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(H, G)-Coincidence theorems for manifolds

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Let X be a paracompact space, let G be a finite group acting freely on X and let H a cyclic subgroup of G of prime order p. Let $f : X \to M$ be a continuous map, where M is a connected m-manifold (orientable, if $p > 2$) such that $f^*(V_k) = 0$, for all $k \geq 1$, where V_k denotes the Wu classes of M. Suppose that $\operatorname{ind} X \geq n > (|G| - r)m$, where $r = \frac{|G|}{p}$. In this work, we estimate the cohomological dimension of the set $A(f, H, G)$ of (H, G)-coincidence points of f. May, 2012 ICMC-USP

1. INTRODUCTION

Let G be a finite group which acts freely on a space X and let $f : X \to Y$ be a continuous map from X into another space Y. If H is a subgroup of G, then H acts on the right on each orbit Gx of G as follows: if $y \in Gx$ and $y = gx$, with $g \in G$, then $h \cdot y = gh^{-1}x$. A point $x \in X$ is said to be a (H, G)- *coincidence point of* f (as introduced by Gonçalves, Pergher and Jaworowski in [5]) if f sends every orbit of the action of H on the G-orbit of x to a single point. Of course, if H is the trivial subgroup, then every point of X is a (H, G)-coincidence. If $H = G$, this is the usual definition of G-coincidence, that is, $f(x) = f(gx)$, for all $g \in G$. If $G = \mathbb{Z}_p$, with p prime, then a nontrivial (H, G)-coincidence point is a G-coincidence point. Let us denote by $A(f, H, G)$ the set of all (H, G)-coincidence points. Borsuk-Ulam type theorems consist in estimating the cohomological dimension of the set $A(f, H, G)$. Two main directions considered of this problem are either when the

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target space Y is a manifold or Y is a CW complex. In the first direction are the papers of Borsuk [1] (the classical theorem of Borsuk-Ulam, for $H = G = \mathbb{Z}_2$, $X = S^n$ and $Y = R^n$), Conner and Floyd [3] (for $H = G = \mathbb{Z}_2$ and Y a n-manifold), Munkholm [9] (for $H = G = \mathbb{Z}_p$, $X = S^n$ and $Y = R^m$), Nakaoka [10] (for $H = G = \mathbb{Z}_p$, X under certain (co)homological conditions and Y a m-manifold) and the following more general version proved by Volovikov [12] using the index of a free $\mathbb{Z}_p$-space X ($\operatorname{ind} X$, see Definition 2.1).

Theorem A. [12, Theorem 1.2] *Let X be a paracompact free $\mathbb{Z}_p$-space of $\operatorname{ind} X \geq n$, and $f : X \to M$ a continuous mapping from X into a m-dimensional connected manifold M (orientable, if $p > 2$). Assume that:*

(1) *$f^*(V_i) = 0$, for $i \geq 1$, where the V_i are the Wu classes of M; and*

(2) *$n > m(p-1)$.*

Then, $\operatorname{ind} A(f) \geq n - m(p-1) > 0$.

In the second direction are the papers of Izydorek and Jaworowski [7] (for $H = G = \mathbb{Z}_p$, $X = S^n$ and Y a CW-complex), Gonçalves and Pergher [4] (for $H = G = \mathbb{Z}_p$, $X = S^n$ and Y a CW-complex) and for proper nontrivial subgroup H of G, Gonçalves, Jaworowski and Pergher [5] (for $H = \mathbb{Z}_p$ subgroup of a finite group G, X a homotopy sphere and Y a CW-complex) and Gonçalves, Jaworowski, Pergher and Volovikov [6](for $H = \mathbb{Z}_p$ subgroup of a finite group G, X under certain (co)homological assumptions and Y a CW-complex).

In this work, considering the target space $Y = M$ a manifold and H a proper non-trivial subgroup of G, we prove the following formulation of the Borsuk-Ulam theorem for manifolds in terms of (H, G)-coincidence.

