_{E_{1+\varepsilon}}\}\leqslant R\} \tag{2.7}$$

and the function $d:K(R,\tau)\times K(R,\tau)\to[0,\infty)$

$$d(v,\tilde v)=\sup_{t\in(\tau,\tau+\delta_0]}\{(t-\tau)^\varepsilon\|v(t)-\tilde v(t)\|_{E_{1+\varepsilon}}\},\ v,\tilde v\in K(R,\tau).$$

One can easily see that d is a metric in $K(R,\tau)$ and that $(K(R,\tau),d)$ is a complete metric space. We will show that there exists a unique function in $K(R,\tau)$, which satisfies variation of constants formula (1.12) on $(\tau,\tau+\delta_0]$.

Choosing arbitrary $u_\tau\in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0,r)$ as in (1.18) we will consider the map

$$(\mathcal{T}v)(t)=U(t,\tau)u_\tau+\int_\tau^t U(t,s)F(s,v(s))ds,\ v\in K(R,\tau),\ t\in(\tau,\tau+\delta_0].$$

It follows from (1.18), (2.2) and (2.5)-(2.6) that

$$\begin{aligned}
\|(t-\tau)^\varepsilon(\mathcal{T}v)(t)\|_{E_{1+\varepsilon}}&\leqslant(t-\tau)^\varepsilon\|U(t,\tau)u_\tau+\int_\tau^t U(t,s)F(s,v(s))ds\|_{E_{1+\varepsilon}}\\
&\leqslant\|(t-\tau)^\varepsilon U(t,\tau)u_\tau\|_{E_{1+\varepsilon}}+cM(t-\tau)^\varepsilon\int_\tau^t(t-s)^{-1+\gamma(\varepsilon)-\varepsilon}(\eta\|v(s)\|^\rho_{E_{1+\varepsilon}}+C_\eta)ds\\
&\leqslant\|(t-\tau)^\varepsilon U(t,\tau)(u_\tau-\omega_0)\|_{E_{1+\varepsilon}}+\|(t-\tau)^\varepsilon U(t,\tau)\omega_0\|_{E_{1+\varepsilon}}\\
&\qquad+cM\eta B_{\varepsilon,\rho}R^\rho+cMC_\eta B_{\varepsilon,\rho}\delta_0^{\gamma(\varepsilon)-\rho\varepsilon}\\
&\leqslant r+\|(t-\tau)^\varepsilon U(t,\tau)\omega_0\|_{E_{1+\varepsilon}}+cM\eta B_{\varepsilon,\rho}R^\rho+cMC_\eta B_{\varepsilon,\rho}\delta_0^{\gamma(\varepsilon)-\rho\varepsilon}\leqslant R,
\end{aligned}$$

which yields that $\mathcal{T}$ takes $K(R,\tau)$ into $K(R,\tau)$. Furthermore, applying (2.3)-(2.5) we get

$$d(\mathcal{T}v_1,\mathcal{T}v_2)\leqslant\frac{1}{2}d(v_1,v_2).$$

Consequently, due to the Banach fixed point theorem, we infer that $\mathcal{T}$ has the unique fixed point $u(\cdot,\tau,u_\tau)$ in $K(R,\tau)$.

Note that, for any $u_\tau\in B_{E_{1+\alpha}}(w_0,r)$ and $\tau\in[-\tau_0,\tau_0]$, $u(\cdot,\tau,u_\tau)$ is defined on an interval of the same length δ_0.

Step 3. For the fixed point $u(\cdot)=u(\cdot,\tau,u_\tau)$ we now show that $\lim_{t\to\tau^+}\|(t-\tau)^\varepsilon u(t)\|_{E_{1+\varepsilon}}=0$.

Adapting (2.2), we have for each $t \in (\tau, \tau+\delta_0]$

$$(t-\tau)^{\varepsilon}\|u(t)\|_{E_{1+\varepsilon}} \leqslant (t-\tau)^{\varepsilon}\|U(t,\tau)u_\tau\|_{E_{1+\varepsilon}} + cMB_{\varepsilon,\rho}[C_\eta(t-\tau)^{\gamma(\varepsilon)} + \eta R^{\rho-1}\lambda(u,t)],$$

where by assumption, given any $\xi > 0$, we can choose $h \in (0,\xi)$ such that

$$(t-\tau)^{\varepsilon}\|U(t,\tau)u_\tau\|_{E_{1+\varepsilon}} < \xi \ \text{ for } \ t \in (\tau, \tau+h).$$

Hence, we get

$$(t-\tau)^{\varepsilon}\|u(t)\|_{E_{1+\varepsilon}} \leqslant \xi + cMB_{\varepsilon,\rho}[C_\eta \xi^{\gamma(\varepsilon)} + \eta R^{\rho-1}\lambda(u,t)],\ t \in (\tau,\tau+h).$$

Since the right hand side above is a nondecreasing function of t, we obtain

$$\lambda(u,t) \leqslant \xi + cMB_{\varepsilon,\rho}[C_\eta \xi^{\gamma(\varepsilon)} + \eta R^{\rho-1}\lambda(u,t)],\ t \in (\tau,\tau+h),$$

and, via (2.5),

$$\frac{7}{8}\lambda(u,t) \leqslant \xi + cMB_{\varepsilon,\rho}C_\eta \xi^{\gamma(\varepsilon)},\ t \in (\tau,\tau+h).$$

This yields

$$\lambda(u,t) = \sup_{s\in(\tau,t]}\{(s-\tau)^{\varepsilon}\|u(s)\|_{E_{1+\varepsilon}}\} \to 0 \ \text{ as } \ t \to \tau^+ \tag{2.8}$$

and ensures that $(t-\tau)^{\varepsilon}u(t) \to 0$ in $E_{1+\varepsilon}$ as $t \to \tau^+$.

Thus the fixed point $u(\cdot)u(\cdot,\tau,u_\tau)$ obtained above is an $E_{1+\varepsilon}$-solution of (1.6) on $[\tau,\tau+\delta_0]$. The proof of the existential part of Theorem 1.2 is thus complete.

2.1.2. *Part II: the uniqueness of $E_{1+\varepsilon}$-solution*

In what follows we turn our attention to the uniqueness of $E_{1+\varepsilon}$-solution of (1.6). Given $-\infty < \tau < T < \infty$ we define a linear subspace $\mathfrak{M}_\tau^T$ of $L^\infty_{loc}((\tau,T],E_{1+\varepsilon})$

$$\mathfrak{M}_\tau^T := \{\varphi \in L^\infty_{loc}((\tau,T],E_{1+\varepsilon}) : \lim_{t\to\tau^+}(t-\tau)^{\varepsilon}\|\varphi(t)\|_{E_{1+\varepsilon}} = 0\}. \tag{2.9}$$

Lemma 2.1. *If $\varphi, \tilde\varphi \in \mathfrak{M}_\tau^T$ satisfy a. e. in $[\tau,T]$ the variation of constants formula*

$$u(t) = U(t,\tau)u_\tau + \int_\tau^t U(t,s)F(s,u(s))ds$$

with a certain $u_\tau \in E_0$ then $\varphi,\tilde\varphi$ are identical elements of $\mathfrak{M}_\tau^T$.

Proof: By assumption we have

$$\|\varphi(t)-\tilde\varphi(t)\|_{E_{1+\varepsilon}} \leqslant cC_\eta M\int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\varepsilon}\|\varphi(s)-\tilde\varphi(s)\|_{E_{1+\varepsilon}}ds$$
$$+ c\eta M\int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\varepsilon}\|\varphi(s)-\tilde\varphi(s)\|_{E_{1+\varepsilon}}(\|\varphi(s)\|^{\rho-1}_{E_{1+\varepsilon}} + \|\tilde\varphi(s)\|^{\rho-1}_{E_{1+\varepsilon}}),\ t \in (\tau,T].$$

Since $\varphi, \tilde{\varphi} \in \mathfrak{M}_\tau^T$, given $\xi \in (0,1)$, there is a certain $h \in (0,\xi)$ such that

$$(t-\tau)^\varepsilon \|\varphi(t)\|_{E_{1+\varepsilon}} + (t-\tau)^\varepsilon \|\tilde{\varphi}(t)\|_{E_{1+\varepsilon}} \leqslant \xi,\ t \in (\tau, \tau+h).$$

Using this and restricting t to the interval $(\tau, \tau+h)$ we obtain the estimate

$$\begin{aligned}(t-\tau)^\varepsilon &\|\varphi(t)-\tilde{\varphi}(t)\|_{E_{1+\varepsilon}}\\ &\leqslant cC_\eta MB(\gamma(\varepsilon)-\varepsilon, 1-\varepsilon)\xi^{\gamma(\varepsilon)-\varepsilon} \sup_{s\in(\tau,\tau+h)} (s-\tau)^\varepsilon \|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}}\\ &+ \xi^{\rho-1+\gamma(\varepsilon)-\varepsilon\rho} 2c\eta MB(1-\varepsilon\rho, \gamma(\varepsilon)-\varepsilon) \sup_{s\in(\tau,\tau+h)} (s-\tau)^\varepsilon \|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}}.\end{aligned}$$

We remark that the inequality above will hold true if we replace its left hand side by $\sup_{s\in(\tau,\tau+h)}(s-\tau)^\varepsilon\|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}}$. On the other hand note that since $\rho > 1$ and $\gamma(\varepsilon) > \varepsilon$, then $\xi > 0$ can be chosen small enough to ensure that the right hand side is less than $\frac{1}{2}\sup_{s\in(\tau,\tau+h)}(s-\tau)^\varepsilon\|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}}$. This yields that $\sup_{s\in(\tau,\tau+h)}(s-\tau)^\varepsilon\|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}} = 0$ and thus $\varphi = \tilde{\varphi}$ in $[\tau, \tau+h]$ for some $h > 0$.

Now, if $\tau^* \in (\tau, \tau+h]$ is such that $\varphi(\tau^*) = \tilde{\varphi}(\tau^*)$ then applying the variation of constants formula with the initial time τ^* and with the initial value $\varphi(\tau^*) = \tilde{\varphi}(\tau^*)$ we get

$$\varphi(t) - \tilde{\varphi}(t) = \int_{\tau^*}^t U(t,s)(F(s,\varphi(s) - F(s,\tilde{\varphi}(s))ds \ \text{ in } \ [\tau^*, T]$$

and hence, letting $c^* = \sup_{s\in[\tau^*,T]}(\|\varphi(s)\|^{\rho-1}_{E_{1+\varepsilon}} + \|\tilde{\varphi}(s)\|^{\rho-1}_{E_{1+\varepsilon}})$,

$$\|\varphi(t)-\tilde{\varphi}(t)\|_{E_{1+\varepsilon}} \leqslant cM(C_\eta + \eta c^*) \int_{\tau^*}^t (t-s)^{\gamma(\varepsilon)-1-\varepsilon} \|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}} ds.$$

Consequently, by singular Gronwall's inequality, we obtain that φ and $\tilde{\varphi}$ coincide. ∎

Lemma 2.1 yields the following uniqueness result.

Corollary 2.1. *Let I_τ^a denote an interval of the form $[\tau, a]$, $[\tau, a)$ or $[\tau, \infty)$ and suppose that φ and $\tilde{\varphi}$ are $E_{1+\varepsilon}$-solutions of* (1.6) *on I_τ^a and $I_\tau^{\tilde{a}}$ respectively.*

Then φ and $\tilde{\varphi}$ coincide on $I_\tau^a \cap I_\tau^{\tilde{a}}$.

The existence uniqueness result stated in Theorem 1.2 i) is thus proved.

2.1.3. Part III: uniform time of existence with respect to u_τ or/and τ

In this part we prove the statements ii)-iii).

Proof of ii). The result will follow from Corollary 2.2 below.

Corollary 2.2. *Suppose that Assumption 1.1 holds, $F : \mathbb{R} \times E_{1+\varepsilon} \to E_{\gamma(\varepsilon)}$ is continuous and there exist constants $\rho > 1$, $0 < \varepsilon < \min\{\frac{1}{\rho}, \mu\}$, $\gamma(\varepsilon) \in [\rho\varepsilon, 1)$ such that for each*

$\eta > 0$ there is a certain $C_\eta > 0$ and, moreover, for any bounded time interval $I \subset \mathbb{R}$ there exists some $c > 0$ for which we have

$$\|F(t,v) - F(t,w)\|_{E_{\gamma(\varepsilon)}} \leqslant c\|v-w\|_{E_{1+\varepsilon}}(\eta\|v\|^{\rho-1}_{E_{1+\varepsilon}} + \eta\|w\|^{\rho-1}_{E_{1+\varepsilon}} + C_\eta),\ v,w \in E_{1+\varepsilon},\ t \in I,$$

and

$$\|F(t,v)\|_{E_{\gamma(\varepsilon)}} \leqslant c(\eta\|v\|^{\rho}_{E_{1+\varepsilon}} + C_\eta),\ v,w \in E_{1+\varepsilon},\ t \in I.$$

Then, given any $t_0 \in \mathbb{R}$, τ in a certain interval $\mathcal{J} \subset \mathbb{R}$ centered at t_0 and given any $r_0 > 0$, there exists $\delta_0 > 0$ such that for any initial condition $u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(0,r_0)$ there exists the unique $E_{1+\varepsilon}$-solution $u = u(\cdot,\tau,u_\tau)$ of (1.6) on $[\tau,\tau+\delta_0]$.

Proof: Letting $w_0 = 0$ and coming back to the proof of Theorem 1.2 we observe that given any $r_0 > 0$ one can now choose $\eta > 0$ such that r in (1.19) satisfies $r > r_0$. Then, proceeding as in the proof of Theorem 1.2, for any $u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(0,r)$ we obtain the existence of $E_{1+\varepsilon}$-solution $u = u(\cdot,\tau,u_\tau)$ of (1.6) on $[\tau,\tau+\delta_0]$. ∎

Proof of iii). We now exploit the additional assumptions on $\{E_\sigma, \sigma \in [0,\mu)\}$ to ensure that the time of existence can be chosen uniformly in a certain neighborhood of a given point $t_0 \in \mathbb{R}$. Actually, following the proof of existential part of Theorem 1.2 in §2.1.1, it suffices to ensure that the number $\bar{\delta}_0(R)$ in (2.6) can be chosen uniformly with respect to $\tau \in \mathcal{J}$.

Note that this is obvious if $w_0 = 0$. Also note that if $0 \neq w_0 \in E_{1+\varepsilon}$ then using (1.9) we obtain this via estimate

$$\sup_{\tau < t \leqslant \tau+\bar{\delta}_0} \|(t-\tau)^\varepsilon U(t,\tau)w_0\|_{E_{1+\varepsilon}} \leqslant \bar{M}\bar{\delta}_0\|w_0\|_{E_{1+\varepsilon}} \leqslant \frac{R}{2},\ \tau \in \mathcal{J}, \tag{2.10}$$

which holds for some constant $\bar{M} = M(1+\varepsilon)$ and $\bar{\delta}_0 = \frac{R}{2\bar{M}\|w_0\|_{E_{1+\varepsilon}}}$.

On the other hand, if $w_0 \in E_1$ and $E_{1+\varepsilon}$ is dense in E_1, then via (1.9) for any $\phi \in E_{1+\varepsilon}$ we have

$$\begin{aligned}\sup_{\tau < t \leqslant \tau+\bar{\delta}_0} \|(t-\tau)^\varepsilon U(t,\tau)w_0\|_{E_{1+\varepsilon}} &\leqslant \sup_{\tau < t \leqslant \tau+\bar{\delta}_0} \|(t-\tau)^\varepsilon U(t,\tau)(w_0-\phi)\|_{E_{1+\varepsilon}} \\ &+ \sup_{\tau < t \leqslant \tau+\bar{\delta}_0} \|(t-\tau)^\varepsilon U(t,\tau)\phi\|_{E_{1+\varepsilon}} \leqslant \hat{M}\|w_0-\phi\|_{E_1} + \bar{\delta}_0\|\phi\|_{E_{1+\varepsilon}}\ \tau \in \mathcal{J}.\end{aligned} \tag{2.11}$$

Since ϕ can be chosen such that $\hat{M}\|w_0-\phi\|_{E_1} \leqslant \frac{R}{4}$ and then $\bar{\delta}_0$ can be chosen such that $\bar{\delta}_0\|\phi\|_{E_{1+\varepsilon}} \leqslant \frac{R}{4}$ we again conclude that $\sup_{\tau<t\leqslant\tau+\bar{\delta}_0}\|(t-\tau)^\varepsilon U(t,\tau)w_0\|_{E_{1+\varepsilon}} \leqslant \frac{R}{2}$ for any $\tau \in \mathcal{J}$, and some $\bar{\delta}_0(R)$ which does not depend on $\tau \in \mathcal{J}$.

Both statements ii) and iii) in Theorem 1.2 are thus proved.

2.1.4. Part IV: further properties of $E_{1+\varepsilon}$-solution

In this part we prove the conditions (1.20), (1.21) and (1.22).

