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Exteriors of codimension one embeddings of product of three spheres into spheres

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Let $f : S^p \times S^q \times S^r \to S^{p+q+r+1}$ be a smooth embedding with $2 \leq p \leq q \leq r$. The first and the third authors have shown that if $p+q \neq r$ or $p+q=r$ is even, then the closure of one of the two components of $S^{p+q+r+1} \setminus f(S^p \times S^q \times S^r)$ is diffeomorphic to the product of two spheres and a disk, and that otherwise, there are infinitely many "exotic" embeddings f_n which do not satisfy such a property. On the other hand, Fox has shown that for every embedding of a closed connected orientable surface into S^3, the closure of each component of the complement is homeomorphic to the exterior of a handlebody embedded in S^3. In this paper,we show that for $p+q=r$ odd, the exotic embeddings f_n mentioned above do not satisfy such "Fox's property": in other words, the closure of a component of $S^{p+q+r+1} \setminus f_n(S^p \times S^q \times S^r)$ cannot be obtained as the exterior of an embedding of N for any component N of the exterior of a standard embedding. October, 2010 ICMC-USP

1. INTRODUCTION

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Let $f : S^p \times S^q \times S^r \to S^{p+q+r+1}$ be a smooth embedding with $p \leq q \leq r$. For $p \geq 2$, the first and the third authors have shown in [8] that if $p+q \neq r$ or $p+q=r$ is even, then the closure of one of the two components of $S^{p+q+r+1} \setminus f(S^p \times S^q \times S^r)$ is diffeomorphic to the product of two spheres and a disk, and that otherwise, there are infinitely many exotic embeddings f_n which do not satisfy such a property.

Let us consider the case $(p,q,r) = (1,1,1)$. It is known that for the standard embedding $S^1 \times S^1 \times S^1 \to S^4$, the closure of one of the two components of the complement is diffeomorphic to $S^1 \times S^1 \times D^2$, while the closure of the other component is the so-called Montesinos' twin, which is a regular neighborhood of the union of two embedded 2-spheres intersecting transversely at exactly two points. It has been shown in [9] that for every locally flat topological embedding $f : S^1 \times S^1 \times S^1 \to S^4$, the closures of the two components of $S^4 \setminus f(S^1 \times S^1 \times S^1)$ are homeomorphic to the exterior of $S^1 \times S^1 \times D^2$ or Montesinos' twin topologically embedded in S^4.

This result for $S^1 \times S^1 \times S^1$ can be considered as a 4-dimensional analogue of the following classical result of Fox. Let F be a closed connected orientable surface. Then for the standard smooth embedding $F \to S^3$, the closure of each of the two components of the complement is diffeomorphic to the handlebody. Fox [3] has shown that for every smooth embedding $f : F \to S^3$, the closure of each of the two components of $S^3 \setminus f(F)$ is diffeomorphic to the exterior of a handlebody embedded in S^3. In view of this result, we may say that every locally flat topological embedding of $S^1 \times S^1 \times S^1$ into S^4 has the Fox property (in the topological category).

In this paper, we first formulate the Fox property in a more general setting. In fact, such a property has already been formulated in [5] for dimension four, and we just generalize the dimension. After that, we show that the exotic embeddings $f_n : S^p \times S^q \times S^r \to S^{p+q+r+1}$ constructed in [8] for $r = p+q$ odd do NOT satisfy the Fox property, provided $n \neq 0, -1$. More precisely, let us consider a standard embedding of $S^p \times S^q \times S^r$ into $S^{p+q+r+1}$. Then, the closure N_1 of one of the two components of the complement is diffeomorphic to a product of two spheres and a disk. Let us denote by N_2 the closure of the other component. Let C_1 be the closure of the component of $S^{p+q+r+1} \setminus f_n(S^p \times S^q \times S^r)$ such that $H_*(C_1) \cong H_*(S^p \times S^r)$. Then we show that there exists no embedding of N_i into $S^{p+q+r+1}$ whose exterior is diffeomorphic to C_1 both for $i=1$ and for $i=2$. This means that the embeddings f_n, $n \neq 0, -1$, constructed in [8] are highly knotted. This answers the question posed at the end of [9].

Throughout the paper all homology and cohomology groups are with coefficients in $\mathbf{Z}$.

The symbol "$[*]$" denotes the homology class represented by $*$.

2. FOX PROPERTY

In the following, the *exterior* of an embedding (or of its image) is the closure of the complement to the regular (or tubular) neighborhood of the image of the embedding.

