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Nonlinear parabolic problems in thin domains with a highly oscillatory boundary

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In this paper we study the behavior of the solutions of nonlinear parabolic problems posed in a domain that degenerates into a line segment (thin domain) which has an oscillating boundary. We combine methods from linear homogenization theory for reticulated structures and from the theory on nonlinear dynamics of dissipative systems to obtain the limit problem for the elliptic and parabolic problem and analyze the convergence properties of the solutions and of the attractors of the evolutionary equations. October, 2010 ICMC-USP

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1. INTRODUCTION

In this paper we are interested in analyzing the asymptotic behavior of solutions of parabolic PDEs in a thin domain with a highly oscillatory behavior in its boundary, as depicted in Figure 1.

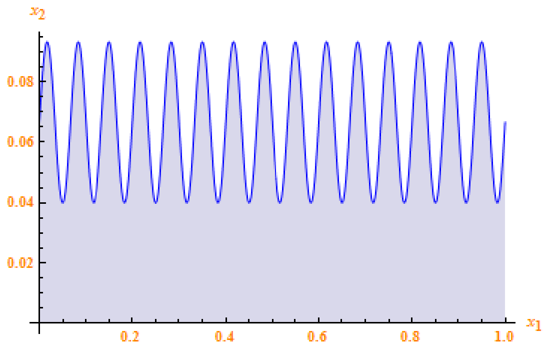

FIG. 1. Thin domain with a highly oscillatory boundary.

To state the problem, let $g : \mathbb{R} \mapsto \mathbb{R}$ be a $\mathcal{C}^1$, L-periodic positive function (see FIGURE 1) with $0 < g_0 \le g(x) \le g_1$ for all $x \in \mathbb{R}$ where $g_1 = \max_{x\in\mathbb{R}}\{g(x)\}$ and consider the following bounded open set

$$R^\epsilon = \{(x, y) \in \mathbb{R}^2 \mid x \in I \text{ and } 0 < y < \epsilon g(x/\epsilon)\} \tag{1.1}$$

where $\epsilon > 0$ is arbitrary and $I = (0, 1)$. For sake of notation, let us denote the *oscillatory* part of the boundary by $\partial_o R^\varepsilon = \{(x, \varepsilon g(x/\varepsilon)) : 0 < x < 1\}$ the *fixed* part of the boundary by $\partial_f R^\varepsilon = \{(x, 0)) : 0 < x < 1\}$ and the *lateral* part of the boundary as $\partial_l R^\varepsilon = \{(0, y)) : 0 < y < \varepsilon g(0)\} \cup \{(1, y)) : 0 < y < \varepsilon g(1/\varepsilon)\}$. If we need to distinguish between the two parts of the lateral boundary (left and right) we will write $\partial_{ll} R^\varepsilon$ and $\partial_{lr} R^\varepsilon$.

In the thin domain R^ϵ we consider the following semilinear parabolic evolution equation

$$\begin{cases} w_t^\epsilon - \Delta w^\epsilon + w^\epsilon = f(w^\epsilon) & \text{in } R^\epsilon, \quad t > 0 \\ \dfrac{\partial w^\epsilon}{\partial \nu^\epsilon} = 0 & \text{on } \partial R^\epsilon \end{cases} \tag{1.2}$$

where ν^ϵ is the unit outward normal to ∂R^ϵ, $\frac{\partial}{\partial \nu^\epsilon}$ is the outside normal derivative and the function $f : \mathbb{R} \mapsto \mathbb{R}$ is a $\mathcal{C}^2$-function with bounded derivatives. Moreover, since we are interested in the behavior of solutions as $t \to \infty$ and its behavior with respect to the small parameter ε, we will require that the solutions of (1.2) be bounded for large values of time. A natural assumption to obtain this boundedness of the solutions is expressed in the following dissipative condition

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < 0. \tag{1.3}$$

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Observe that the domain R^ϵ shrinks in the vertical direction and it also has a very highly oscillatory behavior at the top boundary. Moreover, the amplitude and period of the oscillations are of the same order ϵ, which also coincides with the order of thickness of the thin domain. This scaling makes the problem very resonant and the determination of the limiting problem not straight forward.

In order to analyze problem (1.2) and its related linear elliptic and parabolic problem we first perform a simple change of variables which consists in stretching in the y-direction by a factor of $1/\epsilon$ (that is, $x_1 = x, x_2 = y/\varepsilon$), transforming the domain R^ϵ into the domain

$$\Omega^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in I \text{ and } 0 < x_2 < g(x_1/\epsilon)\}.$$

By doing so, we obtain a domain which is not thin anymore although it presents very highly oscillatory behavior (given by the fact that the upper boundary is the graph of the function $x_1 \to g(x_1/\epsilon)$. Under this change of variables, equation (1.2) is transformed into

$$\begin{cases} u_t^\epsilon - \dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f(u^\epsilon) & \text{in } \Omega^\epsilon, \quad t > 0 \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon \end{cases} \tag{1.4}$$

where $N^\varepsilon = (N_1^\varepsilon, N_2^\varepsilon)$ is the outward normal to the boundary of Ω^ϵ. Observe the factor $1/\varepsilon^2$ in front of the derivative in the x_2 direction which means a very fast diffusion in the vertical direction. In some sense, we have sustituted the thin domain R^ε with a non-thin domain Ω^ε but with a very strong diffusion mechanism in the x_2-direction.

Notice also that domain Ω^ε "converges" in certain sense to the rectangule $\Omega = (0, 1) \times (0, g_1)$ (where $g_1 = \max_{x\in\mathbb{R}}\{g(x)\}$). Hence, we expect that equation (1.4) will converge in certain sense to an equation posed in Ω. But because of the presence of a very strong diffusion mechanism in the x_2-direction we expect the solutions of (1.4) to become homogeneous in the x_2-direction so that the limiting solution will not have a dependence in the x_2 direction and therefore the limitting problem will be a one dimensional problem. This fact is in agreement with the intuitive idea that an equation in a thin domain should approach an equation in a line segment.

We start analyzing the behavior of the solutions of the linear elliptic problem associated to (1.4), that is

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f^\varepsilon & \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon \end{cases} \tag{1.5}$$

where we assume that $f^\varepsilon \in L^2(\Omega^\varepsilon)$ and $\|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C$, with C independent of ϵ. As mentioned above, we expect that the limiting problem is posed in Ω.

Since we need to compare functions in Ω^ε with functions in Ω we will need some extension operators which will transform a function defined in Ω^ε to functions defined in Ω. We will be able to construct an extension operator $P_\varepsilon : H^1(\Omega^\varepsilon) \to H^1(\Omega)$, see Section 3, and by doing so, we show that the solution $u^\epsilon \in H^1(\Omega^\epsilon)$ of the linear elliptic problem associated to equation (1.4) satifies

$$P_\epsilon u^\epsilon \rightharpoonup u_0 \quad w - H^1(\Omega)$$

where $u_0(x_1, x_2) = u_0(x_1)$ is the unique solution of

$$\begin{cases} -\, q_0\, u_{0xx} + u_0 = f_0 & \text{in } (0,1) \\ u_0'(0) = u_0'(1) = 0 \end{cases} \tag{1.6}$$

$q_0 > 0$ is the *homogenization coefficient*, which is obtained in a nontrivial way (see (38)) and f_0 is an appropriate limit of the sequence f^ε (see (46)). We refer to Theorem 4.1 for a complete statement on this result.

To pass to the limit from (1.5) to (1.6) we will use homogenization theory. First we will formally obtain the equation through the multiple scale method (see Section 2), and then, using the oscillating test function method of L. Tartar, we will rigorously prove the convergence result, see Section 4

Let us remark that in our results we also obtain the convergence of resolvent operators associated to problems (1.5) and (1.6), that is, looking at problem (1.5) as $A_\epsilon u = f^\epsilon$ and (1.6) as $A_0 u = f^0$ with an appropriate functional setting we also show the convergence of A_ε^{-1} to A_0^{-1} as operators, see Section 5. This in turn show that, among other things, the spectral properties of A_ε converge to the spectral properties of A_0 and in particular, the eigenvalues and eigenfunctions of the operator A_ε converge to the eigenvalues and eigenfunctions of A_0.

Once we have identified the homogenized operator we will consider the nonlinear evolution problem associated to the homogenized linear elliptic operator,

$$\begin{cases} u_t - q_0 u_{xx} + u = f(u) & x \in (0,1), \quad t > 0 \\ u_0'(0) = u_0'(1) = 0 \end{cases} \tag{1.7}$$

and we will show that this problem is the homogenized nonlinear evolution problem. Actually, we will show that if $C > 0$ is a positive constant and $\varphi_\varepsilon \in L^2(\Omega^\varepsilon)$ with $\|\varphi^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C$, and if we denote by $\hat{\varphi}^\varepsilon \in L^2(0,1)$, the averaged function of φ^ε in the x_2-direction, then if $u^\varepsilon(x_1, x_2, t)$ is the solution of (1.5) with initial condition φ^ε and $u(x_1, t)$ is the solution of (1.7) with initial condition $\hat{\varphi}^\varepsilon$, then for each $\beta < 1$, we have

$$\|P_\varepsilon u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{H^\beta(\Omega)} \leq \nu(\varepsilon) t^{-\gamma} e^{-wt}, \quad t > 0,$$

where the function $\nu(\varepsilon) \to 0$ as $\varepsilon \to 0+$, (recall that P^ε is the extension operator mentioned above). This shows the convergence of the nonlinear parabolic problems, see Section 6 for an analysis on the linear evolution problem and Section 7 for the nonlinear problem

Moreover we will show that the function $\nu(\varepsilon)$ depends only on C, that is, the convergence is uniform for all $\varphi^\varepsilon \in L^2(\Omega^\varepsilon)$ with $\|\varphi^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C$. This uniformity will allow us to analyze the behavior of the asymptotic dynamics of equation (1.4) as the parameter $\varepsilon \to 0$. Hence, if the nonlinearity f satisfies appropriate dissipative conditions, like (1.3), then

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both equations (1.4) and (1.7) have attracrors $\mathcal{A}_\varepsilon \subset H^1(\Omega^\varepsilon)$ $\mathcal{A}_0 \subset H^1(0,1)$ respectively. Recall that an attractor is a compact invariant set which attracts all bounded sets of the phase space. It contains all the assymptotic dynamics of the system and all global bounded solution will lie in the attractor. We will show that the attractors are upper semicontinuous, in the sense that the union of all the attractors is bounded and for each sequence $w^\varepsilon \in \mathcal{A}_\varepsilon$ there exists a subsequence and an element $w^0 \in \mathcal{A}_0$ such that $w^\varepsilon \to w^0$ in an appropriate way. As a matter of fact we will show (see also Section 7), that for any $\beta < 1$,

$$\sup_{\varphi^\epsilon \in \mathcal{A}_\epsilon} \left[\inf_{\varphi \in \mathcal{A}_0} \left\{ \|P_\epsilon \varphi^\epsilon - \varphi\|_{H^\beta(\Omega)} \right\} \right] \to 0, \text{ as } \epsilon \to 0.$$

In this paper we combine two different theories to be able to say something meaningful about the behavior of a nonlinear parabolic equation and its asymptotic states in a thin domain which presents a highly oscillatory behavior. On one hand *homogenization theory*, specially the part related to reticulated structures, is needed to obtain the linear elliptic limiting (homogenized) problem. We refer to [5, 25, 33] for a general introduction to the theory of homogenization and to [10] for a general treatise on reticulated structures. On the other hand we need to use concepts and techniques from the theory of *dissipative systems and attractors* to be able to conclude the continuity of the attractors of the systems involved. We refer to [16] for a general introduction and to [17, 23, 9, 12, 13, 22, 24] for more concrete developments of nonlinear dynamics in thin domains (with no oscillating behavior). There are not many results on the behavior of global attractors of dissipative evolutionary equations under a perturbation related to homogenization. We would like to cite some of them: [6, 7, 14, 15].

One of the key ingredients that will allow us to combine both theories in an effective way is the fact that the convergence of the linear problems obtained in the theory of homogenization is interpreted in terms of convergence of the resolvents of the linear operators involved, (see [26, 27, 28, 29, 30, 8] for a theory of convergence of operators and resolvent of operators and its applications to elliptic and parabolic problems). This is translated to analyze the convergence of the corresponding linear semigroups through the well known expresion of the linear semigroup as an appropriate integral of the resolvent operators. Finally this information is trasferred to the nonlinear dynamics trough the variation of constant formula. This method has been applied to other situations (different from homogenization type problems) and it has been proved to be succesful to study perturbation problems for nonlinear evolutionary equations (see [1, 2, 3, 4, 8, 21] etc.). In this paper, this technique is also proved to be valid for perturbation problems related to homogenization.

2. THE MULTIPLE-SCALE METHOD

In this section we want to apply the method of multiple scales to obtain formally the limit homogenized problem of

$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = f^\epsilon & \text{in } R^\epsilon \\ \dfrac{\partial w^\epsilon}{\partial N^\epsilon} = 0 & \text{on } \partial R^\epsilon \end{cases} \tag{2.1}$$

on the oscillating thin domain R^ϵ given by (1.1) with $f^\epsilon \in L^2(R^\epsilon)$ satisfying

$$\|f^\epsilon\|_{L^2(R^\epsilon)} \leq C,$$

for some $C > 0$ independent of ϵ. Assume to simplify that the nonhomogeneous term f^ϵ is given by

$$f^\epsilon = f \text{ satisfying } f(x_1, x_2) = f(x_1)$$

for all $x_1 \in I$ and $\epsilon > 0$.