Using a similar argument as in (2.2), for $\theta \in (0,\gamma(\varepsilon)) \cap (0,\mu)$ and for the unique $E_{1+\varepsilon}$-solution $u(\cdot) = u(\cdot,\tau,u_\tau)$ of (1.6) we get

$$\begin{aligned}(t-\tau)^\theta \|u(t)\|_{E_{1+\theta}} &\leqslant (t-\tau)^\theta \|U(t,\tau)u_\tau\|_{E_{1+\theta}} + (t-\tau)^\theta \int_\tau^t \|U(t,s)F(s,u(s))\|_{E_{1+\theta}} ds \\ &\leqslant (t-\tau)^\theta \|U(t,\tau)u_\tau\|_{E_{1+\theta}} + cMC_\eta(\gamma(\varepsilon)-\theta)^{-1}(t-\tau)^{\gamma(\varepsilon)} \\ &\qquad + \eta cMB(1-\varepsilon\rho,\gamma(\varepsilon)-\theta)(\sup_{\tau<s\leqslant t}\{(s-\tau)^\varepsilon\|u(s)\|_{E_{1+\varepsilon}}\})^\rho.\end{aligned}$$

Recalling that $u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0,r) \cap \mathfrak{E}^\tau_\theta$ and using (2.8) we conclude that $(t-\tau)^\theta\|u(t)\|_{E_{1+\theta}} \to 0$ as $t \to \tau^+$, which proves (1.20).

For $\theta \in (0,\gamma(\varepsilon)) \cap (0,\mu)$ analogously as in (2.4) we next have

$$\begin{aligned}(t-\tau)^\theta \|u(t,\tau,u^1_\tau) - u(t,\tau,u^2_\tau)\|_{E_{1+\theta}} &\leqslant (t-\tau)^\theta \|U(t,\tau)(u^1_\tau - u^2_\tau)\|_{E_{1+\theta}} \\ &\quad + (t-\tau)^\theta \int_\tau^t \|U(t,s)[F(s,u(s,\tau,u^1_\tau)) - F(s,u(s,\tau,u^2_\tau))]\|_{E_{1+\theta}} ds \\ &\leqslant (t-\tau)^\theta \|U(t,\tau)(u^1_\tau - u^2_\tau)\|_{E_{1+\theta}} \\ &\quad + \Gamma_\theta(t) \sup_{\tau<s\leqslant\tau+\delta_0} \{(s-\tau)^\varepsilon \|u(s,\tau,u^1_\tau) - u(s,\tau,u^2_\tau)\|_{E_{1+\varepsilon}}\},\end{aligned} \tag{2.12}$$

where

$$\Gamma_\theta(t) = cM(1+\theta,\gamma(\varepsilon),T)\max\{B(\gamma(\varepsilon)-\theta,1-\varepsilon),B(1-\rho\varepsilon,\gamma(\varepsilon)-\theta)\}[C_\eta(t-\tau)^{\gamma(\varepsilon)-\theta}+2\eta R^{\rho-1}].$$

Taking $\theta = \varepsilon$ we get

$$\begin{aligned}(t-\tau)^\varepsilon \|u(t,\tau,u^1_\tau) - u(t,\tau,u^2_\tau)\|_{E_{1+\varepsilon}} &\leqslant (t-\tau)^\varepsilon \|U(t,\tau)(u^1_\tau - u^2_\tau)\|_{E_{1+\varepsilon}} \\ &\quad + \Gamma_\varepsilon(t) \sup_{\tau<s\leqslant\tau+\delta_0} \{(s-\tau)^\varepsilon \|u(s,\tau,u^1_\tau) - u(s,\tau,u^2_\tau)\|_{E_{1+\varepsilon}}\}\end{aligned}$$

Since $\Gamma_\varepsilon(t)$ is increasing with respect to t and, by (2.5), $\Gamma_\varepsilon(\tau+\delta_0) \leqslant \frac{1}{2}$ we conclude that

$$\sup_{\tau<s\leqslant\tau+\delta_0} \{(s-\tau)^\varepsilon \|u(s,\tau,u^1_\tau) - u(s,\tau,u^2_\tau)\|_{E_{1+\varepsilon}}\} \leqslant 2 \sup_{\tau<s\leqslant\tau+\delta_0} (s-\tau)^\varepsilon \|U(s,\tau)(u^1_\tau - u^2_\tau)\|_{E_{1+\varepsilon}}.$$

Consequently, using the above inequality and (2.12) we get (1.21) with constant

$$C(\theta_0) = 1 + 2cM(1+\theta,\gamma(\varepsilon),T)B_{\varepsilon,\rho,\theta}[C_\eta(t-\tau)^{\gamma(\varepsilon)-\theta} + 2\eta R^{\rho-1}],$$

where $B_{\varepsilon,\rho,\theta} = \max\{B(\gamma(\varepsilon)-\theta,1-\varepsilon),B(1-\rho\varepsilon,\gamma(\varepsilon)-\theta)\}$. If furthermore $u^1_\tau, u^2_\tau \in E_1 \subset E_0$ then using (1.9) we also conclude that

$$\|u^1_\tau - u^2_\tau\|^{\mathfrak{E}^\tau_\theta}_{\delta_0} + \|u^1_\tau - u^2_\tau\|^{\mathfrak{E}^\tau_\epsilon}_{\delta_0} \leqslant \bar{M}\|\tau^1 - u^2_\tau\|_{E_1}, \tag{2.13}$$

where $\bar{M} = \max\{M(1+\theta) + M(1+\varepsilon)\}$.

Assuming that $0 \leqslant \theta < \min\{\gamma(\varepsilon), \mu\}$ and $\lim_{t\to\tau^+}(t-\tau)^\theta\|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}} = 0$ we now show that $\lim_{t\to\tau^+}(t-\tau)^\theta\|u(t,\tau,u_\tau)-u_\tau\|_{E_{1+\theta}} = 0$, for which we first use the variation of constants formula and (1.9), (1.11) to get for each $t \in (\tau, \tau+\delta_0]$

$$\begin{aligned}
&(t-\tau)^\theta\|u(t)-u_\tau\|_{E_{1+\theta}}\\
&\leqslant (t-\tau)^\theta\|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}} + (t-\tau)^\theta\int_\tau^t \|U(t,s)F(s,u(s))\|_{E_{1+\theta}}ds\\
&\leqslant (t-\tau)^\theta\|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}}\\
&\qquad + cM(t-\tau)^\theta\int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\theta}(C_\eta + \eta(s-\tau)^{-\varepsilon\rho}\|(s-\tau)^\varepsilon u(s)\|^\rho_{E_{1+\varepsilon}})ds\\
&\leqslant (t-\tau)^\theta\|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}} + cC_\eta MB(1-\rho\varepsilon, \gamma(\varepsilon)-\theta)(t-\tau)^{\gamma(\varepsilon)}\\
&\qquad + cM\eta B(1-\rho\varepsilon,\gamma(\varepsilon)-\theta)\lambda^\rho(u,t).
\end{aligned} \tag{2.14}$$

Thus (1.22) is a consequence of (2.14) and (2.8).

The proof of Theorem 1.2 is now complete. ▮

2.1.5. Proof of Remark 1.3

In what follows we show that the $E_{1+\varepsilon}$-solution of (1.6) will be bounded in $E^{1+\varepsilon}$ up to the initial time τ provided that $u_\tau \in E_{1+\varepsilon}$.

Given $u_\tau \in E_{1+\varepsilon}$ and numbers R_0, $\delta_0(R)$ as in the proof of Theorem 1.2 one can define

$$\mathcal{K}_\xi(R) = \{\psi \in L^\infty((\tau,\tau+\xi], E_{1+\varepsilon}) : \sup_{t\in(\tau,\tau+\xi]} \|\psi(t)-u_\tau\| \leqslant R\},\ R \in (0,R_0),\ \xi > 0.$$

Then there exists some suitably small $\xi_0 \in (0, \delta_0(R)]$, such that the Banach fixed point theorem will yield the existence of $\tilde{u} \in \mathcal{K}_{\xi_0}(R)$ satisfying variation of constants formula (1.12) a. e. in $[\tau,\xi]$. We also have that $\tilde{u} \in \mathfrak{M}_\tau^{\tau+\xi}$ (see (2.9)) and, since the solution $u(\cdot,\tau,u_\tau)$ from Theorem 1.2 belong to $\mathfrak{M}_\tau^{\tau+\xi}$ as well, we conclude via Lemma 2.1 that u and $\tilde{u}$ coincide on $(\tau,\tau+\xi]$. Condition (1.23) now follows easily.

2.1.6. Certain extension of the local well posedness result

It does happen in applications that F decomposes into a sum of maps that belong to a class described in Definition 1.1 although with some different parameters (see [8, pp. 385-386]). We remark that if in Theorem 1.2, instead of assuming that F is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ relative to $\{E_\alpha, \alpha\in[0,\mu)\}$, we will assume decomposition condition $(\mathcal{D})$ as below:

$(\mathcal{D})$ there exist maps F_i of the class $\mathcal{L}(\varepsilon(i),\rho(i),\gamma(\varepsilon(i)),\eta,C_\eta)$ relative to $\{E_\alpha,\alpha\in[0,\mu)\}$, $i=1,\ldots,n$, such that

$$F = \sum_{i=1}^n F_i, \tag{2.15}$$

$$\min\{\gamma(\varepsilon(i));\ 1 \leqslant i \leqslant n\} =: \underline{\gamma} > \overline{\varepsilon} := \max\{\varepsilon(i);\ 1 \leqslant i \leqslant n\}, \tag{2.16}$$

then, given $t_0 \in \mathbb{R}$ and $w_0 \in \mathfrak{E}^\tau_\varepsilon$, there exist $\bar{\delta}_0 \in (0,1]$ such that for any τ in a certain interval $\mathcal{J} \subset \mathbb{R}$ centered at t_0 and for any initial condition u_τ satisfying

$$u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r) \tag{2.17}$$

where $\delta_0 \in (0, \bar{\delta}_0]$ and $r = r(\eta) \to \infty$ as $\eta \to 0^+$, there exists the unique $E_{1+\overline{\varepsilon}}$ -solution $u = u(\cdot, \tau, u_\tau)$ of (1.6) on $[\tau, \tau + \delta_0]$.

Furthermore, for any $0 \leqslant \theta < \min\{\underline{\gamma}, \mu\}$ and $u_\tau, u^1_\tau, u^2_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r) \cap \mathfrak{E}^\tau_\theta$ we have (1.20) and (1.21). Also implication (1.22) is satisfied for any $0 \leqslant \theta < \min\{\underline{\gamma}, \mu\}$, $u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r)$.

2.2. Hölder continuity of $E_{1+\varepsilon}$-solution: proof of Theorem 1.4

Theorem 1.4 will be a consequence of the following lemma.

LEMMA 2.2. *Suppose that Assumption 1.1 is satisfied, F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha, \alpha \in [0, \mu)\}$, $\tau \in \mathbb{R}$, $u_\tau \in \mathfrak{E}^\tau_\varepsilon$ and $u = u(\cdot, \tau, u_\tau)$ is the unique $E_{1+\varepsilon}$-solution of (1.6) on the interval $[\tau, T)$ for some $T \in (0, \infty)$. Suppose also that Assumption 1.3 holds.*

Then, given $\delta \in (\tau, T)$, we have that $u \in C^\nu([\delta, T), E_{1+\theta})$ for any $\nu \in (0, \nu_*)$, where $\nu_* = \min\{\gamma(\varepsilon), \mu\} - \theta > 0$.

Proof: Due to Assumptions 1.1, 1.3, given a bounded time interval $[-T, T] \subset \mathbb{R}$ and any $0 \leqslant \zeta \leqslant \sigma < 1 + \mu$, one can choose a positive constant M for which we have

$$\|U(t, \tau)\|_{L(E_\zeta, E_\sigma)} \leqslant M(t - \tau)^{-(\sigma - \zeta)},\ T \geqslant t > \tau \geqslant -T. \tag{2.18}$$

and, if $1 \geqslant \sigma - \zeta \geqslant 0$,

$$\|U(t, \tau) - Id\|_{L(E_\sigma, E_\zeta)} \leqslant M(t - \tau)^{\sigma - \zeta},\ T \geqslant t > \tau \geqslant -T. \tag{2.19}$$

On the other hand, due to (1.20), for any $\delta > \tau$ close enough to τ we have

$$\|u(t, \tau, u_\tau)\|_{E_{1+\varepsilon}} \leqslant (t - \tau)^{-\varepsilon},\ t \in (\tau, \delta),$$

in which case letting $\tilde{c} = c(\eta + C_\eta)$ we deduce from (1.11) that

$$\|F(s, u(s))\|_{E_{\gamma(\varepsilon)}} \leqslant \tilde{c}((s - \tau)^{-\varepsilon\rho} + 1),\ t \in (\tau, \delta). \tag{2.20}$$

Not loosing generality we will assume that $\delta > \tau$ is close enough to τ and (2.20) holds.

Since u is $E_{1+\varepsilon}$-solution and $\delta > \tau$ then $\|u\|_{L^\infty((\delta,T),E_{1+\varepsilon})} \leqslant c_\delta$ and by (1.11) we conclude that

$$m_\delta := \|F(t, u(t))\|_{L^\infty((\delta,T),E_{\gamma(\varepsilon)})} < \infty. \tag{2.21}$$

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From the variation of constants formula, for $\tau < \delta \leqslant t \leqslant t+h < T$ we infer

$$\begin{aligned}
&\|u(t+h)-u(t)\|_{E_{1+\theta}} \leqslant \|(U(t+h,\tau)-U(t,\tau))u_\tau\|_{E_{1+\theta}} \\
&\quad + \int_t^{t+h} \|U(t+h,s)F(s,u(s))\|_{E_{1+\theta}} ds \\
&\quad + \int_\delta^t \|(U(t+h,s)-U(t,s))F(s,u(s))\|_{E_{1+\theta}} ds \\
&\quad + \int_\tau^\delta \|(U(t+h,s)-U(t,s))F(s,u(s))\|_{E_{1+\theta}} ds =: J_1+J_2+J_3+J_4.
\end{aligned} \tag{2.22}$$

Choosing arbitrary

$$\hat{\varepsilon} \in (\theta,\mu) \cap (\theta,\gamma(\varepsilon)) \tag{2.23}$$

and applying (2.18)-(2.19) we get

$$\begin{aligned}
J_1 &= \|(U(t+h,t)-Id)U(t,\tau)u_\tau\|_{E_{1+\theta}} \\
&\leqslant \|U(t+h,t)-Id\|_{L(E_{1+\hat{\varepsilon}},E_{1+\theta})}\|U(t,\tau)\|_{L(E_0,E_{1+\hat{\varepsilon}})}\|u_\tau\|_{E_0} \\
&\leqslant M^2 h^{\hat{\varepsilon}-\theta}(t-\tau)^{-1-\hat{\varepsilon}}\|u_\tau\|_{E_0} \leqslant M^2 h^{\hat{\varepsilon}-\theta}(\delta-\tau)^{-1-\hat{\varepsilon}}\|u_\tau\|_{E_0}.
\end{aligned} \tag{2.24}$$

Using (2.18), (2.21), (2.23) we also obtain

$$\begin{aligned}
J_2 &= \int_t^{t+h} \|U(t+h,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\theta})}\|F(s,u(s))\|_{E_{\gamma(\varepsilon)}} ds \\
&\leqslant M m_\delta \int_t^{t+h} (t+h-s)^{\gamma(\varepsilon)-\theta-1} ds \leqslant M m_\delta (\gamma(\varepsilon)-\theta)^{-1}(T-\tau)^{\gamma(\varepsilon)-\hat{\varepsilon}} h^{\hat{\varepsilon}-\theta}.
\end{aligned} \tag{2.25}$$

On the other hand, by (2.19), (2.21) and (2.23) we have

$$\begin{aligned}
J_3 &\leqslant \int_\delta^t \|U(t+h,t)-Id\|_{L(E_{1+\hat{\varepsilon}},E_{1+\theta})}\|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\hat{\varepsilon}})}\|F(s,u(s))\|_{E_{\gamma(\varepsilon)}} ds \\
&\leqslant M^2 m_\delta \int_\delta^t h^{\hat{\varepsilon}-\theta}(t-s)^{\gamma(\varepsilon)-1-\hat{\varepsilon}} ds \leqslant M^2(\gamma(\varepsilon)-\hat{\varepsilon})^{-1} m_\delta h^{\hat{\varepsilon}-\theta}(t-\delta)^{\gamma(\varepsilon)-\hat{\varepsilon}} \\
&\leqslant M^2(\gamma(\varepsilon)-\hat{\varepsilon})^{-1} m_\delta h^{\hat{\varepsilon}-\theta}(T-\tau)^{\gamma(\varepsilon)-\hat{\varepsilon}},
\end{aligned} \tag{2.26}$$

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whereas using (2.20) we obtain

$$\begin{aligned} J_4 &\leqslant \int_\tau^\delta \|U(t+h,t)-Id\|_{L(E_{1+\hat\varepsilon},E_{1+\theta})}\|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\hat\varepsilon})}\|F(s,u(s))\|_{E_{\gamma(\varepsilon)}}ds \\ &\leqslant \tilde c M^2\int_\tau^\delta h^{\hat\varepsilon-\theta}(t-s)^{\gamma(\varepsilon)-1-\hat\varepsilon}((s-\tau)^{-\varepsilon\rho}+1)ds \\ &\leqslant \tilde c M^2 h^{\hat\varepsilon-\theta}\left(\int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\hat\varepsilon}(s-\tau)^{-\varepsilon\rho}ds+\int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\hat\varepsilon}ds\right) \\ &\leqslant \tilde c M^2 h^{\hat\varepsilon-\theta}\left(B(\gamma(\varepsilon)-\hat\varepsilon,1-\varepsilon\rho)\frac{(T_{u_\tau}-\tau)^{\gamma(\varepsilon)-\hat\varepsilon}}{(\delta-\tau)^{\varepsilon\rho}}+(\gamma(\varepsilon)-\hat\varepsilon)^{-1}(T_{u_\tau}-\tau)^{\gamma(\varepsilon)-\hat\varepsilon}\right). \end{aligned} \tag{2.27}$$

As a consequence of the above estimates for any $\delta>\tau$ close enough to τ there exists $\bar c>0$ such that for each $\tau<\delta\leqslant t\leqslant t+h<T$, we have

$$\|u(t+h)-u(t)\|_{E_{1+\theta}}\leqslant \bar c h^{\hat\varepsilon-\theta}.$$

Recalling that $\hat\varepsilon$ could be any number satisfying (2.23) we get the result. ▮

2.3. Proofs of continuation results

In this subsection we prove the results concerning continuation properties of $E_{1+\varepsilon}$-solution of (1.6).