Theorem 1.1. *Let X be a paracompact space, G a finite group acting freely on X and H a cyclic subgroup of G of prime order p. Let $f : X \to M$ be a continuous map, where M is a connected m-manifold (orientable, if $p > 2$) such that $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M. Suppose that* $\operatorname{ind} X \geq n > (|G| - r)m$*, where $r = \frac{|G|}{p}$. Then,* $\operatorname{ind} A(f, H, G) \geq n - (|G| - r)m$*, and consequently,*

$$\operatorname{cohom.dim} A(f, H, G) \geq n - (|G| - r)m > 0.$$

Let us observe that if $H = G = \mathbb{Z}_p$, we have $(|G| - r)m = (p-1)m$, and therefore, Theorem 1.1 generalizes Theorem A obtained by Volovikov in [12] in terms of (H, G)-coincidences. For the case $n = (|G| - r)m$, p an odd prime, if we consider X a *modp* homology n-sphere in Theorem 1.1(in this case, the continuous map f can be arbitrary), it follows from [10, Theorem 8] that $A(f, H, G) \neq \emptyset$. Also, Theorem 1.1 is a version for manifolds of the main result due to Gonçalves, Jaworowski and Pergher in [5].

Finally, we prove the following nonsymmetric theorem for (H, G)-coincidences, which is a version for manifolds of the main theorem in [8].

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THEOREM 1.2. *Let X be a compact Hausdorff space, G a finite group acting freely on S^n and H a cyclic subgroup of G of order prime p. Let $\varphi : X \to S^n$ be an essential map*[1] *and $f : X \to M$ a continuous map, where M is a connected m-manifold (orientable, if $p > 2$) such that $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M. Suppose that $n > (|G| - r)m$, then*

$$\text{cohom.dim}\, A_\varphi(f, H, G) \geq n - (|G| - r)m,$$

where $r = \frac{|G|}{p}$ and $A_\varphi(f, H, G)$ denotes the (H, G)-coincidence points of f relative to φ.

2. PRELIMINARIES

We start by introducing some definitions and notations. In what follows, we use Čech cohomology with coeficients in the field $\mathbb{Z}_p$. The symbol " $\cong$ " denotes an appropriate isomorphism between algebraic objects.

2.1. The $\mathbb{Z}_p$-index

Suppose that the cyclic group $\mathbb{Z}_p$ acts freely on a paracompact Hausdorff space X, where p is a prime number and denote by $[X]^*$ the space of orbits of X by the action of $\mathbb{Z}_p$. Then, $X \to [X]^*$ is a principal $\mathbb{Z}_p$-bundle and we can consider a classifying map $c : [X]^* \to B\mathbb{Z}_p$.

Remark 2. 1. It is well known that if $\hat{c}$ is another classifying map for the principal $\mathbb{Z}_p$-bundle $X \to [X]^*$, then there is a homotopy between c and $\hat{c}$.

DEFINITION 2.1. We say that the $\mathbb{Z}_p$-index of X is greater than or equal to l if the homomorphism

$$c^* : H^l(B\mathbb{Z}_p; \mathbb{Z}_p) \to H^l([X]^*; \mathbb{Z}_p)$$

is nontrivial. We say that the $\mathbb{Z}_p$-index of X is equal to l if it is greater than or equal to l, and furthermore, $c^* : H^i(B\mathbb{Z}_p; \mathbb{Z}_p) \to H^i([X]^*; \mathbb{Z}_p)$ is zero, for all $i \geq l + 1$. We denote the $\mathbb{Z}_p$-index of X by $\text{ind}\, X$.

2.2. The Wu classes

Let us consider the additives maps *reduced powers* and the *Steenrod squares* (see [11] and [2, Chapter II])

$$P^k : H^{m-2k(p-1)}(M; \mathbb{Z}_p) \to H^m(M; \mathbb{Z}_p)$$

[1] A map $\varphi : X \to S^n$ is said to be an essential map if φ induces nonzero homomorphism $\varphi^* : H^n(S^n; \mathbb{Z}_p) \to H^n(X; \mathbb{Z}_p)$.

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and $Sq^k : H^{m-k}(M;\mathbb{Z}_p) \to H^m(M;\mathbb{Z}_p)$, respectively, for all $k \geq 1$. These maps are also defined in the cohomology of pairs of spaces, and therefore induce maps

$$P^k : H_c^{m-2k(p-1)}(M;\mathbb{Z}_p) \to H_c^m(M;\mathbb{Z}_p)$$

and $Sq^k : H_c^{m-k}(M;\mathbb{Z}_p) \to H_c^m(M;\mathbb{Z}_p)$, which satisfy the same properties that the *reduced powers* and the *Steenrod squares*.