We look for a formal asymptotic expansion of the form

$$w^\epsilon(x_1, x_2) = w_0(x_1, x_2, \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) + \epsilon w_1(x_1, x_2, \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) + \epsilon^2 w_2(x_1, x_2, \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) + \dots$$

But observe that the thin domain R^ϵ degenerates to a line segment when $\epsilon \to 0$. This suggests that w^ϵ tends not to depend on the "macroscopic" variable x_2. Hence, we are assuming that w_i is an independent function of x_2 and therefore we assume the asymptotics

$$w^\epsilon(x_1, x_2) = w_0(x_1, \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) + \epsilon w_1(x_1, \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) + \epsilon^2 w_2(x_1, \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) + \dots \tag{2.2}$$

We denote $x = x_1$, $y = x_1/\epsilon$, $z = x_2/\epsilon$, and we observe that

$$\frac{\partial}{\partial x_1} = \partial_x + \frac{1}{\epsilon}\partial_y, \qquad \frac{\partial}{\partial x_2} = \frac{1}{\epsilon}\partial_z.$$

Let us denote by Y^* the representative cell

$$Y^* = \{(y, z) \in \mathbb{R}^2 \; : \; 0 < y < L \text{ and } 0 < z < g(y)\} \tag{2.3}$$

and we call B_0 the lateral boundary, B_1 the upper boundary and B_2 the lower boundary of Y^*. So that, $\partial Y^* = B_0 \cup B_1 \cup B_2$. Also, we suppose $w_i(x, y, z)$ defined for $x \in I = (0, 1)$ and $(y, z) \in Y^*$, L-periodic in the variable y.

Introducing the formal expansion (2.2) in the problem (2.1), we can obtain after some calculations the following problems for the functions w_0, w_1, w_2.

$$\begin{cases} -\Delta_{y,z} w_0(x, y, z) = 0 & \text{in } Y^* \\ \dfrac{\partial w_0}{\partial N} = 0 & \text{on } B_1 \cup B_2 \\ w_0(x, 0, z) = w_0(x, L, z) & z \in B_0. \end{cases}$$

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where N is the unit outward normal to ∂Y^*.

Consequently, we have that $w_0(x,y,z) = w_0(x)$.

$$\begin{cases} -\Delta_{y,z} w_1(x,y,z) = 2\partial_x\partial_y w_0 \quad \text{in } Y^* \\ \dfrac{\partial w_1}{\partial N}(x,y,g(y)) = \dfrac{g'(y)}{\sqrt{1+(g'(y))^2}}\dfrac{dw_0}{dx} \quad \text{on } B_1 \\ \dfrac{\partial w_1}{\partial N}(x,y,0) = 0 \quad \text{on } B_2 \\ w_1(x,0,z) = w_1(x,L,z) \quad z \in B_0. \end{cases}$$

Thus, taking into account $\partial_x\partial_y w_0 = 0$ and denoting for $X(y,z)$ the solution of

$$\begin{cases} -\Delta_{y,z} X(y,z) = 0 \quad \text{in } Y^* \\ \dfrac{\partial X}{\partial N}(y,g(y)) = -\dfrac{g'(y)}{\sqrt{1+(g'(y))^2}} \quad \text{on } B_1 \\ \dfrac{\partial X}{\partial N}(y,0) = 0 \quad \text{on } B_2 \\ X(0,z) = X(L,z) \quad z \in B_0, \end{cases}$$

we have that $w_1(x,y,z) = -X(y,z)\dfrac{dw_0}{dx}(x)$.

As for w_2, it should satisfy

$$\begin{cases} -\Delta_{y,z} w_2(x,y,z) = f(x) - w_0(x) + 2\partial_x\partial_y w_1(x,y,z) + \partial_x^2 w_0(x) \text{ in } \quad Y^* \\ \dfrac{\partial w_2}{\partial N}(x,y,g(y)) = \dfrac{g'(y)}{\sqrt{1+(g'(y))^2}}\dfrac{\partial w_1}{\partial x}(x,y,g(y)) \quad \text{on } B_1 \\ \dfrac{\partial w_2}{\partial N}(x,y,0) = 0 \quad \text{on } B_2 \\ w_2(x,0,z) = w_2(x,L,z) \quad z \in B_0 \end{cases}$$

or equivalently

$$\begin{cases} -\Delta_{y,z} w_2(x,y,z) = f(x) - w_0(x) + (1 - 2\partial_y X(y,z))\dfrac{d^2 w_0}{dx^2}(x) \quad \text{in } Y^* \\ \dfrac{\partial w_2}{\partial N}(x,y,g(y)) = -\dfrac{g'(y)}{\sqrt{1+(g'(y))^2}} X(y,g(y))\dfrac{d^2 w_0}{dx^2}(x) \quad \text{on } B_1 \\ \dfrac{\partial w_2}{\partial N}(x,y,0) = 0 \quad \text{on } B_2 \\ w_2(x,0,z) = w_2(x,L,z) \quad z \in B_0. \end{cases} \tag{2.4}$$

Since w_2 is a solution of (2.4), it follows from the Fredholm Alternative that

$$0 = \int_{Y^*} \phi(x) \Big\{ f(x) - w_0(x) + (1 - 2\partial_y X(y,z)) \frac{d^2 w_0}{dx^2}(x) \Big\} dydz + \int_{\partial Y^*} \phi(x) \frac{\partial w_2}{\partial N}\, dS. \quad (5)$$

for all test function $\phi(x,y,z) = \phi(x)$. On the other hand,

$$\begin{aligned} \int_{Y^*} \phi(x) \frac{d^2 w_0}{dx^2}(x) \partial_y X(y,z) dydz &= \int_{Y^*} \phi(x) \frac{d^2 w_0}{dx^2}(x) \nabla_{y,z} X(y,z) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dydz \\ &= \int_{\partial Y^*} \phi\, \frac{d^2 w_0}{dx^2}(x)\, X\, N_1\, dS \\ &= \int_{\partial Y^*} \phi\, \frac{\partial w_2}{\partial N}\, dS. \end{aligned}$$

Replacing this expression in (2.4), we get

$$0 = \int_{Y^*} \phi(x) \Big\{ f(x) - w_0(x) + (1 - \partial_y X(y,z))\, \frac{d^2 w_0}{dx^2}(x) \Big\} dydz \quad (6)$$

for any function ϕ defined in I.

Thus, since X depends only on the variables y and z, we can conclude that w_0 must satisfy

$$\begin{cases} -r \dfrac{d^2 w_0}{dx^2}(x) + w_0(x) = f(x), & x \in (0,1) \\ w_0'(0) = w_0'(1) = 0 \end{cases} \quad (7)$$

where

$$r = \frac{1}{|Y^*|} \int_{Y^*} \Big\{ 1 - \frac{\partial X}{\partial y}(y,z) \Big\} dydz.$$

This second-order differential equation in w_0 defined on the interval I is called the *homogenized equation* of the problem (2.1).

3. EXTENSION OPERATOR

In this section we are going to construct an extension operator that will be very important in the proof of the general result in the following sections. Hence, let us consider the following open sets:

$$\mathcal{O} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in I \text{ and } 0 < x_2 < G_1\}$$
$$\mathcal{O}^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in I \text{ and } 0 < x_2 < G_\epsilon(x_1)\}$$

where $I \subset \mathbb{R}$ is an open interval, $G_\epsilon : I \mapsto \mathbb{R}$ is a $\mathcal{C}^1$-function satisfying $0 < G_0 \leq G_\epsilon(x_1) \leq G_1$ for all $x \in I$ and $\epsilon > 0$. Notice that $\mathcal{O}^\epsilon \subset \mathcal{O}$.

LEMMA 3.1.
With the notation above, there exists an extension operator

$$P_\epsilon \in \mathcal{L}(L^p(\mathcal{O}^\epsilon), L^p(\mathcal{O})) \cap \mathcal{L}(W^{1,p}(\mathcal{O}^\epsilon), W^{1,p}(\mathcal{O})) \cap \mathcal{L}(W^{1,p}_{\partial_l}(\mathcal{O}^\epsilon), W^{1,p}_{\partial_l}(\mathcal{O}))$$

(where $W^{1,p}_{\partial_l}$ is the set of functions in $W^{1,p}$ which are zero in the lateral boundary ∂_l) and a constant K independent of ϵ and p such that

$$\begin{gathered} \|P_\epsilon\varphi\|_{L^p(\mathcal{O})} \leq K\,\|\varphi\|_{L^p(\mathcal{O}^\epsilon)} \\ \Big\|\frac{\partial P_\epsilon\varphi}{\partial x_1}\Big\|_{L^p(\mathcal{O})} \leq K\Big\{\Big\|\frac{\partial\varphi}{\partial x_1}\Big\|_{L^p(\mathcal{O}^\epsilon)} + \eta(\epsilon)\Big\|\frac{\partial\varphi}{\partial x_2}\Big\|_{L^p(\mathcal{O}^\epsilon)}\Big\} \\ \Big\|\frac{\partial P_\epsilon\varphi}{\partial x_2}\Big\|_{L^p(\mathcal{O})} \leq K\Big\|\frac{\partial\varphi}{\partial x_2}\Big\|_{L^p(\mathcal{O}^\epsilon)} \end{gathered} \tag{1}$$

for all $\varphi \in W^{1,p}(\mathcal{O}^\epsilon)$ where $1 \leq p \leq \infty$ and

$$\eta(\epsilon) = \sup_{x\in I}\{|G'_\epsilon(x)|\}.$$

Proof. Observe first that the set $\mathcal{O}^0 = (0,1)\times(0,G_0) \subset \mathcal{O}^\epsilon$ for all ε. Hence, if we have that $G_1 \leq 2G_0$, which implies that $G_\varepsilon(x_1) \leq 2G_0$, we can define the operator:

$$(P_\epsilon\varphi)(x_1,x_2) = \begin{cases} \varphi(x_1,x_2) & (x_1,x_2) \in \mathcal{O}^\epsilon \\ \varphi(x_1, 2G_\epsilon(x_1) - x_2) & (x_1,x_2) \in \mathcal{O}/\mathcal{O}^\epsilon. \end{cases}$$

Observe that this operator is obtained through a "reflection" procedure in the x_2 direction along the oscillating boundary. It is straigth forward to check that this operator satisfies (1).

If we are in the case where $G_1 > 2G_0$, we will need to extend first the function $\varphi_{|\mathcal{O}^0}$ in the direction of negative x_2, with a finite number of successive reflections. That is, if φ_0 is defined in $\mathcal{O}^\epsilon$ then we extend φ_0 to the set $(0,1)\times(-G_0,0)$ with the reflecting along the line $x_2 = 0$. This produces the function

$$\varphi_1(x_1,x_2) = \begin{cases} \varphi_0(x_1,x_2), & (x_1,x_2) \text{ with } 0 < x_2 < G_\varepsilon(x_1) \\ \varphi_0(x_1,-x_2), & (x_1,x_2) \text{ with } -G_0 < x_2 \leq 0. \end{cases}$$

We can continue producing these reflections inductively as follows

$$\varphi_n(x_1,x_2) = \begin{cases} \varphi_{n-1}(x_1,x_2), & (x_1,x_2) \text{ with } -(n-1)G_0 < x_2 \leq G_\varepsilon(x_1) \\ \varphi_{n-1}(x_1,-x_2-2(n-1)G_0), & (x_1,x_2) \text{ with } -nG_0 < x_2 \leq -(n-1)G_0 \end{cases}$$

Choosing n large enough so that $nG_0 > G_1$, constructing φ_n and applying to φ_n the procedure by reflection in the x_2 direction along the oscillating boundary, we obtain the extension operator P_ε which satisfies (1). ▮

Remark 3. 1. 1) This operator preserves periodicity in the x_1 variable. That is, if the function φ_ε is periodic in x_1, then the extended function $P_\varepsilon\varphi_\varepsilon$ is also periodic in x_1.

2) This result can be applied to the case $G_\varepsilon(x) = G(x)$ independent of ε. In particular, we can apply the extension operator to the basic cell.

Remark 3. 2. If we denote by $|||\cdot|||$ the norm

$$|||u|||_{W^{1,p}(\mathcal{O}^\epsilon)} = \|u\|_{L^p(\mathcal{O}^\epsilon)} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{L^p(\mathcal{O}^\epsilon)} + (1+\eta(\epsilon))\Big\|\frac{\partial u}{\partial x_2}\Big\|_{L^p(\mathcal{O}^\epsilon)}$$

for all $u \in W^{1,p}(\mathcal{O}^\epsilon)$, then we obtain that the extension operator P_ϵ must satisfy

$$|||P_\epsilon u|||_{W^{1,p}(\mathcal{O})} \leq K|||u|||_{W^{1,p}(\mathcal{O}^\epsilon)}$$

with K a constant independent of ϵ. Note that the norm $|||\cdot|||_{W^{1,p}(\mathcal{O}^\epsilon)}$ is equivalent to the usual one.

4. CONVERGENCE THEOREM FOR THE HOMOGENIZED LIMIT

In this section we use the technique of oscillating test functions of L. Tartar, see Tartar [31] [32] and Cioranescu & Saint Jean Paulin [10] to prove a convergence theorem for the solutions w^ϵ of the problem (2.1). We will consider the case where the whole boundary has a Neumann type condition. We will also indicate how to proceed under other boundary conditions.

To study the convergence of (2.1) on the thin oscillating domain R^ϵ, with Neumann boundary conditions in the lateral boundary, we consider the change of variables stated in the introduction which transforms problem (2.1)into the equivalent linear elliptic problem

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f^\varepsilon & \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1}N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2}N_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon \end{cases} \tag{1}$$

on the open bounded oscillating domain $\Omega^\epsilon \subset \Omega$ given by

$$\Omega^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in I \text{ and } 0 < x_2 < g(x_1/\epsilon)\} \tag{2}$$

with

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in I \text{ and } 0 < x_2 < g_1\}$$

and $f^\varepsilon \in L^2(\Omega^\epsilon)$ with $\|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C$, with C independent of ε.