2.3.1. Proof of part i) in Theorem 1.5

Recalling the statements of Remarks 1.1 we assume that $T_{u_\tau}<\infty$, $\limsup_{t\to T_{u_\tau}^-}\|u(t,\tau,u_\tau)\|_{E_1}<r^*$ for some $r^*>0$ and for any $n\in\mathbb N$ large enough we define $\tau_n:=T_{u_\tau}-\frac1n$, $u_{\tau_n}:=u(T_{u_\tau}-\frac1n,\tau,u_\tau)$. Let us consider now the Cauchy problem

$$\begin{cases}\dot u(t)+A(t)u(t)=F(t,u(t)),\ t>\tau_n,\\ u(\tau_n)=u_{\tau_n}.\end{cases} \tag{2.28}$$

Note that all initial conditions u_{τ_n} defined above belong both to $E_{1+\varepsilon}$ and to a ball $B_{E_1}(0,r^*)$ in E_1 of radius r^*. Also, the initial times τ_n converge to T_{u_τ}.

We then have $\sup_{s\in(\tau_n,\tau_n+\delta]}(s-\tau_n)^\varepsilon\|U(s,\tau_n)u_{\tau_n}\|_{E_{1+\varepsilon}}\leqslant M\|u_{\tau_n}\|_{E_1}\leqslant Mr^*$ and hence $u_{\tau_n}\in B^\delta_{\mathfrak{E}_\varepsilon^{\tau_n}}(0,Mr^*)$.

Due to Corollary 2.2 there is the unique bounded $E_{1+\varepsilon}$-solution of (2.28) on $[\tau_n,\tau_n+\delta_0]$, where δ_0 does not depend on n. Also, due to Lemma 2.1, the latter solution coincides with $u(\cdot,\tau,u_\tau)$ on $[\tau_n,T_{u_\tau}]$ for each n sufficiently large. From this we infer that, by concatenation, $u=u(\cdot,\tau,u_\tau)$ can be continued as an $E_{1+\varepsilon}$-solution of (1.6) onto the interval $[\tau,T_{u_\tau}+\delta_0)$, which contradicts the definition of T_{u_τ}. ▮

2.3.2. Proof of part ii) in Theorem 1.5

Recalling the statements of Remark 1.1 assume that $T_{u_\tau} < \infty$ and $\tau_n \to T_{u_\tau}^-$ is such that $u(\tau_n, \tau, u_\tau) \to w_0$ in E_1 as $n \to \infty$. We then have $\sup_{s\in(\tau_n,\tau_n+\delta]}(s-\tau_n)^\varepsilon \|U(s,\tau_n)(u_{\tau_n} - w_0)\|_{E_{1+\varepsilon}} \leqslant M\|u_{\tau_n} - w_0\|_{E_1}$. Therefore, if r is chosen relative to w_0 as in Theorem 1.2 and $N \in \mathbb{N}$ is such that $\|u_{\tau_n} - w_0\|_{E_1} \leqslant \frac{r}{M}$ for each $n \geqslant N$ then what was said above yields that $u_{\tau_n} \in B^{\delta}_{\mathfrak{E}_\varepsilon^{\tau_n}}(\omega_0, r)$ for every $n \geqslant N$ and any $\delta > 0$.

Due to Theorem 1.2 there is the unique bounded $E_{1+\varepsilon}$-solution of (2.28) on $[\tau_n, \tau_n + \delta_0]$, where δ_0 does not depend on n. Also, due to Lemma 2.1, the latter solution coincides with $u(\cdot, \tau, u_\tau)$ on $[\tau_n, T_{u_\tau}]$ for each n sufficiently large. From this we infer that, by concatenation, $u = u(\cdot, \tau, u_\tau)$ can be continued as an $E_{1+\varepsilon}$-solution of (1.6) onto the interval $[\tau, T_{u_\tau} + \delta_0)$, which contradicts the definition of T_{u_τ}. ▮

2.3.3. *Proof of Proposition 1.1*

Recalling statements of Remark 1.1 assume that $T_{u_\tau} < \infty$ and $\limsup_{t\to T_{u_\tau}^+} \|u(t,\tau,u_\tau)\|_{E_{1+\varepsilon}} < r^*$ for some $r^* > 0$. For any $n \in \mathbb{N}$ large enough we define $\tau_n := T_{u_\tau} - \frac{1}{n}$, $u_{\tau_n} := u(T_{u_\tau} - \frac{1}{n}, \tau, u_\tau)$ and consider the Cauchy problem (2.28).

Since u_{τ_n} belongs to a ball $B_{E_{1+\varepsilon}}(0, r^*)$ in $E_{1+\varepsilon}$ centered at zero and of radius $r^* > 0$, then for any $\delta > 0$ small enough we have that $\sup_{s\in(\tau_n,\tau_n+\delta]}(s-\tau_n)^\varepsilon \|U(s,\tau_n)u_{\tau_n}\|_{E_{1+\varepsilon}} \leqslant M\delta\|u_{\tau_n}\|_{E_{1+\varepsilon}} \leqslant M\delta r^*$. Hence, if $r > 0$ is chosen relatively to $w_0 = 0$ as in Theorem 1.2 and $\delta \in (0, \frac{r}{r^* M})$ then we observe that u_{τ_n} belongs to $B^{\delta}_{\mathfrak{E}_\varepsilon^{\tau_n}}(0, r)$.

As a consequence of Theorem 1.2 the problem (2.28) has the unique $E_{1+\varepsilon}$-solution on $[\tau_n, \tau_n + \delta_0]$, where δ_0 does not depend on n. Also, due to Lemma 2.1, the latter solution coincides with $u(\cdot, \tau, u_\tau)$ on $[\tau_n, T_{u_\tau})]$ for each n sufficiently large. From this we infer that, by concatenation, $u = u(\cdot, \tau, u_\tau)$ can be continued as an $E_{1+\varepsilon}$-solution of (1.6) onto the interval $[\tau, T_{u_\tau} + \delta_0)$, which contradicts the definition of T_{u_τ}. ▮

2.3.4. *Proof of Theorem 1.6*

By assumption, given $\tau \in \mathbb{R}$ and $u_\tau \in E_1$, we obtain from Theorem 1.2 that there exists the unique $E_{1+\varepsilon}$-solution u of (1.6) on the maximal interval of existence $[\tau, T_{u_\tau})$, for which we denote $u_0 := u_\tau$, $T_{u_\tau} := T_{u_0}$.

If $T_{u_0} < \infty$ then, using (1.32)-(1.33) and reflexivity of E_1, we conclude that there exists a certain $u_1 \in E_1$ such that $\lim_{t\to T_{u_0}^-} \|u(t,\tau,u_\tau) - u_1\|_{E_0} = 0$ and $u(t,\tau,u_\tau) \overset{t\to T_{u_0}^-}{\rightharpoonup} u_1$ weakly in E_1. Thus $u(t,\tau,u_\tau)$ can be extended to a function $\mathcal{U}_0 : [\tau, T_{u_0}] \to E_{1+\varepsilon} \cup E_0$ satisfying $\mathcal{U}_0 \in L^\infty_{loc}((\tau, T_{u_0}), E_{1+\varepsilon})$, $\mathcal{U}_0(\tau) = u_\tau = u_0$, $\mathcal{U}_0(t) \overset{E_1}{\rightharpoonup} \mathcal{U}_0(T_{u_0}) = u_1 \in \mathfrak{E}_\varepsilon^{T_{u_0}}$ as $t \to T_{u_0}^-$ and

$$\mathcal{U}_0(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)F(s,\mathcal{U}_0(s))ds \quad \text{for a. e.} \quad t \in [\tau, T_{u_0}].$$

By Theorem 1.2 there exists the unique $E_{1+\varepsilon}$-solution $u(\cdot, T_{u_0}, u_1)$ of the Cauchy problem

$$\begin{cases} \dot{u}(t) + A(t)u(t) = F(t, u(t)), \ t > T_{u_0}, \\ u(T_{u_0}) = u_1, \end{cases}$$

which can be continued on the maximal interval of existence $[T_{u_0}, T_{u_1})$. Now, if $T_{u_1} < \infty$, repeating the above argument we find $u_2 \in E_1$ such that $\lim_{t\to T_{u_1}^-} \|u(t,T_{u_0},u_1)-u_2\|_{E_0} = 0$ and $u(t,T_{u_0},u_1) \overset{t\to T_{u_1}^-}{\rightharpoonup} u_2$ weakly in E_1. Thus $u(t,T_{u_0},u_1)$ can be extended to a function $\mathcal{U}_1 : [T_{u_0}, T_{u_1}] \to E_{1+\varepsilon} \cup E_0$ satisfying $\mathcal{U}_1 \in L^\infty_{loc}((T_{u_0},T_{u_1}),E_{1+\varepsilon})$, $\mathcal{U}_1(T_{u_0}) = u_1$, $\mathcal{U}_1(t) \overset{E_1}{\to} \mathcal{U}_1(T_{u_1}) = u_2 \in \mathfrak{E}_\varepsilon^{T_{u_1}}$ as $t \to T_{u_1}^-$ and

$$\mathcal{U}_1(t) = U(t,T_{u_0})u_1 + \int_{T_{u_0}}^t U(t,s)F(s,\mathcal{U}_1(s))ds \text{ for a. e. } t \in [T_{u_0}, T_{u_1}].$$

If there is $k \in \mathbb{N}$ such that, proceeding as above, we obtain in a $(k+1)$-th step that $T_{u_k} = \infty$, then function $\mathcal{U} : [0,\infty) \to E_{1+\varepsilon} \cup E_0$ given by concatenations of the maps $\mathcal{U}_j$, $j = 0,\ldots,k+1$ is an extension of u into a piecewise-$E_{1+\varepsilon}$-solution on $[\tau,\infty)$.

Otherwise, proceeding inductively we will obtain a sequence of maps $\mathcal{U}_j : [\tau, T_{u_j}] \to E_{1+\varepsilon} \cup E_0$, $j = 0,1,\ldots$, and by concatenations we will again define a piecewise-$E_{1+\varepsilon}$-solution on $[\tau, a)$, with $a := \sum_{j=0}^\infty T_{u_j}$. In this latter case it is evident that either $a = \infty$, or if $a < \infty$ then a is accumulation time of singular times $T_j := \sum_{l=0}^j T_{u_l}$, $j \in \mathbb{N}$.

The above construction ensures that the extension of $E_{1+\varepsilon}$-solution to a piecewise-$E_{1+\varepsilon}$-solution is uniquely defined and hence the proof is complete. ▮

3. SINGULARLY NONAUTONOMOUS LINEAR PARABOLIC PROBLEMS

3.1. Families of sectorial operators in Banach spaces

In what follows X denote a Banach space. Our concern will be a family of unbounded operators in X generating C^0 analytic semigroups, which are all of the same type. Specifically we will show that Assumption 1.1 and Assumption 1.3 do hold under some general conditions on the operators appearing in the main part of (1.6).

Definition 3.1.
The family $\{A(t) : t \in \mathbb{R}\}$ of closed operators $A(t) : D_X \subset X \longrightarrow X$ in a Banach space X, which are defined on the same dense subset D_X of X, is *locally uniformly sectorial* (of the class $\mathcal{LUS}(D_X, X)$ for short) if and only if for each $t \in \mathbb{R}$ the complex half plane $\{\lambda \in \mathbb{C} : Re\lambda \leqslant 0\}$ is contained in the resolvent set $\rho(A(t))$ of $A(t)$ and for any bounded time interval $I \subset \mathbb{R}$ there exists a certain $M > 0$ such that

$$\|(\lambda I - A(t))^{-1}\|_{L(X)} \leqslant \frac{M}{1+|\lambda|}, \; Re\lambda \leqslant 0, \; t \in I. \tag{3.1}$$

If $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ then, for each $s \in \mathbb{R}$, $-A(s)$ generates a C^0 analytic semigroup $\{e^{-A(s)t} : t \geqslant 0\}$ in X, which is exponentially asymptotically decaying as $t \to \infty$. Actually, for a family $\{A(t) : t \in \mathbb{R}\}$ of the class $\mathcal{LUS}(D_X, X)$ we have that

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$Re\sigma(A(s)) > a > 0$ and

$$\|e^{-A(t)s}\|_{L(X)} \leqslant Ce^{-as}, \ s \geqslant 0, \quad \|A(t)e^{-A(t)s}\|_{L(X)} \leqslant \frac{C_1}{s}e^{-as}, \ s > 0,$$

where constants $a, C, C_1 > 0$ are independent of $s > 0$ and on t varying in bounded time intervals (see [23, §1.1]. Consequently, as in [18], the fractional powers $A^{\alpha}(t)$ can be defined as the inverse of $A^{-\alpha}(t) : X \to R(X)$,

$$A^{-\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} s^{\alpha-1} e^{-A(t)s} ds, \ \alpha > 0. \tag{3.2}$$

Also, one can consider the associated fractional power spaces $X^{\alpha}(t)$, $\alpha \geqslant 0$,

$$X^{\alpha}(t) := D(A^{\alpha}(t)) \ \text{ with the norm } \ \|\phi\|_{X^{\alpha}(t)} = \|A^{\alpha}(t)\phi\|_X \ \text{ for } \phi \in X^{\alpha}, \ \alpha > 0, \tag{3.3}$$

where for $\alpha = 0$ we set

$$A^0(t) := Id, \quad X^0(t) := X.$$

As in [23, §1.9, (1.56)] we then have

$$\|A^{\alpha}(t)e^{-A(t)s}\|_{L(X)} \leqslant c_{\alpha} e^{-as} s^{-\alpha}, \ s > 0, \tag{3.4}$$

where c_{α} neither depends on $s > 0$ nor on t varying on bounded time intervals.

Since $A(t)$ coincides with the inverse of $A^{-1}(t)$, then $X^1(t)$ coincides as a set with the domain D_X for every $t \in \mathbb{R}$. Concerning topologies we have the following result.

Proposition 3.1. *If $\{A(t) : \ t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and $I \subset \mathbb{R}$ is any set such that*

$$\sup_{t,s \in I} \|A(t)A^{-1}(s)\|_{L(X)} < \infty, \tag{3.5}$$

then the corresponding spaces $X^1(t)$ are independent of t, except for norms, which are uniformly equivalent on I.

Proof: Evidently, for the graph norms we have

$$\|\phi\|_{X^1(t)} = \|A(t)\phi\|_X = \|A(t)A^{-1}(s)A(s)\phi\|_X \leqslant c\|A(s)\phi\|_X = c\|\phi\|_{X^1(s)}, \ t, s \in I,$$

which gives the result. ∎

If $A(t)$ belongs to a class of operators having locally uniformly bounded purely imaginary powers, that is if $A(t)$ is a positive operator satisfying

$$\exists_{\epsilon > 0} \sup_{s \in [-\epsilon, \epsilon]} \|A^{is}(t)\|_{L(X)} < \infty, \tag{3.6}$$

then the associated fractional power spaces can be characterized with the aid of complex interpolation as

$$X^{(1-\theta)\alpha+\theta\beta} = [X^{\alpha}(t), X^{\beta}(t)]_{\theta},\ 0<\theta<1,\ 0\leqslant\alpha<\beta<\infty; \tag{3.7}$$

see [25], also [5].