Let us consider a connected 1-dimensional polyhedron PL embedded in S^3. Then, considering the boundary of its regular neighborhood, we get a PL embedding of a closed connected orientable surface F into S^3. Conversely, let $f : F \to S^3$ be an arbitrary PL embedding. Then Fox [3] has shown that the closure of each of the two connected com-

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ponents of $S^3 \smallsetminus f(F)$ is PL homeomorphic to the exterior of a connected 1-dimensional polyhedron PL embedded in S^3.

Based on Fox's result above, let us consider the following situation (see [5]). Let Γ and Δ be compact connected polyhedrons. We suppose that there exist piecewise smooth 'Mstandard" embeddings

$$\varphi_0 : \Gamma \to S^n, \quad \psi_0 : \Delta \to S^n$$

such that

$$\varphi_0(\Gamma) \cap \psi_0(\Delta) = \emptyset,\ N_\Gamma \cup N_\Delta = S^n,\ N_\Gamma \cap N_\Delta = \partial N_\Gamma = \partial N_\Delta,$$

where N_Γ and N_Δ are regular neighborhoods of $\varphi_0(\Gamma)$ and $\psi_0(\Delta)$ in S^n, respectively, whose boundaries are appropriately smoothed. Set $M = \partial N_\Gamma = \partial N_\Delta$, which is assumed to be a smooth closed connected orientable $(n-1)$-dimensional manifold.

Definition 2.1. Let $f : M \to S^n$ be a smooth embedding. Note that $S^n \smallsetminus f(M)$ has exactly two components by the Alexander duality. We say that f has the *Fox property* (with respect to the decomposition $S^n = N_\Gamma \cup N_\Delta$) if there exist smooth embeddings $\varphi : N_\Gamma \to S^n$ and $\psi : N_\Delta \to S^n$ such that the closures of the two components of $S^n \smallsetminus f(M)$ are diffeomorphic to the closures of $S^n \smallsetminus \varphi(N_\Gamma)$ and $S^n \smallsetminus \psi(N_\Delta)$.

We can also define the same notion in the PL or the topological category.

Some examples of manifolds whose codimension one embeddings always have the Fox property can be found in the literature: spheres S^{n-1} [1, 2], products of two spheres $S^p \times S^q$ [1, 4, 6, 7, 10, 11], the quaternion space [5], the 3-dimensional torus [9], etc.

3. PRODUCT OF THREE SPHERES

Let $f : S^p \times S^q \times S^r \to S^{p+q+r+1}$ be the *standard embedding*: i.e., $f(S^p \times S^q \times S^r)$ is the boundary of a tubular neighborhood of a standardly embedded $S^p \times S^q$, $S^p \times S^r$ or $S^q \times S^r$ in $S^{p+q+r+1}$ (for details, see [8]). Let N_1 and N_2 be the closures of the two components of $S^{p+q+r+1} \smallsetminus f(S^p \times S^q \times S^r)$. Then one of them, say N_1, is clearly diffeomorphic to $S^p \times S^q \times D^{r+1}$, which is a tubular neighborhood of a standardly embedded $S^p \times S^q$ in $S^{p+q+r+1}$, after an appropriate permutation of p, q and r if necessary. The other one, say N_2, which is diffeomorphic to

$$(((D^{p+1} \times S^r) - \operatorname{Int} D^{p+r+1}) \times S^q) \cup (S^{p+r} \times D^{q+1}),$$

is in fact a regular neighborhood of the union of two embedded spheres S^{p+r} and S^{q+r} in $S^{p+q+r+1}$ intersecting each other transversely along S^{r-1}. (In particular, N_2 has the homotopy type of the bouquet $S^{p+r} \vee S^{q+r} \vee S^r$.)

Let us discuss the Fox property for embeddings of $S^p \times S^q \times S^r$ into $S^{p+q+r+1}$ with respect to the decomposition $S^{p+q+r+1} = N_1 \cup N_2$. According to [8], when $2 \le p \le q \le r$, if $r \ne p+q$, or if $r = p+q$ is even, then every codimension one smooth embedding of $S^p \times S^q \times S^r$ has the Fox property, since the closure of one of the two components of the complement is diffeomorphic to the product of two spheres and a disk. On the other hand, if $r = p + q$ is odd, then we have constructed a family of "exotic" embeddings $f_n : S^p \times S^q \times S^r \to S^{p+q+r+1}$, $n \in \mathbf{Z}$, $n \neq 0$, such that the closure of none of the components of $S^{p+q+r+1} \smallsetminus f_n(S^p \times S^q \times S^r)$ is diffeomorphic to a product of two spheres and a disk. Recall that if $m \neq n$, then there exists no diffeomorphism of $S^{p+q+r+1}$ which takes the image of the embedding f_n to that of f_m.