The variational formulation of (1) is the following: find $u^\epsilon \in H^1(\Omega^\epsilon)$ such that

$$\int_{\Omega^\epsilon} \Big\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \Big\} dx_1 dx_2 = \int_{\Omega^\epsilon} f^\varepsilon \varphi dx_1 dx_2, \ \forall \varphi \in H^1(\Omega^\epsilon). \tag{3}$$

Taking $\varphi = u^\epsilon$ in (3), we get the following apriori estimates for u^ε

$$\Big\| \frac{\partial u^\epsilon}{\partial x_1} \Big\|^2_{L^2(\Omega^\epsilon)} + \frac{1}{\epsilon^2}\Big\| \frac{\partial u^\epsilon}{\partial x_2} \Big\|^2_{L^2(\Omega^\epsilon)} + \|u^\epsilon\|^2_{L^2(\Omega^\epsilon)} \le \|f^\varepsilon\|_{L^2(\Omega^\epsilon)} \|u^\epsilon\|_{L^2(\Omega^\epsilon)}. \tag{4}$$

Now, consider the representative cell Y^* defined in (2.3)

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < L \text{ and } 0 < y_2 < g(y_1)\}$$

and let χ be its characteristic function. We extend χ periodically on the variable $y_1 \in \mathbb{R}$ and denote this extension again by χ.

If we denote by χ^ϵ the characteristic function of the set Ω^ϵ, we easily see that

$$\chi^\epsilon(x_1, x_2) = \chi(\frac{x_1}{\epsilon}, x_2), \text{ for } (x_1, x_2) \in \Omega^\epsilon \tag{5}$$

Let us also consider the following families of isomorphisms $T^\epsilon_k : A^\epsilon_k \mapsto Y$ given by

$$T^\epsilon_k(x_1, x_2) = (\frac{x_1 - \epsilon k L}{\epsilon}, x_2) \tag{6}$$

where

$$A^\epsilon_k = \{(x_1, x_2) \in \mathbb{R}^2 \mid \epsilon k L < x_1 < \epsilon L(k+1) \text{ and } 0 < x_2 < g_1\}$$
$$Y = (0, L) \times (0, g_1)$$

with $k \in \mathbb{N}$ satisfying $0 \le k < |I|/(\epsilon L)$.

Let us consider the following auxiliary problem given by

$$\begin{cases} -\Delta X = 0 \text{ in } Y^* \\ \dfrac{\partial X}{\partial N} = 0 \text{ on } B_2 \\ \dfrac{\partial X}{\partial N} = -\dfrac{g'(y_1)}{\sqrt{1 + g'(y_1)^2}} \text{ on } B_1 \\ X(0, y_2) = X(L, y_2) \text{ on } B_0 \\ \displaystyle\int_{Y^*} X \, dy_1 dy_2 = 0. \end{cases} \tag{7}$$

where B_0 is the lateral boundary, B_1 is the upper boundary and B_2 is the lower boundary of ∂Y^*.

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Taking the isomorphism (6) and the extension operator

$$P \in \mathcal{L}(H^1(Y^*), H^1(Y)) \cap \mathcal{L}(L^2(Y^*), L^2(Y))$$

defined by Lemma 3.1 with $G_\epsilon(x_1) = g(x_1)$ independent of ϵ and $Y = (0, L) \times (0, g_1)$, see Remark 3.1, we define ω_k^ϵ in $(x_1, x_2) \in A_k^\epsilon$ by

$$\begin{aligned} \omega_k^\epsilon(x_1, x_2) &= x_1 - \epsilon\Big(PX \circ T_k^\epsilon(x_1, x_2)\Big) \\ &= x_1 - \epsilon\Big(PX(\frac{x_1 - \epsilon Lk}{\epsilon}, x_2)\Big). \end{aligned}$$

Observe that for any $(x_1, x_2) \in \Omega^\epsilon$ there exists k such that $(x_1, x_2) \in A_k^\epsilon$. Therefore, the function w^ϵ defined by $w^\epsilon(x_1, x_2) = \omega_k^\epsilon(x_1, x_2)$ is well defined and $w^\epsilon \in H^1(\Omega)$.

We introduce now the vector $\eta^\epsilon = (\eta_1^\epsilon, \eta_2^\epsilon)$ defined by

$$\eta_i^\epsilon(x_1, x_2) = \frac{\partial \omega^\epsilon}{\partial x_i}(x_1, x_2), \quad (x_1, x_2) \in \Omega^\epsilon. \tag{8}$$

Since $\frac{\partial}{\partial x_1} = \frac{1}{\epsilon}\frac{\partial}{\partial y_1}$ and $\frac{\partial}{\partial x_2} = \frac{\partial}{\partial y_2}$ we have that

$$\begin{aligned} \eta_1^\epsilon(x_1, x_2) &= 1 - \frac{\partial X}{\partial y_1}(\frac{x_1 - \epsilon kL}{\epsilon}, x_2) = 1 - \frac{\partial X}{\partial y_1}(\frac{x_1}{\epsilon}, x_2) := \eta_1(y_1, y_2), \\ \eta_2^\epsilon(x_1, x_2) &= -\epsilon\frac{\partial X}{\partial y_2}(\frac{x_1 - \epsilon kL}{\epsilon}, x_2) = -\epsilon\frac{\partial X}{\partial y_2}(\frac{x_1}{\epsilon}, x_2) := \eta_2(y_1, y_2) \end{aligned}$$

for $(y_1, y_2) = (\frac{x_1 - \epsilon kL}{\epsilon}, x_2) \in Y^*$ and $(x_1, x_2) \in \Omega^\epsilon$. We note that η_i is a periodic function on the variable y_1 and $\eta_i^\epsilon(x_1, x_2) = \eta_i(\frac{x_1}{\epsilon}, x_2)$ for $i = 1, 2$.

It follows from definition of X that the functions η_1^ϵ and η_2^ϵ satisfy

$$\begin{aligned} \frac{\partial \eta_1^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial \eta_2^\epsilon}{\partial x_2} &= 0 \text{ in } \Omega^\epsilon \\ \eta_1^\epsilon N_1^\epsilon + \frac{1}{\epsilon^2}\eta_2^\epsilon N_2^\epsilon &= 0 \text{ on } (x_1, g\Big(\frac{x_1}{\epsilon}\Big)) \\ \eta_1^\epsilon N_1^\epsilon + \frac{1}{\epsilon^2}\eta_2^\epsilon N_2^\epsilon &= 0 \text{ on } (x_1, 0) \\ \eta^\epsilon(0, x_2) = \eta^\epsilon(1, x_2) &\text{ for } 0 < x_2 < g(0) \end{aligned} \tag{9}$$

where

$$N^\epsilon = (N_1^\epsilon, N_2^\epsilon) = \left(-\frac{g'(\frac{x_1}{\epsilon})}{(\epsilon^2 + g'(\frac{x_1}{\epsilon})^2)^{\frac{1}{2}}}, \frac{\epsilon}{(\epsilon^2 + g'(\frac{x_1}{\epsilon})^2)^{\frac{1}{2}}}\right) \text{ on } (x_1, g\Big(\frac{x_1}{\epsilon}\Big))$$
$$N^\epsilon = (0, -1) \text{ on } (x_1, 0)$$

with $x_1 \in I$. In fact, by (7) we have

$$\frac{\partial \eta_1^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial \eta_2^\epsilon}{\partial x_2} = -\frac{1}{\epsilon}\Big(\frac{\partial^2 X}{\partial {y_1}^2}(\frac{x_1}{\epsilon}, x_2) + \frac{\partial^2 X}{\partial {y_2}^2}(\frac{x_1}{\epsilon}, x_2)\Big) = 0 \text{ in } \Omega^\epsilon$$

$$\begin{aligned}
\eta_1^\epsilon N_1^\epsilon + \frac{1}{\epsilon^2}\eta_2^\epsilon N_2^\epsilon &= -\frac{g'(\frac{x_1}{\epsilon})\Big(1 - \frac{\partial X}{\partial y_1}(\frac{x_1}{\epsilon}, x_2)\Big) + \frac{\partial X}{\partial y_2}(\frac{x_1}{\epsilon}, x_2)}{\sqrt{\epsilon^2 + g'(\frac{x_1}{\epsilon})^2}} \\
&= -\frac{g'(\frac{x_1}{\epsilon}) + (\frac{\partial X}{\partial y_1}(\frac{x_1}{\epsilon}, x_2), \frac{\partial X}{\partial y_2}(\frac{x_1}{\epsilon}, x_2)) \cdot (-g'(\frac{x_1}{\epsilon}), 1)}{\sqrt{\epsilon^2 + g'(\frac{x_1}{\epsilon})^2}} \\
&= -\frac{g'(\frac{x_1}{\epsilon}) - \frac{g'(\frac{x_1}{\epsilon})}{\sqrt{1+g'(\frac{x_1}{\epsilon})^2}}\sqrt{1 + g'(\frac{x_1}{\epsilon})^2}}{\sqrt{\epsilon^2 + g'(\frac{x_1}{\epsilon})^2}} = 0
\end{aligned}$$

on $(x_1, g\Big(\frac{x_1}{\epsilon}\Big))$ and

$$\eta_1^\epsilon N_1^\epsilon + \frac{1}{\epsilon^2}\eta_2^\epsilon N_2^\epsilon = (1 - \frac{\partial X}{\partial y_1}(\frac{x_1}{\epsilon}, x_2)) \cdot 0 + \frac{1}{\epsilon}\frac{\partial X}{\partial y_2}(\frac{x_1}{\epsilon}, x_2) = 0 \text{ on } (x_1, 0).$$

Therefore, multiplying (9) by a test function $\varphi \in H^1(\Omega)$ with $\varphi = 0$ in neighborhood of the lateral boundaries and integrating by parts, we obtain

$$\begin{aligned}
0 &= \int_{\Omega^\epsilon} \varphi \left(\frac{\partial \eta_1^\epsilon}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial \eta_2^\epsilon}{\partial x_2}\right) dx_1 dx_2 \\
&= \int_{\partial\Omega^\epsilon} \varphi \left(\eta_1^\epsilon N_1^\epsilon + \frac{1}{\epsilon^2}\eta_2^\epsilon N_2^\epsilon\right) dS - \int_{\Omega^\epsilon} \left(\frac{\partial \varphi}{\partial x_1}\eta_1^\epsilon + \frac{1}{\epsilon^2}\frac{\partial \varphi}{\partial x_2}\eta_2^\epsilon\right) dx_1 dx_2 \\
&= \int_{(x_1, g(\frac{x_1}{\epsilon}))} \varphi \left(\eta_1^\epsilon N_1^\epsilon + \frac{1}{\epsilon^2}\eta_2^\epsilon N_2^\epsilon\right) dS - \int_{\Omega^\epsilon} \left(\frac{\partial \varphi}{\partial x_1}\eta_1^\epsilon + \frac{1}{\epsilon^2}\frac{\partial \varphi}{\partial x_2}\eta_2^\epsilon\right) dx_1 dx_2 \\
&= 0 - \int_{\Omega^\varepsilon} \left(\frac{\partial \varphi}{\partial x_1}\eta_1^\epsilon + \frac{1}{\epsilon^2}\frac{\partial \varphi}{\partial x_2}\eta_2^\epsilon\right) dx_1 dx_2.
\end{aligned}$$

That is

$$\int_{\Omega^\varepsilon} \left(\eta_1^\epsilon \frac{\partial \varphi}{\partial x_1} + \eta_2^\epsilon \frac{1}{\epsilon^2}\frac{\partial \varphi}{\partial x_2}\right) dx_1 dx_2 = 0 \tag{10}$$

Let $\phi = \phi(x_1) \in \mathcal{C}_0^\infty(0,1)$ and considering the test function $\varphi = \phi\, \omega^\epsilon$ in (3) and in (10), we obtain

$$
\begin{aligned}
&\int_{\Omega^\varepsilon} f^\varepsilon(\phi\,\omega^\epsilon)dx_1dx_2\\
=\ &\int_{\Omega^\varepsilon}\Big\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial}{\partial x_1}(\phi\,\omega^\epsilon)+\frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial}{\partial x_2}(\phi\,\omega^\epsilon)+u^\epsilon(\phi\,\omega^\epsilon)\Big\}dx_1dx_2\\
=\ &\int_{\Omega^\varepsilon}\Big\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial}{\partial x_1}(\phi\,\omega^\epsilon)+\frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial}{\partial x_2}(\phi\,\omega^\epsilon)+u^\epsilon(\phi\,\omega^\epsilon)\Big\}dx_1dx_2\\
&-\int_{\Omega^\varepsilon}\Big\{\eta_1^\epsilon\frac{\partial}{\partial x_1}(\phi\,w^\epsilon)+\frac{1}{\epsilon^2}\eta_2^\epsilon\frac{\partial}{\partial x_2}(\phi\,w^\epsilon)\Big\}dx_1dx_2\\
=\ &\int_{\Omega^\varepsilon}\Big\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \phi}{\partial x_1}\omega^\epsilon+\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \omega^\epsilon}{\partial x_1}\phi+\frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \omega^\epsilon}{\partial x_2}\phi+u^\epsilon\phi\omega^\epsilon\\
&-\eta_1^\epsilon\frac{\partial \phi}{\partial x_1}u^\epsilon-\eta_1^\epsilon\frac{\partial u^\epsilon}{\partial x_1}\phi-\frac{1}{\epsilon^2}\eta_2^\epsilon\frac{\partial u^\epsilon}{\partial x_2}\phi\Big\}dx_1dx_2.
\end{aligned}
\tag{11}
$$