Definition 3.2. We will say that the family of positive operators $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{BIP}(X)$ if and only if, given any $t \in \mathbb{R}$, $A(t)$ has the property (3.6).

Corollary 3.1. *If $\{A(t) :\ t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$ and $I \subset \mathbb{R}$ is any set such that* (3.5) *holds then $X^{\theta}(t)$, $\theta \in [0,1]$, are independent of $t \in I$, except for norms, which are uniformly equivalent on I.*

Following [3], given $\{A(t) :\ t \in \mathbb{R}\}$ of the class $\mathcal{LUS}(D_X, X)$, we will consider the extrapolated space $X^{-1}(t)$ generated by $(X, A(t))$, where

$$X^{-1}(t) \ \text{ is the completion of } (X, \|A^{-1}(t)\cdot\|_X).$$

If this is the case, we will also extend $A(t)$ to a closed operator in $X^{-1}(t)$, for which we will use the same notation.

Whenever $t, s \in \mathbb{R}$ are such that $A^{-1}(s)A(t), A^{-1}(t)A(s) : D_X \subset X \to X$ are bounded operators, which happens in particular when the domains of the adjoint operators $A'(t)$ and $A'(s)$ are the same, then $X^{-1}(t)$ coincide with $X^{-1}(s)$ algebraically and topologically as for some constants $c_1, c_2 > 0$ we have

$$c_1\|A^{-1}(s)x\|_X \leqslant \|A^{-1}(t)x\|_X \leqslant c_2\|A^{-1}(s)x\|_X,\ x\in X,$$

(see [4]). This ensures that the following counterpart of Proposition 3.1 holds for the extrapolated spaces.

Proposition 3.2. *If $\{A(t) :\ t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and $I \subset \mathbb{R}$ is any set such that the operators $A^{-1}(t)A(s) : D_X \subset X \to X$ are uniformly bounded for $t, s \in I$; that is for the closure $\overline{A^{-1}(t)A(s)}$ we have*

$$\sup_{t,s\in I}\|\overline{A^{-1}(t)A(s)}\|_{L(X)} < \infty, \tag{3.8}$$

then the associated spaces $X^{-1}(t)$ are independent of $t \in I$, except for norms, which are uniformly equivalent on I.

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On the other hand, due to [3, Proposition V.1.31], if $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ then (the closed extension of) each $A(t)$ belongs to a class $Lis(X, X^{-1}(t))$ of linear isomorphisms from X into $X^{-1}(t)$. Furthermore, we have that $\{\lambda \in \mathbb{C} : Re\lambda \leqslant 0\} \subset \rho(A(t))$ and for any bounded time interval I there exists $M > 0$ such that

$$\|(\lambda I - A(t))^{-1}\|_{L(X^{-1}(t))} \leqslant \frac{M}{1+|\lambda|},\ Re\lambda \leqslant 0,\ t \in I. \tag{3.9}$$

Letting $Y(t) = X^{-1}(t)$ and applying (3.2) we then associate with $(Y(t), A(t))$ the fractional power scale $\{Y^\alpha(t) : \alpha \geqslant 0\}$ and, as in [3, p. 266], we consider

$$X^\alpha(t) := Y^{\alpha+1}(t),\ \alpha \in [-1, \infty), \tag{3.10}$$

which is the extrapolated fractional power scale of order 1 generated by $(X, A(t))$.

Corollary 3.2. *If $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$ and $I \subset \mathbb{R}$ is any set such that* (3.5) *and* (3.8) *hold, then for each $\theta \in [-1, 1]$ spaces $Y^{\theta+1}(t) = X^\theta(t)$ are independent of $t \in I$, except for norms, which are uniformly equivalent on I; that is for every $\theta \in [-1, 1]$ there exists $c > 0$ such that*

$$\|\phi\|_{X^\theta(t)} \leqslant c\|\phi\|_{X^\theta(s)},\ s, t \in I,$$

for every ϕ from the set $X^\theta(t) = X^\theta(s)$.

In what follows, given $t_0 \in \mathbb{R}$, $\alpha_0 \in [0, 1)$ and letting $\mu_0 := 1 - \alpha_0$ we consider a family of Banach spaces

$$E_\alpha := Y^{\alpha+\alpha_0}(t_0), \quad \|\cdot\|_{E_\alpha} = \|A^{\alpha+\alpha_0}(t_0)\cdot\|_{Y(t_0)}, \quad \alpha \in [0, 1+\mu_0]. \tag{3.11}$$

Lemma 3.1. *Suppose that $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$, conditions* (3.5), (3.8) *hold on each bounded time interval $I \subset \mathbb{R}$ and $\{E_\alpha, \alpha \in [0, 1+\mu_0]\}$ is defined as in* (3.11)*, where μ_0 is a strictly positive number.*

Then,

i) $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_{E_0}, E_0)$ with $D_{E_0} = E_1$,

ii) for any bounded time interval $I \subset \mathbb{R}$ and $\sigma \in [0, 1+\mu_0]$ there exist constants $c, c', c'' > 0$ such for each $t, s \in I$ we have

$$\|\phi\|_{E_\sigma} \leqslant c''\|A^\sigma(t)\phi\|_{E_0} \leqslant c\|A^\sigma(s)\phi\|_{E_0} \leqslant c'\|\phi\|_{E_\sigma},\ \phi \in E_\sigma. \tag{3.12}$$

Proof: Recall that $\{Y^\alpha(t) : \alpha \geqslant 0\}$ is the fractional power scale generated by $(Y(t), A(t))$. Hence (realization of) $A(t)$ can be viewed as a closed densely defined operator in $Y^{\alpha_0}(t)$ with the domain $Y^{\alpha_0+1}(t)$. The resolvent set of $A(t)$ in this setting will still contain the

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half-plane $\{\lambda \in \mathbb{C} : Re\lambda \leqslant 0\}$ and for each bounded time interval I there will be a certain constant $M > 0$ such that

$$\|(\lambda I - A(t))^{-1}\phi\|_{Y^{\alpha_0}(t)} \leqslant \frac{M}{1+|\lambda|}\|\phi\|_{Y^{\alpha_0}(t)},\ Re\lambda \leqslant 0,\ t \in I,\ \phi \in Y_0^{\alpha}(t). \tag{3.13}$$

Part i) is thus a consequence of Corollary 3.2 and (3.9).

Concerning part ii) we first observe that, due to Corollary 3.2, if $\phi \in E^\sigma$ then ϕ belongs to each of the sets $Y^{\sigma+\alpha_0}(t)$ and $Y^{\sigma+\alpha_0}(s)$ as these sets coincide for each $t, s \in \mathbb{R}$ and $A^\sigma(t)\phi, A^\sigma(s)\phi$ are the elements of $E^0 = Y^{\alpha_0}(t_0)$ (actually $A^\sigma(t), A^\sigma(s)$ are one-to-one from E_σ onto E_0). Given a bounded time interval $I \subset \mathbb{R}$ we can thus use equivalence of norms stated in Corollary 3.2 to get, for some constants $\bar{c}, \tilde{c}, \hat{c}$ depending on I but not on $t, s \in I$, that

$$\begin{aligned}\|A^\sigma(t)\phi\|_{E_0} &= \|A^\sigma(t)\phi\|_{Y^{\alpha_0}(t_0)} \leqslant \bar{c}\|A^\sigma(t)\phi\|_{Y^{\alpha_0}(t)} = \bar{c}\|\phi\|_{Y^{\sigma+\alpha_0}(t)} \leqslant \tilde{c}\|\phi\|_{Y^{\sigma+\alpha_0}(s)} \\ &= \tilde{c}\|A^\sigma(s)\phi\|_{Y^{\alpha_0}(s)} \leqslant \hat{c}\|A^\sigma(s)\phi\|_{Y^{\alpha_0}(t_0)} = \hat{c}\|A^\sigma(s)\phi\|_{E_0}\end{aligned}$$

whenever $t, s \in I$. Similarly, using again the equivalence of norms, we also have

$$\|\phi\|_{E_\sigma} = \|\phi\|_{Y^{\sigma+\alpha_0}(t_0)} \leqslant \tilde{c}\|\phi\|_{Y^{\sigma+\alpha_0}(t)} = \tilde{c}\|A^\sigma(t)\phi\|_{Y^{\alpha_0}(t)} \leqslant \hat{c}\|A^\sigma(t)\phi\|_{Y^{\alpha_0}(t_0)} = \hat{c}\|A^\sigma(t)\phi\|_{E_0}$$

and

$$\|A^\sigma(s)\phi\|_{E_0} = \|A^\sigma(s)\phi\|_{Y^{\alpha_0}(t_0)} \leqslant \tilde{c}\|A^\sigma(s)\phi\|_{Y^{\alpha_0}(s)} = \tilde{c}\|\phi\|_{Y^{\sigma+\alpha_0}(s)} \leqslant \hat{c}\|\phi\|_{Y^{\sigma+\alpha_0}(t_0)} = \hat{c}\|\phi\|_{E_\sigma},$$

which proves ii). ▌

Corollary 3.3. *Under the assumptions of Lemma 3.1 we have that for anyh bounded time interval $I \subset \mathbb{R}$ and $\sigma \in [0, 1+\mu_0]$ there exists a constant $c > 0$ such that*

$$\|A^\sigma(t)A^{-\sigma}(s)\|_{L(E_0)} \leqslant c \ \ \textit{for each} \ \ t, s \in I, \tag{3.14}$$

where μ_0 is a strictly positive number.

Proof: It suffices to note that $A^\sigma(t)$, $A^\sigma(s)$ are one-to-one from E_σ onto E_0 and then use (3.12). ▌

3.2. Smoothing properties of singularly nonautonomous linear problems

We will now assume that a family $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and that, in addition,

$$\exists_{\mu\in(0,1]}\ \forall_{T>0}\ \exists_{C>0}\ \forall_{t,\tau,s\in[-T,T]}\ \|(A(t) - A(\tau))A^{-1}(s)\|_{L(X)} \leqslant C|t-\tau|^\mu. \tag{3.15}$$

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Under these assumptions, following [3, 13, 17, 20, 23], we will consider in X a singularly nonautonomous linear problem

$$\begin{cases} \dot{u}(t) + A(t)u(t) = 0, \ t > \tau, \\ u(\tau) = u_\tau. \end{cases} \tag{3.16}$$

Recall that a continuous function $[\tau, \infty) \ni t \to u(t) \in E_0$ is a *classical solution* of (3.16) if it is continuously differentiable in (τ, ∞), $u(t) \in D_X$ for each $t > \tau$ and both relations in (3.16) are satisfied.

Concerning solutions of (3.16) the following result is known (see [13, §2] for the proof).

Proposition 3.3. *Suppose that $\{A(t) : \ t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and* (3.15) *holds.*

Then, there exists a continuous process $\{U(t,\tau) : \ t \geqslant \tau \in \mathbb{R}\} \subset L(X)$ defined by (3.16) *in X such that given $\tau \in \mathbb{R}$ and $u_\tau \in X$, the map $[\tau, \infty) \ni t \to u(t) = U(t,\tau)u_\tau \in X$ is a classical solution of* (3.16).

We remark that $U(t,\tau)$ can be viewed as a fundamental solution of the evolution equation $\dot{u}(t) + A(t)u(t) = 0$, see [17, p. 109], which nonetheless we will not pursue here. Instead we will turn our attention towards the smoothing action of $U(t,s)$.

Proposition 3.4. *Under the assumptions of Proposition 3.3 for each bounded time interval $I = [-T, T]$ there is a positive constant N such that*

$$\|A^\sigma(t)U(t,\tau)A^{-\zeta}(\tau)\|_{L(X)} \leqslant N(t-\tau)^{\zeta-\sigma}, \ 0 \leqslant \zeta \leqslant \sigma < 1+\mu, \ -T \leqslant \tau < t \leqslant T, \tag{3.17}$$

where μ is as in (3.15).

For the proof of (3.17) we refer the reader to the classical work [23] (see also [13, Theorem 2.2]).

In Proposition 3.5 below we will focus on another smoothing property of the linear process. For this we will need the following *additional assumption*:

$$\forall_{1+\mu>\xi>0} \ \forall_{T>0} \ \exists_{c>0} \ \forall_{t,\tau\in[-T,T]} \ \|A^\xi(t)A^{-\xi}(\tau)\|_{L(X)} \leqslant c. \tag{3.18}$$

Proposition 3.5. *If $\{A(t) : \ t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and* (3.15) *holds then*

$$\forall_{T>0} \ \forall_{\substack{1\geqslant\zeta>\sigma\geqslant0 \\ 1>\zeta-\sigma>\delta>0}} \ \exists_{N>0} \ \forall_{-T\leqslant\tau\leqslant t\leqslant T} \ \|A^\sigma(t)[U(t,\tau) - Id]A^{-\zeta}(\tau)\|_{L(X)} \leqslant N(t-\tau)^\delta. \tag{3.19}$$

Actually, if also (3.18) *is satisfied then*

$$\forall_{T>0} \ \forall_{\substack{1+\mu>\zeta>\sigma\geqslant0 \\ 1>\zeta-\sigma}} \ \exists_{N>0} \ \forall_{-T\leqslant\tau\leqslant t\leqslant T} \ \|A^\sigma(t)[U(t,\tau) - Id]A^{-\zeta}(\tau)\|_{L(X)} \leqslant N(t-\tau)^{\zeta-\sigma}. \tag{3.20}$$

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Proof: From [23, (1.53)] we infer that

$$U(t,\tau)A^{-\zeta}(\tau) = e^{(t-\tau)A(t)}A^{-\zeta}(\tau) + \int_\tau^t e^{(t-s)A(t)}[A(s) - A(t)]U(s,\tau)A^{-\zeta}(\tau)ds,$$

where the function under the integral above is integrable in $L(X)$ over $s \in (0,t)$, which will be clear from the estimates below.

We next rewrite $A^\sigma(t)[U(t,\tau) - Id]A^{-\zeta}(\tau)$ as a sum $J_1 + J_2$, where

$$J_1 = A^\sigma(t)[e^{(t-\tau)A(t)} - Id]A^{-\zeta}(\tau) \text{ and } J_2 = \int_\tau^t A^\sigma(t)e^{(t-s)A(t)}[A(s) - A(t)]U(s,\tau)A^{-\zeta}(\tau)ds.$$

Note that taking $s = \tau$ in (3.15) we obtain that (3.5) holds on any bounded time interval $I \subset \mathbb{R}$. Thus note that, under the assumptions that $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and (3.15) holds, one obtains as in [23, §1.9, (1.59)] that

$$\forall_{1\geqslant\zeta>\xi\geqslant 0}\ \forall_{T>0}\ \exists_{c>0}\ \forall_{t,\tau\in[-T,T]}\ \|A^\xi(t)A^{-\zeta}(\tau)\|_{L(X)} \leqslant c. \tag{3.21}$$

Without (3.18), if $1 \geqslant \zeta > \sigma \geqslant 0$ and $0 < \delta < \zeta - \sigma < 1$ then applying [18, Theorem 1.4.3] we can estimate $\|J_1 v\|_X$ for every $v \in X$ by $\frac{1}{\delta}c_{1-\delta}(t-\tau)^\delta\|A^{\delta+\sigma}(t)A^{-\zeta}(\tau)v\|_X$, which via (3.21) can be bounded on $[-T,T]$ by $\frac{1}{\delta}cc_{1-\delta}(t-\tau)^\delta\|v\|_X$.

On the other hand, if $1 + \mu > \zeta > \sigma \geqslant 0$ and (3.18) holds, then choosing $\tilde\delta = \zeta - \sigma$ and applying again [18, Theorem 1.4.3] we estimate $\|J_1 v\|_X$ for each $v \in X$ by $\frac{1}{\tilde\delta}c_{1-\tilde\delta}(t-\tau)^{\tilde\delta}\|A^{\tilde\delta+\sigma}(t)A^{-\zeta}(\tau)v\|_X = \frac{1}{\zeta-\sigma}c_{1-\zeta+\sigma}(t-\tau)^{\zeta-\sigma}\|A^\zeta(t)A^{-\zeta}(\tau)v\|_X$, which is now bounded on $[-T,T]$ by $\frac{1}{\zeta-\sigma}cc_{1-\zeta+\sigma}(t-\tau)^{\zeta-\sigma}\|v\|_X$ due to (3.18).