The Wu classes are defined, for $p > 2$, as follows. By formula of the universals coefficients,

$$H^{2k(p-1)}(M;\mathbb{Z}_p) \cong Hom(H_c^{m-2k(p-1)}(M;\mathbb{Z}_p),\mathbb{Z}_p).$$

This isomorphism carries $V \mapsto \overline{V}$, with $\overline{V}(x) = \langle V \smile x, [M]\rangle$, where $[M] \in H_m^c(M;\mathbb{Z}_p)$ is the fundamental class of M. Consider the homomorphism

$$h : H_c^{m-2k(p-1)}(M;\mathbb{Z}_p) \to \mathbb{Z}_p,$$

defined by $h(x) = \langle P^k(x), [M]\rangle$. Then, there is an unique element $V_k \in H^{2k(p-1)}(M;\mathbb{Z}_p)$ such that $h = \overline{V}_k$. The element V_k is the k-th Wu class of M, for $k \geq 0$. Similarly, the Wu classes are defined for $p = 2$; in this case are used the maps *Steenrod squares*.

3. THE *WU* CLASSES FOR PRODUCT OF MANIFOLDS

Let W and M be manifolds of dimensions w and m, respectively, $w \geq m$, both orientables if $p > 2$. In next lemma we obtain a characterization of the Wu classes of the product $W \times M$, in terms of the Wu classes of W and M.

LEMMA 3.1. *Let W and M be connected manifolds of dimensions w and m respectively, with $w \geq m$ and W, M both orientables if $p > 2$. Then, the k-th Wu class of $W \times M$ is given by:*

$$v_k = \sum_{s=0}^{k} (-1)^{\Lambda(s)} U_s \times V_{k-s}, \tag{1}$$

where U_s and V_{k-s} are the k–th and $(k-s)$–th Wu classes of W and M, respectively, and

$$\begin{cases} (-1)^{\Lambda(s)} = (-1)^{[w-s](k-s)} \equiv 1(\mathrm{mod}_2), & \text{if } p = 2, \\ (-1)^{\Lambda(s)} = (-1)^{[w-2s(p-1)](k-s)}, & \text{if } p > 2. \end{cases}$$

First, we will need of the following result to prove Lemma 3.1.

LEMMA 3.2. *Let W and M be manifolds and let R be a commutative ring with unity. If $u_1 \in H^p(W;R)$, $v_1 \in H^r(M;R)$, $u_2 \in H_c^q(W;R)$ and $v_2 \in H_c^s(M;R)$, then*

$$(u_1 \times v_1) \smile (u_2 \times v_2) = (-1)^{qr}(u_1 \smile u_2) \times (v_1 \smile v_2).$$

Proof. We denote by $\pi : W \times M \to W$ and $p : W \times M \to M$ the natural projections, which are proper maps. We have

$$u_1 \times v_1 = \pi^*(u_1) \smile p^*(v_1) \in H^{p+r}(W \times M; R),$$
$$u_2 \times v_2 = \pi_c^*(u_2) \smile p_c^*(v_2) \in H_c^{q+s}(W \times M; R),$$

where π_c^* and p_c^* denote the induced maps in cohomology with compact support.

Thus, by [2, Chapter II Proposition 7.3 and Corollary 7.2], it follows that:

$$(u_1 \times v_1) \smile (u_2 \times v_2) = (-1)^{qr}(\pi^*(u_1) \smile \pi_c^*(u_2)) \smile (p^*(v_1) \smile p_c^*(v_2)).$$

By [2, Chapter II, Section 8.2], we can conclude that

$$\begin{aligned} (u_1 \times v_1) \smile (u_2 \times v_2) &= (-1)^{qr}\pi_c^*(u_1 \smile u_2) \smile p_c^*(v_1 \smile v_2) \\ &= (-1)^{qr}(u_1 \smile u_2) \times (v_1 \smile v_2). \end{aligned}$$

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Proof. of Lemma 3.1. Firstly, we will show that for $p = 2$,

$$Sq^k(x) = \left[\sum_{s=0}^{k} U_s \times V_{k-s}\right] \smile x, \quad \text{for all } x \in H_c^{(w+m)-k}(W \times M; \mathbb{Z}_2),$$

where $U_s \in H^s(W; \mathbb{Z}_2)$ and $V_{k-s} \in H^{k-s}(M; \mathbb{Z}_2)$ are the s–th and $(k-s)$–th Wu classes of W and M, respectively. Thus, by uniqueness $v_k = \sum_{s=0}^{k} U_s \times V_{k-s}$ is the k–th Wu class of $W \times M$.