Using that $\eta_i^\epsilon=\frac{\partial\omega^\epsilon}{\partial x_i}$ we cancel the appropriate terms and obtain

$$
\int_{\Omega^\varepsilon}\Big\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \phi}{\partial x_1}\,\omega^\epsilon-\eta_1^\epsilon\frac{\partial \phi}{\partial x_1}\,u^\epsilon+u^\epsilon\phi\,\omega^\epsilon\Big\}dx_1dx_2=\int_{\Omega^\varepsilon} f^\varepsilon\phi\,\omega^\epsilon dx_1dx_2,\ \forall\phi\in C_0^\infty(0,1) \tag{12}
$$

On the other hand, we have obtained before the weak formulation of the problem (3), that is

$$
\int_{\Omega^\epsilon}\Big\{\frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1}+\frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2}+u^\epsilon\varphi\Big\}dx_1dx_2=\int_{\Omega^\epsilon} f^\varepsilon\varphi dx_1dx_2,\ \forall\varphi\in H^1(\Omega^\epsilon). \tag{13}
$$

Now we need to pass to the limit in (12) and (13). In order to accomplish this we need to write both expressions as integrals in the same domain. For this, we will use the extension P_ε constructed in Lemma 3.1, the standard extension by zero, that we denote by$\tilde{\ }$, and the characteristic function χ^ε of Ω^ε as follows:

$$
\begin{aligned}
&\int_{\Omega}\Big\{\widetilde{\frac{\partial u^\epsilon}{\partial x_1}}\frac{\partial \phi}{\partial x_1}\,\omega^\epsilon-\widetilde{\eta_1^\epsilon}\frac{\partial \phi}{\partial x_1}\,P_\varepsilon u^\epsilon+\chi^\varepsilon P_\varepsilon(u^\epsilon)\phi\,\omega^\epsilon\Big\}dx_1dx_2\\
&\qquad\qquad=\int_\Omega \chi^\varepsilon f^\varepsilon\phi\,\omega^\epsilon dx_1dx_2,\qquad \forall\phi\in\mathcal{C}_0^\infty(0,1)
\end{aligned}
\tag{14}
$$

$$
\int_{\Omega}\Big\{\widetilde{\frac{\partial u^\epsilon}{\partial x_1}}\frac{\partial \varphi}{\partial x_1}+\chi^\varepsilon P_\varepsilon u^\epsilon\varphi\Big\}dx_1dx_2=\int_\Omega \chi^\varepsilon f^\varepsilon\varphi dx_1dx_2,\ \forall\varphi\in H^1(0,1). \tag{15}
$$

Observe that in this last equality we have taken $\varphi\in H^1(0,1)$ and the term including partial derivatives with respect to x_2 do not appear.

We want to pass to the limit in the expressions above, (14) and (15) . In order to accomplish this, we pass to the limit in the different functions that form the integrands.

(a). Limit of χ_ε.
From (5), we have

$$\chi^\epsilon(\cdot, x_2) \overset{\varepsilon\to 0}{\rightharpoonup} \theta(x_2) := \frac{1}{L}\int_0^L \chi(s, x_2)ds \quad w^* - L^\infty(I), \quad \forall x_2 \in (0, g_1). \tag{16}$$

Observe that the limit θ does not dependent on the variable x_1 and we can get the area of the open set Y^* with the formula

$$L\int_0^{g_1} \theta(x_2)dx_2 = |Y^*|. \tag{17}$$

Also, from (16) we have that

$$H^\epsilon(x_2) = \int_I \varphi(x_1, x_2)\left\{\chi^\epsilon(x_1, x_2) - \theta(x_2)\right\}dx_1 \to 0 \text{ as } \epsilon \to 0$$

a.e. $x_2 \in (0, g_1)$ and for all $\varphi \in L^1(\Omega)$. So, due to

$$\int_\Omega \varphi(x_1, x_2)\left\{\chi^\epsilon(x_1, x_2) - \theta(x_2)\right\}dx_1dx_2 = \int_0^{g_1} H^\epsilon(x_2)dx_2$$
$$\text{and } |H^\epsilon(x_2)| \leq \int_I \varphi(x_1, x_2)dx_1,$$

we can get by Lebesgue's Dominated Convergence Theorem that

$$\chi^\epsilon \overset{\varepsilon\to 0}{\rightharpoonup} \theta \quad w^* - L^\infty(\Omega). \tag{18}$$

(b). Limit in the tilde functions
Since $\|f^\epsilon\|_{L^2(\Omega)}$ is uniformly bounded, we get from (4) that there exists M independent of ϵ such that

$$\|\widetilde{u^\epsilon}\|_{L^2(\Omega)}, \left\|\widetilde{\frac{\partial u^\epsilon}{\partial x_1}}\right\|_{L^2(\Omega)} \text{ and } \frac{1}{\epsilon}\left\|\widetilde{\frac{\partial u^\epsilon}{\partial x_2}}\right\|_{L^2(\Omega)} \leq M \text{ for all } \epsilon > 0. \tag{19}$$

Then, we can extract a subsequence, still denoted by $\widetilde{u^\epsilon}$, $\widetilde{\frac{\partial u^\epsilon}{\partial x_1}}$ and $\widetilde{\frac{\partial u^\epsilon}{\partial x_2}}$, such that

$$\begin{aligned} \widetilde{u^\epsilon} &\rightharpoonup u^* \quad w - L^2(\Omega) \\ \widetilde{\frac{\partial u^\epsilon}{\partial x_1}} &\rightharpoonup \xi^* \quad w - L^2(\Omega) \text{ and} \\ \widetilde{\frac{\partial u^\epsilon}{\partial x_2}} &\to 0 \quad s - L^2(\Omega) \end{aligned} \tag{20}$$

as $\epsilon \to 0$ for some u^* and $\xi^* \in L^2(\Omega)$.

Moreover, since $\|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C$ independent of ε, we have $\|\tilde{f}^\varepsilon\|_{L^2(\Omega)} \le C$ and therefore, the function $\hat{f}^\varepsilon$ defined by

$$\hat{f}^\varepsilon(x_1) \equiv \int_0^{g_1} \tilde{f}^\varepsilon(x_1, x_2)dx_2 = \int_0^{g(x_1/\varepsilon)} f^\varepsilon(x_1, x_2)dx_2 \tag{21}$$

satisfies that $\hat{f}^\varepsilon \in L^2(0,1)$. Hence, via subsequences, we have the existence of a function $\hat{f} = \hat{f}(x_1) \in L^2(0,1)$ such that

$$\hat{f}_\varepsilon \rightharpoonup \hat{f} \qquad w - L^2(0,1) \tag{22}$$

Remark 4. 1. Observe that in the case where $f^\varepsilon(x_1, x_2) = f(x_1)$ then

$$\hat{f}(x_1) = \Big(\int_0^{g_1} \theta(x_2)dx_2 \Big) f(x_1) = \frac{|Y^*|}{L} f(x_1)$$

where we have used (17).

(c). Limit in the extended functions

Using priori estimate (4), the fact that $u^\varepsilon \in H^1(\Omega^\varepsilon)$ and using the results from Lemma 3.1 on the extension operator P_ϵ we get that

$$\|P_\epsilon u^\epsilon\|_{L^2(\Omega)}, \Big\| \frac{\partial P_\epsilon u^\epsilon}{\partial x_1} \Big\|_{L^2(\Omega)} \text{ and } \frac{1}{\epsilon} \Big\| \frac{\partial P_\epsilon u^\epsilon}{\partial x_2} \Big\|_{L^2(\Omega)} \le \tilde{M} \text{ for all } \epsilon > 0 \tag{23}$$

where $\tilde{M}$ is a positive constant independent of ϵ given by estimate (19) and Lemma 3.1. Then, we can extract a subsequence, still denoted by $P_\epsilon u^\epsilon$ and a function $u_0 \in H^1(\Omega)$, such that

$$\begin{aligned} P_\epsilon u^\epsilon &\rightharpoonup u_0 \quad w - H^1(\Omega) \\ P_\epsilon u^\epsilon &\to u_0 \quad s - L^2(\Omega) \\ \text{and } \frac{\partial P_\epsilon u^\epsilon}{\partial x_2} &\to 0 \quad s - L^2(\Omega). \end{aligned} \tag{24}$$

A consequence of the limits (24) is that $u_0(x_1, x_2) = u_0(x_1)$ on Ω. More precisely,

$$\frac{\partial u_0}{\partial x_2}(x_1, x_2) = 0 \text{ a.e. } \Omega. \tag{25}$$

In fact, for all $\varphi \in C_0^\infty(\Omega)$, we have by (24) that

$$\begin{aligned} \int_\Omega u_0 \frac{\partial \varphi}{\partial x_2} dx_1 dx_2 &= \lim_{\epsilon \to 0} \int_\Omega P_\epsilon u^\epsilon \frac{\partial \varphi}{\partial x_2} dx_1 dx_2 \\ &= -\lim_{\epsilon \to 0} \int_\Omega \frac{\partial P_\epsilon u^\epsilon}{\partial x_2} \varphi \, dx_1 dx_2 = 0. \end{aligned}$$

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In particular, we may consider that the function $u_0 \in H^1(0,1)$.

Also, we note that $\widetilde{u^\epsilon} = \chi^\epsilon P_\epsilon u^\epsilon$ a.e. Ω. Thus, it follows from (18), (20) and (24) that we have the following relationship between u^* and u_0

$$u^*(x_1,x_2) = \theta(x_2)\, u_0(x_1) \qquad \text{a.e. } \Omega. \tag{26}$$

(d). Limit in ω^ε.

With the definition of ω^ε, we have

$$\int_{A_k^\epsilon} |\omega^\epsilon - x_1|^2 dx_1 dx_2 = \int_Y \epsilon^3 |(PX)(y_1,y_2)|^2 dy_1 dy_2 \leq \int_{Y^*} C\epsilon^3 |X(y_1,y_2)|^2 dy_1 dy_2$$

and so,

$$\begin{aligned}\int_\Omega |\omega^\epsilon - x_1|^2 dx_1 dx_2 &\approx \sum_{k=1}^{\frac{1}{\epsilon L}} \int_{Y^*} C\epsilon^3 |X(y_1,y_2)|^2 dy_1 dy_2 \\ &\approx \epsilon^2 \int_{Y^*} C|X(y_1,y_2)|^2 dy_1 dy_2 \to 0 \text{ as } \epsilon \to 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{A_k^\epsilon} \Big| \frac{\partial}{\partial x_1}(\omega^\epsilon - x_1) \Big|^2 dx_1 dx_2 &= \int_Y \Big| \frac{\partial (PX)}{\partial y_1}(y_1,y_2) \Big|^2 \epsilon\, dy_1 dy_2 \\ &\leq \epsilon \int_{Y^*} C \Big| \frac{\partial X}{\partial y_1}(y_1,y_2) \Big|^2 dy_1 dy_2\end{aligned}$$

and

$$\begin{aligned}\int_{A_k^\epsilon} \Big| \frac{\partial}{\partial x_2}(\omega^\epsilon - x_1) \Big|^2 dx_1 dx_2 &= \int_Y \Big| \epsilon \frac{\partial (PX)}{\partial y_2}(y_1,y_2) \Big|^2 \epsilon dy_1 dy_2 \\ &\leq \epsilon^3 \int_{Y^*} C \Big| \frac{\partial X}{\partial y_2}(y_1,y_2) \Big|^2 dy_1 dy_2.\end{aligned}$$

Also, we have

$$\begin{aligned}\int_\Omega \Big| \frac{\partial}{\partial x_1}(\omega^\epsilon - x_1) \Big|^2 dx_1 dx_2 &\approx \sum_{k=1}^{\frac{1}{\epsilon L}} \int_{Y^*} C\epsilon | \frac{\partial X}{\partial y_1}(y_1,y_2)|^2 dy_1 dy_2 \\ &\approx \int_{Y^*} \tilde{C} \Big| \frac{\partial X}{\partial y_1}(y_1,y_2) \Big|^2 dy_1 dy_2\end{aligned}$$

for all $\epsilon > 0$ and

$$\int_\Omega \Big| \frac{\partial}{\partial x_2}(\omega^\epsilon - x_1) \Big|^2 dx_1 dx_2 \leq \epsilon^2 \int_{Y^*} \tilde{C} \Big| \frac{\partial X}{\partial y_2}(y_1,y_2) \Big|^2 dy_1 dy_2 \to 0 \text{ as } \epsilon \to 0.$$

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Then, we can conclude

$$\omega^\epsilon \to x_1 \quad s - L^2(\Omega) \text{ and } w - H^1(\Omega) \tag{27}$$

with

$$\frac{\partial \omega^\epsilon}{\partial x_2} \to 0 \quad s - L^2(\Omega).$$

(e). Limit of η_i^ε

Let $\widetilde{\eta}^\epsilon = \eta^\epsilon \chi^\epsilon$ be the extension by zero of the vector η^ϵ to the whole Ω. Clearly, we may extend $\widetilde{\eta}^\epsilon = \eta^\epsilon \chi^\epsilon$ periodically on $\mathbb{R} \times (0, g_1)$. So, $\widetilde{\eta}_1^\epsilon(x_1, x_2) = \widetilde{\eta}_1(\frac{x_1}{\epsilon}, x_2)$ and we can obtain by the Average Theorem that

$$\widetilde{\eta}_1^\epsilon(x_1, x_2) \rightharpoonup \frac{1}{L}\int_0^L \Big(1 - \widetilde{\frac{\partial X}{\partial y_1}}(s, x_2)\Big)\chi(s, x_2)ds := q(x_2) \quad w^* - L^\infty(I). \tag{28}$$

where χ is the characteristic function of Y^*.