Consequently, for J_1 we obtain by assumption that

$$\|J_1\|_{L(X)} \leqslant \begin{cases} \frac{1}{\delta}cc_{1-\delta}(t-\tau)^\delta, & 0 < \delta < \zeta - \sigma, \text{ not assuming (3.18)}, \\ \frac{1}{\zeta-\sigma}cc_{1-\zeta+\sigma}(t-\tau)^{\zeta-\sigma}, & \text{assuming (3.18)}. \end{cases}$$

J_2 is equal to $\int_\tau^t A^\sigma(t)e^{(t-s)A(t)}[(A(s) - A(t))A^{-1}(s)]A(s)U(s,\tau)A^{-\zeta}(\tau)ds$, where by (3.4) and (3.15) we have

$$\|A^\sigma(t)e^{(t-s)A(t)}[(A(s) - A(t))A^{-1}(s)]\|_{L(X)} \leqslant c(t-s)^{-\sigma}(t-s)^\mu.$$

Note that if $0 \leqslant \zeta \leqslant 1$ we obtain from (3.17) that

$$\|A(s)U(s,\tau)A^{-\zeta}(\tau)\|_{L(X)} \leqslant c(s-\tau)^{\zeta-1}.$$

On the other hand, if $1+\mu > \zeta > 1$, then $A(s)U(s,\tau)A^{-\zeta}(\tau) = A(s)U(s,\tau)A^{-1}(\tau)A^{1-\zeta}(\tau)$ and

$$\|A(s)U(s,\tau)A^{-\zeta}(\tau)\|_{L(X)} \leqslant \|A(s)U(s,\tau)A^{-1}(\tau)\|_{L(X)}\|A^{1-\zeta}(\tau)\|_{L(X)} \leqslant c,$$

as in this case $A^{1-\zeta}(\tau) = A^{-(\zeta-1)}(\tau) = \frac{1}{\Gamma(\zeta-1)}\int_0^\infty s^{\zeta-2}e^{-A(\tau)s}ds$ is a bounded operator and

$$\|A^{1-\zeta}(\tau)\|_{L(X)} \leqslant \frac{C}{\Gamma(\zeta-1)}\int_0^\infty s^{\zeta-2}e^{-as}ds = Ca^{1-\zeta}.$$

Since t,τ vary in bounded interval for $0\leqslant\zeta\leqslant 1$ we thus infer that

$$\begin{aligned}\|J_2\|_{L(X)} &\leqslant \tilde{c}\int_\tau^t (t-s)^{\mu-\sigma}(s-\tau)^{\zeta-1}ds \leqslant \tilde{c}(t-\tau)^{\mu-\sigma+\zeta}B(1+\mu-\sigma,\zeta)\\ &= \tilde{c}B(1+\mu-\sigma,\zeta)(t-\tau)^{\mu}(t-\tau)^{\zeta-\sigma} \leqslant \bar{c}(t-\tau)^{\zeta-\sigma},\end{aligned}$$

whereas for $1+\mu>\zeta>1$ we have

$$\|J_2\|_{L(X^\alpha)} \leqslant \tilde{c}\int_\tau^t (t-s)^{\mu-\sigma}ds = \hat{c}(t-\tau)^{1+\mu-\sigma} = \hat{c}(t-\tau)^{1+\mu-\zeta}(t-\tau)^{\zeta-\sigma} \leqslant \bar{c}(t-\tau)^{\zeta-\sigma}.$$

Combining the above estimates we get the result. ▮

3.3. Validity of Assumption 1.1 and Assumption 1.3

In this subsection our concern is to describe a general mechanism that leads to validity of Assumption 1.1 and Assumption 1.3.

THEOREM 3.1. *Suppose that* $\{A(t): t\in\mathbb{R}\}$ *is of the class* $\mathcal{LUS}(D_X,X)\cap\mathcal{BIP}(X)$, *conditions* (3.5), (3.8) *hold on each bounded time interval* $I\subset\mathbb{R}$ *and* $\{E_\alpha,\alpha\in[0,1+\mu_0]\}$ *is defined as in* (3.11). *Suppose furthermore that*

$$\exists_{\mu\in(0,\mu_0]}\ \forall_{T>0}\ \exists_{C>0}\ \forall_{t,\tau,s\in[-T,T]}\ \|(A(t)-A(\tau))A^{-1}(s)\|_{L(E^0)} \leqslant C|t-\tau|^{\mu}. \tag{3.22}$$

Under these assumptions:
i) there exists a continuous process $\{U(t,\tau):\ t\geqslant\tau\in\mathbb{R}\}\subset L(E_0)$ *defined by* (3.16) *in* E_0 *such that given* $\tau\in\mathbb{R}$ *and* $u_\tau\in E_0$, *the map* $[\tau,\infty)\ni t\to u(t)=U(t,\tau)u_\tau\in E^0$ *is a classical solution of* (3.16); *furthermore,*
ii) $\|U(t,\tau)\|_{L(E_\zeta,E_\sigma)}\leqslant M(t-\tau)^{\zeta-\sigma}$, $0\leqslant\zeta\leqslant\sigma<1+\mu$, $-T\leqslant\tau<t\leqslant T$, *and*
iii) $\|U(t,\tau)-Id\|_{L(E_\zeta,E_\sigma)}\leqslant M(t-\tau)^{\zeta-\sigma}$, $1+\mu>\zeta>\sigma\geqslant 0$, $1\geqslant\zeta-\sigma>0$, $-T\leqslant\tau<t\leqslant T$,
where M *in ii)-iii) is a certain positive constant which can depend on* ζ,σ,T *but does not depend on* $t,\tau\in[-T,T]$.

Proof: Due to Lemma 3.1 we obtain that $\{A(t): t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_{E_0},E_0)$ with $D_{E_0}=E_1$. From this and (3.22) we obtain via Proposition 3.3 (applied with $X=E_0$) the statements in part i).

Then, from Proposition 3.4 (applied with $X=E_0$), for each bounded time interval $I\subset\mathbb{R}$ there is a positive constant N such that

$$\|A^\sigma(t)U(t,\tau)A^{-\zeta}(\tau)\|_{L(X)} \leqslant N(t-\tau)^{\zeta-\sigma},\ 0\leqslant\zeta\leqslant\sigma<1+\mu,\ t\in I. \tag{3.23}$$

Since $A^{\zeta}(\tau)$ is an one-to-one map from E_σ onto E_0, the inequality in (3.23) can be rewritten equivalently as

$$\|A^{\sigma}(t)U(t,\tau)\phi\|_{E_0} \leqslant N(t-\tau)^{\zeta-\sigma}\|A^{\zeta}(\tau)\phi\|_{E^0},\ \phi \in E_\zeta, \tag{3.24}$$

and by (3.12) also as

$$\|[U(t,\tau)\phi\|_{E_\sigma} \leqslant M(t-\tau)^{\zeta-\sigma}\|\phi\|_{E^\zeta},\ \phi \in E_\zeta, \tag{3.25}$$

which gives ii).

Finally, taking into account the result of Corollary 3.3 we can apply Proposition 3.5 with $X = E_0$ and obtain from (3.20) that

$$\forall_{T>0}\ \forall_{\substack{1+\mu>\zeta>\sigma\geqslant 0\\ 1>\zeta-\sigma}}\ \exists_{N>0}\ \forall_{-T\leqslant\tau\leqslant t\leqslant T}\ \|A^{\sigma}(t)[U(t,\tau)-Id]A^{-\zeta}(\tau)\|_{L(E_0)} \leqslant N(t-\tau)^{\zeta-\sigma}. \tag{3.26}$$

Again the inequality in (3.26) can be rewritten equivalently as

$$\|A^{\sigma}(t)[U(t,\tau)-Id]\phi\|_{E_0} \leqslant N(t-\tau)^{\zeta-\sigma}\|A^{\zeta}(\tau)\phi\|_{E_0},\ \phi \in E_\zeta,$$

and by (3.12) also as

$$\|[U(t,\tau)-Id]\phi\|_{E_\sigma} \leqslant M(t-\tau)^{\zeta-\sigma}\|\phi\|_{E_\zeta},\ \phi \in E_\zeta,$$

which gives iii). ∎

Let us remark that condition (3.22) can be expressed equivalently as in the proposition below.

PROPOSITION 3.6. *Suppose that* $\{A(t) : t \in \mathbb{R}\}$ *is of the class* $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$, *conditions* (3.5), (3.8) *hold on each bounded time interval* $I \subset \mathbb{R}$ *and* $\{E_\alpha, \alpha \in [0, 1+\mu_0]\}$ *is defined as in* (3.11).

Then (3.22) *is equivalent to*

$$A(\cdot) \in C^{\mu}_{loc}(\mathbb{R}, L(E_1, E_0)). \tag{3.27}$$

Proof: On any bounded time interval $I \subset \mathbb{R}$ condition (3.22) implies that

$$\|(A(t)-A(\tau))\phi\|_{E_0} \leqslant C|t-\tau|^{\mu}\|A(s)\phi\|_{E_0} \quad \text{and} \quad \|A(t)\phi\|_{E_0} \leqslant c\|A(s)\phi\|_{E_0}$$

whenever $t, \tau, s \in I$, $\phi \in D_{E_0}$. Due to (3.12) we then have $\|(A(t)-A(\tau))\phi\|_{E_0} \leqslant \tilde{C}|t-\tau|^{\mu}\|\phi\|_{E_1}$ for $t, \tau \in I$, which proves that $A(\cdot) \in C^{\mu}(I, L(E_1, E_0))$.

On the other hand, if $A(\cdot) \in C^{\mu}_{loc}(I, L(E_1, E_0))$ then, given a bounded time interval $I \subset \mathbb{R}$, we have that $A(\cdot) \in C^{\mu}(I, L(E_1, E_0))$. Combining this with (3.12) we obtain that

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$\|(A(t)-A(\tau))\psi\|_{E_0} \leqslant C|t-\tau|^{\mu}\|\psi\|_{E_1} \leqslant \tilde{C}|t-\tau|^{\mu}\|A(s)\psi\|_{E_0}$, $t,\tau,s \in I$, $\psi \in E_1$, and letting $\phi = A^{-1}(s)\psi$ we get (3.22). ∎

Note that under the assumptions of Theorem 3.1 both Theorem 1.2 and Theorem 1.4 apply provided that the required assumption on F holds. Actually in applications we often have some $\nu_0 \in (0,1)$ such that for each bounded time interval $I \subset \mathbb{R}$ and any B bounded in $E_{1+\varepsilon}$ there exist constant $c > 0$ such that

$$\|F(t,v)-F(s,w)\|_{E_0} \leqslant c(|t-s|^{\nu_0} + \|v-w\|_{E_{1+\varepsilon}}),\ t,s \in I,\ v,w \in B, \tag{3.28}$$

in which case $E_{1+\varepsilon}$-solution of (1.6) will have the properties of a classical solution (see Proposition 3.7 below).

Proposition 3.7. *Suppose that $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$, conditions* (3.5), (3.8) *hold on each bounded time interval $I \subset \mathbb{R}$ and $\{E_\alpha, \alpha \in [0, 1+\mu_0]\}$ is defined as in* (3.11). *Suppose also that* (3.27), (3.28) *are satisfied and that F is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$.*

Then $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_{E_0}, E_0)$ with $D_{E_0} = E_1$ and Theorems 1.2 and 1.4 apply. The unique $E_{1+\varepsilon}$-solution, $u = u(\cdot,\tau,u_\tau)$, is of the class $C^1((\tau,\tau+\delta_0], E_0)$, $u(t) \in D_{E_0}$ for $t \in (\tau,\tau+\delta_0]$ and $\dot{u}(t) + A(t)u(t) = F(t,u(t))$ for each $t \in (\tau,\tau+\delta_0]$.

Proof: By Theorem 3.1 we know that Theorems 1.2 and 1.4 apply. Hence there is the unique $E_{1+\varepsilon}$-solution of (1.6), $u(\cdot) = u(\cdot,\tau,u_\tau)$ and $u \in C^{\nu}_{loc}((0,T_{u_\tau}), E_{1+\varepsilon})$ for some $\nu \in (0,1)$. The latter property and (3.28) yield that $F(\cdot,u(\cdot)) \in C^{\sigma}_{loc}((0,T_{u_\tau}),E_0)$ for $\sigma = \min\{\nu_0,\nu\}$. The result now follows as in [20, §5.7, Theorem 7.1] and [13, §2.3]). ∎

Remark 3. 1. i) Following [13, Lemma 3.5] one can prove that $\lim_{t\to\tau^+}\|U(t,\tau)u_\tau - u_\tau\|_{E_1} = 0$ in which case (1.22) yields that $\lim_{t\to\tau^+}\|u(t,\tau,u_\tau)-u_\tau\|_{E_{1+\theta}} = 0$. Applying this and (1.21) with $\theta = 0$ we obtain that $E_{1+\varepsilon}$-solution $u(t,\tau,u_\tau)$ of (1.6) is continuous in E_1 with respect to $(t,u_\tau) \in [\tau,\tau+\delta_0]\times E_1$.
ii) Following [27, Theorem 3.10] we obtain that, under the assumptions of Proposition 3.7, the $E_{1+\varepsilon}$-solution $u(\cdot) = u(\cdot,\tau,u_\tau)$ of (1.6) possesses the following properties

$$A(\cdot)u(\cdot) \in C((\tau,\tau+\delta_)],E_0)$$
$$\frac{d}{dt}u(\cdot) \in \mathcal{F}^{1,\sigma}(\tau,\tau+\delta_0],E_0)$$

where $\mathcal{F}^{1,\sigma}(\tau,\tau+\delta_0],E_0)$ is defined in [27, page 05].

3.4. "Singularly nonautonomous" makes difference

Note that the fact that we deal with singularly nonautonomous problems matters a lot. A strongly damped wave equation

$$\begin{cases} u_{tt} + \eta(-\Delta)u_t + (-\Delta)u = f(t,u), \ t > 0, \ x \in \Omega, \\ u(0,x) = u_0(x), \ u_t(0,x) = v_0(x), \ x \in \Omega, \\ u(t,x) = 0, \ t \geqslant 0, \ x \in \partial\Omega, \end{cases} \tag{3.29}$$

considered in a bounded smooth domain $\Omega \subset \mathbb{R}^n$ can serve here as an example.

For fixed $\eta > 0$ the problem (3.29) is known to fall into a class of abstract parabolic problems in the "energy" space $X = H^1_0(\Omega) \times L^2(\Omega)$. Namely, (3.29) can be rewritten as

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 & -I \\ -\Delta & -\eta\Delta \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ f^e(t,u) \end{bmatrix}, \ t > 0, \qquad \begin{bmatrix} u \\ v \end{bmatrix}_{t=\tau} = \begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix}, \tag{3.30}$$

where f^e is a Nemitskiĭ operator associated with f and $\begin{bmatrix} 0 & -I \\ -\Delta & -\eta\Delta \end{bmatrix}$ can be viewed as a negative generator of a C^0 analytic semigroup in $H^1_0(\Omega) \times L^2(\Omega)$. In addition, $\begin{bmatrix} 0 & -I \\ -\Delta & -\eta\Delta \end{bmatrix}$ will also has bounded purely imaginary powers. Then one can show as in [9, 10] that, if η is a constant, the problem (3.30) is well posed in $H^1_0(\Omega) \times L^2(\Omega)$ provided that f grows with respect to u not faster than $|u|^{\frac{N+2}{N-2}}$).

When η depends on time variable t, then the domain of the main part operator $A(t) = \begin{bmatrix} 0 & -I \\ -\Delta & -\eta(t)\Delta \end{bmatrix}$ will in the above setting depend on t and hence (3.30) requires a more advance approach (see e.g. [4]). On the other hand note that if one tried to verify the condition (3.15) this would fail anyway as recalling the precise definition of $A(t)$ in $X = H^1_0(\Omega) \times L^2(\Omega)$ we have

$$A(t)\begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} -\psi \\ -\Delta(\varphi + \eta(t)\psi) \end{bmatrix}, \tag{3.31}$$

where

$$\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D_X(t) := \{\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in H^1_0(\Omega) \times H^1_0(\Omega); \ \varphi + \eta(t)\psi \in H^2(\Omega \cap H^1_0(\Omega)\}.$$

Hence, given $\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}$, one might not be able to apply $A(t) - A(s)$ to $A^{-1}(\tau)\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}$ whilst formal calculations would indicate anyway that $(A(t) - A(s))A^{-1}(\tau)$ might not be bounded in $L(X)$.

4. APPLICATIONS

In what follows we show how the abstract results apply in sample problems. This involves singularly nonautonomous wave equations with a structural damping and a sample $2m$-th order singularly nonautonomous parabolic equation.

4.1. Singularly nonautonomous wave equation with structural damping

In this example, following [9, 11, 10], we consider the initial boundary value problem of the form:

$$\begin{cases} u_{tt} + \eta(t)(-\Delta)^{\frac{1}{2}} u_t + \nu u_t + (-\Delta)u = f(t,u), \quad t > 0, \quad x \in \Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x), \quad x \in \Omega, \\ u(t,x) = 0, \quad t \geqslant 0, \quad x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $(u_0, v_0) \in H^1_0(\Omega) \times L^2(\Omega)$.