From the Kunneth's formula [2, Chapter II Theorem 15.2],

$$H_c^{(w+m)-k}(W \times M; \mathbb{Z}_2) \cong \bigoplus_{\beta=0}^{(w+m)-k} H_c^{\beta}(W; \mathbb{Z}_2) \otimes H_c^{(w+m)-k-\beta}(M; \mathbb{Z}_2).$$

Therefore, for each $x \in H_c^{(w+m)-k}(M \times W; \mathbb{Z}_2)$ we can write

$$x = \sum_{\beta=0}^{(w+m)-k} x_\beta \times y_{(w+m)-k-\beta},$$

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where $x_\beta \in H_c^\beta(W;\mathbb{Z}_2)$ and $y_{(w+m)-k-\beta} \in H_c^{(w+m)-k-\beta}(M;\mathbb{Z}_2)$. Thus, applying the k–th Steenrod square Sq^k on x and using the Cartan formula in [2], we obtain

$$\begin{aligned} Sq^k(x) &= \sum_{\beta=0}^{(w+m)-k} Sq^k[x_\beta \times y_{(w+m)-k-\beta}] \\ &= \sum_{\beta=0}^{(w+m)-k} \left(\sum_{s=0}^{k} Sq^s(x_\beta) \times Sq^{k-s}(y_{(w+m)-k-\beta}) \right) \\ &= \sum_{s=0}^{k} \left(\sum_{\beta=0}^{(w+m)-k} Sq^s(x_\beta) \times Sq^{k-s}(y_{(w+m)-k-\beta}) \right). \end{aligned} \tag{2}$$

Now, let us consider $\beta \neq w-s$ for each s fixed, $0 \leq s \leq k$.
If $\beta > w-s$, so $\beta + s > w$ and we have

$$Sq^s(x_\beta) \in H^{\beta+s}(W;\mathbb{Z}_2) = 0. \tag{3}$$

If $\beta < w-s$, so $m < w+m-\beta-s$ and we have

$$Sq^{k-s}(y_{(w+m)-k-\beta}) \in H^{(w+m)-\beta-s}(M;\mathbb{Z}_2) = 0. \tag{4}$$

Therefore, from (2), (3) and (4), using that

$$Sq^s(x_{(w-s)}) = U_s \smile x_{w-s} \text{ and } Sq^{k-s}(y_{m-(k-s)}) = V_{k-s} \smile y_{m-(k-s)},$$

we conclude that

$$\begin{aligned} Sq^k(x) &= \sum_{s=0}^{k} Sq^s(x_{w-s}) \times Sq^{k-s}(y_{m-(k-s)}) \\ &= \sum_{s=0}^{k} [U_s \smile x_{w-s}] \times \left[V_{k-s} \smile y_{m-(k-s)}\right]. \end{aligned}$$

On the other hand, using Lemma 3.2 we have

$$\begin{aligned} v_k \smile x &= \left[\sum_{s=0}^{k} U_s \times V_{k-s}\right] \smile \left[\sum_{\beta=0}^{(w+m)-k} x_\beta \times y_{(w+m)-k-\beta}\right] \\ &= \sum_{s=0}^{k} \sum_{\beta=0}^{(w+m)-k} [U_s \times V_{k-s}] \smile [x_\beta \times y_{(w+m)-k-\beta}] \\ &= \sum_{s=0}^{k} \sum_{\beta=0}^{(w+m)-k} [U_s \smile x_\beta] \times [V_{k-s} \smile y_{(w+m)-k-\beta}]. \end{aligned}$$

Again, let us consider $\beta \neq w - s$ for each s fixed, $0 \leq s \leq k$.

If $\beta > w - s$, then $U_s \smile x_\beta \in H_c^{\beta+s}(W; \mathbb{Z}_2) = \{0\}$.

If $\beta < w - s$, then $V_{k-s} \smile y_{(w+m)-k-\beta} \in H_c^{(w+m)-\beta-s}(M; \mathbb{Z}_2) = \{0\}$.