Hence, arguing as (18) we can prove

$$\widetilde{\eta}_1^\epsilon \rightharpoonup q \quad w^* - L^\infty(\Omega). \tag{29}$$

Now, by the convergences shown in (a)-(e) above, we can pass to the limit in (14) and in (15). We obtain

$$\int_\Omega \Big\{\xi^* \frac{\partial \phi}{\partial x_1} x_1 - q\frac{\partial \phi}{\partial x_1} u_0 + \theta u_0 \phi\, x_1\Big\} dx_1 dx_2 = \int_0^1 \hat{f}(x_1)\phi(x_1)\, x_1 dx_1, \qquad \forall \phi \in \mathcal{C}_0^\infty(0,1)$$

Observe that $\xi^* \frac{\partial}{\partial x_1}(\phi\, x_1) = \xi^* x_1 \frac{\partial \phi}{\partial x_1} + \xi^* \phi$. Consequently, we have

$$\int_\Omega \Big\{\xi^* \frac{\partial}{\partial x_1}(\phi\, x_1) - \phi\, \xi^* - q\frac{\partial \phi}{\partial x_1} u_0 + \theta u_0 \phi\, x_1\Big\} dx_1 dx_2 = \int_0^1 \hat{f}(x_1)\phi(x_1)\, x_1 dx_1 \tag{30}$$

for all $\phi \in \mathcal{C}_0^\infty(0,1)$. From (15), we get

$$\int_\Omega \Big\{\xi^* \frac{\partial \varphi}{\partial x_1} + \theta u_0 \varphi\Big\} dx_1 dx_2 = \int_0^1 \hat{f}\varphi dx_1, \qquad \forall \varphi \in H^1(0,1). \tag{31}$$

In particular, via iterated integration and (17), we get

$$\begin{aligned} \int_0^1 \Big\{\Big(\int_0^{g_1} \xi^*(x_1, x_2)dx_2\Big)\frac{\partial \varphi(x_1)}{\partial x_1} + \frac{|Y^*|}{L} u_0(x_1)\varphi(x_1)\Big\} dx_1 \\ = \int_0^1 \hat{f}(x_1)\varphi(x_1)dx_1, \qquad \forall \varphi \in H^1(0,1) \end{aligned} \tag{32}$$

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Taking $\varphi = \phi x_1$ in (31), we get

$$\int_\Omega \Big\{ \xi^* \frac{\partial}{\partial x_1}(\phi\, x_1) + \theta u_0 \phi\, x_1 \Big\} dx_1 dx_2 = \int_0^1 \hat{f} \phi x_1 dx_1. \tag{33}$$

Hence, it follows from (30) and (33) that, for all $\phi \in \mathcal{C}_0^\infty(0,1)$,

$$0 = \int_\Omega \Big\{ \phi\, \xi^* + q \frac{\partial \phi}{\partial x_1}\, u_0 \Big\} dx_1 dx_2 = \int_\Omega \Big\{ \phi\, \xi^* - q\phi \frac{\partial u_0}{\partial x_1} \Big\} dx_1 dx_2 \tag{34}$$

where we have performed an integration by parts to obtain the last integral. Hence, if we define

$$\hat{q} \equiv \int_0^{g_1} q(s) ds = \frac{1}{L} \int_{Y^*} \Big\{ 1 - \frac{\partial X}{\partial y_1}(y_1, y_2) \Big\}\, dy_1 dy_2 \tag{35}$$

and performing an iterated integration in (34) we obtain

$$\int_0^1 \phi(x_1) \Big(\int_0^{g_1} \xi^*(x_1, x_2) dx_2 - \hat{q} \frac{\partial u_0(x_1)}{\partial x_1} \Big) dx_1 = 0, \quad \forall \phi \in \mathcal{C}_0^\infty(0,1)$$

which implies that

$$\int_0^{g_1} \xi^*(x_1, x_2) dx_2 = \hat{q} \frac{\partial u_0(x_1)}{\partial x_1}, \qquad \text{a.e. } x_1 \in (0,1) \tag{36}$$

Plugging this last equality in (32) we get

$$\int_0^1 \hat{q} \frac{\partial u_0}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{|Y^*|}{L}\, u_0\, \varphi dx_1 = \int_0^1 \hat{f}\, \varphi\, dx_1, \quad \forall \varphi \in H^1(0,1) \tag{37}$$

Hence, if we define $q_0 = \frac{L}{|Y^*|} \hat{q}$, that is,

$$q_0 = \frac{1}{|Y^*|} \int_{Y^*} \Big\{ 1 - \frac{\partial X}{\partial y_1}(y_1, y_2) \Big\}\, dy_1 dy_2 \tag{38}$$

and also

$$f_0(x_1) = \frac{L}{|Y^*|} \hat{f}(x_1) \tag{39}$$

we get from (37)

$$\int_0^1 \Big(q_0 \frac{\partial u_0}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + u_0\, \varphi \Big) dx_1 = \int_0^1 f_0\, \varphi\, dx_1, \quad \forall \varphi \in H^1(0,1) \tag{40}$$

which is the weak formulation of

$$\begin{cases} -q_0\, u_{0xx} + u_0 = f_0 & \text{in } (0,1) \\ u_0'(0) = u_0'(1) = 0 \end{cases}. \tag{41}$$

Remark 4. 2. Observe that, in relation with Remark (4.1), we have that if originally the nonhomogeneous term f does not deppend on x_2, that is, $f(x_1, x_2) = f(x_1)$ then with the definition of f_0, see (39), and with (17), we have

$$f_0(x_1) = \frac{L}{|Y^*|}\hat{f}(x_1) = \frac{L}{|Y^*|}\int_0^{g_1} \theta(s) f(x_1) ds = \frac{L}{|Y^*|}\frac{|Y^*|}{L} f(x_1) = f(x_1).$$

Hence, equation (41) is in agreement with the one found via the method of Multiple Scales in Section 2.

We show now that as a matter of fact, problem (41) is well posed in the sense that the diffusion coefficient $q_0 > 0$. For this, we use the variational formulation of the auxiliary problem (7), that is, the elliptic bilinear form

$$b : H_* \times H_* \mapsto \mathbb{R} : (\phi, \varphi) \to \int_{Y^*} \nabla\phi \cdot \nabla\varphi \, dy_1 dy_2$$

where

$$H_* = \{\phi \in H^1(Y^*) \,|\, \int_{Y^*} \phi \, dy_1 dy_2 = 0 \text{ and } \phi(0, y_2) = \phi(L, y_2) \text{ with } 0 < y_2 < g(0)\}.$$

So, for all $\phi \in H_*$ we obtain that X satisfies

$$b(X, \phi) = -\int_{B_1} \frac{g'}{\sqrt{1 + {g'}^2}} \phi \, dS.$$

Recall that B_1 is the upper boundary of the basic, cell, that is, $B_1 = \{(y_1, y_2) \in \mathbb{R}^2 \,|\, 0 < y_1 < L \text{ and } y_2 = g(y_1)\} \subset \partial Y^*$. Consequently, $y_1 - X$ satisfies

$$b(y_1 - X, \phi) = \int_{B_1} N_1 \phi \, dS + \int_{B_1} \frac{g'}{\sqrt{1 + {g'}^2}} \phi \, dS = 0 \tag{42}$$

for all $\phi \in H_*$ where the normal vector N on B_1 is given by $N = (N_1, N_2) = \frac{(-g', 1)}{\sqrt{1+{g'}^2}}$. Also, we have by (38)

$$\begin{aligned} q_0 \, |Y^*| &= \int_{Y^*} \frac{\partial}{\partial y_1}(y_1 - X) \frac{\partial y_1}{\partial y_1} \, dy_1 dy_2 = \int_{Y^*} \nabla(y_1 - X) \cdot \nabla y_1 \, dy_1 dy_2 \\ &= b(y_1 - X, y_1). \end{aligned} \tag{43}$$

Hence, due to relation (42) with $\phi = -X$ and identity (42), we get

$$q_0 \, |Y^*| = b(y_1 - X, y_1) + b(y_1 - X, -X) = b(y_1 - X, y_1 - X) > 0. \tag{44}$$

Thus, since $|Y^*| > 0$ we have that $q_0 > 0$ and the problem above is well posed. In particular u_0 is the unique solution of problem (41).

So, we may summarize in the following theorem the results from this section:

THEOREM 4.1. *Let u^ϵ be the solution of* (1) *with $f^\varepsilon \in L^2(\Omega^\varepsilon)$ and $\|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C$ with C independent of the parameter ε. Then,*

i) If we have a sequence $\varepsilon \to 0$ such that

$$\hat{f}^\varepsilon(\cdot) = \int_0^{g(x_1/\varepsilon)} f^\varepsilon(\cdot, x_2)dx_2 \rightharpoonup \hat{f}(\cdot), \qquad w - L^2(0,1)$$

then

$$P_\epsilon u^\epsilon \rightharpoonup u_0 \quad w - H^1(\Omega) \tag{45}$$

where $P_\epsilon \in \mathcal{L}(L^2(\Omega^\epsilon), L^2(\Omega)) \cap \mathcal{L}(H^1(\Omega^\epsilon), H^1(\Omega))$ is the extension operator constructed in Lemma 3.1 *(with $\mathcal{O}_\varepsilon = \Omega_\varepsilon$ and $\mathcal{O} = \Omega$) and $u_0(x_1,x_2) = u_0(x_1)$ for all $(x_1,x_2) \in \Omega$ is the unique solution of*

$$\begin{cases} -q_0\, u_{0xx} + u_0 = f_0 \quad \text{in } (0,1) \\ \qquad u_0'(0) = u_0'(1) = 0 \end{cases} \tag{46}$$

with $q_0 > 0$ is given by (38) *and $f_0(\cdot) = \frac{L}{|Y^*|}\hat{f}(\cdot)$.*

ii) For any sequence $\varepsilon \to 0$, we have another subsequence, still denoted by $\varepsilon \to 0$, such that the functions $\hat{f}^\varepsilon(\cdot) \rightharpoonup \hat{f}(\cdot)$ w-$L^2(0,1)$ for some function $\hat{f}$. In particular, for this subsequence we have the results from i).

Remark 4. 3. i) The extension operator P_ε in (45) does not need to be exactly the one constructed in Lemma 3.1. We just need that P_ε satisfies (1) with $\mathcal{O}_\varepsilon = \Omega_\varepsilon$ and $\mathcal{O} = \Omega$.
ii) The convergence in (45) implies that $P_\varepsilon u_\varepsilon \to u_0$ s-$H^\alpha(\Omega)$ for all $0 \leq \alpha < 1$.

5. CONVERGENCE OF RESOLVENT

In this section we consider the problems (1) and (41) in its abstract form and showed that under a proper notion of convergence, solution of (1) converges to solution of (41) as $\epsilon \to 0$. This brings us to the concept of compact convergence developed in the works [26, 27, 28, 29, 30]. See also [1, 2, 3, 4, 8, 24] for applications to concrete perturbation problems. For this, let $\{L_\epsilon\}_{\epsilon>0}$ be a family of Hilbert spaces defined by $L_\epsilon = L^2(\Omega^\epsilon)$ with the canonical inner product

$$(u,v)_\epsilon = \int_{\Omega^\epsilon} u(x)v(x)\,dx$$

and let $L_0 = L^2(I)$ be the limiting Hilbert space with the inner product $(\cdot,\cdot)_0$ given by

$$(u,v)_{L_0} = \mu(g)\int_0^1 u(x)v(x)\,dx$$

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where

$$\mu(g) = \frac{1}{L}\int_0^L g(s)\,ds.$$

We can write (1) as an abstract equation $A_\epsilon u = f^\epsilon$, where $A_\epsilon : \mathcal{D}(A_\epsilon) \subset L^2(\Omega^\epsilon) \mapsto L^2(\Omega^\epsilon)$ is the self adjoint, positive linear operator with compact resolvent defined by

$$\begin{aligned} \mathcal{D}(A_\epsilon) &= \{u \in H^2(\Omega^\epsilon) \mid \frac{\partial u}{\partial x_1} N_1^\epsilon + \frac{1}{\epsilon^2}\frac{\partial u}{\partial x_2} N_2^\epsilon = 0 \text{ on } \partial\Omega^\epsilon\} \\ A_\epsilon u &= -\frac{\partial^2 u}{\partial {x_1}^2} - \frac{1}{\epsilon^2}\frac{\partial^2 u}{\partial {x_2}^2} + u \quad u \in \mathcal{D}(A_\epsilon). \end{aligned} \tag{1}$$

Similarly, we associate the limit problem (41) to the *limit linear operator* $A_0 : \mathcal{D}(A_0) \subset L_0 \mapsto L_0)$ defined by

$$\begin{aligned} \mathcal{D}(A_0) &= \{u \in H^2(I) \mid u'(0) = u'(1) = 0\} \\ A_0 u &= -q_0\, u_{x_1 x_1} + u \quad u \in \mathcal{D}(A_0) \end{aligned} \tag{2}$$

where q_0 is the positive constant given in (38). Clearly, A_0 is a positive self adjoint operator with compact resolvent.

In order to simplify the writing, we denote by L_ϵ^α the fractional power scale associated to operators A_ϵ with $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \epsilon \leqslant 1$. We also will write $L_\epsilon := L_\epsilon^0$ for all $0 \leqslant \epsilon \leqslant 1$. Observe that $L_\epsilon^{\frac{1}{2}}$ is the Sobolev Space $H^1(\Omega^\epsilon)$ with equivalent norm

$$\|u\|_{L_\epsilon^{\frac{1}{2}}} = \left\|\frac{\partial u}{\partial x_1}\right\|_{L_\epsilon} + \frac{1}{\epsilon}\left\|\frac{\partial u}{\partial x_2}\right\|_{L_\epsilon} + \|u\|_{L_\epsilon}.$$

Remark 5. 1. It follows from Remark 3.2 that the extension operators $P_\epsilon \in \mathcal{L}(L_\epsilon^{\frac{1}{2}}, H^1(\Omega)) \cap \mathcal{L}(L_\epsilon, L^2(\Omega))$ are uniform bounded in ϵ. Therefore we obtain by interpolation

$$\sup_{0\leqslant\epsilon\leqslant 1} \|P_\epsilon\|_{\mathcal{L}(L_\epsilon^\alpha, H^{2\alpha}(\Omega))} < \infty, \quad 0 \leqslant \alpha \leqslant \frac{1}{2}.$$

In the previous section we have passed to limit in the variational formulation of the problem (1) as $\epsilon \to 0$ getting the limit equation (41). Here, we apply the concept of *compact convergence* to obtain convergence properties of the eigenvalues and eigenfunctions when the oscillating domain degenerates to a line segment. See [1], [4] and [8] for others examples.