Assumption 4.1. *We assume that Ω is a bounded smooth domain in $\mathbb{R}^N$ with $N \geqslant 3$, ν is a nonnegative parameter and*

$$\eta \in C^\mu_{loc}(\mathbb{R}, (0,\infty)) \ \ \text{for some} \ \ \mu \in (0,1]. \tag{4.2}$$

Note that due to (4.2), given any bounded time interval $I \subset \mathbb{R}$, there are constants $\kappa_1, \kappa_2 > 0$, such that $\eta(t) \in [\kappa_1, \kappa_2]$ for each $t \in I$.

Letting $v = \dot{u}$ we rewrite (4.1) in the form (1.6)

$$\frac{d}{dt}\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] + A(t)\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = F(t, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]), \ t > 0, \qquad \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_{t=\tau} = \left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right], \tag{4.3}$$

where $A(t)$ and $F(t, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right])$ can be viewed in matrix form as

$$A(t) = \begin{bmatrix} 0 & -I \\ -\Delta & \eta(t)(-\Delta)^{\frac{1}{2}} + \nu I \end{bmatrix}, \quad F(t, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) = \begin{bmatrix} 0 \\ f^e(t,u) \end{bmatrix} \tag{4.4}$$

and f^e denotes a Nemitskiĭ operator associated with f.

We set in this example

$$X = H^1_0(\Omega) \times L^2(\Omega), \quad D_X = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$$

and, referring to [12, proof of Lemma 1 iii)] and [9, Proposition 1], we conclude that $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$. Furthermore, since for each $t \in \mathbb{R}$ we have $A^{-1}(t) = \begin{bmatrix} \eta(t)(-\Delta)^{-\frac{1}{2}} & (-\Delta)^{-1} \\ -I & 0 \end{bmatrix}$, we obtain for any $t, s \in \mathbb{R}$ that

$$A(t)A^{-1}(s) = \begin{bmatrix} 0 & -I \\ -\Delta & \eta(t)(-\Delta)^{\frac{1}{2}} + \nu I \end{bmatrix} \begin{bmatrix} \eta(s)(-\Delta)^{-\frac{1}{2}} + \nu(-\Delta)^{-1} & (-\Delta)^{-1} \\ -I & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ (\eta(s)-\eta(t))(-\Delta)^{\frac{1}{2}} & I \end{bmatrix}$$

and

$$A^{-1}(s)A(t) = \begin{bmatrix} \eta(s)(-\Delta)^{-\frac{1}{2}} + \nu(-\Delta)^{-1} & (-\Delta)^{-1} \\ -I & 0 \end{bmatrix} \begin{bmatrix} 0 & -I \\ -\Delta & \eta(t)(-\Delta)^{\frac{1}{2}} + \nu I \end{bmatrix} = \begin{bmatrix} I & (\eta(s)-\eta(t))(-\Delta)^{-\frac{1}{2}} \\ 0 & I \end{bmatrix}.$$

Consequently, for any bounded time interval $I \subset \mathbb{R}$, we have

$$\sup_{t,s \in I} \|A(t)A^{-1}(s)\|_{L(X)} = \sup_{t,s \in I} \sup_{\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\|_X = 1} \| \begin{bmatrix} \phi \\ (\eta(s)-\eta(t))(-\Delta)^{\frac{1}{2}}\phi + \psi \end{bmatrix} \|_X \leqslant (1 + 2\kappa_2) \tag{4.5}$$

and

$$\sup_{t,s\in I}\|\overline{A^{-1}(s)A(t)}\|_{L(X)} = \sup_{t,s\in I}\sup_{\|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_X=1}\|\left[\begin{smallmatrix}\phi+(\eta(s)-\eta(t))(-\Delta)^{-\frac{1}{2}}\psi\\ \psi\end{smallmatrix}\right]\|_X \leqslant (1+2\kappa_2), \quad (4.6)$$

which are counterparts of (3.5) and (3.8).

Let us denote by $\{Z^\alpha, \alpha \geqslant -1\}$ the extrapolated fractional power scale generated by $(L^2(\Omega), -\Delta)$. As in (3.11), choosing $\alpha_0 = 0$, we define the spaces E_α, $\alpha \in [0,2]$. Due to [9, Theorem 2] they are characterized as

$$E_\alpha := Y^{\alpha+\alpha_0}(t_0) = Z^{\frac{\alpha}{2}} \times Z^{\frac{\alpha-1}{2}}, \qquad \alpha \in [0,2]. \quad (4.7)$$

Note that, following [3, Theorem V.1.38], $Z^{-\alpha}(t)$ can be viewed for $\alpha \in (0,1)$ as a completion of $(L^2(\Omega), \|(-\Delta)^{-\alpha}\cdot\|_{L^2(\Omega)})$.

With this set-up we now prove that

$$A(\cdot) \in C^\mu_{loc}(\mathbb{R}, L(E_1,E_0)) \text{ with } E_1 = H^1_0(\Omega)\times L^2(\Omega) \text{ and } E_0 = L^2(\Omega)\times H^{-1}(\Omega), \quad (4.8)$$

where μ is as in (4.2). Indeed, given $t,s \in [-T,T]$ we immediately have

$$\begin{aligned}\sup_{\|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{E_1}=1}\|[A(t)-A(s)]\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{E_0} &= \sup_{\|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{E_1}=1}\left\|\left[\begin{smallmatrix}0\\ [\eta(t)-\eta(s)](-\Delta)^{\frac{1}{2}}\psi\end{smallmatrix}\right]\right\|_{E_0}\\ &= |\eta(t)-\eta(s)|\sup_{\|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{E_1}=1}\|(-\Delta)^{\frac{1}{2}}\psi\|_{Z^{-\frac{1}{2}}} \leqslant c|t-s|^\mu.\end{aligned}$$

Due to Proposition 3.6, (4.8) is equivalent with (3.22) and what was said above allows us to apply Theorem 3.1. Consequently, Assumption 1.1 and Assumption 1.3 are both satisfied, which leads to the following conclusion.

PROPOSITION 4.1. *Suppose that Assumption 4.1 holds and let* $E_\alpha = Z^{\frac{\alpha}{2}} \times Z^{\frac{\alpha-1}{2}}$ *for* $\alpha \in [0,\mu)$, *where* μ *is as in* (4.2).

Then there exists a continuous process $\{U(t,\tau) : (t,\tau) \in \mathbb{R}^2, t \geqslant \tau \in \mathbb{R}\} \subset L(E_0)$ *associated in* $E_0 = L^2(\Omega)\times H^{-1}_0(\Omega)$ *with a singularly nonautonomous linear problem*

$$\frac{d}{dt}\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right] + \begin{bmatrix}0 & -I\\ -\Delta & \eta(t)(-\Delta)^{\frac{1}{2}}+\nu I\end{bmatrix}\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right] = 0,\ t>0, \qquad \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right]_{t=\tau} = \left[\begin{smallmatrix}u_\tau\\ v_\tau\end{smallmatrix}\right], \quad (4.9)$$

and $\{U(t,\tau) : (t,\tau) \in \mathbb{R}^2, t \geqslant \tau \in \mathbb{R}\}$ *enjoys the smoothing properties* (1.9), (1.24).

Remark 4. 1. Besides (4.8) we also have

$$A(\cdot) \in C^\mu_{loc}(\mathbb{R}, L(E_2,E_1)) \text{ with } E_2 = H^2(\Omega)\cap H^1_0(\Omega)\times H^1_0(\Omega),\ E_1 = H^1_0(\Omega)\times L^2(\Omega)$$

as (4.2) yields

$$\sup_{\|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{E_2}=1} \|[A(t)-A(s)]\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{E_1} = \sup_{\|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{E_2}=1} \left\|\left[\begin{smallmatrix}0\\ [\eta(t)-\eta(s)](-\Delta)^{\frac{1}{2}}\psi\end{smallmatrix}\right]\right\|_{E_1}$$

$$= |\eta(t)-\eta(s)| \sup_{\|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{E_2}=1} \|(-\Delta)^{\frac{1}{2}}\psi\|_{Z^0} \leqslant c|t-s|^{\mu}.$$

whenever $t, s \in [-T, T]$ for some $T > 0$.

We now turn our attention towards Lipschitz properties of the nonlinear term. Assuming $N \geqslant 3$ we define a number

$$\rho_c := \frac{N+2}{N-2},$$

which in this example plays a role of a critical exponent.

Remark 4. 2. To keep the notation short let us adapt throughout the rest of the paper the Landau symbols $O(\varphi)$, $o(\varphi)$. Namely we will write that $h(t,x,s) = O(\varphi(s))$ if, given a bounded time interval $I \subset \mathbb{R}$, $|h(t,x,s)| \leqslant c|\varphi(s)|$ for some constant $c > 0$, which does not depend on $s \in \mathbb{R}$, $x \in \Omega$ and $t \in I$. We will also write that $h(t,x,s) = o(\varphi(s))$ if, given a bounded time interval $I \subset \mathbb{R}$, $\lim_{|s|\to\infty} \frac{|h(t,x,s)|}{\varphi(s)} = 0$ uniformly with respect to $x \in \Omega$ and $t \in I$.

PROPOSITION 4.2. *Suppose that $N \geqslant 3$, $f \in C(\mathbb{R}^2, \mathbb{R})$, $f'_s \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $E_\alpha = Z^{\frac{\alpha}{2}} \times Z^{\frac{\alpha-1}{2}}$ for $\alpha \in [0, \mu)$, where μ is as in (4.2).*
i) If $f'_s(t,s) = O(c_\eta + \eta|s|^{\rho-1})$ for some $\eta > 0$ and $\rho \in (1, \rho_c)$, then the map $F(t, \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right])$ in (4.4) is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha, \alpha \in [0,\mu)\}$ and is subcritical.
ii) If $f'_s(t,s) = O(c_\eta + \eta|s|^{\rho_c-1})$ for some $\eta > 0$ and i) does not apply, then the map $F(t, \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right])$ in (4.4) is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha, \alpha \in [0,\mu)\}$ and is critical.
iii) If $f'_s(t,s) = o(|s|^{\rho_c-1})$ and i) does not apply, then $F(t, \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right])$ in (4.4) is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha, \alpha \in [0,\mu)\}$ and is almost critical.

Proof: Parts i)-ii) follow in a similar manner as [9, Lemma 3 and Corollary 2]. Part iii) can be proved analogously as in [10, Lemma 3.1 and Corollary 3.1]. ∎

COROLLARY 4.1. *Suppose that Assumption 4.1 holds and let $E_\alpha = Z^{\frac{\alpha}{2}} \times Z^{\frac{\alpha-1}{2}}$ for $\alpha \in [0, \mu)$, where μ is as in (4.2). Suppose also that the assumptions of Proposition 4.2 are satisfied; in particular that $f'_s(t,s) = O(c_\eta + \eta|s|^{\rho_c-1})$ for some $\eta > 0$.*

Then Theorem 1.2 applies and, given any $\tau \in \mathbb{R}$, $\left[\begin{smallmatrix}u_\tau\\ v_\tau\end{smallmatrix}\right] \in H^1_0(\Omega) \times L^2(\Omega)$, the abstract counterpart (4.3)-(4.4) of (4.1) has the unique $E_{1+\varepsilon}$-solution $\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right] = \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](\cdot, \tau, \left[\begin{smallmatrix}u_\tau\\ v_\tau\end{smallmatrix}\right])$ defined on the maximal interval of existence $[\tau, T_{u_\tau, v_\tau})$.

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With additional assumption on f there will exist an "energy" functional $\mathcal{L}$,

$$\mathcal{L}\left(\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]\right) = \frac{1}{2}\|w_2\|^2_{L^2(\Omega)} + \frac{1}{2}\|(-\Delta)^{\frac{1}{2}} w_1\|^2_{L^2(\Omega)} - \int_\Omega \int_0^{w_1} f(s) ds dx, \quad \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \in E_1, \tag{4.10}$$

decreasing along $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \tau, \left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right])$.

Lemma 4.1. *Suppose that $f = f(u)$, that is f does not depend on t. Then $\mathcal{L}$ in* (4.10) *takes bounded subsets of E_1 into bounded subsets of $\mathbb{R}$ and, given any sufficiently smooth solution $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ of* (4.3) *on the interval I_τ, $\mathcal{L}\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right)$ is a nonincreasing function of time variable $t \in I_\tau$.*

If λ_1^D denotes the first positive eigenvalue of the negative Dirichlet Laplacian $-\Delta$ in $L^2(\mathbb{R}^N)$ and

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1 \tag{4.11}$$

then $\mathcal{L}$ is also bounded below. Consequently, all sufficiently smooth solutions of (4.3)*, as long as they exists, satisfy the estimate*

$$\|\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \tau, \left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right])\|_{E_1} \leqslant d_1 \mathcal{L}(\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right]) + d_2, \tag{4.12}$$

for some constants $d_1, d_2 > 0$.

Proof: Multiplying the first equation in (4.1) by $v = u_t$, we have

$$\frac{d}{dt}\left(\mathcal{L}\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right)\right) = -\eta(t)\|(-\Delta)^{\frac{1}{4}} v\|^2_{L^2(\Omega)} - \nu\|v\|^2_{L^2(\Omega)} \leqslant 0, \tag{4.13}$$

which yields that $\mathcal{L}(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) \leqslant \mathcal{L}(\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right])$ as long as the solution exists.

On the other hand, using (4.11), we obtain for any $\delta > 0$ small enough that

$$-\int_\Omega \int_0^{w_1} f(s) ds dx \geqslant -\frac{\lambda_1 - \delta}{2}\|w_1\|^2_{L^2(\Omega)} - N_\delta|\Omega|,$$

where N_δ is a certain positive constant. Consequently we get the estimate

$$\mathcal{L}(\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]) \geqslant \frac{\delta}{2\lambda_1}\|(-\Delta)^{\frac{1}{2}} w_1\|^2_{L^2(\Omega)} + \frac{1}{2}\|w_2\|^2_{L^2(\Omega)} - N_\delta|\Omega|, \ \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \in E_1,$$

and the result now follows easily. ∎

Theorem 1.5 now leads to the following conclusion.

Corollary 4.2. *Suppose that Assumption 4.1 holds and assume for $f \in C^1(\mathbb{R}, \mathbb{R})$ that* (4.11) *is satisfied and $f'_s(s) = o(|s|^{\rho_c - 1})$; in particular assume that f does not depend on time variable.*

Then, given $\tau \in \mathbb{R}$ *and* $\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right] \in E_1 = H^1_0(\Omega) \times L^2(\Omega)$*, there exists the unique global* $E_{1+\varepsilon}$*-solution of* (4.1).

Suppose finally that we have $f'_s(s) = O(1 + |s|^{\rho_c - 1})$ but not $f'_s(s) = o(|s|^{\rho_c-1})$. Note that (1.27) is rather uneasy to verify as the estimate in $E_{1+\varepsilon}$-norm can hardly be derived. Nonetheless, since we know (4.12) and, in addition,

$$(-\Delta)^{-\frac{1}{2}}\dot{v} + \eta(t)v + \nu(-\Delta)^{-\frac{1}{2}}v + (-\Delta)^{\frac{1}{2}}u = (-\Delta)^{-\frac{1}{2}}f(u),$$

we infer that

$$u \in W^{1,1}((0, T_{u_\tau,v_\tau}), L^2(\Omega)), \quad \dot{u} \in W^{1,1}((0, T_{u_\tau,v_\tau}), H^{-1}(\Omega)).$$

Hence we conclude that

$$[0, T_{u_\tau,v_\tau}) \ni t \longrightarrow \begin{bmatrix} u(t,u_\tau,v_\tau) \\ v(t,u_\tau,v_\tau) \end{bmatrix} \in E_0 = L^2(\Omega) \times H^{-1}(\Omega) \text{ is uniformly continuous,}$$

in which case (1.33) holds and thus Theorem 1.6 applies.

Corollary 4.3. *Suppose that Assumption 4.1 holds and assume for* $f \in C^1(\mathbb{R},\mathbb{R})$ *that* $f'_s(s) = O(1 + |s|^{\rho_c-1})$ *and* (4.11) *is satisfied; in particular assume that* f *does not depend on time variable.*

Then, whenever $\tau \in \mathbb{R}$ *and* $u_\tau \in E_1$ *are such that* T_{u_τ,v_τ} *is finite, there exist* $a \in (T_{u_\tau}, \infty]$ *and the unique extension* $\mathcal{U} : [\tau, a) \to E_1$ *of maximally defined* $E_{1+\varepsilon}$*-solution* u *of* (1.6) *such that* $\mathcal{U}$ *is a piecewise-*$E_{1+\varepsilon}$*-solution of* (1.6) *on the interval* $[\tau, a)$ *and either* $a = \infty$ *or* a *is an accumulation time of singular times.*

4.2. Singularly nonautonomous $2m$-th order parabolic problems

Consider singularly nonautonomous initial boundary value problem

$$\begin{cases} u_t + A(t)u = f(t,x,u),\ t > 0,\ x \in \Omega \subset \mathbb{R}^N,\ N > 2m \geqslant 2, \\ B_0 u = \ldots = B_{m-1}u = 0,\ t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x),\ x \in \Omega, \end{cases} \tag{4.14}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, f is a suitably smooth real valued function, $A(t)$ is a $2m$-th order linear differential operator

$$A(t) = (-1)^m \sum_{|\sigma| \leqslant 2m} a_\sigma(t,x) D^\sigma,\ t \in \mathbb{R}, \tag{4.15}$$

and

$$B_j = \sum_{|\sigma| \leqslant m_j} b^j_\sigma(x) D^\sigma,\ j = 1, \ldots, m, \tag{4.16}$$

are boundary operators of order $m_j \in \{0, 1, \ldots, 2m-1\}$.