Therefore,

$$v_k \smile x = \sum_{s=0}^{k} [U_s \smile x_{w-s}] \times [V_{k-s} \smile y_{m-k-s}].$$

Now, in the analogous way, we will show that for $p > 2$,

$$P^k(x) \;=\; \left[\sum_{s=0}^{k} (-1)^{\Lambda(s)} U_s \times V_{k-s}\right] \smile x, \quad \text{for all } x \in H_c^{(w+m)-2k(p-1)}(W \times M; \mathbb{Z}_p),$$

where $U_s \in H^{2s(p-1)}(W; \mathbb{Z}_p)$ and $V_{k-s} \in H^{2(k-s)(p-1)}(M; \mathbb{Z}_p)$ are the s–th and $(k-s)$–th Wu classes of W and M, respectively. So, by uniqueness, we conclude that the class $v_k = \sum_{s=0}^{k} (-1)^{\Lambda(s)} U_s \times V_{k-s}$ is the k-th Wu class of $W \times M$.

By Kunneth's formula,

$$H_c^{(w+m)-2k(p-1)}(W \times M; \mathbb{Z}_p) \cong \bigoplus_{\beta=0}^{(w+m)-2k(p-1)} H_c^{\beta}(W; \mathbb{Z}_p) \otimes H_c^{(w+m)-2k(p-1)-\beta}(M; \mathbb{Z}_p).$$

Therefore, for each $x \in H_c^{(w+m)-2k(p-1)}(W \times M; \mathbb{Z}_p)$ we can write

$$x = \sum_{\beta=0}^{(w+m)-2k(p-1)} x_\beta \times y_{(w+m)-2k(p-1)-\beta},$$

where $x_\beta \in H_c^{\beta}(W; \mathbb{Z}_p)$ and $y_{(w+m)-2k(p-1)-\beta} \in H_c^{(w+m)-2k(p-1)-\beta}(M; \mathbb{Z}_p)$.

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Thus, applying the k–th reduced power P^k on x and using the Cartan formula, we obtain

$$\begin{aligned} P^k(x) &= \sum_{\beta=0}^{(w+m)-2k(p-1)} P^k[x_\beta \times y_{(w+m)-2k(p-1)-\beta}] \\ &= \sum_{\beta=0}^{(w+m)-2k(p-1)} \left(\sum_{s=0}^{k} P^s(x_\beta) \times P^{k-s}(y_{(w+m)-2k(p-1)-\beta}) \right) \\ &= \sum_{s=0}^{k} \left(\sum_{\beta=0}^{(w+m)-2k(p-1)} P^s(x_\beta) \times P^{k-s}(y_{(w+m)-2k(p-1)-\beta}) \right). \end{aligned} \tag{5}$$

Now, let us consider $\beta \neq w - 2s(p-1)$ for each s fixed, $0 \leq s \leq k$.

If $\beta > w - 2s(p-1)$, so $\beta + 2s(p-1) > w$ and we have

$$P^s(x_\beta) \in H^{\beta+2s(p-1)}(W;\mathbb{Z}_p) = 0. \tag{6}$$

If $\beta < w - 2s(p-1)$, so $m < w + m - \beta - 2s(p-1)$ and we have

$$P^{k-s}(y_{(w+m)-2k(p-1)-\beta}) \in H^{(w+m)-\beta-2s(p-1)}(M;\mathbb{Z}_p) = 0. \tag{7}$$

Therefore, from (5), (6) and (7), using that

$$\begin{aligned} P^s(x_{w-2s(p-1)}) &= U_s \smile x_{w-2s(p-1)} \text{ and} \\ P^{k-s}(y_{m-2(k-s)(p-1)}) &= V_{k-s} \smile y_{m-2(k-s)(p-1)}, \end{aligned}$$

we conclude

$$\begin{aligned} P^k(x) &= \sum_{s=0}^{k} P^s(x_{w-2s(p-1)}) \times P^{k-s}(y_{m-2(k-s)(p-1)}) \\ &= \sum_{s=0}^{k} \left[U_s \smile x_{w-2s(p-1)}\right] \times \left[V_{k-s} \smile y_{m-2(k-s)(p-1)}\right]. \end{aligned}$$