To this end, let $E_\epsilon : L_0 \mapsto L_\epsilon$ be a family of linear continuous operator given by

$$(E_\epsilon u)(x_1, x_2) = u(x_1) \text{ on } \Omega^\epsilon \tag{3}$$

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for $u \in L_0$. Since

$$\begin{aligned}\|E_\epsilon u\|^2_{L_\epsilon} &= \int_{\Omega^\epsilon} u^2(x_1)\, dx_1 dx_2 \\ &= \int_0^1 \int_0^{g(x_1/\epsilon)} u^2(x_1)\, dx_2 dx_1 \\ &= \int_0^1 g(x_1/\epsilon) u^2(x_1)\, dx_1 \\ &\overset{\epsilon\to 0}{\to} \left(\frac{1}{L}\int_0^L g(s) ds\right)\left(\int_0^1 u^2(x_1)\, dx_1\right) = \|u\|^2_{L_0},\end{aligned}$$

we have that

$$\|E_\epsilon u\|_{L_\epsilon} \to \|u\|_{L_0} \text{ as } \epsilon \to 0.$$

Similarly, we can consider $E_\epsilon : L^1_0 \to L^1_\epsilon$. Considering in L^1_0 the equivalent norm $\|u\|_{L^1_0} = \| - q_0 u_{x_1x_1} + u\|_{L_0}$, we obtain that

$$\|E_\epsilon u\|_{L^1_\epsilon} \to \|u\|_{L^1_0}. \tag{4}$$

Also

$$\sup_{0\leqslant\epsilon\leqslant 1} \{\|E_\epsilon\|_{\mathcal{L}(L_0,L_\epsilon)}, \|E_\epsilon\|_{\mathcal{L}(L^1_0,L^1_\epsilon)}\} < \infty. \tag{5}$$

Also, we can obtain by interpolation that

$$C = C(\alpha) = \sup_\epsilon \|E_\epsilon\|_{\mathcal{L}(L^\alpha_0,L^\alpha_\epsilon)} < \infty \text{ for } 0 \leqslant \alpha \leqslant 1.$$

For more details see [24].

Let us consider the following concepts of convergence, compactness and compact convergence of operators associated to the family of operators $\{E_\epsilon\}_{\epsilon>0}$.

Definition 5.1. We say that a sequence of elements $\{u^\epsilon\}_{\epsilon>0}$ with $u^\epsilon \in L_\epsilon$ is *E-convergent* to $u \in L_0$ if $\|u^\epsilon - E_\epsilon u^\epsilon\|_{L_\epsilon} \to 0$ as $\epsilon \to 0$. We write $u^\epsilon \overset{E}{\to} u$.

Definition 5.2. A sequence $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \in L_{\epsilon_n}$ is said to be *E-precompact* if for any subsequence $\{u_{n'}\}$ there exist a subsequence $\{u_{n''}\}$ and $u \in L_0$ such that $u_{n''} \overset{E}{\to} u$ as $n'' \to \infty$. A family $\{u^\epsilon\}_{\epsilon>0}$ is said *pre-compact* if each sequence $\{u_{\epsilon_n}\}$, with $\epsilon_n \to 0$, is pre-compact.

Definition 5.3. We say that a family of operators $\{B_\epsilon \in \mathcal{L}(L_\epsilon) \,|\, \epsilon > 0\}$ *E-converges* to $B \in \mathcal{L}(L_0)$ as $\epsilon \to 0$, if $B_\epsilon f^\epsilon \overset{E}{\to} Bf$ whenever $f^\epsilon \overset{E}{\to} f \in L_0$. We write $B_\epsilon \overset{EE}{\to} B$.

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DEFINITION 5.4. We say that a family of compact operators $\{B_\epsilon \in \mathcal{L}(L_\epsilon) \,|\, \epsilon > 0\}$ converges compactly to a compact operator $B \in \mathcal{L}(L_0)$ if for any family $\{f^\epsilon\}_{\epsilon>0}$ with $\|f^\epsilon\|_{L_\epsilon} \leq 1$, we have that the family $\{B_\epsilon f^\epsilon\}$ is E-precompact and $B_\epsilon \stackrel{EE}{\to} B$. We write $B_\epsilon \stackrel{CC}{\to} B$.

The following corollary is a key result to prove that the spectrum of A_ϵ approaches the spectrum of A_0 as ϵ goes to zero.

COROLLARY 5.1. *The family of compact operators* $\{A_\epsilon^{-1} \in \mathcal{L}(L_\epsilon)\}_{\epsilon>0}$ *converges compactly to the compact operator* $A_0^{-1} \in \mathcal{L}(L_0)$ *as* $\epsilon \to 0$.

Proof. Take $\{f^\epsilon\}_{\epsilon>0}$ with $\|f^\epsilon\|_{L_\epsilon} \leq 1$ and define $u^\epsilon = A_\epsilon^{-1} f^\epsilon$. Thus $A_\epsilon u^\epsilon = f^\epsilon$ and u^ϵ satisfies the problem (1). Hence, it follows from Theorem 4.1 that there exist $f_0 \in L_0$ and $u_0 \in H^1(\Omega)$ such that $P_\epsilon u^\epsilon \rightharpoonup u_0$ in $H^1(\Omega)$ with $u_0(x_1, x_2) = u_0(x_1)$ a.e. Ω satisfying $A_0 u_0 = f_0$, where P_ϵ is the extension operator given by Lemma 3.1 . So, we can conclude from the inequality

$$\|u^\epsilon - E_\epsilon u_0\|_{L_\epsilon} = \|\,(P_\epsilon u^\epsilon - u_0)\,|_{\Omega^\epsilon}\|_{L_\epsilon} \leq \|P_\epsilon u^\epsilon - u_0\|_{L^2(\Omega)}$$

that $u^\epsilon \stackrel{E}{\to} u_0$ proving that the family $\{u^\epsilon = A_\epsilon^{-1} f^\epsilon\}_{\epsilon>0}$ is E-precompact.

Now, let us show that $A_\epsilon^{-1} \stackrel{EE}{\to} A_0^{-1}$. To do this, we will assume

$$f^\epsilon \stackrel{E}{\to} f_0 \tag{6}$$

Therefore there exists a positive constant C, independent of ϵ, such that $\|\widetilde{f}_\epsilon\|_{L^2(\Omega)} \leq C$. Following (21) and (22) we conclude by Theorem 4.1 that $A_\epsilon^{-1} f^\epsilon \to A_0^{-1} f_0$, which proves the result. ▮

Let us consider the operator $M_\epsilon : L^p(\Omega^\epsilon) \mapsto L^p(0,1)$, $1 \leq p \leq \infty$, given by

$$(M_\epsilon f^\epsilon)(x_1) = \frac{L}{|Y^*|} \int_0^{g(x_1/\epsilon)} f^\epsilon(x_1, x_2)\, dx_2 \quad x_1 \in (0,1). \tag{7}$$

It is easy to see, from Fubini-Tonelli Theorem and Holder Inequality, that M_ϵ is a well-defined bounded linear operator with $\|M_\epsilon f^\epsilon\|_{L^p(0,1)} \leq \frac{L}{|Y^*|}\, g_1^{1-1/p}\, \|f^\epsilon\|_{L^p(\Omega^\epsilon)}$. In particular we have that

$$\|M_\epsilon f^\epsilon\|_{L_0} \leq \frac{L}{|Y^*|}\, \sqrt{\mu_g\, g_1}\, \|f^\epsilon\|_{L_\epsilon}.$$

A similar operator was considered in [2, 3]. Observe that M_ϵ is a multiple of operator $\hat{f}$ defined by (21).

In this setting we point out to the Theorem 4.1 to prove:

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LEMMA 5.1. *Let $\{f^\epsilon\} \subset L_\epsilon$ a sequence and suppose $\|f^\epsilon\|_{L_\epsilon} \leqslant C$, for some C independent of ϵ. Then there exists a subsequence such that*

$$\|A_\epsilon^{-1} f^\epsilon - E_\epsilon A_0^{-1} M_\epsilon f^\epsilon\|_{L_\epsilon} \to 0 \text{ as } \epsilon \to 0.$$

Proof. We have $M_\epsilon f^\epsilon \rightharpoonup f_0$ w-$L^2(0,1)$, where f_0 is like in the Theorem 4.1. Again from Theorem 4.1, $P_\epsilon A_\epsilon^{-1} f^\epsilon \to A_0^{-1} f_0$ s-$L^2(\Omega)$. The continuity of A_0^{-1} implies the result. ▌

Now, we estimate the difference of the resolvent operators in the following way.

COROLLARY 5.2. *There exist $\epsilon_0 > 0$ and a function $\nu : (0, \epsilon_0) \mapsto \mathbb{R}^+$ with $\nu(\epsilon) \to 0$ as $\epsilon \to 0$ such that*

$$\|A_\epsilon^{-1} - E_\epsilon A_0^{-1} M_\epsilon\|_{\mathcal{L}(L_\epsilon)} \leq \nu(\epsilon) \quad \forall \epsilon \in (0, \epsilon_0).$$

Proof. We prove this result by contradiction. For this, suppose there exist a $\delta > 0$ and sequences $\{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $\{f^n\}_{n\in\mathbb{N}} \subset L_{\epsilon_n}$ with $\|f^n\|_{L_{\epsilon_n}} = 1$ with $\epsilon_n \to 0$ and

$$\|A_{\epsilon_n}^{-1} f^n - E_{\epsilon_n} A_0^{-1} M_{\epsilon_n} f^n\|_{L_{\epsilon_n}} \geqslant \delta$$

as $n \to \infty$. From the Lemma 5.1 there exists a subsequence satisfying that $\|A_{\epsilon_n}^{-1} f^n - E_{\epsilon_n} A_0^{-1} M_{\epsilon_n} f^n\|_{L_{\epsilon_n}} \stackrel{n\to\infty}{\longrightarrow} 0$ which give us a contradiction and completes the proof. ▌

Due to Corollary 5.1, we have that the operators A_ϵ satisfy the following hypotheses

(**H1**) A_ϵ is closed, has compact resolvent, $0 \in \rho(A_\epsilon)$, $\epsilon \in [0,1]$ and $A_\epsilon^{-1} \stackrel{CC}{\to} A_0^{-1}$.

We already know that the spectrum of A_ϵ or A_0 consists of isolated eigenvalues only. Let us consider an isolated point $\lambda_0 \in \sigma(A_0)$ and associate its generalized eigenspace $W(\lambda_0, A_0) = Q(\lambda, A_0)L_0$, where

$$Q(\lambda_0, A_0) = \frac{1}{2\pi i} \int_{S_\delta} (\xi I - A_0)^{-1} d\xi,$$

$S_\delta = \{\xi \in \mathbb{C} \,|\, |\xi - \lambda_0| = \delta\}$ and δ is chosen so small that there is not other point $\sigma(A_0)$ in the disc $\{\xi \in \mathbb{C} \,|\, |\xi - \lambda_0| \leq \delta\}$. By hypotheses (**H1**) and [2, Lemma 4.9] we have there exists $\epsilon_0 > 0$ such that $\rho(A_\epsilon) \supset S_\delta$ for all $\epsilon \in (0, \epsilon_0)$. Thus, we can denote by $W(\lambda_0, A_\epsilon) = Q(\lambda_0, A_\epsilon)L_\epsilon$ where

$$Q(\lambda_0, A_\epsilon) = \frac{1}{2\pi i} \int_{S_\delta} (\xi I - A_\epsilon)^{-1} d\xi.$$

Consequently, we obtain from [2, Lemma 4.10] the next result about the spectrum convergence of the operator A_ϵ.

THEOREM 5.1. *Let A_ϵ and A_0 as above. Then, the following statements holds:*

(i)For any $\lambda_0 \in \sigma(A_0)$, there is a sequence $\lambda_\epsilon \in \sigma(A_\epsilon)$, such that $\lambda_\epsilon \to \lambda_0$ as $\epsilon \to 0$.
(ii)If $\lambda_\epsilon \to \lambda_0$, with $\lambda_\epsilon \in \sigma(A_\epsilon)$, then $\lambda_0 \in \sigma(A_0)$.
(iii)There is $\epsilon_0 > 0$ such that $\dim W(\lambda_0, A_\epsilon) = \dim W(\lambda_0, A_0)$ for all $0 < \epsilon \leqslant \epsilon_0$.
(iv)For any $u \in W(\lambda_0, A_0)$, there is a sequence $u^\epsilon \in W(\lambda_0, A_\epsilon)$, such that $u^\epsilon \xrightarrow{E} u$.
(v)If $u^\epsilon \in W(\lambda_0, A_\epsilon)$ satisfies $\|u^\epsilon\|_{L_\epsilon} = 1$, then $\{u^\epsilon\}$ has an E-convergent subsequence and any limit point of this sequence belongs to $W(\lambda_0, A_0)$.

6. CONVERGENCE OF LINEAR SEMIGROUPS

In this section we study the convergence properties of the linear semigroups generated by the operators A_ϵ and A_0 considered in the last section.