Without being too exhaustive we will require of each $A(t)$ with $t \in \mathbb{R}$ to satisfy the so called *α-root condition* for some $\alpha \in (0, \frac{\pi}{2})$, see [6, p. 650], which implies in particular that, given $t \in \mathbb{R}$, the operator $A(t) + (-1)^m e^{i\theta} D_s^{2m}$ is for each $\theta \in [-\alpha - \frac{\pi}{2}, \alpha + \frac{\pi}{2}]$ uniformly elliptic in $\Omega \times \mathbb{R}$. This can be achieved requiring that each $A(t)$ is a uniformly strongly elliptic operator, that is for each $t \in \mathbb{R}$ we have

$$\exists_{c>0} \, \forall_{x\in\Omega} \, \forall_{\xi\in\mathbb{R}^N} \; Re\Big(\sum_{|\sigma|=2m} a_\sigma(x,t)\xi^\sigma \Big) \geqslant c|\xi|^{2m}, \tag{4.17}$$

(see [6, p. 659]).

In what follows we suitably adapt from [6, p. 661] the notion of a regular parabolic initial boundary value problem.

Definition 4.1. We say that a family $\{(\mathcal{A}(t), \{B_j\}, \Omega, \partial\Omega), \; t \in \mathbb{R}\}$ is of the class $\mathcal{RPIBVP}$ of regular parabolic initial boundary value problems of order $2m$ if $(\mathcal{A}(t), \{B_j\}, \Omega, \partial\Omega, \alpha)$ is a strongly α-regular elliptic boundary value problem of class C^0 and order $2m$ for every $t \in \mathbb{R}$ as in [6, p. 655] and, in addition, (4.18) below holds.

$$\begin{aligned}&\text{There exists } \mu \in (0,1] \text{ such that for each bounded time interval } I \subset \mathbb{R}\\ &\text{and for any } |\sigma| \leqslant 2m \text{ map } \; I \ni t \to a_\sigma(t,\cdot) \in C(\overline{\Omega}, \mathbb{R}) \; \text{ is of the class}\\ &C^\mu(I, C(\overline{\Omega}, \mathbb{R})); \text{ in addition, for any } |\sigma| = 2m, \text{ modulus of continuity}\\ &\text{of maps } \; \overline{\Omega} \ni x \to a_\sigma(t,x) \in \mathbb{R} \; \text{ can be chosen uniformly for } t \in I.\end{aligned} \tag{4.18}$$

We will next consider spaces $H^s_p(\Omega)$ as in [25]. For $p = 2$ they are Hilbert spaces and will be denoted by $H^s(\Omega)$. Following [25] we also define

$$H^s_{p,\{B_j\}}(\Omega) = \{\phi \in H^s_p(\Omega) : \; \forall_{i \in \{j:\; m_j < s - \frac{1}{p}\}} \; B_i \phi_{|\partial\Omega} = 0\}.$$

Assuming that $\{(\mathcal{A}(t), \{B_j\}, \Omega, \partial\Omega), \; t \in \mathbb{R}\}$ is of the class $\mathcal{RPIBVP}$ we have the following estimate

$$\|\varphi\|_{H^{2m}_p(\Omega)} \leqslant c^*(\|A(t)\varphi\|_{L^p(\Omega)} + \|\varphi\|_{L^p(\Omega)}), \; \varphi \in H^{2m}_{p,\{B_j\}}(\Omega), \; t \in I, \tag{4.19}$$

where $I \subset \mathbb{R}$ is arbitrarily chosen bounded time interval. We emphasize that $c^* > 0$ actually depends on Ω, m, N, p, α, moduli of continuity of the top order coefficients of $A(t)$ with $t \in I$, coefficients of boundary operators B_j and certain constants related to the notion of α-regular elliptic boundary value problem which are specified in [6, Theorems 12.1] (see also [1, 2]). Thus for the problems as in Definition 4.1 the constant c^* in (4.19) is independent of t varying in a bounded time interval $I \subset \mathbb{R}$.

We remark that, due to (4.15), (4.18) and properties of $H^{2m}_p(\Omega)$-norm we also have

$$\|A(t)\varphi\|_{L^p(\Omega)} \leqslant c_* \|\varphi\|_{H^{2m}_p(\Omega)}, \; \varphi \in H^{2m}_p(\Omega), \; t \in I, \tag{4.20}$$

where c_* depends on Ω, m and $L^\infty(I, C(\overline{\Omega}, \mathbb{R}))$-norms of coefficients of $A(t)$.

We now summarize the conditions on (4.14) that we will need throughout the rest of this subsection.

Assumption 4.2. *We assume that $N > 2m$, Ω is a bounded C^2-domain in $\mathbb{R}^N$, $A(t)$, $B_j(t)$ ($t \in \mathbb{R}$, $j = 0, \ldots, m-1$) are given by* (4.15)-(4.16) *and* $\{(\mathcal{A}(t), \{B_j\}, \Omega, \partial\Omega),\ t \in \mathbb{R}\}$ *is of the class $\mathcal{RPIBVP}$. In addition $A(t)$, $t \in \mathbb{R}$, are all selfadjoint operators in $L^2(\Omega)$ and they are bounded from below by a positive constant that can be chosen uniformly for t in bounded time intervals $I \subset \mathbb{R}$; that is,*

$$\langle A(t)\phi, \phi\rangle_{L^2(\Omega)} \geqslant s_* \|\phi\|^2_{L^2(\Omega)}, \tag{4.21}$$

where $s_* > 0$ can depend on I but not on $t \in I$.

Example 4.1. There are number of problems, for which Assumption 4.2 holds. This happens e.g. when

$$\begin{aligned} &p = 2,\ m = 2,\ N > 4,\ A(t) = \theta(t)\Delta^2,\ t \in \mathbb{R}, \\ &\theta \in C^\mu_{loc}(\mathbb{R}, (0, \infty)) \ \text{ for a certain } \ \mu \in (0, 1] \\ &\text{and also } \ \{B_j,\ j = 0, 1\} = \Big\{\frac{\partial^j}{\partial \nu^j},\ j = 0, 1\Big\}. \end{aligned} \tag{4.22}$$

Proposition 4.3. *Suppose that Assumption 4.2 is satisfied and that*

$$E_\alpha = \begin{cases} ([L^2(\Omega), H^{2m}_{\{B_j\}}(\Omega)]_{1-\alpha})',\ \alpha \in [0, 1), \\ [L^2(\Omega), H^{2m}_{\{B_j\}}(\Omega)]_{\alpha-1},\ \alpha \in [1, 2], \end{cases} \tag{4.23}$$

where μ is as in (4.18).

Then there exists a continuous process associated in $E_0 = (H^{2m}_{\{B_j\}}(\Omega))'$ with a singularly nonautonomous linear problem

$$\begin{cases} u_t + A(t)u = 0,\ t > 0,\ x \in \Omega \subset \mathbb{R}^N,\ N > 2m \geqslant 2, \\ B_0 u = \ldots = B_{m-1} u = 0,\ t > 0,\ x \in \partial\Omega, \\ u(0, x) = u_0 \in L^2(\Omega), \end{cases} \tag{4.24}$$

and possessing smoothing properties (1.9), (1.24).

Proof: We will ensure that Theorem 3.1 applies with

$$X = L^2(\Omega) \ \text{ and } \ E_0 = (H^{2m}_{\{B_j\}}(\Omega))'.$$

First note that, proceeding as in [14, Proposition 1.3.3], we get $\|(\lambda I - A(t))\phi\|_X \geqslant 2^{-\frac{1}{2}}|\lambda - s_*|$ whenever $\phi \in H^2_{\{B_j\}}(\Omega)$, $t \in I$, $Re(\lambda) \leqslant s_*$. From this we conclude that $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$. On the other, since by assumption purely imaginary powers are unitary operators, we also have that $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{BIP}(X)$.

We now fix a bounded time interval $I \subset \mathbb{R}$ and concentrate on points $t \in I$. Using (4.21), Schwartz's inequality and (4.19) we get

$$\|\varphi\|_{H^{2m}(\Omega)} \leqslant c^*(1 + s_*^{-1})\|A(t)\varphi\|_{L^2(\Omega)}, \varphi \in H^{2m}_{\{B_j\}}(\Omega),\ t \in I, \tag{4.25}$$

where s_* is as in (4.21).

Now, to obtain (3.5), we apply (4.20) with $p = 2$, $\varphi = A^{-1}(s)\psi$, $\psi \in L^2(\Omega)$, and then use (4.25) with $t = s$ and $\varphi = A^{-1}(s)\psi$ to conclude that

$$\|A(t)A^{-1}(s)\psi\|_{L^2(\Omega)} \leqslant c_*\|A^{-1}(s)\psi\|_{H^{2m}(\Omega)} \leqslant c_*c^*(1 + s_*^{-1})\|\psi\|_{L^2(\Omega)},\ \psi \in L^2(\Omega),\ t, s \in I.$$

In the proof of (3.8) we adapt the idea of [4, Remark 6.6 (c)]. Note that from above consideration we have $\sup_{t,s\in I}\|A(s)A^{-1}(t)\|_{L(L^2(\Omega))} \leqslant N$. Combining this with the fact that the operators are selfadjoint we get

$$|\langle\phi, A^{-1}(t)A(s)\psi\rangle_{L^2(\Omega)}| \leqslant N\|\phi\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)},\ \phi \in L^2(\Omega),\ \psi \in H^{2m}_{\{B_j\}}(\Omega),\ t, s \in I.$$

This ensures that the set $\{A^{-1}(t)A(s)\psi : t, s \in I, \psi \in H^{2m}_{\{B_j\}}(\Omega), \|\psi\|_{L^2(\mathbb{R}^N)} \leqslant 1\}$ is bounded in $L^2(\Omega)$ and hence

$$\|\overline{A^{-1}(t)A(s)}\|_{L(L^2(\Omega))} \leqslant \bar{c},$$

where constant $\bar{c} > 0$ does not depend on $t, s \in I$.

Letting $\alpha_0 = 0$, we define next spaces E_α, $\alpha \in [0, 1 + \mu_0] = [0, 2]$ as in (3.11), which characterized here as in (4.23).

To ensure that

$$A(\cdot) \in C^\mu_{loc}(\mathbb{R}, L(E_1, E_0)) \ \text{ with } E_1 = L^2(\Omega) \text{ and } E_0 = (H^{2m}_{\{B_j\}}(\Omega))' \tag{4.26}$$

we observe that, whenever $\phi \in L^2(\mathbb{R}^N)$ and $t, s \in \mathbb{R}$,

$$\|(A(t) - A(s))\phi\|_{(H^{2m}_{\{B_j\}}(\Omega))'} = \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1} |\int_\Omega \phi(A(t) - A(s))\psi|.$$

Hence, for any t, s in a bounded time interval $I \subset \mathbb{R}$, using (4.15) and (4.18) we get via Schwartz's inequality that

$$\sup_{\|\phi\|_{E_1}=1} \|[A(t) - A(s)]\phi\|_{E_0} = \sup_{\|\phi\|_{L^2(\Omega)}=1} \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1} \Big| \int_\Omega \phi(A(t) - A(s))\psi \Big|$$

$$\leqslant \sup_{\|\phi\|_{L^2(\Omega)}=1} \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1} \sum_{|\sigma|\leqslant 2m} \int_\Omega |\phi||a_\sigma(t,x) - a_\sigma(s,x)||D^\sigma\psi|$$

$$\leqslant \sup_{\|\phi\|_{L^2(\Omega)}=1} \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1} \sum_{|\sigma|\leqslant 2m} \|a_\sigma(t,\cdot) - a_\sigma(s,\cdot)\|_{C(\overline{\Omega},\mathbb{R})} \|\phi\|_{L^2(\Omega)} \|D^\sigma\psi\|_{L^2(\Omega)} \leqslant c|t-s|^\mu.$$

This proves that the assumptions of Theorem 3.1 are satisfied. Hence, applying Theorem 3.1 we get the result. ∎

Remark 4. 3. Besides (4.26) we also have

$$A(\cdot) \in C^\mu_{loc}(\mathbb{R}, L(E_2, E_1)) \ \text{ with } E_2 = H^{2m}_{\{B_j\}}(\Omega),\ E_1 = L^2(\Omega) \tag{4.27}$$

as (4.18) yields

$$\|(A(t) - A(s))\phi\|_{E_1} = \|(-1)^m \sum_{|\sigma|\leqslant 2m} (a_\sigma(t,x) - a_\sigma(s,x))D^\sigma\phi\|_{E_1}$$

$$\leqslant \|a_\sigma(t,\cdot) - a_\sigma(s,\cdot)\|_{C(\overline{\Omega})} \sum_{|\sigma|\leqslant 2m} \|D^\sigma\phi\|_{E_1} \leqslant |t-s|^\mu \|\phi\|_{E_2}$$

whenever $\phi \in E_2$ and s, t vary in a bounded time interval $I \subset \mathbb{R}^N$.

Having obtained the relevant information about the linear part in (4.14) we now turn our attention towards the analysis of a nonlinear term. For this we define

$$\rho_c := \frac{N+4m}{N},$$

which will be a critical exponent for (4.14) with initial data in $L^2(\Omega)$. To describe properties of the Nemitskiĭ operator F associated with f we again use the Landau symbols $O(\varphi)$, $o(\varphi)$ as in Remark 4.2.

PROPOSITION 4.4. *Suppose that* $N > 2m$, $f, f'_s \in C(\mathbb{R}^{N+2}, \mathbb{R})$ *and* E_α *are as in* (4.23) *for* $\alpha \in [0, \mu)$ *and* μ *is as in* (4.18). *Suppose additionally that*

$$\mu > \frac{N}{4m}\Big(1 - \frac{2}{\rho_c}\Big). \tag{4.28}$$

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i) If $f'_s(t,x,s) = O(c_\eta + \eta|s|^{\rho-1})$ for some $\eta > 0$ and $\rho \in (1,\rho_c)$, then the map $F(t,u)$ in (4.14) is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ relative to $\{E_\alpha, \alpha \in [0,\mu)\}$ and is subcritical.
ii) If $f'_s(t,x,s) = O(c_\eta + \eta|s|^{\rho_c-1})$ for some $\eta > 0$ and i) does not apply, then the map $F(t,u)$ in (4.14) is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ relative to $\{E_\alpha, \alpha \in [0,\mu)\}$ and is critical.
iii) If $f'_s(t,x,s) = o(|s|^{\rho_c-1})$ and i) does not apply, then $F(t,u)$ in (4.14) is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ relative to $\{E_\alpha, \alpha \in [0,\mu)\}$ an is almost critical.

Proof: Restricting time variable t to a fixed bounded time interval I we will show that there are constants $c > 0$, $C_\eta > 0$ and $\varepsilon \in (0, \frac{1}{\rho})$, $\varepsilon < \mu$, $\rho\varepsilon \leqslant \gamma(\varepsilon) < 1$ such that

$$\|F(t,v) - F(t,w)\|_{E_{\gamma(\varepsilon)}} \leqslant c\|v-w\|_{E_\varepsilon}\big(C_\eta + \eta\|v\|^{\rho-1}_{E_\varepsilon} + \eta\|w\|^{\rho-1}_{E_\varepsilon}\big),\ v,w \in E_\varepsilon. \tag{4.29}$$

Furthermore, we will describe admissible triples $(\rho,\varepsilon,\gamma(\varepsilon))$ for which (4.29) holds and prove that the map is indeed critical for $\rho = \rho_c$ whilst it is subcritical for $\rho \in (1,\rho_c)$.