Remark 3. 1. For an embedding $f : S^p \times S^q \times S^r \to S^{p+q+r+1}$, $1 \leq p \leq q \leq r$, let us denote by C_1 and C_2 the closures of the two components of $S^{p+q+r+1} \smallsetminus f(S^p \times S^q \times S^r)$. Then, as has been observed in [8], each C_i has the homology of $S^p \times S^q$, $S^p \times S^r$ or $S^q \times S^r$.

When $r = p + q$ is odd, for the exotic embeddings f_n, C_1 and C_2 have the homologies of $S^p \times S^r$ and $S^q \times S^r$. Recall that by [8, Proposition 7.1] if $H_*(C_1) \cong H_*(S^p \times S^q)$, $p, q \geq 2$, then C_1 is diffeomorphic to $S^p \times S^q \times D^{r+1}$: consequently, such an f has the Fox property.

Remark 3. 2. In [8], exotic embeddings f_n have been constructed for $2 \leq p \leq q$ with $r = p + q$ odd. It is not difficult to observe that exactly the same construction works also for $p = 1$ and $r = q + 1$ odd.

Our main result of this paper is the following observation.

THEOREM 3.1. *Suppose $1 \leq p \leq q \leq r$, and $r = p+q$ is odd. Then the exotic embeddings $f_n : S^p \times S^q \times S^r \to S^{p+q+r+1}$, $n \neq 0, -1$, do not have the Fox property with respect to the decomposition $S^{p+q+r+1} = N_1 \cup N_2$ coming from the standard embedding of $S^p \times S^q \times S^r$ into $S^{p+q+r+1}$.*

4. DIFFEOMORPHISMS OF $S^P \times S^Q \times S^R$

For the proof of Theorem 3.1, we need a lemma concerning diffeomorphisms of $S^p \times S^q \times S^r$.

Set $M = S^p \times S^q \times S^r$ with $r = p+q$. Let us study those automorphisms of $H_r(M) \cong \mathbf{Z} \oplus \mathbf{Z}$ which are induced by diffeomorphisms of M. Set

$$\alpha_1 = [S^p \times S^q \times \{*\}], \quad \alpha_2 = [\{*\} \times \{*\} \times S^r],$$

which constitute a base of $H_r(M)$, where we choose orientations of the representative manifolds once and for all. To each diffeomorphism $\varphi : M \to M$ we associate the 2×2 integer matrix $[\varphi_*] = (a_{ij}) \in GL(2, \mathbf{Z})$ by

$$\varphi_*(\alpha_i) = \sum_{i=1}^{2} a_{ij} \alpha_i, \quad i = 1, 2.$$

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Lemma 4.1. *The matrix $[\varphi_*]$ is of the form*

$$\begin{pmatrix} \pm 1 & 0 \\ * & \pm 1 \end{pmatrix}.$$

Proof. Let $\eta_p \in H^p(M)$ and $\eta_q \in H^q(M)$ be the cohomology classes Poincaré dual to the homology classes $[\{*\} \times S^q \times S^r] \in H_{q+r}(M)$ and $[S^p \times \{*\} \times S^r] \in H_{p+r}(M)$, respectively. Note that $H^p(M)$ and $H^q(M)$ are infinite cyclic groups generated by η_p and η_q, respectively. Since $\varphi^* : H^*(M) \to H^*(M)$ is an automorphism, we have $\varphi^*(\eta_p) = \pm\eta_p$ and $\varphi^*(\eta_q) = \pm\eta_q$. In particular, we have

$$\varphi^*(\eta_p \smile \eta_q) = \pm\eta_p \smile \eta_q \in H^{p+q}(M) = H^r(M).$$

Let ξ_1 and $\xi_2 \in H^r(M)$ be the cohomology classes Poincaré dual to the homology classes $[\{*\} \times \{*\} \times S^r]$ and $[S^p \times S^q \times \{*\}] \in H_r(M)$, respectively. Note that we may assume $\eta_p \smile \eta_q = \xi_1$, since $(\{*\} \times S^q \times S^r) \cap (S^p \times \{*\} \times S^r) = \{*\} \times \{*\} \times S^r$. Note also that $\{\xi_1, \xi_2\}$ is the base of $H^r(M) \cong \mathrm{Hom}(H_r(M), \mathbf{Z})$ dual to $\{\alpha_1, \alpha_2\}$.