Using standard arguments, we can see that there exists $\epsilon_0 > 0$ such that the numerical range of the operators $-A_\epsilon$ are contained in $(-\infty, -1] \subset \mathbb{C}$ for all $\epsilon \in (0, \epsilon_0)$. So, it follows from [20, Theorem 3.9 p.12] that there exists constants M and $\frac{\pi}{2} < \phi < \pi$, independent of ϵ, such that

$$\| (\mu + A_\epsilon)^{-1} \|_{\mathcal{L}(L_\epsilon)} \leqslant \frac{\mathrm{M}}{|\mu + 1|} \quad \forall \mu \in \Sigma_{-1,\phi} \tag{1}$$

where $\Sigma_{-1,\phi} = \{\mu \in \mathbb{C} : 0 < |\arg(\mu + 1)| < \phi\}$. Here, we are replacing L_ϵ by L_0 as $\epsilon = 0$. Consequently the operators A_ϵ, $\epsilon \in [0, \epsilon_0]$, are sectorial operators with uniform estimates in ϵ of the resolvent $(\mu - A_\epsilon)^{-1}$ on the sector $\Sigma_{-1,\pi-\phi}$.

We recall that if $\lambda \in \rho(A_0)$, we can choose $\epsilon_0 > 0$ such that $\lambda \in \rho(A_\epsilon)$ for all $0 \leqslant \epsilon < \epsilon_0$. So, we can apply [4, Lemma 3.5] with $V_\epsilon = \lambda$ to obtain the relationship

$$\begin{aligned}(\lambda - A_\epsilon)^{-1} &- E_\epsilon(\lambda - A_0)^{-1} M_\epsilon = \\ &= [I - \lambda(\lambda - A_\epsilon)^{-1}][E_\epsilon A_0^{-1} M_\epsilon - A_\epsilon^{-1}][I - \lambda E_\epsilon(\lambda - A_0)^{-1} M_\epsilon].\end{aligned}$$

It follows from (1) that

$$\begin{aligned}\|I - \lambda(\lambda - A_\epsilon)^{-1}\|_{\mathcal{L}(L_\epsilon)} &\leq 1 + M \\ \|I - \lambda E_\epsilon(\lambda - A_0)^{-1} M_\epsilon\|_{\mathcal{L}(L_\epsilon)} &\leq 1 + \|E_\epsilon\| \, \|M_\epsilon\| \, M.\end{aligned}$$

Consequently, due to Corollary 5.2, we have there exists a function $\nu : (0, \epsilon_0) \to R^+$, $\nu(\epsilon) \to 0$ as $\epsilon \to 0$, such that

$$\|(\lambda - A_\epsilon)^{-1} - E_\epsilon(\lambda - A_0)^{-1} M_\epsilon\|_{\mathcal{L}(L_\epsilon)} \leqslant \nu(\epsilon). \tag{2}$$

Also if $\{\mathrm{e}^{-A_\epsilon t} : t \geqslant 0\}$ denote the exponentially decaying analytic semigroup in L_ϵ generated by A_ϵ, then we obtain from [18, Theorem 1.3.4] that for any $0 < \omega < 1$, there exists a constant $\rho = \rho(\omega)$, independent of ϵ, such that

$$\|e^{-A_\epsilon t}\|_{\mathcal{L}(L_\epsilon, L_\epsilon^\alpha)} \leqslant \rho\, t^{-\alpha}\, e^{-\omega t} \text{ for all } t > 0,\ 0 \leqslant \alpha \leqslant 1 \text{ and } 0 \leqslant \epsilon \leqslant \epsilon_0. \tag{3}$$

The continuity of resolvent operators allow us to obtain the continuity of linear semigroups.

Theorem 6.1. *Suppose* $0 \leqslant \alpha < \frac{1}{2}$. *Then there exists a function* $\nu_\alpha : (0, \epsilon_0] \to \mathbb{R}^+$, $\nu_\alpha(\epsilon) \overset{\epsilon \to 0}{\longrightarrow} 0$, *such that*

$$\|e^{-A_\epsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon\|_{\mathcal{L}(L_\epsilon, L_\epsilon^\alpha)} \leqslant \nu_\alpha(\epsilon) e^{-\omega t} t^{\alpha - 1} \text{ for all } t > 0.$$

Consequently there exists a constant K *independent of* ϵ *such that*

$$\|P_\epsilon e^{-A_\epsilon t} - e^{-A_0 t} M_\epsilon\|_{\mathcal{L}(L^2(\Omega^\epsilon), H^{2\alpha}(\Omega))} \leq K \nu_\alpha(\epsilon) e^{-\omega t} t^{\alpha - 1} \text{ for all } t > 0.$$

Proof. Since the operators A_ϵ are sectorial, we have for $\bar{\omega} < 1$,

$$e^{(-A_\epsilon + \bar{\omega} I)t} = \frac{1}{2\pi i} \int_{\tilde{\Gamma}} e^{(\mu + \bar{\omega})t} (\mu + \bar{\omega} + A_\epsilon - \bar{\omega})^{-1} \mathrm{d}\mu$$

where $\tilde{\Gamma}$ is the boundary of $\Sigma_{-1,\phi} = \{\mu \in \mathbb{C} : |\arg(\mu + 1)| \leq \phi\}$ with $\frac{\pi}{2} > \phi > \pi$ oriented in such a way that the imaginary part of μ increases as μ runs $\tilde{\Gamma}$.

We change variables $\mu + \bar{\omega} \mapsto \mu$ and we denote $B_\epsilon := A_\epsilon - \bar{\omega}$. Our aim is to estimate the difference

$$2\pi \|e^{-B_\epsilon t} u^\epsilon - E_\epsilon e^{-B_0 t} M_\epsilon u^\epsilon\|_{L_\epsilon^\alpha} = \left\| \int_{\Gamma_0} e^{\mu t} [(\mu + B_\epsilon)^{-1} u^\epsilon - E_\epsilon (\mu + B_0)^{-1} M_\epsilon u^\epsilon] \mathrm{d}\mu \right\|_{L_\epsilon^\alpha} \tag{4}$$

where Γ_0 is the boundary of $\Sigma_{0,\phi}$.

It follows from (1) that $\|(\mu + B_\epsilon)^{-1}\|_{\mathcal{L}(L_\epsilon)} \leq \dfrac{M}{|\mu|}$ for all $\mu \in \Gamma_0$ and $\epsilon \in [0, \epsilon_0]$. Therefore

$$\begin{aligned} \|(\mu + B_\epsilon)^{-1} u^\epsilon - E_\epsilon (\mu + B_0)^{-1} M_\epsilon u^\epsilon\|_{L_\epsilon} &\leq \frac{M + \|E_\epsilon\| \, \|M_\epsilon\|}{|\mu|} \|u\|_{L_\epsilon} \\ &\leq \frac{M_1}{|\mu|} \|u\|_{L_\epsilon}. \end{aligned} \tag{5}$$

Also we have

$$\begin{aligned} \|B_\epsilon (\mu + B_\epsilon)^{-1} u^\epsilon\|_{L_\epsilon} &= \|(I - \mu(\mu + B_\epsilon)^{-1}) u^\epsilon\|_{L_\epsilon} \\ &\leq \|u^\epsilon\|_{L_\epsilon} + |\mu| \|(\mu + B_\epsilon)^{-1} u^\epsilon\|_{L_\epsilon} \\ &\leq (1 + M) \|u^\epsilon\|_{L_\epsilon}. \end{aligned}$$

Consequently, we obtain from moment's inequality

$$\begin{aligned}\|(\mu+B_\epsilon)^{-1}u^\epsilon\|_{L_\epsilon^{\frac{1}{2}}} &\leq \|(\mu+B_\epsilon)^{-1}u^\epsilon\|_{L_\epsilon}^{\frac{1}{2}}\,\|(\mu+B_\epsilon)^{-1}u^\epsilon\|_{L_\epsilon^1}^{\frac{1}{2}}\\ &\leq \frac{M^{\frac{1}{2}}}{|\mu|^{\frac{1}{2}}}(1+M)^{\frac{1}{2}}\|u^\epsilon\|_{L_\epsilon}.\end{aligned}$$

Let us observe that for $u^\epsilon \in L_\epsilon$

$$\begin{aligned}\|B_\epsilon^{\frac{1}{2}}E_\epsilon(\mu+B_0^{-1})M_\epsilon u^\epsilon\|_{L_\epsilon} &= \|E_\epsilon(\mu+B_0^{-1})M_\epsilon u^\epsilon\|_{L_\epsilon^{\frac{1}{2}}}\\ &\leq g_1^{1/2}\|(\mu+B_0^{-1})M_\epsilon u^\epsilon\|_{H^1(0,1)}\\ &= g_1^{1/2}\|B_0^{\frac{1}{2}}(\mu+B_0^{-1})M_\epsilon u^\epsilon\|_{L_0}\\ &\leq g_1^{1/2}\frac{M^{1/2}}{|\mu|^{1/2}}\|M_\epsilon\|\,\|u^\epsilon\|_{L_\epsilon}.\end{aligned}$$

So we can conclude that

$$\|(\mu+B_\epsilon)^{-1}u^\epsilon - E_\epsilon(\mu+B_0)^{-1}M_\epsilon u^\epsilon\|_{L_\epsilon^{\frac{1}{2}}} \leq \frac{M_2}{|\mu|^{\frac{1}{2}}}\|u^\epsilon\|_{L_\epsilon}. \tag{6}$$

Let us call $x=(\mu+B_\epsilon)^{-1}u^\epsilon - E_\epsilon(\mu+B_0)^{-1}M_\epsilon u^\epsilon$. Using the moment's inequality again, we obtain

$$\|x\|_{L_\epsilon^\alpha} \leq M_3\|x\|_{L_\epsilon^{\frac{1}{2}}}^{2\alpha}\,\|x\|_{L_\epsilon}^{1-2\alpha}. \tag{7}$$

Hence, due to estimates (6), (4) and (2) we get

$$\|(\mu+B_\epsilon)^{-1} - E_\epsilon(\mu+B_0)^{-1}M_\epsilon\|_{\mathcal{L}(L_\epsilon,L_\epsilon^\alpha)} \leqslant \frac{M_4\,\nu(\epsilon)^{(1-2\alpha)}}{|\mu|^\alpha}. \tag{8}$$

With the change of variables $\beta=\mu t$ in (4) we have the equivalent integral

$$\left\|\int_{\Gamma_0} e^\beta\left[\left(\beta t^{-1}+B_\epsilon\right)^{-1}E_\epsilon u - E_\epsilon\left(\beta t^{-1}+B_0\right)^{-1}u\right]\frac{d\beta}{t}\right\|_{L_\epsilon^\alpha}.$$

Now, it follows from (8) that

$$\|\left(\beta t^{-1}+B_\epsilon\right)^{-1}u^\epsilon - E_\epsilon\left(\beta t^{-1}+B_0\right)^{-1}M_\epsilon u^\epsilon\|_{L_\epsilon^\alpha} \leq \frac{M_4\,\nu(\epsilon)^{(1-2\alpha)}}{|\beta t^{-1}|^\alpha}\|u^\epsilon\|_{L_\epsilon}.$$

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Thus

$$\Big\| t^{-1} \int_{\Gamma_0} e^{\beta} \big[\left(\beta t^{-1} + B_\epsilon\right)^{-1} u^\epsilon - E_\epsilon \left(\beta t^{-1} + B_0\right)^{-1} M_\epsilon u^\epsilon \big] \mathrm{d}\beta \Big\|_{L_\epsilon^\alpha}$$
$$\leq M_4 t^{\alpha-1} \nu(\epsilon)^{(1-2\alpha)} \|u^\epsilon\|_{L_\epsilon} \int_{\Gamma_0} \frac{|e^\beta|}{|\beta|^\alpha} \mathrm{d}|\beta|.$$

That is,

$$\big\| e^{-B_\epsilon t} u^\epsilon - E_\epsilon e^{-B_0 t} M_\epsilon u^\epsilon \big\|_{L_\epsilon^\alpha} \leq M_5 t^{\alpha-1} \nu(\epsilon)^{(1-2\alpha)} \|u^\epsilon\|_{L_\epsilon} \text{ for } \; t > 0.$$

Therefore, for any $\alpha \in [0, \frac{1}{2})$ and $\omega \in (0,1)$, there exists a function denoted by $\nu_\alpha : (0, \epsilon_0] \to \mathbb{R}^+$ with $\nu_\alpha(\epsilon) \stackrel{\epsilon \to 0}{\longrightarrow} 0$, such that

$$\|e^{-A_\epsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon\|_{\mathcal{L}(L_\epsilon, L_\epsilon^\alpha)} \leq \nu_\alpha(\epsilon) e^{-\omega t} t^{\alpha - 1} \text{ for all } t > 0.$$

To conclude the proof, observe that Remark 5.1, implies the existence of a constant K such that

$$\begin{aligned} \|P_\epsilon e^{-A_\epsilon t} - e^{-A_0 t} M_\epsilon \;\|_{\mathcal{L}(L_\epsilon, H^{2\alpha}(\Omega))} &= \|P_\epsilon e^{-A_\epsilon t} - P_\epsilon E_\epsilon e^{-A_0 t} M_\epsilon\|_{\mathcal{L}(L_\epsilon, H^{2\alpha}(\Omega))} \\ &\leqslant \|P_\epsilon\|_{\mathcal{L}(L_\epsilon^\alpha, H^{2\alpha}(\Omega))} \|e^{-A_\epsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon\|_{\mathcal{L}(L_\epsilon, L_\epsilon^\alpha)} \\ &\leqslant K \|e^{-A_\epsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon\|_{\mathcal{L}(L_\epsilon, L_\epsilon^\alpha)}. \end{aligned} \tag{9}$$

▮

Corollary 6.1. *Suppose* $0 \leqslant \alpha < \frac{1}{2}$ *and* $u^\epsilon \stackrel{E}{\longrightarrow} u$. *Then there is a function* $\nu : (0, \epsilon_0] \to \mathbb{R}^+,\ \nu(\epsilon) \stackrel{\epsilon \to 0}{\longrightarrow} 0$, *such that*

$$\big\| e^{-A_\epsilon t} u^\epsilon - E_\epsilon e^{-A_0 t} u \big\|_{L_\epsilon^\alpha} \leq \nu(\epsilon) e^{-\omega t} t^{-\alpha} \textit{ for all } t > 0. \tag{10}$$

Proof. Adding an appropriate term we have

$$\|e^{-A_\epsilon t} u^\epsilon - E_\epsilon e^{-A_0 t} u\|_{L_\epsilon^\alpha} \leqslant \|e^{-A_\epsilon t} u^\epsilon - E_\epsilon e^{-A_0 t} M_\epsilon u^\epsilon\|_{L_\epsilon^\alpha} + \|E_\epsilon e^{-A_0 t} \left(M_\epsilon u^\epsilon - u\right)\|_{L_\epsilon^\alpha}.$$

Let us observe that $M_\epsilon u^\epsilon - u = M_\epsilon \left(u^\epsilon - E_\epsilon u\right)$. Hence, we have

$$\|M_\epsilon u^\epsilon - u\|_{L_0} \leq \sqrt{\frac{\mu_g}{g_0}} \|u^\epsilon - E_\epsilon u\|_{L_\epsilon}.$$

Consequently, the result follows from Theorem 6.1 and $u^\epsilon \stackrel{E}{\longrightarrow} u$. ▮

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7. UPPER SEMICONTINUITY OF ATTRACTORS

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a $\mathcal{C}^2$-function with bounded derivatives satisfying the dissipative property (1.3) and let Ω^ϵ be the perturbed domain defined in (2).