Observe that due to (4.23) we have

$$\begin{aligned} &E_{1+\varepsilon} \hookrightarrow L^s(\Omega),\ \varepsilon \in [0,1],\ 2m\varepsilon - \frac{N}{2} \geqslant -\frac{N}{s},\ s \geqslant 2,\\ &E_{\gamma(\varepsilon)} \hookleftarrow L^\sigma(\Omega),\ \gamma(\varepsilon) \in [0,1),\ \frac{2N}{N+4m(1-\gamma(\varepsilon))} \leqslant \sigma \leqslant 2,\ \sigma > 1, \end{aligned} \tag{4.30}$$

where $\frac{2N}{N+4m(1-\gamma(\varepsilon))} > 1$ provided that

$$\gamma(\varepsilon) > \frac{4m-N}{4m} =: \tilde{\gamma} > 0. \tag{4.31}$$

Using (4.30) and taking into account that $f'_s(t,x,s) = O(c_\eta + \eta|s|^{\rho_c-1})$ we have

$$\begin{aligned} \|F(t,v) - F(t,w)\|_{E_{\gamma(\varepsilon)}} &\leqslant \hat{c}\|F(t,v) - F(t,w)\|_{L^{\frac{2N}{N+4m(1-\gamma(\varepsilon))}}(\Omega)}\\ &\leqslant \tilde{c}\||v-w|(c_\eta + \eta|v|^{\rho-1} + \eta|w|^{\rho-1})\|_{L^{\frac{2N}{N+4m(1-\gamma(\varepsilon))}}(\Omega)}. \end{aligned}$$

Applying next Hölder's inequality with $q = \frac{N+4m(1-\gamma(\varepsilon))}{N-4m\varepsilon}$, $q' = \frac{N+4m(1-\gamma(\varepsilon))}{4m(1-\gamma(\varepsilon)+\varepsilon)}$, $\frac{N}{4m} > \varepsilon$, recalling the embedding $H^{2m\varepsilon}(\Omega) \hookrightarrow L^{\frac{2N}{N-4m\varepsilon}}(\Omega)$, and assuming that

$$H^{2m\varepsilon}(\Omega) \hookrightarrow L^{\frac{N(\rho-1)}{2m(1-\gamma(\varepsilon)+\varepsilon)}}(\Omega), \tag{4.32}$$

we obtain

$$\begin{aligned} &\|F(t,v) - F(t,w)\|_{E_{\gamma(\varepsilon)}} \leqslant \tilde{c}\|v-w\|_{L^{\frac{2N}{N-4m\varepsilon}}(\Omega)}\|c_\eta + \eta|v|^{\rho-1} + \eta|w|^{\rho-1}\|_{L^{\frac{N}{2m(1-\gamma(\varepsilon)+\varepsilon)}}(\Omega)}\\ &\tilde{c}\|v-w\|_{L^{\frac{2N}{N-4m\varepsilon}}(\Omega)}\big(\|c_\eta\|_{L^{\frac{N}{2m(1-\gamma(\varepsilon)+\varepsilon)}}(\Omega)} + \|\eta|v|^{\rho-1}\|_{L^{\frac{N}{2m(1-\gamma(\varepsilon)+\varepsilon)}}(\Omega)} +\\ &\qquad\qquad + \|\eta|w|^{\rho-1}\|_{L^{\frac{N}{2m(1-\gamma(\varepsilon)+\varepsilon)}}(\Omega)}\big)\\ &\leqslant c\|v-w\|_{E_\varepsilon}(C_\eta + \eta\|v\|^{\rho-1}_{E_\varepsilon} + \eta\|w\|^{\rho-1}_{E_\varepsilon}),\ v,w \in E_\varepsilon, \end{aligned}$$

where (4.32) requires that

$$\overline{\gamma} := \frac{(4m\varepsilon - N)(\rho-1) + 4m(1+\varepsilon)}{4m} \geqslant \gamma(\varepsilon) \geqslant \frac{4m(1+\varepsilon) - N(\rho-1)}{4m} =: \underline{\gamma}. \tag{4.33}$$

We remark that for $\rho \in (1, 1+\frac{4m}{N}]$ and $\varepsilon > 0$ we have $\overline{\gamma} > \underline{\gamma} > 0$ and $\overline{\gamma} \geqslant \varepsilon\rho$. We also have $1 > \overline{\gamma}$ if $\varepsilon \in (0, \frac{N(\rho-1)}{4m\rho})$. Furthermore, $\overline{\gamma} > \tilde{\gamma}$ and $\frac{N}{4m} > \varepsilon > 0$ if and only if $\varepsilon \in \big(\max\{0, \frac{N(\rho-2)}{4m\rho}\}, \frac{N}{4m}\big)$.

The above analysis ensures that any triple $(\rho, \varepsilon, \gamma(\varepsilon))$ such that $\rho \in (1, 1+\frac{4m}{N}]$, $\varepsilon \in (\max\{0, \frac{N(\rho-2)}{4m\rho}\}, \min\{\mu, \frac{N(\rho-1)}{4m\rho}\})$ and $\gamma(\varepsilon) \in [\rho\varepsilon, \overline{\gamma}] \cap [\underline{\gamma}, \overline{\gamma}] \cap (\tilde{\gamma}, \overline{\gamma}] =: \mathcal{I}(\varepsilon)$ is admissible. Therefore the proof of parts i)-ii) will be complete if we justify that $\mathcal{I}(\varepsilon)_{|\rho=\rho_c} = \{\varepsilon\rho_c\}$.

Note that for any admissible triple $(\rho, \varepsilon, \gamma(\varepsilon))$ the left hand side inequality in (4.33) implies

$$\rho \leqslant \frac{N + 4m - 4m\gamma(\varepsilon)}{N - 4m\varepsilon},$$

and since $\gamma(\varepsilon) \geqslant \rho\varepsilon$ we then have $\rho \leqslant \frac{N+4m-4m\rho\varepsilon}{N-4m\varepsilon}$, which holds if and only if $\rho \leqslant \frac{N+4m}{N} = \rho_c$.

From the above consideration it becomes clear that $\rho = \rho_c$ cannot be attained for any $\gamma(\varepsilon) > \rho_c\varepsilon$ and therefore $\rho = \rho_c$ necessitates $\gamma(\varepsilon) = \varepsilon\rho_c$. Note that $\overline{\gamma}_{|\rho=\rho_c} = \varepsilon\rho_c$; that is for $\rho = \rho_c$ we have $\mathcal{I}(\varepsilon) = \{\varepsilon\rho_c\}$. This completes the proof of i)-ii).

Part iii) now follows easily as having $|f'(t,x,s)| \leqslant O(c_\eta + \eta|s|^{\rho_c - 1})$ for each $\eta > 0$ we obtain (4.29) for any $\eta > 0$. ▮

Corollary 4.4. *Suppose that $m \in \mathbb{N}$, $N > 2m$, $f, f'_s \in C(\mathbb{R}^{N+2}, \mathbb{R})$ and E_α are as in (4.23) for $\alpha \in [0, \mu)$ and μ as in (4.18).*

Suppose in addition that

$$1 - \frac{2}{\rho_c} < 0, \tag{4.34}$$

that is $N > 4m$.

Then in either case i), ii) or iii) of Proposition 4.4 number $\varepsilon > 0$ can be chosen as small as we wish. Furthermore, whenever t varies in a bounded time interval $I \subset \mathbb{R}$, there exists a certain $c > 0$ such that

$$\|F(t,\phi)\|_{E_0} \leqslant c(1 + \|\phi\|_{E_1}^{\rho_c}), \ \phi \in E_1.$$

Remark 4. 4. It can be seen from the proof of Proposition 4.4 that $\varepsilon > 0$ may not be in general close enough to zero. Namely, to ensure in the critical case that the set of admissible triples therein is nonvoid one needs to assume that $\varepsilon > \max\{0, \frac{N(\rho_c-2)}{4m\rho_c}\}$. This in turn implies that the number μ, which measures Hölder regularity of the coefficients in the main part of the equation cannot be too small either (see (4.28)).

Corollary 4.5. *Suppose that the conditions in Assumption 4.2 are satisfied and* E_α *are as in* (4.23) *for* $\alpha \in [0, \mu)$*, where* μ *in* (4.18) *satisfies also* (4.28).

Then Theorem 1.2 applies and, hence, given any $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$*, the initial boundary value problem* (4.14) *has the unique* $E_{1+\varepsilon}$*-solution* $u = u(\cdot, \tau, u_\tau)$ *defined on the maximal interval of existence* $[\tau, T_{u_\tau})$.

We will now derive an $L^2(\Omega)$-estimate of the solutions.

Lemma 4.2. *Suppose that*

$$sf(t,x,s) \leqslant C(t,x)s^2 + D(t,x),\ t \in \mathbb{R},\ x \in \Omega, \tag{4.35}$$

for some $C \in L^\infty_{loc}(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ *and* $D \in L^1_{loc}(\mathbb{R}, L^1(\Omega))$.

If $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$*, then for each* $T > \tau$ *and for any sufficiently smooth solution* u *of* (4.14) *which exists for all* $t \in [\tau, T)$ *we have that*

$$\|u(t,\tau,u_\tau)\|^2_{L^2(\mathbb{R}^N)} \leqslant g(\tau, \|u_\tau\|_{L^2(\Omega)}, T),\ t \in [\tau, T), \tag{4.36}$$

where $g : \mathbb{R} \times [0,\infty) \times (0,\infty) \to [0,\infty)$ *is a certain continuous function.*

Proof: Note that we restrict here time variable to the interval $[\tau, T)$, which in turn allows us to choose the constant s_* such that (4.21) holds uniformly for $t \in [\tau, T)$. In a similar vein we also define

$$C^* := \sup_{(t,x)\in[\tau,T]\times\Omega} 2|C(t,x)|.$$

Multiplying now the first equation in (4.14) by u in $L^2(\Omega)$ and using (4.21), (4.35) we obtain for any $\lambda \in (0, s_*)$ the estimate of the form

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + (s_* - C^*)\|u(t)\|^2_{L^2(\Omega)} \leqslant \|D(t,\cdot)\|_{L^1(\Omega)},\ t \in [\tau, T).$$

Solving the above inequality we get

$$\|u(t)\|^2_{L^2(\Omega)} \leqslant \|u_\tau\|^2_{L^2(\Omega)} e^{-2t(s_*-C^*)} + 2\int_\tau^t \|D(s,\cdot)\|_{L^1(\Omega)} e^{-2(t-s)(s_*-C^*)} ds,\ t \in [\tau, T),$$

which completes the proof. ∎

Theorem 1.5 now implies the following result.

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Corollary 4.6. *Suppose that the assumptions of Corollary 4.5 and Lemma 4.2 are satisfied.*

If $f_s'(t,x,s) = o(|s|^{\rho_c - 1})$ *then, given any* $\tau \in \mathbb{R}$ *and* $u_\tau \in L^2(\Omega)$*, the unique* $E_{1+\varepsilon}$*-solution of* (4.14) *exists globally in time.*

Proof: If suffices to note that if the maximal time of existence T_{u_τ} of $E_{1+\varepsilon}$-solution $u(\cdot,\tau,u_\tau)$ was finite, then Lemma 4.2 would imply that $\sup_{[\tau,T_{u_\tau})} \|u(t,\tau,u_\tau)\|_{L^2(\mathbb{R}^N)} < \infty$. Then, applying Theorem 1.5 with $E_1 = L^2(\Omega)$, one would reach contradiction. ∎

Note that in the critical case $\rho = \rho_c$ some better estimate of the solutions can be sometimes obtained if additional conditions are imposed on (4.14). For example in the autonomous case $H^1(\Omega)$-estimate can be found as in [26] or if the maximum principle applies, which does happen if $m = 1$, then $L^\infty(\Omega)$-bound on the solutions may be known. However, without any such specific assumption, one can hardly find the estimate of the solutions in $E_{1+\varepsilon}$-norm needed to apply Proposition 1.1. On the other hand, Theorem 1.6 will yield the existence of a piecewise-$E_{1+\varepsilon}$-solution on some larger time interval than the maximal interval of existence of $E_{1+\varepsilon}$-solution.

Lemma 4.3. *Suppose that the assumptions of Corollary 4.5 and Lemma 4.2 are satisfied. Assume in addition that* (4.34) *holds.*

Under these assumptions, whenever $\tau \in \mathbb{R}$, $u_\tau \in E_1 = L^2(\Omega)$ *and* $T_{u_\tau} < \infty$*, the map* $[\tau, T_{u_\tau}) \ni t \to u(t) \in E_0 = (H^{2m}_{\{B_j\}}(\Omega))'$*, where* u *is* $E_{1+\varepsilon}$*-solution of the problem* (4.14)*, is uniformly continuous.*

Proof: From the differential equation in (4.14) we infer that

$$\|u_t(t)\|_{(H^{2m}_{\{B_j\}}(\Omega))'} \leqslant \|A(t)u(t)\|_{(H^{2m}_{\{B_j\}}(\Omega))'} + \|f(t,\cdot,u)\|_{(H^{2m}_{\{B_j\}}(\Omega))'},\ t \in (\tau, T_{u_\tau}).$$

Using (4.15) we have

$$\|A(t)u\|_{(H^{2m}_{\{B_j\}}(\Omega)')} = \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1} \Big|\int_\Omega uA(t)\psi\Big| \leqslant \|u(t)\|_{L^2(\Omega)} \max_{|\sigma|\leqslant 2m} \|a_\sigma(t,\cdot)\|_{C(\overline{\Omega},\mathbb{R})},$$

$t \in (\tau, T_{u_\tau})$ and by (4.18), (4.36) we then get

$$\|A(t)u\|_{L^\infty((\tau,T_{u_\tau}),(H^{2m}_{\{B_j\}}(\Omega))')} \leqslant cg(\tau, \|u_\tau\|_{L^2(\Omega)}, T_{u_\tau}).$$

From Corollary 4.4 we deduce that $\|f(t,\cdot,u)\|_{(H^{2m}_{\{B_j\}}(\Omega))'}$ is bounded by a multiple of $(1 + \|u(t)\|^{\rho_c}_{L^2(\Omega)})$ and therefore, using again (4.36), we conclude that

$$\|f(t,\cdot,u)\|_{L^\infty((\tau,T_{u_\tau}),(H^{2m}_{\{B_j\}}(\Omega))')} \leqslant c(1 + [g(\tau, \|u_\tau\|_{L^2(\Omega)}, T_{u_\tau})]^{\rho_c}).$$

Since the above estimates ensure that $u(\cdot,\tau,u_\tau) \in W^{1,1}((\tau,T),(H^{2m}_{\{B_j\}}(\Omega))')$ the proof is complete. ∎

Theorem 1.6 and Lemmas 4.2, 4.3 now lead to the following conclusion.

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Corollary 4.7. *Suppose that the assumptions of Corollary 4.5 and Lemma 4.2 are satisfied. Assume in addition that* (4.34) *holds.*

Then, whenever $\tau \in \mathbb{R}$ *and* $u_\tau \in L^2(\Omega)$ *are such that* $T_{u_\tau} < \infty$*, there exist* $a \in (T_{u_\tau}, \infty]$ *and the unique extension* $\mathcal{U} : [\tau, a) \to E_1$ *of* $E_{1+\varepsilon}$*-solution* $u(\cdot, \tau, u_\tau)$ *of* (4.14) *such that* $\mathcal{U}$ *is a piecewise-*$E_{1+\varepsilon}$*-solution of* (4.14) *on the interval* $[\tau, a)$ *and either* $a = \infty$ *or* a *is an accumulation time of singular times.*

Remark 4. 5. It remains an open problem whether the result in Corollary 4.7 holds without the assumption (4.34) that is under the assumptions of Corollary 4.5 and Lemma 4.2 only.

4.3. Closing comments

To present our ideas better we have used in applications a Hilbert space setting, in which case needed assumptions are much easier to verify. This is due to the fact that the purely imaginary powers of selfadjoint operators are unitary operators and that, in general, the estimates of the resolvent operators in Hilbert spaces become more straightforward. Without being too exhaustive we nonetheless mention that in a similar manner one can consider in applications a Banach space setting.

Specifically we remark that the singularly perturbed wave equation with a structural damping, (4.1), can be locally well posed in $W_0^{1,p}(\Omega) \times L^p(\Omega)$, which generalizes the results from autonomous case obtained in [12]. On the other hand well posedness of the $2m$-th order singularly perturbed parabolic equation, (4.14), could be studied in $H^{2m}_{p,\{B_j\}}(\Omega)$ as in the classical work [23]. It is true that the usage of a general $L^p(\Omega)$-setting would lead to a better regularity properties of the solutions as, by embedding, the solution would "enter" some other scales with suitably larger exponents p. Nonetheless this is the matter we will not pursue here.

We finally remark that, in contrary to the autonomous or nonautonomous case, for singularly nonautonomous problems the modulus of continuity of the linear part of the equation may indeed determine possibility of solving the semilinear problem. Note that the main idea that drives the argument in the above approach lies in the smoothing properties of the linear evolution process. This is exhibited in estimates (1.9), (1.24), which are valid for $\zeta, \sigma \in (1, 1+\mu)$ with μ determined in applications by the moduli of continuity of the coefficients in the linear part of the equation. As shown in the analysis carried out in Proposition 4.4, if values of the moduli of continuity of the coefficients are not large enough, then in the critical case the order of magnitude of nonlinearity may become too large to handle the problem in the described manner; in particular condition (4.28) may fail for small μ. Then, for critically growing nonlinearities, local well posedness of (4.14) with initial data in $L^2(\Omega)$ becomes an open problem.

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