On the other hand, using Lemma 3.2 we have

$$\begin{aligned} v_k \smile x &= \left[\sum_{s=0}^{k} (-1)^{\Lambda(s)} U_s \times V_{k-s}\right] \smile \left[\sum_{\beta=0}^{(w+m)-2k(p-1)} x_\beta \times y_{(w+m)-2k(p-1)-\beta}\right] \\ &= \sum_{s=0}^{k} \sum_{\beta=0}^{(w+m)-2k(p-1)} [(-1)^{\Lambda(s)} U_s \times V_{k-s}] \smile [x_\beta \times y_{(w+m)-2k(p-1)-\beta}] \\ &= \sum_{s=0}^{k} \sum_{\beta=0}^{(w+m)-2k(p-1)} (-1)^{(\Lambda(s)+\Gamma(s))} [U_s \smile x_\beta] \times [V_{k-s} \smile y_{(w+m)-2k(p-1)-\beta}], \end{aligned}$$

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where $\Gamma(s) = (k-s)\beta$. Let us consider $\beta \neq w - 2s(p-1)$ for each s fixed, $0 \le s \le k$. If $\beta > w - 2s(p-1)$, then

$$U_s \smile x_\beta \in H_c^{\beta+2s(p-1)}(W; \mathbb{Z}_p) = \{0\}.$$

If $\beta < w - 2s(p-1)$, then

$$V_{k-s} \smile y_{(w+m)-2k(p-1)-\beta} \in H_c^{(w+m)-\beta-2s(p-1)}(M; \mathbb{Z}_p) = \{0\}.$$

Note that for $\beta = w - 2s(p-1)$, $\Gamma(s) = \Lambda(s)$ and therefore,

$$\begin{aligned} v_k \smile x &= \sum_{s=0}^{k} (-1)^{2\Lambda(s)} [U_s \smile x_{w-2s(p-1)}] \times [V_{k-s} \smile y_{m-2(k-s)(p-1)}] \\ &= \sum_{s=0}^{k} [U_s \smile x_{w-2s(p-1)}] \times [V_{k-s} \smile y_{m-2(k-s)(p-1)}]. \end{aligned}$$

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4. PROOFS OF THE MAIN THEOREMS

Now, we denote by $a_1, \ldots, a_r$ a set of representatives of the left lateral classes of G/H, where $r = \frac{|G|}{p}$. We define, for each $i = 1, \ldots, r$, $g_i : X \to X$ by $g_i(x) = a_i x$. Consider the map $F : X \to M^r$ defined by

$$F = (f_1 \times \ldots \times f_r) \circ d,$$

where $d : X \to X^r$ is the diagonal map and $f_i = f \circ g_i$. We prove the following

LEMMA 4.1. *If $f^*(V_k) = 0$, for all $k \ge 1$, where the V_k are the Wu classes of M, then the homomorphism F^* induced by F is such that $F^*(v_k) = 0$, for all $k \geqslant 1$, where v_k are the Wu classes of M^r.*

Proof. It suffices to show that $(f_1 \times \cdots \times f_r)^*(v_k) = 0$, for $k \geqslant 1$. The proof will be done by induction on r. If $r = 1$, then $F = f_1$ and $f_1{}^*(V_k) = g_1{}^* \circ f^*(V_k) = 0$. Now, let us denote by

$$p_1 : M^{r-1} \times M \to M^{r-1},\ p_2 : M^{r-1} \times M \to M,$$
$$q_1 : X^{r-1} \times X \to X^{r-1},\ q_2 : X^{r-1} \times X \to X$$

the natural projections.
If $r = 2$, we have $v_k = \sum_{s=0}^{k} (-1)^{\Lambda(s)} V_s \times V_{k-s}$, then

$$(f_1 \times f_2)^*(v_k) = \sum_{s=0}^{k} (-1)^{\Lambda(s)} (f_1 \times f_2)^*(V_s \times V_{k-s}).$$

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Since $V_s \times V_{k-s} = {p_1}^*(V_s) \smile {p_2}^*(V_{k-s})$, it follows that

$$\begin{aligned}(f_1 \times f_2)^*(v_k) &= \sum_{s=0}^{k} (-1)^{\Lambda(s)} (f_1 \times f_2)^*({p_1}^*(V_s)) \smile (f_1 \times f_2)^*({p_2}^*(V_{k-s})) \\ &= \sum_{s=0}^{k} (-1)^{\Lambda(s)} {q_1}^* \circ {f_1}^*(V_s) \smile {q_2}^* \circ {f_2}^*(V_{k-s}) = 0.\end{aligned}$$