In the previous section, we have studied the asymptotic behavior of the linear parts of the problem

$$\begin{cases} u^\epsilon_t - \dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f(u^\epsilon) \quad \text{in } \Omega^\epsilon \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0 \\ \dfrac{\partial u^\epsilon}{\partial x_1} N^\epsilon_1 + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N^\epsilon_2 = 0 \quad \text{on } \partial\Omega^\epsilon \end{cases} \tag{1}$$

under the degenerated perturbation Ω^ϵ and we have proved a result on the continuity of the linear semigroups, Theorem 6.1 and Corollary 6.1. In this section, we show that the attractors $\mathcal{A}_\epsilon$ of the nonlinear elliptic problem are upper semicontinuous with respect to this perturbations. To this end, we relate the continuity of the linear semigroups with the continuity of the nonlinear semigroups for dissipative parabolic equations by using the variation of constants formula. This in turn will imply the upper semicontinuity of the attractors. See also [1], [4], [8] and [19] for other examples that use a similar technique.

THEOREM 7.1. *Suppose $0 \leqslant \alpha < \frac{1}{2}$, and let $u^\epsilon \in L_\epsilon$ satisfying*

$$\|u^\epsilon\|_{L_\epsilon} \leq C \tag{2}$$

for some positive constant C independent of ϵ.

Then, for each $\tau > 0$, there exists a function $\bar{\nu}_\alpha : (0, \epsilon_0] \to \mathbb{R}^+$, $\bar{\nu}_\alpha(\epsilon) \overset{\epsilon\to 0}{\longrightarrow} 0$, such that

$$\|T_\epsilon(t)u^\epsilon - E_\epsilon T_0(t) M_\epsilon u^\epsilon\|_{L^\alpha_\epsilon} \leq \bar{\nu}_\alpha(\epsilon) e^{-\omega t} t^{\alpha-1} \tag{3}$$

for all $t \in (0, \tau)$.

Moreover, the family of attractors $\{\mathcal{A}_\epsilon : \epsilon \in [0, \epsilon_0]\}$ of the problem (1) is upper semicontinuous at $\epsilon = 0$ in L^α_ϵ for $0 \leq \alpha < 1/2$, in the sense that

$$\sup_{\varphi^\epsilon \in \mathcal{A}_\epsilon} \Big[\inf_{\varphi \in \mathcal{A}_0} \big\{ \|\varphi^\epsilon - E_\epsilon \varphi\|_{L^\alpha_\epsilon} \big\} \Big] \to 0, \ \text{as } \epsilon \to 0. \tag{4}$$

Consequently there exists a constant K independent of ϵ such that

$$\|P_\epsilon T_\epsilon(t) u^\epsilon - T_0(t) M_\epsilon u^\epsilon\|_{H^{2\alpha}(\Omega)} \leq K \bar{\nu}_\alpha(\epsilon) e^{-\omega t} t^{\alpha-1} \tag{5}$$

for all $t \in (0, \tau)$ and all $0 \leqslant \alpha < \frac{1}{2}$. Also,

$$\sup_{\varphi^\epsilon \in \mathcal{A}_\epsilon} \Big[\inf_{\varphi \in \mathcal{A}_0} \big\{ \|P_\epsilon \varphi^\epsilon - \varphi\|_{H^{2\alpha}(\Omega)} \big\} \Big] \to 0, \ \text{as } \epsilon \to 0. \tag{6}$$

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Proof. We use the variation of constants formula on the integral formulation

$$T_\epsilon(t)u^\epsilon = e^{-A_\epsilon t}u^\epsilon + \int_0^t e^{-A_\epsilon(t-s)} f(T_\epsilon(s)u^\epsilon)\, ds \text{ for all } \epsilon \in [0,1].$$

Consequently

$$\begin{aligned} \|T_\epsilon(t)u_\epsilon - E_\epsilon T_0(t)M_\epsilon u^\epsilon\|_{L^\alpha_\epsilon} &\leqslant \|e^{-A_\epsilon t}u^\epsilon - E_\epsilon e^{-A_0 t}M_\epsilon u^\epsilon\|_{L^\alpha_\epsilon} \\ &+ \int_0^t \|e^{-A_\epsilon(t-s)} f(T_\epsilon(s)u^\epsilon) - E_\epsilon e^{-A_0(t-s)} f(T_0(s)M_\epsilon u^\epsilon)\|_{L^\alpha_\epsilon} ds. \end{aligned}$$

It follows from (10) that there exist $\epsilon_0 > 0$ and $\nu : (0, \epsilon_0] \mapsto \mathbb{R}^+$, $\nu \overset{\epsilon \to 0}{\to} 0$, such that

$$\|e^{-A_\epsilon t} - E_\epsilon e^{-A_0 t} M_\epsilon\|_{\mathcal{L}(L_\epsilon, L^\alpha_\epsilon)} \leq \nu(\epsilon) e^{-\omega t} t^{\alpha - 1} \text{ for } t > 0.$$

Furthermore, we have

$$\begin{aligned} &\int_0^t \|e^{-A_\epsilon(t-s)} f(T_\epsilon(s)u^\epsilon) - E_\epsilon e^{-A_0(t-s)} f(T_0(s)M_\epsilon u^\epsilon)\|_{L^\alpha_\epsilon} ds \\ &\quad\leqslant \int_0^t \| \left(e^{-A_\epsilon(t-s)} - E_\epsilon e^{-A_0(t-s)} M_\epsilon\right) f(T_\epsilon(s)u^\epsilon)\|_{L^\alpha_\epsilon} ds \\ &\quad+ \int_0^t \|E_\epsilon e^{-A_0(t-s)} \Big(M_\epsilon f(T_\epsilon(s)u^\epsilon) - f(T_0(s)M_\epsilon u^\epsilon)\Big)\|_{L^\alpha_\epsilon} ds. \end{aligned}$$

Since u^ϵ satisfies (2) for $\epsilon > 0$, we have that $\{f(T_\epsilon(s)u^\epsilon) \in L_\epsilon : s \in [0,t]\}$ is uniformly bounded. Hence, we obtain by Theorem 6.1 that there exists a constant $M = M(\tau, C)$ such that

$$\begin{aligned} &\int_0^t \|\Big(e^{-A_\epsilon(t-s)} - E_\epsilon e^{-A_0(t-s)} M_\epsilon\Big) f(T_\epsilon(s)u^\epsilon)\|_{L^\alpha_\epsilon} ds \\ &\qquad\qquad \leqslant \int_0^t \nu_\alpha(\epsilon) e^{-\omega(t-s)} (t-s)^{\alpha-1} \|f(T_\epsilon(s)u^\epsilon)\|_{L_\epsilon} ds \\ &\qquad\qquad \leqslant M\nu_\alpha(\epsilon) t^{\alpha-1} \text{ for all } t \in (0,\tau). \end{aligned}$$

Also, if L denotes the uniform Lipschitz constant of the Niemitskii operator defined by $\mathcal{C}^2$-function f, we use $E_\epsilon f = f E_\epsilon$ and $M_\epsilon E_\epsilon = I$ to get

$$\begin{aligned}
&\int_0^t \|E_\epsilon e^{-A_\epsilon(t-s)}\Big(M_\epsilon f(T_\epsilon(s)u^\epsilon) - f(T_0(s)M_\epsilon u^\epsilon)\Big)\|_{L^\alpha_\epsilon} ds \\
&= \int_0^t \|E_\epsilon e^{-A_\epsilon(t-s)}\Big(M_\epsilon f(T_\epsilon(s)u^\epsilon) - M_\epsilon E_\epsilon f(T_0(s)M_\epsilon u^\epsilon)\Big)\|_{L^\alpha_\epsilon} ds \\
&= \int_0^t \|E_\epsilon e^{-A_\epsilon(t-s)} M_\epsilon \Big(f(T_\epsilon(s)u^\epsilon) - f(E_\epsilon T_0(s)M_\epsilon u^\epsilon)\Big)\|_{L^\alpha_\epsilon} ds \\
&\leq \int_0^t \rho \,\|E_\epsilon\| \,\|M_\epsilon\| \, e^{-w(t-s)}(t-s)^{-\alpha} L \|T_\epsilon(s)u^\epsilon - E_\epsilon T_0(s)M_\epsilon u^\epsilon\|_{L^\alpha_\epsilon}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
e^{\omega t}\|T_\epsilon(t)u^\epsilon - E_\epsilon T_0(t)M_\epsilon u^\epsilon\|_{L^\alpha_\epsilon} &\leqslant (1+M)\nu_\alpha(\epsilon)t^{\alpha-1} \\
&\quad + \rho\,\|E_\epsilon\|\,\|M_\epsilon\|\, L \int_0^t (t-s)^{-\alpha} e^{\omega s}\|T_\epsilon(s)u^\epsilon - E_\epsilon T_0(s)M_\epsilon u^\epsilon)\|_{L^\alpha_\epsilon} ds.
\end{aligned}$$

If we put $\varphi(t) := e^{\omega t}\,\|T_\epsilon(t)u^\epsilon - E_\epsilon T_0(t)M_\epsilon u^\epsilon\|_{L^\alpha_\epsilon}$, we have

$$\varphi(t) \leqslant (1+M)\nu_\alpha(\epsilon)t^{\alpha-1} + \rho\,\|E_\epsilon\|\,\|M_\epsilon\|\, L \int_0^t (t-s)^{-\alpha}\varphi(s)\, ds \text{ on } (0,\tau).$$

Thus, due to Gronwall's Inequality from [18, Exercise 4* of the Section 7.1], we get the inequality (3) from

$$\varphi(t) \leqslant K\nu_\theta(\epsilon)t^{\alpha-1}$$

where $K = K(M, C, \rho, L, \tau, \|E_\epsilon\|, \|M_\epsilon\|)$ is a constant.

To obtain the upper semicontinuity of the attractors $\mathcal{A}_\epsilon$, we observe first that by the uniform $L^\infty(\Omega^\epsilon)$ bounds of the attractors, proved in [?, Theorem 2.6], we have

$$\bigcup_{0\leq\epsilon\leq\epsilon_0} M_\epsilon \mathcal{A}_\epsilon$$

is a bounded set in $L^\infty(0,1)$. It follows from attractivity properties of $\mathcal{A}_0$ that, for a fixed $\eta > 0$, there exists a $\tau > 0$ such that

$$\operatorname{dist}_{L^\alpha_0}(T(\tau)M_\epsilon\varphi^\epsilon, \mathcal{A}_0) \equiv \inf_{\varphi\in\mathcal{A}_0} \|T(\tau)M_\epsilon\varphi^\epsilon - \varphi\|_{L^\alpha_0} \leq \eta \quad \forall\varphi^\epsilon \in \mathcal{A}_\epsilon \text{ and } 0 \leq \epsilon \leq \epsilon_0$$

that is,

$$\operatorname{dist}_{L^\alpha_\epsilon}(E_\epsilon T(\tau)M_\epsilon\varphi^\epsilon, E_\epsilon\mathcal{A}_0) \leq \eta \quad \forall\varphi^\epsilon \in \mathcal{A}_\epsilon \text{ and } 0 \leq \epsilon \leq \epsilon_0.$$

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Hence, due to (7.1) with $t = \tau$, we have that there exists $\epsilon_1 > 0$ such that

$$\|T_\epsilon(\tau)\varphi^\epsilon - E_\epsilon T(\tau) M_\epsilon \varphi^\epsilon\|_{L^\alpha_\epsilon} \leq \eta \quad \forall \varphi^\epsilon \in \mathcal{A}_\epsilon \text{ and } 0 \leq \epsilon \leq \epsilon_1.$$

Therefore

$$\mathrm{dist}_{L^\alpha_\epsilon}(T_\epsilon(\tau)\varphi^\epsilon, E_\epsilon \mathcal{A}_0) \leq \eta \quad \forall \varphi^\epsilon \in \mathcal{A}_\epsilon \text{ and } 0 \leq \epsilon \leq \epsilon_1.$$

Since $\mathcal{A}_\epsilon$ is invariant, we get

$$\mathrm{dist}_{L^\alpha_\epsilon}(\varphi^\epsilon, E_\epsilon \mathcal{A}_0) \leq \eta \quad \forall \varphi^\epsilon \in \mathcal{A}_\epsilon \text{ and } 0 \leq \epsilon \leq \epsilon_1.$$

Now we can argue as (8) to obtain (5) and (6). ∎

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