Then the matrix representative of the automorphism $\varphi^* : H^r(M) \to H^r(M)$ with respect to the base $\{\xi_1, \xi_2\}$ is of the form

$$\begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}.$$

Therefore, $\varphi_* : H_r(M) \to H_r(M)$ has a matrix representative, with respect to the dual base $\{\alpha_1, \alpha_2\}$, of the form

$$\begin{pmatrix} \pm 1 & 0 \\ * & \pm 1 \end{pmatrix}.$$

This completes the proof. ∎

5. PROOF OF THEOREM 3.1

Let C_1 and C_2 be the closures of the two components of $S^{p+q+r+1} \setminus f_n(S^p \times S^q \times S^r)$. Recall that by changing the indices if necessary, we may assume

$$H_*(C_1) \cong H_*(S^p \times S^r), \quad H_*(C_2) \cong H_*(S^q \times S^r).$$

Note that in the notation of [8, §9], C_1 and C_2 correspond to $\widetilde{X}_-$ and $\widetilde{X}_+$, respectively.

Let us first show that C_1 cannot be diffeomorphic to the exterior of a product of two spheres and a disk embedded in $S^{p+q+r+1}$. Suppose this was the case. Then, by an argument using the Alexander duality, we see that the manifold

$$W = C_1 \cup_\varphi (D^{p+1} \times S^q \times S^r), \tag{1}$$

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obtained by attaching $D^{p+1} \times S^q \times S^r$ to C_1 by using a diffeomorphism $\varphi : \partial(D^{p+1} \times S^q \times S^r) \to \partial C_1$, should be diffeomorphic to $S^{p+q+r+1}$.

Let us consider the Mayer–Vietoris exact sequence associated with the decomposition (1):

$$H_r(S^p \times S^q \times S^r) \xrightarrow{\Phi=(\psi_*,-j_*)} H_r(C_1) \oplus H_r(D^{p+1} \times S^q \times S^r) \longrightarrow H_r(W), \tag{2}$$

where $j : S^p \times S^q \times S^r = \partial(D^{p+1} \times S^q \times S^r) \to D^{p+1} \times S^q \times S^r$ is the inclusion map and $\psi : S^p \times S^q \times S^r \to C_1$ is given by the composition

$$\psi : S^p \times S^q \times S^r = \partial(D^{p+1} \times S^q \times S^r) \xrightarrow{\varphi} S^p \times S^q \times S^r = \partial C_1 \xrightarrow{i} C_1$$

with $i : \partial C_1 \to C_1$ being the inclusion map.

By Lemma 4.1, we have

$$\varphi_*(\alpha_1) = \pm\alpha_1 + b\alpha_2$$

for some $b \in \mathbf{Z}$. Let $\beta \in H_r(C_1) \cong \mathbf{Z}$ be a generator. Furthermore, set $\beta' = [\{*\} \times \{*\} \times S^r] \in H_r(D^{p+1} \times S^q \times S^r) \cong \mathbf{Z}$, which is a generator. Then, by the construction of f_n, we have $i_*(\alpha_1) = \pm 2\beta$ and $i_*(\alpha_2) = \pm(4n+1)\beta$ (for details, see [8]). Therefore, we have

$$\psi_*(\alpha_1) = i_* \circ \varphi_*(\alpha_1) = i_*(\pm\alpha_1 + b\alpha_2) = \pm 2\beta \pm b(4n+1)\beta = (\pm 2 \pm b(4n+1))\beta.$$

On the other hand, by Lemma 4.1, we have $\varphi_*(\alpha_2) = \pm\alpha_2$. Hence, we have

$$\psi_*(\alpha_2) = i_* \circ \varphi_*(\alpha_2) = i_*(\pm\alpha_2) = \pm(4n+1)\beta.$$

Furthermore, we have $j_*(\alpha_1) = 0$, since $\alpha_1 = [S^p \times S^q \times \{*\}]$ and $S^p \times S^q \times \{*\} = \partial(D^{p+1} \times S^q \times \{*\})$. We also have $j_*(\alpha_2) = \pm\beta'$.

Therefore, the matrix representative of

$$\begin{aligned} \Phi = (\psi_*, -j_*) : H_r(S^p \times S^q \times S^r) &\cong \mathbf{Z} \oplus \mathbf{Z} \\ &\to \mathbf{Z} \oplus \mathbf{Z} \cong H_r(C_1) \oplus H_r(D^{p+1} \times S^q \times S^r) \end{aligned}$$

with respect to the bases $\{\alpha_1, \alpha_2\}$ and $\{\beta, \beta'\}$ is of the form

$$\begin{pmatrix} \pm 2 \pm b(4n+1) & \pm(4n+1) \\ 0 & \pm 1 \end{pmatrix}.$$

According to the exact sequence (2), in order to have $H_r(W) = 0$, we must have

$$\pm 2 \pm b(4n+1) = \pm 1$$

for some integer b, which is possible only when $n = 0, -1$. As we have assumed $n \neq 0, -1$, this is impossible. Consequently, the manifold W cannot be diffeomorphic to $S^{p+q+r+1}$.