If $r > 2$, we have

$$\begin{aligned}(f_1 \times \cdots \times f_{r-1}) \circ q_1 &= p_1 \circ (f_1 \times \cdots \times f_r) \\ f_r \circ q_2 &= p_2 \circ (f_1 \times \cdots \times f_r).\end{aligned}$$

By Lemma 3.1, we have $v_k = \sum_{s=0}^{k} (-1)^{\Lambda(s)} U_s \times V_{k-s}$ and assuming inductively that $(f_1 \times \cdots \times f_{r-1})^*(U_s) = 0$, we conclude that

$$\begin{aligned}(f_1 \times \cdots \times f_r)^*(v_k) &= (f_1 \times \cdots \times f_r)^*(\sum_{s=0}^{k} (-1)^{\Lambda(s)} U_s \times V_{k-s}) \\ &= \sum_{s=0}^{k} (-1)^{\Lambda(s)} (f_1 \times \cdots \times f_r)^*({p_1}^*(U_s)) \smile (f_1 \times \cdots \times f_r)^*({p_2}^*(V_{k-s})) \\ &= \sum_{s=0}^{k} (-1)^{\Lambda(s)} {q_1}^* \circ (f_1 \times \cdots \times f_{r-1})^*(U_s) \smile {q_2}^* \circ {f_r}^*(V_{k-s}) = 0.\end{aligned}$$

∎

4.1. Proof of Theorem 1.1 and its consequences

In this section, we present the proofs of Theorems 1.1 and its consequences, as follows.

Proof. of Theorem 1.1. We consider the map F defined previously. We have

$$A(f, H, G) \supset A_F = \{x \in X : F(x) = F(hx), \forall h \in H\}.$$

In fact, let x be a point in the set A_F, then

$$(f(a_1x), \ldots, f(a_rx)) = (f(a_1hx), \ldots, f(a_rhx)),$$

for all $h \in H$. Thus, $f(a_ix) = f(a_ihx)$, for all $h \in H$ and $i = 1, \ldots, r$. According to the definition of the action of H on the orbit Gx, $h^{-1} \cdot a_ix := a_i(h^{-1})^{-1}x = a_ihx \in a_iHx$, for

$i = 1, \dots, r$. Thus, f collapses each orbit $a_i Hx$ determined by the action of H on $a_i x$, for $i = 1, \dots, r$, therefore $x \in A(f, H, G)$.

Now we observe that $H \cong \mathbb{Z}_p$ acts freely on X by restriction and by hypothesis $\text{ind}\, X \geq n > n - (p-1)rm$. By Lemma 4.1, $F^*(v_k) = 0$, for all $k \geq 1$, where v_k are the Wu classes of M^r. Thus, according to [12, Theorem 1.2]

$$\text{ind}\, A_F \geq n - (p-1)rm = n - (|G| - r)m.$$

Let us consider the inclusion $i : A_F \to A(f, H, G)$, which is an equivariant map, and so it induces $\bar{i} : [A_F]^* \to [A(f, H, G)]^*$ a map between the orbit spaces. Therefore, if $c : [A(f, H, G)]^* \to B\mathbb{Z}_p$ is any classifying map, we have that $c \circ \bar{i} : [A_F]^* \to B\mathbb{Z}_p$ is a classifying map. Thus,

$$\text{ind}\, A(f, H, G) \geq \text{ind}\, A_F \geq n - (|G| - r)m.$$

∎

Corollary 4.1. *Let X be a paracompact space and let G be a finite group acting freely on X. Let M be a orientable m-manifold, and p a prime number that divide $|G|$. Suppose that $\text{ind}X \geq n > (|G| - r)m$, where $r = \frac{|G|}{p}$. Then, for a continuous map $f : X \to M$ such that $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M, there exists a non-trivial subgroup H of G, such that*

$$\text{cohom.dim}\, A(f, H, G) \geq n - (|G| - r)m.$$

Proof. Let p be a prime number such that divide $|G|$. By Cauchy Theorem, there is a cyclic subgroup H of G of order p. Then, we apply Theorem 1.1. ∎

Remark 4. 1. Let us observe that, if $f^* : H^i(M; \mathbb{Z}_p) \to H^i(X; \mathbb{Z}_p)$ is trivial, for $i \geq 1$, and p is the smallest prime number dividing $|G|$, then $r = \frac{|G|}{p} \geq \frac{|G|}{q}$, where q can be any other prime number dividing $|G|$. Thus, $n > (|G| - \frac{|G|}{q})m$, therefore for each prime number q dividing $|G|$, there exists a cyclic subgroup of order q, H_q of G such that $\text{ind}\, A(f, H_q, G) \geq n - (|G| - r)m$.

The following theorem is a version for manifolds of the main result in [5].

Theorem 4.1. *Let G be a finite group which acts freely on the n-sphere S^n and let H be a cyclic subgroup of G of prime order p. Let $f : S^n \to M$ be a continous map, where M is a m-manifold (orientable, if $p > 2$). If $n > (|G| - r)m$, where $r = \frac{|G|}{p}$, then*

$$\text{cohom.dim}(A(f, H, G)) \geq n - (|G| - r)m.$$

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Proof. Since $n > (|G|-r)m \geq m$, $f^*(V_k) = 0$, for all $k \geq 1$. Moreover, $\operatorname{ind} S^n = n$ and thus we apply Theorem 1.1. ∎

4.2. Proof of Theorem 1.2

Now, let us consider X a compact Hausdorff space and an essential map $\varphi : X \to S^n$. Suppose that G is a finite group of order s, which acts freely on S^n, and H is a subgroup of order p of G. Let $G = \{g_1, ..., g_s\}$ be a fixed enumeration of elements of G, where g_1 is the identity of G. A nonempty space X_φ can be associated with the essential map $\varphi : X \to S^n$ as follows:

$$X_\varphi = \{(x_1, ..., x_s) \in X^s : g_i\varphi(x_1) = \varphi(x_i), i = 1, ..., s\},$$

where X^s denotes the s-fold cartesian product of X. The set X_φ is a closed subset of X^s and so it is compact. We define a G-action on X_φ as follows: for each $g_i \in G$ and for each $(x_1, ..., x_s) \in X_\varphi$,

$$g_i(x_1, ..., x_s) = (x_{\sigma_{g_i}(1)}, ..., x_{\sigma_{g_i}(s)}),$$

where the permutation σ_{g_i} is defined by $\sigma_{g_i}(k) = j$, $g_k g_i = g_j$. We observe that if $x = (x_1, ..., x_s) \in X_\varphi$, then $x_i \neq x_j$, for any $i \neq j$, and therefore G acts freely on X_φ.

Let us consider a continuous map $f : X \to M$, where M is a topological space and $\widetilde{f} : X_\varphi \to M$ is given by $\widetilde{f}(x_1, ..., x_s) = f(x_1)$.

Definition 4.1. The set $A_\varphi(f, H, G)$ of (H, G)-coincidence points of f relative to φ is defined by

$$A_\varphi(f, H, G) = A(\widetilde{f}, H, G).$$

Proof. of Theorem 1.2. Let $\widetilde{f} : X_\varphi \to M$ given by $\widetilde{f}(x_1, ..., x_r) = f(x_1)$, that is, $\widetilde{f} = f \circ \pi_1$, where π_1 is the natural projection on the 1-th coordinate. By hypothesis, $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M, then we have $\widetilde{f}^*(V_k) = 0$, for all $k \geq 1$. Moreover, the $\mathbb{Z}_p$-index of X_φ is equal to n by [8, Theorem 3.1]. In this way, X_φ and $\widetilde{f}$ satisfy the hypothesis of Theorem 1.1, which implies that the $\mathbb{Z}_p$-index of the set $A(\widetilde{f}, H, G)$ is greater than or equal to $n - (|G| - r)m$. By definition, $A_\varphi(f, H, G) = A(\widetilde{f}, H, G)$, and then

$$\operatorname{cohom.dim} A_\varphi(f, H, G) \geq n - (|G| - r)m.$$

∎

By a similar argument to that used in the proof of Corollary 4.1 we have the following corollary of Theorem 1.2

Corollary 4.2. *Let X be a compact Hausdorff space and G a finite group acting freely on S^n. Let M be a orientable m-manifold and p a prime number dividing $|G|$. Suppose that*

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$n > (|G| - r)m$, where $r = \frac{|G|}{p}$. Then, for a continuous map $f : X \to M$, with $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M, there exists a non-trivial subgroup H of G, such that

$$\text{cohom.dim}\, A_\varphi(f, H, G) \geq n - (|G| - r)m.$$

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