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Let us now show that C_1 cannot be diffeomorphic to the exterior of N_2 embedded in $S^{p+q+r+1}$, where N_2 is the exterior of a product of two spheres and a disk embedded standardly in $S^{p+q+r+1}$. For this, let us consider the manifold

$$W' = C_1 \cup_{\varphi'} N_2,$$

obtained by attaching N_2 to C_1 by using a diffeomorphism $\varphi' : \partial N_2 \to \partial C_1$. Note that by an argument using the Alexander duality, we have only to consider the case where N_2 is the exterior of $S^p \times S^r$ standardly embedded in $S^{p+q+r+1}$:

$$N_2 = (((D^{p+1} \times S^q) - \operatorname{Int} D^{p+q+1}) \times S^r) \cup (S^{p+q} \times D^{r+1}) \simeq S^{p+q} \vee S^{q+r} \vee S^q.$$

Let us consider the Mayer–Vietoris exact sequence:

$$H_r(S^p \times S^q \times S^r) \xrightarrow{\Phi=(\psi'_*, -j'_*)} H_r(C_1) \oplus H_r(N_2) \longrightarrow H_r(W'),$$

where $j' : S^p \times S^q \times S^r = \partial N_2 \to N_2$ is the inclusion map and $\psi' : S^p \times S^q \times S^r \to C_1$ is given by the composition

$$\psi' : S^p \times S^q \times S^r = \partial N_2 \xrightarrow{\varphi'} S^p \times S^q \times S^r = \partial C_1 \xrightarrow{i} C_1.$$

By an argument similar to the above, we can show that a matrix representative of Φ is of the form

$$\begin{pmatrix} \pm 2 \pm b(4n+1) & \pm(4n+1) \\ \pm 1 & 0 \end{pmatrix}$$

for some integer b. Therefore, we have $H_r(W') = 0$ only if $\pm(4n+1) = \pm 1$, which is possible only if $n = 0$. Since we have assumed $n \neq 0$, this shows that W' cannot be diffeomorphic to $S^{p+q+r+1}$.

Consequently, the embedding f_n, $n \neq 0, -1$, does not satisfy the Fox property. This completes the proof. ▮

Remark 5. 1. In [8, §9], we have used the matrix

$$\kappa_n = \begin{pmatrix} 4n+1 & 2n \\ 2 & 1 \end{pmatrix}$$

for constructing exotic embeddings f_n. Analyzing the proof, we see that we could as well use the matrix

$$\rho_n = \begin{pmatrix} 4n+1 & 8n \\ 2n & 4n-1 \end{pmatrix}$$

for constructing exotic embeddings. Let the resulting embedding be denoted by $g_n : S^p \times S^q \times S^r \to S^{p+q+r+1}$. Then we can show that for g_n, $n = 0, -1$, the closures C_1 and C_2 of the two components of $S^{p+q+r+1} \smallsetminus g_n(S^p \times S^q \times S^r)$ both violate the Fox property. In other words, we can show that none of $C_1 \cup (D^{p+1} \times S^q \times S^r)$, $C_1 \cup N_2$, $C_2 \cup (S^p \times D^{q+1} \times S^r)$

or $C_2 \cup N_2$ is diffeomorphic to $S^{p+q+r+1}$, where N_2' is the exterior of $S^q \times S^r$ standardly embedded in $S^{p+q+r+1}$.

Remark 5. 2. In [9], "exotic" embeddings of $S^1 \times S^q \times S^r$ into S^{q+r+2} have been constructed: the closure of none of the two components of the complement is homotopy equivalent to the product of two spheres. However, they DO satisfy the Fox property, since the closure of one of the two components of the complement is diffeomorphic to the closure of one of the two components of the complement of the standard embedding. So, in a sense, the embeddings constructed for $p = 1$ in [9] are weakly exotic. We do not know if there exists an embedding of $S^1 \times S^q \times S^r$ into S^{q+r+2} which do not satisfy the Fox property when $r \neq q + 1$ or $r = q + 1$ is even.

Conjecture 5.1. Suppose $1 \leq q \leq r$, and $r \neq q+1$ or $r = q+1$ is even. Then embeddings of $S^1 \times S^q \times S^r$ into $S^{p+q+r+1}$ have the Fox property.

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